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AN INVESTIGATION OF NONLINEAR XENON OSCILLATION BY METHOD OF AVERAGING

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Abstract—A nonlinear analysis of xenon-temperature controlled nuclear reactor dynamics is presented. The set of equations in question belongs to a general class of rate equations with quadratic nonlinearities. Boundedness of the solutions is examined. The mean value of periodic solutions for the flux is shown to be always less than the equilibrium value. The Bogoliubov's method of averaging as extended by Case is applied to obtain approximate solutions. The mechanism of the existence of relaxation oscillations in the linear stability region is analyzed. Computer calculations are performed and found in good agreement with the approximate solutions obtained by means of the method of averaging.

INTRODUCTION

THE dynamics of xenon-temperature controlled nuclear reactors can be described by a set of ordinary differential equations which belongs to a general class of "rate equations" (GOEL *et al.*, 1971) with quadratic nonlinearities. Various systems in biology, chemistry and even sociology may be shown to belong to this class.

Considerable work on the dynamics of xenon-temperature controlled nuclear reactors has been done in the past by several investigators. Its extensive linear and computer analyses were performed by CHERNICK, LELLOUCHE and WOLLMAN (1961). They were also the first to notice the boundedness of the solutions to the equation of this problem. Its nonlinear stability was investigated by AKCASU and AKHTAR (1967) and later by LELLOUCHE (1967) using a stability criterion by AKCASU and DALFES (1960). The criterion gives sufficient conditions for the asymptotic stability in the large. Although it provides a powerful tool to investigate the asymptotic stability of a nuclear reactor, it does not yield any information about the dynamic behavior of the reactor when the obtained sufficient conditions are not satisfied.

In this paper, first the boundedness of the solutions as well as the positiveness is examined. An exact relationship between the mean and mean square value of any periodic solution is obtained. Then we construct approximate solutions by means of the Bogoliubov's method of averaging (KRYLOV and BOGOLIUBOV, 1947) as extended by CASE (unpublished, 1963) and investigate the stability of the reactor in terms of these approximate solutions. The method of averaging with Case's prescription was first applied to the nuclear reactor dynamics by NOBLE (1965, unpublished) in the case of a linear feedback. For the two temperature feedback model, he compared his analytical results to SHOTKIN'S computer calculation (1963) with a satisfactory agreement. Our problem differs from Noble's in that the feedback mechanism in the xenon-temperature controlled reactor is nonlinear.

Since the solutions are bounded, there exist isolated periodic solutions or quasi-periodic solutions, provided all the attainable equilibria are linearly unstable. In this work we are mainly concerned with the cases where the system has only one

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attainable equilibrium. CHERNICK, LELLOUCHE, and WOLLMAN (1961) demonstrated numerically that periodic solutions can exist even if the only attainable equilibrium is linearly stable provided the initial perturbation is sufficiently large. If the phase space in question were two-dimensional, this phenomenon could be explained in terms of an unstable limit cycle enclosing the linearly stable limit cycle. However, since the phase space of our present problem is three-dimensional, such an explanation is not applicable. The investigation of the mechanism of this phenomenon is one of the main objectives of this paper.

GENERAL DISCUSSIONS ON SOME NONLINEAR PROPERTIES OF THE KINETICS EQUATION

The dynamics of xenon-temperature controlled nuclear reactor can be described (AKCASU *et al.*, 1971) by

$$l^* \frac{d\phi}{dt} = \left(\rho_{ex} - \frac{\sigma_X}{C\sigma_f} X - \gamma\phi\right)\phi \equiv \rho\phi \tag{1}$$

$$\frac{dI}{dt} = y_I \sigma_f \phi - \lambda_I I \tag{2}$$

$$\frac{dX}{dt} = y_X \sigma_f \phi - \lambda_X X + \lambda_I I - \sigma_X X \phi \tag{3}$$

