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On the Statistics of Random Pulse Processes*

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Statistics are obtained for pulse trains in which the pulse shapes as well as the time base are random. The general expression derived for the mean and spectral density of the pulse train require neither independence of intervals between time base points nor independence of the pulses. The spectral density appears as an infinite series that can be summed to closed form in many applications (e.g., pulse duration modulation with skipped and jittered samples). If the time base is a Poisson point process and the pulse shapes are independent, stronger results become available; we are then able to calculate joint characteristic functions for the pulse process, thus providing a more complete statistical description. Examples are given, illustrating use of the above results for pulse duration modulation (with arbitrary pulse shapes) and telephone traffic.

INTRODUCTION

In various applications involving pulse trains, both the pulse shape and the time base are random in nature. As one example, consider pulse duration modulation (PDM) of a random signal with irregular sampling times caused by jitter and the random loss of pulses. A second example concerns disturbances in a receiver due to an electrical storm; the times and effects of lightning bolts are each random. There are also phenomena not ordinarily

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regarded as having random pulse shapes and time base, but which may be interpreted as such a pulse train. An example of this type is the number of telephone lines in use when the length of calls as well as their origination times are random.

In this paper, we show how some of the statistics of pulse trains with random pulse shapes and random time base may be calculated. Two techniques are discussed. The first of these results in general expressions for the means and spectra of such pulse trains; these expressions are valid also for correlated pulses and time bases with intervals between pulses that may be neither independent nor identically distributed. The other technique yields even more information, namely the first and second order probability densities for the pulse process, but under the more restrictive condition that the time base is a Poisson point process.[1]

The second order statistics of the impulse train

$$s(t) = \sum_{-\infty}^{\infty} \alpha_n \delta(t - t_n) \tag{1.1}$$

were considered by the authors in an earlier paper (1968). The time base $\{t_n\}$ was assumed to be a stationary point process [Beutler and Leneman (1966a) and (1966b)], while $\{\alpha_n\}$ was taken to be a wide stationary discrete parameter process with specified covariance. From the second order properties of $s(t)$ it is easy to deduce similar results for $\sum_{-\infty}^{\infty} \alpha_n h(t - t_n)$, thus treating pulses of fixed shape with random amplitude and time base; the pulse train $\sum_{-\infty}^{\infty} \alpha_n h(t - t_n)$ merely represents the $s(t)$ of (1.1) after its passage through a linear time-invariant filter with response function $h(\cdot)$. However, this model is incapable of generalization to pulse trains in which the shapes of the respective pulses may vary also. For that case we must analyze the statistics of

$$y(t) = \sum_{-\infty}^{\infty} h_n(t - t_n), \tag{1.2}$$

in which $\{t_n\}$ is again the random time base (a stationary point process) and $h_n(\cdot)$ is the n-th pulse. A typical pulse train of this type is shown in Fig. 1. It is seen that $\sum_{-\infty}^{\infty} \alpha_n h(t - t_n)$ becomes a special case of $y(t)$ when we take $h_n(t) = \alpha_n h(t)$ in (1.2)

In the next section, we shall find universal formulas for the mean and spectral density of the $y(t)$ of (1.2) under the following hypotheses. It is

[1] For this restricted case, the spectrum (only) is calculated in Mazzetti (1964).

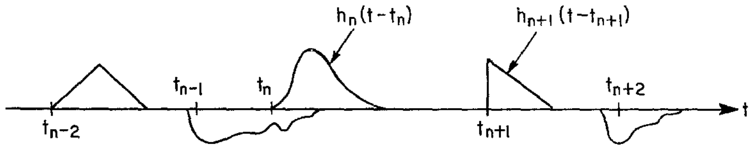

FIG. 1. A realization of a train of randomly shaped pulses.

supposed that $\{t_n\}$ is an ergodic stationary point process, and that the random functions $\{h_n(t)\}$ are independent of $\{t_n\}$ with means

$$E[h_n(t)] = m(t) \tag{1.3}$$

that do not depend on n. The transform covariance for $\{h_n(t)\}$ is

$$\Gamma_n^*(\omega) = E[H_{m+n}(\omega)\,\overline{H_m(\omega)}], \tag{1.4}$$

where the overline denotes complex conjugacy and $H_k(\omega)$ is the Fourier transform of $h_k(t)$. The point of (1.4) is that the indicated expectation satisfies the weak stationarity condition that it does not depend on m, but only on n.

The class of stationary point processes for which the spectral density expression is obtained embraces most $\{t_n\}$ considered to be realistic time bases. Poisson point processes are included, as well as uniformly timed sampling that has been subjected to random jitter and/or random deletion (skipping) of pulses. Other possible variations on the stationary point processes include systematic skipping of some originally existent points, and points at intervals of varying lengths following a planned or random sequence. Stationary point processes are analyzed in Beutler and Leneman, (1966a) and (1966b), and the statistics of point processes required for this paper are calculated therein.

