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HEAT TRANSFER IN THE LAMINAR CREEPING FLOW BETWEEN PARALLEL CIRCULAR DISKS WITH ECCENTRIC INLET

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NOMENCLATURE

b,	space between disks;
b_{ni},	coefficients in the expansions of the eigenfunctions;
c,	defined in equation (3);
$\bar{c}$,	$c/(b/2)$;
c_i,	defined in equation (18);
C_p,	specific heat at constant pressure;
e,	eccentricity;
$\bar{e}$,	e/r_i;
E,	Eckert number, $U_0^2/C_p\Delta T$;
h, h_1, h_2,	the scale factors defined in equation (2);
k,	heat conductivity;
k_i, k_e,	values of x_2 at inlet and outlet radii;
N,	a positive integer;
$Nu, \overline{Nu}$,	local and average Nusselt numbers;
Pe,	Péclét number;
$\overline{Pe}$,	$Pe\log_e(1/\gamma)/(k_i - k_e)$;
q,	heat transfer;
r,	radius of the disk;
$\overline{r_e}$,	$r_e/(b/2)$;
$\overline{r_i}$,	$r_i/(b/2)$;
R,	a function of x_2 only, equation (13);
T,	temperature;
U_0,	reference velocity;
v_1, v_2,	velocity components in the x_1- and x_2-directions;
x_1, x_2, x_3,	bipolar coordinate frame;
$x_2', \bar{x}_2, \bar{x}_3$;	defined in equations (10) and (11);
$\bar{Y}_i$,	defined in equation (19);
Z,	a function of $\bar{x}_3$ only, equation (13).

Greek letters

μ,	dynamic viscosity;
ρ,	density;
γ,	r_i/r_e;
ϕ,	$\bar{e}\gamma/(1-\gamma)$;
λ_i, λ,	eigenvalues;
θ,	$(T_w - T)/(T_w - T_0)$.

Subscripts i, e and w refer to conditions at inlet, exit and wall respectively.

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ANALYSIS

THE ENRGY equation for the laminar, creeping flow of incompressible fluids between parallel circular disks with eccentric inlet (see Fig. 1) can be written, in bipolar coordinates, as

$$\rho C_p\left(\frac{v_1}{h_1}\frac{\partial T}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial T}{\partial x_2}\right) = k\frac{\partial^2 T}{\partial x_3^2} \tag{1}$$

subject to the boundary conditions:

$$T(x_1, x_2, b/2) = T(x_1, x_2, -b/2) = T_w$$
$$T(x_1, k_i, x_3) = T_0.$$

The viscous dissipation terms are neglected, which is justified for small Eckert numbers, i.e. $E(=U_m^2/C_p\Delta T) \ll 1$.

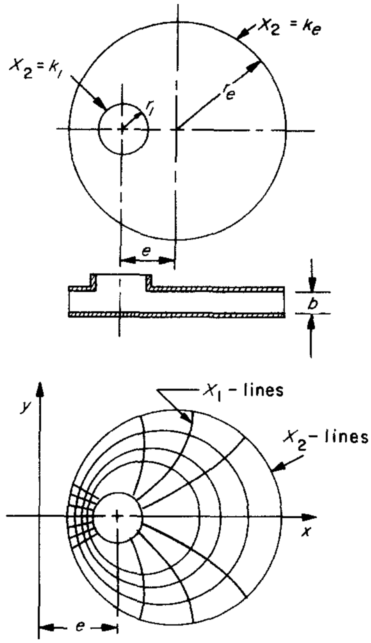

FIG. 1. Bipolar coordinates and the disks.

The conduction in x_1- and x_2-directions are neglected in comparison with conduction in the x_3-direction. Also, the entrance effect can be neglected due to the extremely small clearance between the disks in such problems. The scale factors, h_1 and h_2, of the bipolar coordinates are defined by:

$$h_1 = h_2 = h = \frac{C}{\cosh x_2 - \cos x_1}. \tag{2}$$

The coordinate plane, $x_3 = 0$, is parallel to and midway between the disk surfaces. The constant C is related to the inner and outer radii, r_i and r_e, and the eccentricity e, through the relation:

$$C = r_i \sinh k_i = r_e \sinh k_e \tag{3}$$

where

$$k_i = \cosh^{-1}\frac{1}{\gamma}\frac{\gamma(1+\phi^2) + (1-\phi^2)}{2\phi} \tag{4}$$

$$k_e = \cosh^{-1}\frac{\gamma(1-\phi^2) + (1+\phi^2)}{2\phi} \tag{5}$$

with

$$\gamma = \frac{r_i}{r_e}, \qquad \phi = \frac{e}{r_e - r_i} = \frac{\bar{e}\gamma}{1-\gamma} \tag{6}$$

where $\bar{e} = e/r_i$.