The dependent variables ϕ, I and X are the neutron flux, the iodine density per fuel atom and the xenon density per fuel atom, respectively. The quantity l^* is an effective neutron life in which the effect of delayed neutron is incorporated. The quantities σ_X and σ_f are the microscopic absorption cross section of xenon and the microscopic fission cross section of the fuel, y_I and y_X the iodine and the xenon fission yields, λ_I and λ_X their decay constants, γ the flux coefficient of the reactivity (or the temperature coefficient of the reactivity). When γ is positive, the temperature feedback is stabilizing. The symbol C is a factor converting the local xenon absorption per fission to the overall reactivity. In a steady state operation the external reactivity ρ_{ex} is just compensated by the feedback reactivity:

$$\rho_{ex} = \frac{\sigma_X y}{C} \frac{\phi_0}{\lambda_X + \sigma_X \phi_0} + \gamma\phi_0$$

the subscript "$_0$" referring to the steady state values.

Since all the parameters except for γ and $\phi_0 = \phi_0(\rho_{ex})$ are fixed (see Table 1) as nuclear parameters, the behavior of the solutions can be specified by the position of the point (γ, ϕ_0), which we refer to as an operating point, on the γ–ϕ_0 plane. The γ–ϕ_0 plane can be divided by a curve into two regions, the linear stability and the linear instability regions (CHERNICK *et al.*, 1961). We call this curve the critical curve and denote a point on it by (γ_c, ϕ_c). When the reactor is operated on this curve, the system is neutral and the characteristic equation for the linear part of (1), (2) and (3) has two pure imaginary roots $\pm i\omega_0$ and a real negative root ζ for $\gamma > 0$. Figure 1 shows the critical curve. If an operating point (γ, ϕ_0) lies to the right of the critical curve, then the reactor is linearly stable and otherwise linearly unstable. If it is on the critical curve, i.e., neutral, the linear analysis does not say anything definite about stability of the original nonlinear system.

TABLE 1.—THE VALUES OF THE CONSTANTS IN THIS WORK

$l^* = 0{\cdot}1$ sec
$C = 1{\cdot}6$
$y_I = 0{\cdot}062$
$y_X = 0{\cdot}002$
$\sigma_X = 3 \times 10^{-18}\,\text{cm}^2$
$\lambda_X = 2 \times 10^{-5}\, l\, \text{sec}^{-1}$
$\lambda_I = 3 \times 10^{-5}\, l\, \text{sec}^{-1}$

Without solving the kinetics equation it is possible to discuss some properties of the solutions, which will help us to scrutinize the results to be obtained later by means of the method of averaging. First we will show that if

$$\phi(0) > 0 \tag{4}$$

$\phi(t)$ remain nonnegative for all $t > 0$.

Successive differentiation of (1) with respect to time leads to

$$\phi^{(n)} = \phi F_n \qquad n = 1, 2, \ldots \tag{5}$$

where

$$F_1 = \frac{\rho}{l^*}$$

$$F_n = F_1 F_{n-1} + \frac{\mathrm{d}}{\mathrm{d}t} F_{n-1} \qquad n = 2, 3, \ldots$$

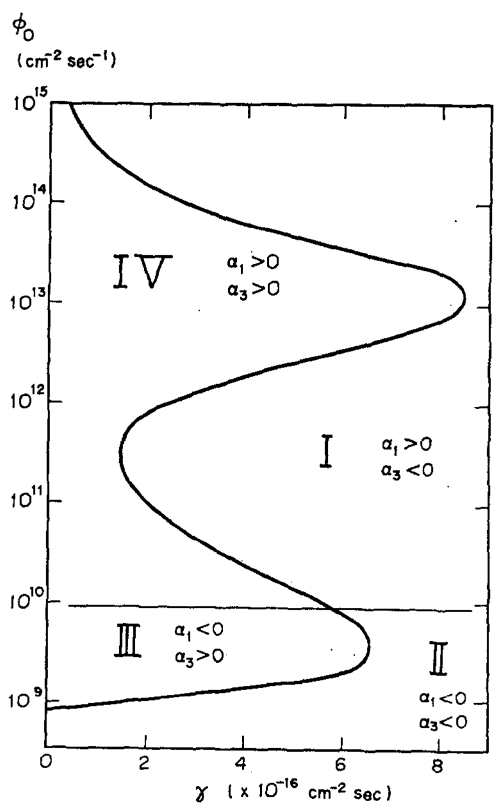

FIG. 1.—The critical curve and the four regions.