FIRST AND SECOND ORDER STATISTICS

The mean and spectral density for the $y(t)$ of (1.2) can be derived in terms of simple closed form expressions. The spectral density appears as an infinite series in terms of Γ_n^* and f_n^*, where the latter is the generating function of the distance separating n successive points, i.e., of $t_{k+n} - t_k$. As will be seen from the examples that follow the derivation, the infinite series representing the spectral density of the pulse train can be summed to an analytical expression for many models of interest in applications. The arguments used will be heuristic, but the validity of the final results can be established either by alternative methods (in the time domain) or by justifying the various types in the calculation.

The mean $y(t)$ is computed by taking expectations of (1.2) successively on $\{h_n(t)\}$ and $\{t_n\}$. Because of the independence of these two random processes

$$E[y(t)] = E\left[\sum_{-\infty}^{\infty} h_n(t-t_n)\right] = E\left[\sum_{-\infty}^{\infty} m(t-t_n)\right]. \tag{2.1}$$

Now if we let $s(t) = \sum \delta(t-t_n)$ it is possible to write

$$m(t-t_n) = \int_{-\infty}^{\infty} m(t-\tau)\, s(\tau)\, d\tau \tag{2.2}$$

and hence

$$E[y(t)] = E\left[\int_{-\infty}^{\infty} m(t-\tau)\, s(\tau)\, d\tau\right]. \tag{2.3}$$

From an interchange of expectation and integration in (2.3) one then obtains

$$E[y(t)] = \beta \int_{-\infty}^{\infty} m(u)\, du. \tag{2.4}$$

In the formula (2.4) $E[s(t)] = \beta$ represents the average number of pulses per unit time. The expectation of $s(t)$ has been derived in Beutler and Leneman (1968), and values of the mean number of points per unit time β are available for a wide variety of stationary point processes in Beutler and Leneman (1966b).

The spectral density S_y can be adduced by first computing the correlation $E[y(t+\tau)\,y(t)]$ and then taking the Fourier transform of the expectation. This method has been used for the $s(t)$ of (1.1) in Beutler and Leneman (1968), but becomes inconvenient when an attempt is made to generalize the same technique to $y(t)$. The same result may be attained more simply by utilizing the direct method [Davenport and Root (1958), p. 108], which means that we use the formula

$$S_y(\omega) = \lim_{T\to\infty} E\left(\frac{1}{T}\left|\int_0^T y(t)\, e^{-i\omega t}\, dt\right|^2\right). \tag{2.5}$$

The indicated limit is best taken by letting $T = \beta^{-1}N$, with N then tending toward infinity through the integers to attain the desired limit. Such T is convenient because it allows us to suppose that for large N approximately N pulses fall into the interval $(0, \beta^{-1}N]$. This is indeed true if $\{t_n\}$ is an ergodic stationary point process as defined in Section 3.6 of Beutler and Leneman (1966a); i.e., if the average number of points in $(0, T]$ tends toward β for almost every realization as $T \to \infty$.[2] We also assume that the interval $(0, \beta^{-1}N]$ is sufficiently large so that the contributions of pulses intruding from outside the interval and tails of pulses lost by restricting the interval are negligible compared with N. Then over $(0, \beta^{-1}N]$ $y(t)$ is approximated $\sum_k^{k+N-1} h_n(t - t_n)$, where t_k is the first point past the origin. Insofar as the statistics of the sum are concerned, k is irrelevant because of the stationarity of $\{t_n\}$ and the assumption (1.4) which permits us to translate the indices of H_n. Accordingly, we take $k = 1$ and obtain asymptotically

$$\int_0^{\beta^{-1}N} y(t)\, e^{-i\omega t}\, dt = \sum_1^N H_n(\omega)\, e^{-i\omega t_n}. \tag{2.6}$$

This expression is substituted into (2.5). On multiplying it by its conjugate and taking its expectation we find that

$$E\left|\int_0^{\beta^{-1}N} y(t)\, e^{-i\omega t}\, dt\right|^2 = \sum_{m,n=1}^{N} \Gamma^*_{n-m}(\omega)\, E\{\exp[-i\omega(t_n - t_m)]\}. \tag{2.7}$$

The right hand expectation represents (for $n \geqslant m$) the characteristic function for $n - m$ successive intervals between points. If the probability density function for the length of k successive intervals[3] is called f_k, we may define $f_k{}^*$ as the corresponding characteristic function and write $f_k{}^*$ as

$$f_k{}^*(i\omega) = \int_0^\infty e^{-i\omega x} f_k(x)\, dx = E\{\exp[-i\omega(t_{j+k} - t_j)]\}. \tag{2.8}$$