It was shown by Cairns *et al.* [1] that for this problem the velocity components v_1 and v_2 can be written as:

$$v_1 = 0 \tag{7}$$

and

$$v_2 = (\cosh x_2 - \cos x_1)^2 \frac{1}{2c\mu}\frac{p_i - p_e}{k_e - k_i}\left\{\left(\frac{b}{2}\right)^2 - x_3^2\right\}. \tag{8}$$

By substituting the velocities from equations (7) and (8) into the energy equation, equation (1) we get:*

$$\rho C_p(\cosh x_2 - \cos x_1)^2 \frac{1}{2c\mu}\frac{p_i - p_e}{k_e - k_i}\left\{\left(\frac{b}{2}\right)^2 - x_3^2\right\}\frac{\partial T}{\partial x_2} = k\frac{\partial^2 T}{\partial x_3^2}. \tag{9}$$

Introducing the transformation

$$x_2' = x_2 - k_i, \qquad \bar{x}_3 = \frac{x_3}{(b/2)}, \qquad \theta = \frac{T - T_w}{T_0 - T_w},$$

$$\bar{c} = \frac{c}{(b/2)}, \qquad \overline{Pe} = Pe\frac{\ln(1/\gamma)}{k_i - k_e},$$

$$Pe = \frac{\rho C_p}{k\mu}\frac{1}{3}\frac{p_i - p_e}{\ln(1/\gamma)}\left(\frac{b}{2}\right)^2 \tag{10}$$

and

$$\bar{x}_2 = -\int_0^{x_2'} \frac{\bar{c}^2\, dy_2}{[\cosh(y_2 + k_i) - \cos x_1]^2}. \tag{11}$$

Equation (9) becomes

$$\frac{\partial^2\theta}{\partial \bar{x}_3^2} = \overline{Pe}(1 - \bar{x}_3^2)\frac{\partial\theta}{\partial \bar{x}_2}, \qquad -1 < \bar{x}_3 < 1, \qquad \bar{x}_2 > 0 \tag{12}$$

subject to the boundary conditions:

$$\bar{x}_2 = 0:\ \theta = 1; \qquad \bar{x}_3 = \pm 1:\ \theta = 0.$$

Since equation (12) is linear, we assume a solution as:

$$\theta(\bar{x}_2, \bar{x}_3) = R(\bar{x}_2)Z(\bar{x}_3). \tag{13}$$

By using the technique of separation of variables, equation (13) is substituted into equation (12) and we get:

$$\frac{3\overline{Pe}}{2}\frac{R'}{R} = \frac{Z''}{(1 - \bar{x}_3{}^2)Z} = -\lambda^2 \tag{14}$$

where λ^2 is an arbitrary constant. The first equation in equation (14) gives the general solution:

$$R(\bar{x}_2) = C_1 \exp\left(-\frac{2\lambda^2}{3\overline{Pe}}\bar{x}_2\right) \tag{15}$$

where C_1 is a constant to be determined by the boundary condition $\theta(0, \bar{x}_3) = 1$ and the eigenvalue λ^2. The second equation from equation (14) is:

$$Z'' + \lambda^2(1 - \bar{x}_3^2)Z = 0 \tag{16}$$

subject to the boundary conditions: $Z(\pm 1) = 0$.

* $\cosh x_2 - \cos x_1 \neq 0$ since $\cosh x_2 > \cos x_1$ for the region $0 \leqslant x_1 \leqslant 2\pi$ and $0 < k_e \leqslant x_2 \leqslant k_i$, in this problem.

Equation (16), which constitutes a Sturm–Lieuville eigenvalue problem, is solved approximately by the well-known Galerkin's method [2]. It is then combined with equation (15) to give the solution of equation (12) as

$$\theta(\bar{x}_2, \bar{x}_3) = \sum_{i=1}^{N} C_i \left[\exp\left(-\frac{2\lambda^2_i}{3\overline{Pe}}\bar{x}_2\right)\right] \bar{Y}_i \tag{17}$$

where C_i and $\bar{Y}_i$ are given by:

$$C_i = \frac{8\lambda_i}{\pi} \frac{\sum_{n=1}^{N} (-1)^{n+1} \frac{b_{ni}}{[(2n-1/2)\pi]^4}}{\sum_{n=1}^{N} b_{ni}^2} \sum_{n=1}^{N} \frac{b_{ni}}{2n-1} \tag{18}$$

and

$$\bar{Y}_i = \frac{\sum_{n=1}^{N} \frac{b_{ni}}{(2n-1)} \cos\frac{2n-1}{2}\pi\bar{x}_3}{\sum_{n=1}^{N} \frac{b_{ni}}{(2n-1)}} \tag{19}$$

respectively. Also, λ_i and b_{ni} are the eigenvalue and the coefficients in the expansion of the eigenfunction respectively. Fifteen sets of values are computed in the numerical solution given below.