and

$$\phi^{(n)} = \frac{d^n}{dt^n} \phi$$

Suppose $\phi(t)$ reaches zero for the first time at $t = t_0 > 0$ after an initial disturbance was introduced at time zero. Then due to (5)

$$\phi^{(n)}(t_0) = 0 \qquad n = 1, 2, \ldots \tag{6}$$

whence it follows, from the continuity of ϕ which we assume, that

$$\phi(t) = 0 \qquad \text{for all } t \geq 0 \tag{7}$$

This is contradictory to (4). Hence if $\phi(0) > 0$, then $\phi(t)$ remains strictly positive for any finite time, although it may vanish at $t = \infty$. To examine the behavior of $\phi(t)$ near its singularity $\phi = 0$ we linearize (1) to obtain

$$\frac{d\phi}{dt} = \frac{\gamma}{l^*} M\phi \tag{8}$$

where

$$M = \frac{\sigma_X}{c\sigma_f \gamma} X_0 + \phi_0.$$

Hence if the condition $\gamma > 0$ is also added $\phi(t)$ remains strictly positive for all $t > 0$. If the initial values $I(0)$ and $X(0)$ are nonnegative then we can show (ASAHI, 1972), making use of the above result, that $I(t)$ and $X(t)$ remain strictly positive for any finite $t > 0$ and moreover if $\gamma > 0$, they remain strictly positive for all $t > 0$.

Next we will show that under the condition (4) and for $\gamma > 0$ the solutions are bounded from above. Since $X(t)$ is positive for all $t > 0$ it follows from (1) that

$$\dot{\phi} < \frac{\gamma}{l^*} \phi(M - \phi) \tag{9}$$

for $t > 0$, M being now positive. The relationship (9) shows that when ϕ exceeds M it is already decreasing and hence that ϕ is bounded from above by $M_0 =$ max $(M, \phi(0))$. We note that ϕ is asymptotically bounded from above by M. The boundedness of I and X from above can be shown[12] on account of the boundedness of ϕ.

We note that in the case of $\gamma > 0$, with which we are primarily concerned, there are two equilibria (0, 0, 0) and (ϕ_0, I_0, X_0) and only the latter is physically attainable. Since then the solutions are bounded, if the only attainable equilibrium (ϕ_0, I_0, X_0) is linearly unstable, the solutions will asymptotically reach isolated periodic solutions or quasi-periodic solutions.

Next we will compare the mean value of periodic solutions with the equilibrium flux level ϕ_0. To this end, following the same procedure as in AKCASU and AKHTAR (1967), we express $\phi(t)$ in (1) as

$$\phi(t) = \int_{-\infty}^{t} du G_1(t - u)\varphi(u) + \int_{-\infty}^{t} du G_2(t - u)\varphi^2(u) \tag{10}$$

where

$$\varphi(t) = \phi(t) - \phi_0$$

$$G_1(t) = \sigma_X \lambda_0 e^{-\lambda_X t} + \lambda_2 y_I \sigma_f e^{-\lambda_I t} - \lambda_1 \delta(t)$$

$$G_2(t) = -\sigma_X \gamma e^{-\lambda_X t}$$

$$\lambda_0 = \rho_{ex} - \frac{1}{c}\left(\frac{\lambda_I \lambda_X}{\lambda_I - \lambda_X} + y_I\right) + \lambda_I l^* + 2\gamma\phi_0$$

$$\lambda_1 = \gamma + \sigma_X l^*$$

$$\lambda_2 = \sigma_X \lambda_I / c\sigma_f(\lambda_I - \lambda_X)$$

We note in (10) that the feedback contains a nonlinear component when $\gamma \neq 0$. The periodicity condition $\phi(t) = \phi(t + T)$ for all t, when applied to the equality