For many stationary point processes of interest in applications, the $f_k{}^*$ have been calculated in Beutler and Leneman (1966b). The negative indices on the right side of (2.8) produce expectations of the complex conjugate for each term, so that it is consistent to define $f^*_{-n}(i\omega) = \overline{f_n{}^*(i\omega)}$ and $\Gamma^*_{-n}(\omega) = \overline{\Gamma_n{}^*(\omega)}$. With this convention the left side of (2.7) becomes

$$E\left|\int_0^{\beta^{-1}N} y(t)\, e^{-i\omega t}\, dt\right|^2 = \sum_{m,n=1}^{N} \Gamma^*_{n-m}(\omega)\, f^*_{n-m}(i\omega). \tag{2.9}$$

[2] Neither this assumption nor the one following are needed to effect the final result, but they greatly facilitate its derivation. The alternative approach through a time domain argument also yields the same final expression for the spectral density, and this provides an additional check on its validity.

[3] There exists stationary point processes for which different sets of lengths of n successive intervals do not all have the same distribution functions, but these do not appear to be of physical interest. See Beutler and Leneman (1966a), Section 4.2.

The calculation of the spectral density S_y is completed by dividing the double sum in (2.9) by $\beta^{-1}N$ and taking the limit on N, in conformance with the expression (2.5) for S_y. Thus

$$S_y(\omega) = \beta \lim_{N\to\infty} \left[\frac{1}{N} \sum_{m,n=1}^{N} \Gamma_{n-m}^{*}(\omega) f_{n-m}^{*}(i\omega) \right]. \tag{2.10}$$

The double sum is replaced by a single sum and the denominator moved inside the summation, viz.,

$$S_y(\omega) = \beta \lim_{N\to\infty} \left[\sum_{-N}^{+N} \left[1 - \frac{|k|}{N}\right] \Gamma_k{}^*(\omega) f_k{}^*(i\omega) \right]. \tag{2.11}$$

The desired spectral density is therefore

$$S_y(\omega) = \beta \sum_{-\infty}^{\infty} \Gamma_k{}^*(\omega) f_k{}^*(i\omega), \tag{2.12}$$

which can also be written

$$S_y(\omega) = \beta \left\{ \Gamma_0{}^*(\omega) + 2 \operatorname{Re}\left[\sum_{1}^{\infty} \Gamma_k{}^*(\omega) f_k{}^*(i\omega) \right] \right\}. \tag{2.13}$$

The first formula (2.12) for S_y is certainly more elegant than the second, but the latter has proved more useful in the actual computation of spectral densities.

The expression (2.12) for the spectral density S_y generalizes the S_s for (1.1) as obtained in Eq. (2.27) of Beutler and Leneman (1968). To see this, we observe that $y(t) = \sum h_n(t - t_n)$ specializes to $s(t) = \sum \alpha_n \delta(t - t_n)$ if we let $h_n(t) = \alpha_n \delta(t)$. Then $H_k(\omega) = \alpha_k$ and so $\Gamma_n{}^*(\omega) = E[\alpha_{m+n}\alpha_m] = \rho(n)$ [in the notation of Beutler and Leneman (1968)]; this yields Eq. (2.27) of Beutler and Leneman (1968) directly.

As a first example, consider pulse duration modulation (PDM) with rectangular unit height pulses whose pulse width a_n is given by a signal $x(t)$ sampled at time t_n, i.e., $a_n = x(t_n)$. Then $y(t) = \sum h_n(t - t_n)$ with

$$h_n(t) = \begin{cases} 1 & 0 \leqslant t < a_n \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

To find $\Gamma_k{}^*$ we first take the Fourier transform of h_n, viz.,

$$H_n(\omega) = \int_0^{a_n} e^{-i\omega t}\, dt = \frac{1 - e^{-i\omega a_n}}{i\omega}. \tag{2.15}$$

We assume that the samples a_n are pairwise independent identically distributed random variables so that we have

$$\Gamma_k^*(\omega) = | E[H_n(\omega)]|^2, \qquad k \neq 0, \tag{2.16}$$

and also

$$\Gamma_0^*(\omega) = E[| H_n(\omega)|^2]. \tag{2.17}$$

Accordingly,

$$\Gamma_k^*(\omega) = \frac{1}{\omega^2} [| 1 - \phi(\omega)|^2] \tag{2.18}$$

for nonzero k, and

$$\Gamma_0^*(\omega) = \frac{2}{\omega^2} \{1 - \mathrm{Re}[\phi(\omega)]\}. \tag{2.19}$$