NUMERICAL SOLUTIONS AND DISCUSSIONS

The heat transfer is defined as

$$q_{w\bar{x}_2} = -k\left(\frac{\partial T}{\partial x_3}\right)_{x_3=b/2} = -\frac{k(T_0 - T_w)}{b/2}\left(\frac{\partial \theta}{\partial \bar{x}_3}\right)_{\bar{x}_3=1} = \bar{h}_{\bar{x}_2}(T_0 - T_w). \tag{20}$$

Thus, the local Nusselt number is given by:

$$Nu_{\bar{x}2} = \frac{\bar{h}_{\bar{x}_2}(b/2)}{k} = -\left(\frac{\partial \theta}{\partial \bar{x}_3}\right)_{\bar{x}_3=1} = \sum_{i=1}^{N}\left\{C_i\left[\exp\left(-\frac{2\lambda_i^2}{3\overline{Pe}}\bar{x}_2\right)\right]\sum_{m=1}^{N} b_{mi}\frac{2m-1}{2}\pi(-1)^{m+1}\right\}. \tag{21}$$

The average Nusselt number can be obtained by integrating over the disk surface, i.e.

$$\overline{Nu} = -\frac{1}{\pi(\bar{r}_e^{\,2} - \bar{r}_i^{\,2})}\int_0^{2\pi}\int_{k_i}^{k_e} Nu_{\bar{x}2}\bar{h}_1\bar{h}_2\,\mathrm{d}x_2\,\mathrm{d}x_1 = \frac{1}{\pi(\bar{r}_e^{\,2} - \bar{r}_i^{\,2})}\int_0^{2\pi}\int_0^{\bar{x}_{2e}} Nu_{\bar{x}2}\,\mathrm{d}\bar{x}_2\,\mathrm{d}x_1. \tag{22}$$

Substituting $Nu\bar{x}_2$ from equation (21) into equation (22), we get:

$$\overline{Nu} = \frac{1}{\pi(\bar{r}_e^{\,2} - \bar{r}_i^{\,2})}\int_0^{2\pi}\sum_{i=1}^{N}\left\{C_i\frac{3\overline{Pe}}{2\lambda_i^2} \times \left[\exp\left(-\frac{2\lambda_i^2}{3\overline{Pe}}\bar{x}_{2e}\right) - 1\right]\sum_{m=1}^{N} b_{mi}\frac{2m-1}{2}\pi(-1)^m\right\}\mathrm{d}x_1. \tag{23}$$

The evaluation of the upper limit in the integration of $\mathrm{d}\bar{x}_2$ in equation (22), $\bar{x}_{2e}$, is given in detail in Appendix I.

Figures 2 and 3 are plots of the average Nusselt number as a function of $\bar{e}$ and γ fo $\bar{r}_i = 50$ and $Pe = 5000$ and 20000, respectively. It is seen that in general the heat transfer is decreased if the inlet is eccentric, as compared with the heat transfer with concentric inlet under the same conditions.

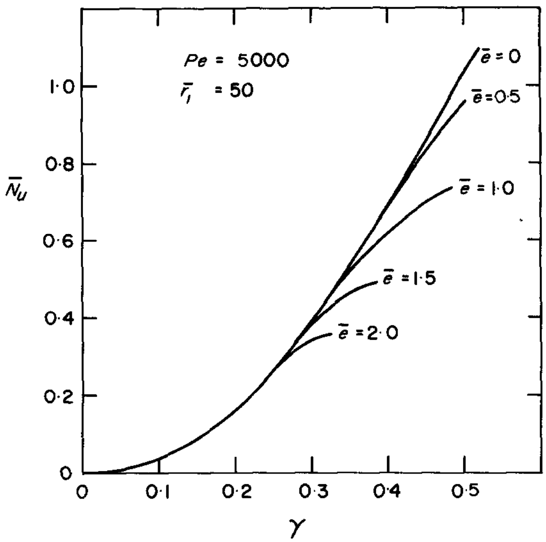

FIG. 2. Average Nusselt number vs. γ ($Pe = 5000$).