$$\phi(t) = (a \text{ const.}) \times \exp\left\{\frac{1}{l}\int_{\infty}^{t} \rho(\tau)\, d\tau\right\}$$

obtained from integration of (1), yields

$$0 = \int_{t}^{t+T} \rho(\tau)\, d\tau$$

which states that the average feedback reactivity vanishes when the flux is a periodic function of time. Substituting (10) into this condition we obtain

$$0 = H(0)\varphi_{av} - \frac{\sigma_X \gamma}{\lambda_X}(\varphi^2)_{av}$$

where

$$H(0) = -\gamma - \frac{\sigma_X}{\lambda_X}\left(\gamma\phi_0 + \frac{y}{c}\frac{1}{\lambda_X + \sigma_X\phi_0}\right)$$

and φ_{av} and $(\varphi^2)_{av}$ are the averages of φ and φ^2 over one period T. For a positive γ there is a unique non-trivial ϕ_0 so that φ_{av} and $(\varphi^2)_{av}$ are also uniquely defined. For a positive γ, $H(0)$ is negative so that φ_{av} is negative, i.e., the flux $\phi(t)$ oscillates with a mean value less than the steady state level ϕ_0. We also note that $\varphi_{av} = 0$ when the feedback is linear, i.e., $\gamma = 0$. These conclusions are consistent with the general results obtained in the case of a linear feedback by others (Akcasu, 1958; Smets, 1960).

Table 2.—A comparison of approximate and computer solutions in case I (asymptotic stability) for $\phi_0 = 10^{11}\ \text{cm}^{-2}\,\text{sec}^{-1}$, $\frac{\gamma - \gamma_c}{\gamma_c} = 0{\cdot}5 \times 10^{-2}$, $x_1(0) = 2{\cdot}0$ and $T_c = \frac{2\pi}{\omega_c} = 143$ hr

	Analytical result	Computer result	
		$e = 2 \times 10^{-5}$	$l = 10^{-3}$
Period	$1{\cdot}002T_c$	$1{\cdot}003T_c$	$1{\cdot}004T_c$
$e^{\alpha_3 T}$	0·988	0·980	0·967

APPLICATION OF METHOD OF AVERAGING

We will use the method of averaging to obtain approximate solutions of our problem. The formulation of the Bogoliubov's method of averaging with CASE's prescription (unpublished paper, 1963) is given in Appendix. In applying the method of averaging, we restrict ourselves to the vicinity of the critical curve. We choose the deviation of the operating point from an appropriate reference point on the critical curve as the small parameter required in the method of averaging. The choice of the reference point is not unique. In this paper, we will choose the reference point associated with an operating point (γ, ϕ_0) as the point specified by the intersection of the horizontal line $\phi_0 = \text{const.}$ (the given operating level) with the critical curve. The reference point (γ_c, ϕ_0) so defined is unique for each operating point $(\gamma_0\ \phi_0)$ and defines the critical flux coefficient γ_c. Given an operating point, there is another choice for the corresponding reference point that is the intersection of $\gamma = \text{const.}$ (the given operating flux coefficient) with the critical curve. In this case the reference point is not unique. If we adopt the reference point as the closest intersection, we can show (ASAHI, 1972) that for the operating point sufficiently close to the critical curve, these two calculation schemes yield identical results.