In the preceding two equations, ϕ denotes the characteristic function of any a_n,

$$\phi(\omega) = \int_0^\infty e^{i\omega a} dG(a). \tag{2.20}$$

It only remains to substitute the Γ_k^* into the appropriate spectral density relation (2.12) or (2.13). Since $f_0^*(i\omega) = 1$, the first of these formulas yields for the spectral density of PDM with rectangular pulses:

$$S_y(\omega) = \frac{\beta}{\omega^2} \left\{ 1 - | \phi(\omega)|^2 + | 1 - \phi(\omega)|^2 \left[\sum_{-\infty}^{\infty} f_k^*(\omega) \right] \right\}. \tag{2.21}$$

The alternative form of S_y which features a one-sided infinite series in the f_k^* is

$$S_y(\omega) = \frac{2\beta}{\omega^2} \left\{ 1 - \mathrm{Re}[\phi(\omega)] + | 1 - \phi(\omega)|^2 \,\mathrm{Re} \left[\sum_{1}^{\infty} f_k^*(\omega) \right] \right\}. \tag{2.22}$$

Although S_y appears to have a singularity at $\omega = 0$ due to the $1/\omega^2$ term, such is not the case because of the behavior of ϕ near the origin when a_n has finite variance. However, $\sum f_k^*(0)$ diverges, reflecting the existence of a delta function at the origin. The intensity of this delta function is the square of the mean of $y(t)$, and this mean is seen to be

$$E[y(t)] = \beta \int_{-\infty}^{\infty} m(t)\, dt = \beta \left\{ E \left[\int_{-\infty}^{\infty} h_n(t)\, dt \right] \right\} = \beta \{E(a_n)\} \tag{2.23}$$

by (1.3) followed by an interchange of integration and expectation. We have also used $E[h_n(t)] = m(t)$ (by definition), and the fact that the integral of $h_n(t)$ is merely a_n.

We continue this example with a generalization to pulses of arbitrary shape. Now

$$h_n(t) = h\left(\frac{t}{a_n}\right), \qquad a_n > 0, \tag{2.24}$$

where $h(\cdot)$ has a Fourier transform, is differentiable (we admit delta function derivatives), and $h(-\infty) = h(\infty) = 0$. Then

$$H_n(\omega) = \int_{-\infty}^{\infty} h\left(\frac{t}{a_n}\right) e^{-i\omega t}\, dt = a_n \int_{-\infty}^{\infty} h(u)\, e^{-ia_n\omega u}\, du, \tag{2.25}$$

which we may integrate by parts to obtain

$$H_n(\omega) = \frac{1}{i\omega} \int_{-\infty}^{\infty} h'(u)\, e^{-ia_n\omega u}\, du. \tag{2.26}$$

The independence of the a_n permits us to find $\Gamma_k{}^*$ easily from (2.16) and (2.17); for nonzero k,

$$\Gamma_k{}^*(\omega) = \frac{1}{\omega^2} \left| \int_{-\infty}^{\infty} h'(u)\, \phi(u\omega)\, du \right|^2; \tag{2.27}$$

and for $k = 0$,

$$\Gamma_0{}^*(\omega) = \frac{1}{\omega^2} \iint_{-\infty}^{\infty} h'(u)\, h'(v)\, \phi(\omega[v - u])\, du dv. \tag{2.28}$$

As before, ϕ is the characteristic function of any a_n, and the prime denotes differentiation of a function with respect to its argument. In some instances it may be more convenient to once more integrate by parts; this rephrases (2.27) in terms of h and ϕ', and (2.28) as a function of h and ϕ''.

The above expressions for $\Gamma_k{}^*$ are somewhat simplified by the change of variable $u - v = s$, $v = t$. If we then put

$$\Lambda(t, \omega) = \frac{1}{\omega} \int_{-\infty}^{\infty} h'(s + t)\, \phi(\omega s)\, ds, \tag{2.29}$$

we shall have

$$\Gamma_0{}^*(\omega) = \frac{1}{\omega} \int_{-\infty}^{\infty} h'(t)\, \Lambda(t, \omega)\, dt \tag{2.30}$$

and

$$\Gamma_k{}^*(\omega) = |\, \Lambda(0, \omega)|^2, \qquad k \neq 0. \tag{2.31}$$

The reader can easily verify that these two formulas agree with the Γ_k^* previously obtained for the rectangular pulse, for which $h'(u) = \delta(u) - \delta(u - 1)$. Moreover, substitution into the formulas (2.12) or (2.13) for the spectral density is routine and will not be carried out here. As in the case of rectangular pulses, the spectral density is obtained in closed form whenever $\sum f_n^*(i\omega)$ can be summed.

The second example is concerned with the theory of telephone traffic, and more particularly with the number of lines in use at any specified time t.[4] We assume that the lengths of calls are independent random variables with identical probability distributions $G(\cdot)$. The n-th call is initiated at time t_n ($n = 0, \pm 1, \pm 2,\ldots$), and $\{t_n\}$ constitutes a Poisson point process (over the entire real line) for which β is the average number of calls initiated in any unit period. Finally, it is supposed that there are a sufficient number of lines to assure that every potential caller can in fact initiate a call whenever desired.