The curves in Figs. 2 and 3 labeled $\bar{e} = 0$, are for concentric inlet. Based on these figures, the following becomes apparent:

1. For a fixed eccentricity, e, and exit radius, r_e, increasing the inlet radius, r_1, will decrease the parameter $\bar{e}$ and increase the parameter γ. The average Nusselt number is thus increased.
2. For a pair of disks with fixed inlet and outlet radii, r_i and r_e, the heat transfer is decreased if the eccentricity is increased.
3. For a pair of disks with fixed inlet radius, r_i, and eccentricity, e, the averaged Nusselt number is increased if the outlet radius r_e is decreased.
4. As γ is decreased, the curves approach to the concentric inlet curve. Since γ is the ratio of r_i to r_e, this means that

the effect of eccentricity is reduced by either drilling a smaller hole or increasing the outlet radius.

5. The average Nusselt number is decreased for smaller Péclét numbers, which is physically reasonable since a smaller Péclét number means a larger thermal conductivity. For the same temperature difference, a decrease in heat transfer is expected.

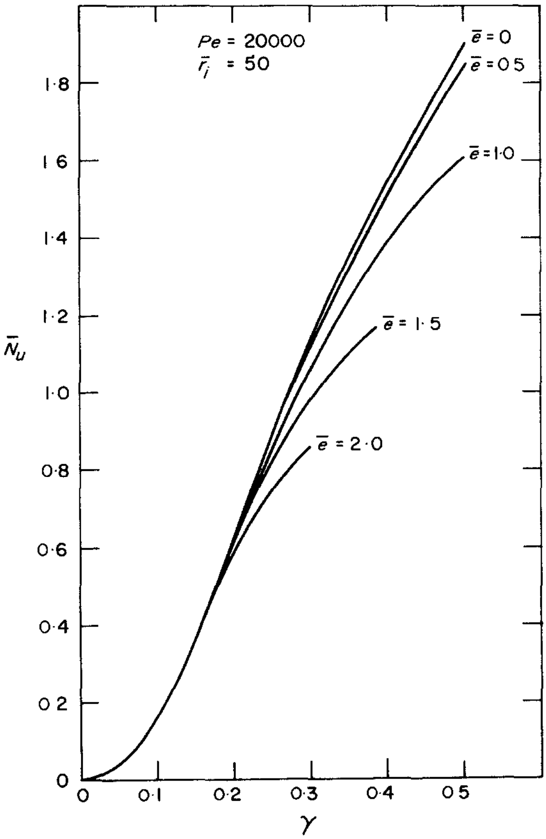

FIG. 3. Average Nusselt number vs. γ ($Pe = 2 \times 10^5$).

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APPENDIX 1

Evaluation of x_{2e}

The upper limit in the integration of $d\bar{x}_2$ in equation (22), $\bar{x}_{2e}$, can be obtained as follows: Integrating equation (11), we get:

$$\bar{x}_2 = \frac{2\bar{c}^2}{\sin^2 x_1}\left\{\frac{1 - e^{x_2}\cos x_1}{X(x_2)} - \frac{1 - e^{k_i}\cos x_1}{X(k_i)} - \frac{\cos x_1}{|\sin x_1|}\left[\tan^{-1}\left(\frac{e^{x_2} - \cos x_1}{|\sin x_1|}\right) - \tan^{-1}\left(\frac{e^{k_i} - \cos x_1}{|\sin x_1|}\right)\right]\right\} \quad (A.1)$$

if $\sin x_1 \neq 0$, and

$$\bar{x}_2 = 4\bar{c}^2\{[(e^{x_2} - 1)^{-2} - (e^{k_i} - 1)^{-2}]/2 + [(e^{x_2} - 1)^{-3} - (e^{k_i} - 1)^{-3}]/3\} \quad (A.2)$$

if $\sin x_1 = 0$, where

$$X(x) = e^{2x} - 2e^x \cos x_1 + 1.$$

At the inner radius, both equations give:

$$\bar{x}_{2i} = 0.$$

At the outer radius, equations (A.1) and (A.2) give:

$$\bar{x}_{2e} = \frac{2\bar{c}^2}{\sin^2 x_1}\left\{\frac{1 - e^{k_e}\cos x_1}{X(k_e)} - \frac{1 - e^{k_i}\cos x_1}{X(k_i)} - \frac{\cos x_1}{|\sin x_1|}\left[\tan^{-1}\left(\frac{e^{k_e} - \cos x_1}{|\sin x_1|}\right) - \tan^{-1}\left(\frac{e^{k_i} - \cos x_1}{|\sin x_1|}\right)\right]\right\}$$

if $\sin x_1 \neq 0$, and

$$\bar{x}_{2e} = 4\bar{c}^2\{[(e^{k_e} - 1)^{-2} - (e^{k_i} - 1)^{-2}]/2 + [(e^{k_e} - 1)^{-3} - (e^{k_i} - 1)^{-3}]/3\}$$

if $\sin x_1 = 0$.