If we substitute

$$x_1(t) = \frac{\phi(t)}{\phi_0} - 1,\ x_2(t) = \frac{I(t)}{I_0} - 1$$

and

$$x_3(t) = \frac{X(t)}{X_0} - 1$$

into (1), (2) and (3) and assume $x_1(t)$, $x_2(t)$ and $x_3(t)$ together with

$$a \equiv \frac{\gamma_0 - \gamma}{\omega_c l^*}\phi_0$$

are small, we obtain

$$\begin{aligned} \dot{x}_1 &= -a_{11}x_1 - a_{13}x_3 - \varepsilon a_{11}x_1^2 - \varepsilon a_{13}x_1x_3 + \varepsilon a x_1 + \varepsilon^2 a x_1^2 \\ \dot{x}_2 &= a_{21}(x_1 - x_2) \\ \dot{x}_3 &= -a_{31}x_1 + a_{32}x_2 - a_{33}x_3 - \varepsilon a_0 x_1 x_3 \end{aligned} \tag{12}$$

where t was replaced by $\omega_c t$ and the parameter a_{ij} and a_0 are defined in Table 3. The parameter a is a measure of the displacement of (γ, ϕ_0) from (γ_0, ϕ_0). In (12) the powers of the symbol ε imply the order of magnitude of each term when x_1, x_2, x_3 and a are assumed to be of order of ε. The bookkeeping parameter ε will be set to unity at the end of the calculation.

To transform (12) to a standard form discussed in Appendix we change the variables (x_1, x_2, x_3) to (z_1, z_2, z_3) such that

$$\begin{aligned} x_1 &= z_1 \cos t + z_2 \sin t + z_3 e^{-kt} \\ x_2 &= a_{21}\left\{z_1 \frac{a_{21}\cos t + \sin t}{a_{21}^2 + 1} + z_2 \frac{a_{21}\sin t - \cos t}{a_{21}^2 + 1} + z_3 \frac{e^{-kt}}{a_{21} - k}\right\} \\ x_3 &= \frac{1}{a_{13}}\{z_1(\sin t - a_{11}\cos t) - z_2(\cos t + a_{11}\sin t) + z_3(k - a_{11})e^{-kt}\} \end{aligned} \tag{13}$$

TABLE 3.—NOMENCLATURE I

$$a_0 = \frac{\gamma_c - \gamma}{\omega_c l^*} \phi_0$$
$$a_{11} = \gamma_c \phi_0 / l^* \omega_c$$
$$a_{21} = \lambda_I / \omega_c$$
$$a_{31} = \left(\frac{y_I}{y} \lambda_{Xe} - \lambda_X\right) \Big/ \omega_c$$
$$a_{32} = y_I \lambda_{Xe} / y \omega_c$$
$$a_{13} = y(\lambda_{Xe} - \lambda_X) / l^* c \lambda_{Xe} \omega_c$$
$$a_{33} = \lambda_{Xe} / \omega_c$$
$$\lambda_{Xe} = \lambda_X + \sigma_X \phi_0$$
$$y = y_x + y_I$$
$$\nu_1 = a \frac{kb_1 + 1}{2(k^2 + 1)} (1 - b_1)$$
$$\nu_2 = a \frac{kc_1 - b_1}{2(k^2 + 1)} (1 - b_1) - \frac{a^2}{8} (b_1^2 + c_1^2)$$
$$b_1 = \{a_{21}^2 + 1 + (k - a_{21})(k - a_{11})\}/(k^2 + 1)$$
$$b_2 = (a_{21} - k)/(k^2 + 1)$$
$$c_1 = \{k(a_{21}^2 + 1) - (k - a_{11})(ka_{21} + 1)\}/(k^2 + 1)$$
$$c_2 = (ka_{21} + 1)/(k^2 + 1)$$
$$k = a_{11} + a_{21} + a_{33}$$
$$\alpha_1 = p_{21} - \frac{p_{15} p_9}{k} + p_{20} p_{15} + \frac{p_{16} p_9}{2}$$
$$\alpha_2 = p_{22} - \frac{p_{15} p_{10}}{k} - \frac{p_{16} p_{20}}{2} + p_{19} p_{15} - \tfrac{3}{8}(p_1^2 + p_2^2)$$
$$\alpha_3 = a \left[\frac{b_1}{2} + \frac{(1 - b_1)(kb_1 + c_1)}{2(k^2 + 1)}\right]$$
$$\alpha_4 = a \left[\frac{c_1}{2} + \frac{(1 - b_1)(kb_1 + c_1)}{2(k^2 + 1)} - \frac{b_1^2 + c_1^2}{8} a\right]$$

where $k = -\zeta/\omega_c > 0$. Substituting (13) into (12) and solving for dz_1/dt, dz_2/dt and dz_3/dt we obtain the standard form

$$\frac{dz_i}{dt} = \varepsilon Y_i^{(1)}(\mathbf{z}, t) + \varepsilon^2 Y_i^{(2)}(\mathbf{z}, t) \qquad i = 1, 2, 3$$