In the example, we determine the mean and autocorrelation of the number of lines in use. This number is modelled by $y(t) = \sum h_n(t - t_n)$, where $h_n(t)$ is taken to be the rectangular function defined by (2.14), with a_n the duration of the n-th call. Hence the general results we have already derived are applicable to this problem. For instance, the mean number of lines in use is

$$E[y(t)] = \beta[E(a_n)] = \beta \int_0^\infty x\, dG(x). \tag{2.32}$$

The autocorrelation function is in theory also available, since we may substitute for the f_k^* in the spectral density formula (2.12), and then take the inverse Fourier transformation for the autocorrelation function. However, we shall find it more instructive to start with the general autocorrelation expression, and to evaluate it directly.

We assert that the autocorrelation $E[y(t + \tau)\, y(t)]$ is

$$R_y(\tau) = \beta \left\{ \int_{-\infty}^{\infty} \Gamma_0(\tau + u, u)\, du + \sum_{k=1}^{\infty} \iint_{-\infty}^{\infty} \Gamma_k(\tau + v, u) f_k(| u - v |)\, du\, dv \right\}, \tag{2.33}$$

in which

$$\Gamma_k(u, v) = E[h_{m+k}(u)\, \overline{h_m(v)}], \tag{2.34}$$

[4] The same model has been used to represent the number of units being serviced by a system with an infinite number of servers, and the number of fibers in a cord consisting of fibers of random lengths. Consequently, the results of this example have been obtained earlier by alternate methods; see Rao (1966) and Haji and Newell (1970).

and f_k represents the density function for k successive intervals between points. Verification of the correctness of (2.33) is accomplished by taking the Fourier transform on τ; the result should be the spectral density S_y as given by (2.12). The double integral in (2.33) may be written

$$\iint \Gamma_k(v, u) f_k(| u - v + \tau |) \, du \, dv,$$

so that its transform on τ gives rise to the integral

$$\begin{aligned} &\int_{-\infty}^{\infty} f_k(| u - v + \tau |) \, e^{-i\omega\tau} \, d\tau \\ &\qquad = e^{i\omega(u-v)} \left[\int_0^{\infty} f_k(x) \, e^{-i\omega x} \, dx + \int_0^{\infty} f_k(x) \, e^{+i\omega x} \, dx \right]. \end{aligned} \tag{2.35}$$

It will be recalled that f_k is the probability density function for k successive intervals, and that $f_k{}^*$ is the corresponding characteristic function according to (2.8). Hence the right side of (2.35) gives rise to $f_k{}^*$ as well as $\overline{f_k{}^*} = f_{-k}^*$. To complete the verification of (2.33) we obbserve that Γ_k and $\Gamma_k{}^*$ [as defined by (2.34) and (1.4), respectively] are connected by the transform relation

$$\Gamma_k{}^*(\omega) = \iint_{-\infty}^{\infty} \Gamma_k(u, v) \, e^{-i\omega(u-v)} \, du \, dv. \tag{2.36}$$

We require that $\Gamma_k(u, v) = \Gamma_k(v, u)$; since, in any case, $\Gamma_k(u, v) = \overline{\Gamma_{-k}(v, u)}$, we have $\Gamma_k{}^*(\omega) = \Gamma_{-k}^*(\omega)$. It follows from the latter and (2.35) that the k index term in (2.33) transforms into the $-k$ and $+k$ terms of (2.12). The single integral term in (2.33) becomes the zero index term of (2.12).

With the aid of (2.33) we now evaluate the correlation R_y for the number of telephone lines in use at any given two times. Because of the symmetry of $R_y(\tau)$, it suffices to consider $\tau \geqslant 0$. Then for the $h_n(\cdot)$ defined by (2.14),

$$\Gamma_0(u + \tau, u) = E[h_m(u + \tau) \, h_m(u)] = E[h_m(u + \tau)], \tag{2.37}$$

if $u \geqslant 0$, and $\Gamma_0(u + \tau, u) = 0$ otherwise. This means that Γ_0 is specified for $u \geqslant 0$ by the distribution function G of the length of any call since

$$\begin{aligned} E[h_m(u + \tau)] = P[(u + \tau) \leqslant a_m] &= 1 - P[a_m < (u + \tau)] \\ &= 1 - G(u + \tau). \end{aligned} \tag{2.38}$$

These considerations, together with the change of variable $u + \tau = x$, permit us to write the first integral in (2.33) as

$$\int_{-\infty}^{\infty} \Gamma_0(\tau + u, u) \, du = \int_{\tau}^{\infty} [1 - G(x)] \, dx, \qquad \tau \geqslant 0. \tag{2.39}$$