The expression of $Y_i^{(1)}(\mathbf{z}, t)$ and $Y_i^{(2)}(\mathbf{z}, t)$ are lengthy and will not be presented here. They can be found in ASAHI (1972). In general, the application of the method of averaging to nonlinear system with three or more state variables requires simple but tedious calculations. In this paper we shall present only the results of the calculations, which can be reproduced by following the general procedure of the method of averaging outlined in Appendix. Interested readers can find the details of the calculations in ASAHI (1972).

Following the prescription given in Appendix we obtain (ASAHI, 1972)

$$G_1^{(1)}(\boldsymbol{\xi}, t) = \frac{a}{2}(b_1 \xi_1 + c_1 \xi_2)$$

$$G_2^{(1)}(\boldsymbol{\xi}, t) = \frac{a}{2}(-c_1 \xi_1 + b_1 \xi_2)$$

$$G_3^{(1)}(\boldsymbol{\xi}, t) = (1 - b_1)a\xi_3 + (1 - b_1)ae^{kt}(\xi_1 \cos t + \xi_2 \sin t) - P_{15}e^{kt}(\xi_1^{\,2} + \xi_2^{\,2})$$
$$+ e^{kt} \sin 2t\left\{\frac{P_{16}}{2}(\xi_2^{\,2} - \xi_1^{\,2}) - 2P_{15}\xi_1\xi_2\right\}$$
$$G_1^{(2)}(\boldsymbol{\xi}, t) = (\alpha_1\xi_1 + \alpha_2\xi_2)(\xi_1^{\,2} + \xi_2^{\,2}) + \nu_1\xi_1 + \nu_2\xi_2$$
$$G_2^{(2)}(\boldsymbol{\xi}, t) = (-\alpha_2\xi_1 + \alpha_1\xi_2)(\xi_1^{\,2} + \xi_2^{\,2}) - \nu_2\xi_1 + \nu_1\xi_2$$

where the parameters α_i and ν_i are defined in Table 3 and p_i in Table 4. Then we obtain the equations for the long time behavior of ξ_1, ξ_2 and ξ_3 [see equation (A.3)]