In the second integral of the autocorrelation (2.33), we have by the assumed independence of the $\{h_n(t)\}$

$$\Gamma_k(x, y) = E[h_n(x)]\, E[h_n(y)], \qquad k \neq 0. \tag{2.40}$$

Thus Γ_k fails to depend on its index, and the summation may be applied only to the f_k. But for a Poisson point process $\{t_n\}$ it follows that whenever $u \neq v$

$$\sum_{k=1}^{\infty} f_k(|\, u - v\,|) = \beta, \tag{2.41}$$

as shown in Beutler and Leneman (1966a), Eq. (4.1.2). Accordingly, the summation of the double integrals in (2.33) becomes

$$\beta \iint_{-\infty}^{\infty} \Gamma_k(\tau + v, u)\, du\, dv = \beta \left(\int_{-\infty}^{\infty} E[h_n(u)]\, du \right)^2 = \beta\{E(a_n)\}^2, \tag{2.42}$$

in which the right side is attained by interchanging expectation and integration, and noting that the integral of $h_n(t)$ is a_n. The values of the integrals (2.39) and (2.42) are substituted into (2.33), the expression for R_y. When use is made of the symmetry of the autocorrelation, the final result is seen to be

$$R_y(\tau) = \beta \int_{|\tau|}^{\infty} [1 - G(x)]\, dx + \{\beta[E(a_n)]\}^2. \tag{2.43}$$

In the above example, the special role played by the Poisson point process and the assumed independence of telephone call durations made it possible to evaluate R_y explicitly. In the absence of these specialized conditions it is generally easier to calculate the spectral density S_y, in large part because the $f_k{}^*$ usually constitute terms of a power series.

Statistics for Poisson Point Processes

When the underlying time base of a pulse process is a Poisson point process, the numbers of pulses in disjoint intervals are mutually independent, and the number of pulses originating in a given interval depends only on the length of the interval. These special properties [which almost serve to specify the Poisson point process (see Beutler and Leneman (1966a), Section 4)] can be exploited to obtain more complete statistics than the first and second order moments found in the preceding section. Indeed, when $\{t_n\}$ is a Poisson point process and the $h_n(t)$ are mutually independent random functions, we can determine the characteristic function for the pulse process $y(t) = \Sigma h_n(t - t_n)$; once this characteristic function is known, any statistic of $y(t)$ is (at least in theory) available. Joint statistics depend on multivariate characteristic functions, and these are complicated indeed. Nevertheless, we will indicate how these are obtained, and give an explicit formula for the joint characteristic function of $y(t_1)$, $y(t_2)$. The utility of these results will then be demonstrated by applying them to the number of telephone lines in use, thereby generalizing the last example in the preceding section.

We consider again the $y(t)$ of the form

$$y(t) = \sum_{-\infty}^{\infty} h_n(t - t_n), \tag{3.1}$$

with the specializing assumptions that $\{t_n\}$ is a Poisson stationary point process and that the $h_n(t)$ are mutually independent. For this $y(t)$ we seek its characteristic function $E\{\exp[i\lambda y(t)]\}$. The indicated expectation is taken in two stages. First, the expectation with respect to $\{h_n(t)\}$ is denoted by

$$E_1[e^{i\lambda y(t)}] = E_1 \left\{ \exp \left[i\lambda \sum_{-\infty}^{\infty} h_n(t - t_n) \right] \right\}. \tag{3.2}$$

Because of the hypothesized independence of the h_n, this expectation can be written

$$E_1[e^{i\lambda y(t)}] = \prod_{-\infty}^{\infty} \Psi(t - t_n, \lambda), \tag{3.3}$$

in which Ψ is defined

$$\Psi(u, \lambda) = E_1\{\exp[i\lambda h_n(u)]\}; \tag{3.4}$$

since the h_n are identically distributed, Ψ does not depend on the index n. We define further

$$g(u, \lambda) = \frac{1}{i\lambda} \log \Psi(u, \lambda) \tag{3.5}$$

so that the expectation (3.3) becomes

$$E_1[e^{i\lambda y(t)}] = \exp \left[i\lambda \sum_{-\infty}^{\infty} g(t - t_n, \lambda) \right]. \tag{3.6}$$

Since $E[e^{i\lambda y(t)}] = E_2 E_1[e^{i\lambda y(t)}]$, where E_2 is the expectation with respect to $\{t_n\}$, completing the calculation of the characteristic function requires that we apply E_2 to the expectation (3.6) immediately above. To facilitate this operation, we observe that the exponential

$$\sum_{-\infty}^{\infty} g(t - t_n, \lambda) = \int_{-\infty}^{\infty} g(t - u, \lambda)\, dN(u), \tag{3.7}$$

where $N(t)$ is the stationary increment process with $N(0) = 0$ and a unit jump at each t_n [compare Beutler and Leneman (1968)]. Since $N(t)$ may be assumed continuous from the right, the integral in (3.7) is of the Riemann–Stieltjes type and may be regarded as the limit (as the intervals between variates tend toward zero) of the sums $\sum_k g(u_k, \lambda)[N(u_{k+1}) - N(u_k)]$, where $\{u_n\}$ is an increasing (nonrandom) sequence on the real line. Now since $\{t_n\}$ is a Poisson point process $\{N(u_{k+1}) - N(u_k)\}$ constitutes a mutually independent set of random variables and so

$$\begin{aligned} &E_2\left(\exp\left\{\sum_k i\lambda g(t - u_k, \lambda)[N(u_{k+1}) - N(u_k)]\right\}\right) \\ &\quad = \prod_k E_2\{\exp\left(i\lambda g(t - u_k, \lambda)[N(u_{k+1}) - N(u_k)]\right)\}. \end{aligned} \tag{3.8}$$

Moreover, $N(t)$ is a stationary increment process, which means that $N(v) - N(u)$ and $N(v - u)$ have the same statistics. Hence

$$E[e^{i\lambda y(t)}] = \lim \prod_k E_2\{\exp[i\lambda g(t - u_k, \lambda)\, N(u_{k+1} - u_k)]\}. \tag{3.9}$$

The right side expectation is easily evaluated because $N(t)$ is known to be a simple Poisson process whose characteristic function [see Parzen (1962), pp. 13 and 30] is

$$E[e^{i\gamma N(t)}] = \exp\{\beta t(e^{i\gamma} - 1)\}. \tag{3.10}$$

Thus, with the role of γ being played by $\lambda g(t - u_k, \lambda)$ and that of t by $(u_{k+1} - u_k)$ the right side expectation in (3.9) becomes

$$\begin{aligned} &\prod_k \exp\left((\beta(u_{k+1} - u_k)\{\exp[i\lambda g(t - u_k, \lambda)] - 1\}\right) \\ &\quad = \exp\left(\beta\left[\sum \{\exp[i\lambda g(t - u_k, \lambda)] - 1\}(u_{k+1} - u_k)\right]\right) \\ &\quad \to \exp\left\{\beta \int_{-\infty}^{\infty} [\Psi(t - u, \lambda) - 1]\, du\right\} \\ &\quad = \exp\left\{\beta \int_{-\infty}^{\infty} [\Psi(u, \lambda) - 1]\, du\right\}. \end{aligned} \tag{3.11}$$

The right side of (3.11) above is the limit of the indicated Riemann sums, and use has been made of the relation between $g(u, \lambda)$ and $\Psi(u, \lambda)$ [see (3.5)] in arriving at the final form. The characteristic function of $y(t)$, according to (3.9) and (3.11), is therefore[5]

$$E[e^{i\lambda y(t)}] = \exp\left\{\beta \int_{-\infty}^{\infty} [\Psi(u, \lambda) - 1]\, du\right\}. \tag{3.12}$$

An extension of the preceding arguments is used to derive higher-order characteristic functions such as

$$E[e^{i\lambda_1 y(t)+i\lambda_2 y(t+\tau)}] = \exp\left\{\beta \int_{-\infty}^{\infty} [\Psi(u, u+\tau, \lambda_1, \lambda_2) - 1]\, du\right\}, \tag{3.13}$$

in which

$$\Psi(x, y, \lambda_1, \lambda_2) = E_1\{\exp[i\lambda_1 h_n(x) + i\lambda_2 h_n(y)]\} \tag{3.14}$$

by definition. From the joint characteristic function the autocorrelation can be obtained, and similar methods are applicable to higher joint moments. In view of the apparent difficulty of dealing with $y(t) = \sum h_n(t - t_n)$ [a very general form of pulse train], the characteristic function forms (3.12) and (3.13) are remarkably simple.

As an example, we continue the telephone line usage problem of the preceding section, with the same assumptions as before. For this process, we obtained the mean and autocorrelation function (2.32) and (2.43), respectively. The techniques of the present section enable us to go further. In particular, we shall derive first- and second-order characteristic functions, and hence in principle *all* first and second-order statistics.

Although the second-order characteristic function (3.13) clearly includes the first order one (3.12) by taking $\lambda_2 = 0$, we shall find it instructive to compute the two separately. To find $E[e^{i\lambda y(t)}]$, we first obtain Ψ as defined by (3.4). Since $h_n(t)$ is specified by (2.14) this random function can take on only the values zero or unity. Now if $h_n(t) = 0$, $\exp[i\lambda h_n(t)] = 1$, and this is the case for $t < 0$ and $t \geqslant a_n$; thus for $t \geqslant 0$, there is the probability $G(t)$ that

[5] This result is not entirely new. For a Poisson renewal process $\{t_n\}$ specified on the positive half-axis and with $y(0) = 0$, the equivalent result for arbitrary $t \geqslant 0$ is found in Parzen (1962); if $t \to \infty$, this result agrees with our (3.12).

$h_n(t) = 0$ and $[1 - G(t)]$ that $h_n(t) = 1$. Here G is again the probability distribution function of any a_n. These considerations imply that

$$\Psi(u, \lambda) = \begin{cases} 1 & u < 0 \\ e^{i\lambda}[1 - G(u)] + G(u) & u \geqslant 0, \end{cases} \tag{3.15}$$

$$E[e^{i\lambda y(t)}] = \exp\left\{\beta(e^{i\lambda} - 1) \int_0^\infty [1 - G(u)]\, du\right\}, \tag{3.16}$$

by substituting in (3.12) for Ψ. The form of the characteristic function (3.16) shows that $y(t)$ is Poisson distributed with parameter $\beta \int_0^\infty [1 - G(u)]\, du$. Essentially the same result is obtained when $\{t_n\}$ is a Poisson renewal process over the positive half-axis and $y(0) = 0$ [see Parzen (1962), Example 5E on pp. 147–148], except that the characteristic function asymptotically approaches ours as $t \to \infty$.

Since $E[h_n(u)] = 1 - G(u)$ from (2.38), the characteristic function (3.16) also takes on either of the alternative forms

$$E[e^{i\lambda y(t)}] = \exp\{\beta(e^{i\lambda} - 1)\, E(a_n)\} = \exp\{(e^{i\lambda} - 1)\, E[y(t)]\} \tag{3.17}$$

through an application of the identity (2.23). Therefore, the probability distribution of $y(t)$ depends on $\{t_n\}$ only through the mean number of calls per unit time, and on a_n only through its mean $E(a_n)$.

The bivariate characteristic function $E[e^{i\lambda_1 y(t) + i\lambda_2 y(t+\tau)}]$ is calculated as follows: for arbitrary τ,

$$E[e^{i\lambda_1 y(t) + i\lambda_2 y(t+\tau)}] = E[e^{i\lambda_2 y(t) + i\lambda_1 y(t-\tau)}] \tag{3.18}$$

from considerations of stationarity and symmetry, so it suffices to get this characteristic function for $\tau \geqslant 0$. This means that the $\Psi(x, y, \lambda_1, \lambda_2)$ of (3.14) is needed only for $x \leqslant y$, since the latter argument is always greater than the former in (3.13), where Ψ is used in the expression for $E[e^{i\lambda_1 y(t) + i\lambda_2 y(t+\tau)}]$. There are three cases. If $y < 0$, then $\Psi(x, y, \lambda_1, \lambda_2) = 1$. If $x < 0 \leqslant y$, then $\Psi(x, y, \lambda_1, \lambda_2)$ coincides with $\Psi(y, \lambda_2)$ as specified by (3.15). Finally, for $x \geqslant 0$, we have both $h_n(x)$ and $h_n(y)$ unity with probability $P[y < a_n] = 1 - G(y)$, $h_n(x)$ unity and $h_n(y)$ zero with $P[x < a_n \leqslant y] = G(y) - G(x)$, and both $h_n(x)$ and $h_n(y)$ zero with probability $P[a_n \leqslant x] = G(x)$. Hence for $-\tau \leqslant u < 0$,

$$\Psi(u, u + \tau, \lambda_1, \lambda_2) = e^{i\lambda_2}[1 - G(u + \tau)] + G(u + \tau), \tag{3.19}$$

and for $u \geqslant 0$

$$\Psi(u, u + \tau, \lambda_1, \lambda_2) = e^{i\lambda_1 + i\lambda_2}[1 - G(u + \tau)] + e^{i\lambda_1}[G(u + \tau) - G(u)] + G(u). \tag{3.20}$$

The final step consists of substituting Ψ from (3.19) and (3.20) into the expression (3.13) for the bivariate characteristic function, viz.,

$$E[e^{i\lambda_1 y(t)+i\lambda_2 y(t+\tau)}] = \exp\Big(\beta \int_0^\infty \{e^{i\lambda_1+i\lambda_2}[1 - G(u+\tau)] + e^{i\lambda_1}[G(u+\tau) - G(u)] + [G(u) - 1]\}\, du + \beta(e^{i\lambda_2} - 1) \int_0^\tau [1 - G(u)]\, du\Big). \tag{3.21}$$

Various specializations of this characteristic function (e.g., $\lambda_2 = 0$ or $\tau = 0$) are left to the reader, as is the computation of the autocorrelation or other moments dependent jointly only on $y(t_1)$ and $y(t_2)$.

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