TABLE 4.—NOMENCLATURE II

$$p_1 = (-b_1 - a_0b_2 + a_0a_{11}c_2)/4$$
$$p_2 = (b_1 + a_0b_2 + 3a_0a_{11}c_2)/4$$
$$p_3 = (-c_1 - a_0c_2 + a_0a_{11}b_2)/4$$
$$p_5 = (-c_1 - a_0c_2 + 3a_0a_{11}b_2)/4$$
$$p_6 = (c_1 + a_0c_2 + a_0a_{11}c_2)/4$$
$$p_7 = (b_1 + a_0b_2 + a_0a_{11}c_2)/4$$
$$p_9 = -\{k(b_1 + a_0b_2) + c_1 + a_0c_2\}/2 + a_0a_{11}b_2$$
$$p_{10} = \{-k(c_1 + a_0c_2) + b_1 + a_0b_2\}/2 + a_0a_{11}c_2$$
$$p_{11} = -\{k(c_1 + a_0c_2) + b_1 + a_0b_2\}/2 + a_0a_{11}c_2$$
$$p_{12} = \{-k(b_1 + a_0b_2) + c_1 + a_0c_2\}/2 + a_0a_{11}b_2$$
$$p_{15} = a_0a_{11}b_2/2$$
$$p_{16} = 1 - b_1 - a_0b_2$$
$$p_{19} = \tfrac{1}{2}(-kp_{11} + 2p_{12})/(k^2 + 4)$$
$$p_{20} = \tfrac{1}{2}(-kp_{12} - 2p_{11})/(k^2 + 4)$$
$$p_{21} = -\tfrac{1}{2}(p_6p_7 + p_2p_3)$$
$$p_{22} = \tfrac{1}{2}(p_3p_6 - p_2p_7 - 2p_5p_6 - 2p_1p_2)$$
$$p_{25} = (1 - b_1)(k + \alpha_3)/\{(k + \alpha_3)^2 + \nu^2\} - \frac{c_1}{4}$$
$$p_{26} = (1 - b_1)\nu/\{(k + \alpha_3)^2 + \nu^2\} + \frac{b_1}{4}$$
$$p_{27} = -(p_1 + p_2)/2 - p_{15}/(k + 2\alpha_3)$$
$$p_{28} = \frac{p_7}{2} - \frac{p_1}{3} + (\nu p_{16} - (k + 2\alpha_3)p_{15}\}/\{(k + 2\alpha_3)^2 + 4\nu^2\}$$
$$p_{29} = \frac{p_3}{2} + \frac{p_6}{3} - \{p_{16}(k + 2\alpha_3)/2 + 2\nu p_{15}\}/\{(k + 2\alpha_3)^2 + 4\nu^2\}$$
$$p_{30} = (1 - b_1)k/(k^2 + \nu^2) - \frac{c_1}{4}$$
$$p_{31} = (1 - b_1)/(k^2 + \nu^2) + \frac{b_1}{4}$$
$$p_{32} = -(p_1 + p_2)/2 - p_{15}/k$$
$$p_{33} = \frac{p_7}{2} - \frac{p_1}{3} + (\nu p_{16} - kp_{15})/(k^2 + 4\nu^2)$$
$$p_{34} = \frac{p_3}{2} + \frac{p_6}{3} - (kp_{16}/2 + 2\nu p_{15})/(k^2 + 4\nu^2)$$

up to the second order in ε, which, after the change of variables $(\xi_1, \xi_2) = (r\cos\theta, r\sin\theta)$ reduce to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \varepsilon r(\alpha_3 + \varepsilon\alpha_1 r^2) \tag{14}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\varepsilon(\alpha_4 + \varepsilon\alpha_2 r^2) \tag{15}$$

$$\begin{aligned}\frac{\mathrm{d}\xi_3}{\mathrm{d}t} &= \varepsilon(1 - b_1)a\xi_3 + \varepsilon r^2 \mathrm{e}^{kt}\left\{-p_{15} + \frac{p_{16}}{2}\sin 2(\theta - t) - p_{15}\cos 2(\theta - t)\right\} \\ &\quad + \varepsilon r \mathrm{e}^{kt}(1 - b_1)a\cos(\theta - t)\end{aligned} \tag{16}$$

where the parameters b_i and c_i are defined in Table 3.

Since $0 < b_1 < 1$ and $kb_1 + c_1$ is positive regardless of ϕ_0, α_3/a is always positive. The fact that (14) does not contain ξ_3 and θ implies quite a simplification of our problem which originally has three dependent variables. Following the prescription in Appendix we now can obtain the solution up to the second order in ε. But we only calculate the terms up to the first order in ε only to avoid the complication. In the equation (16) for the long time behavior of ξ_3, we did not include $G_3^{(2)}(\boldsymbol{\xi}, t)$, since its contribution give rise to terms of the second order in ε. There are two solutions to (14) and (15); $(r, \theta) = [0, -\varepsilon\alpha_4 t + \theta(0)]$ and

$$r^2(t)\frac{\alpha_3}{\varepsilon\alpha_1} = \frac{Q\mathrm{e}^{2\varepsilon\alpha_3 t}}{1 - Q\mathrm{e}^{2\varepsilon\alpha_3 t}} \tag{17}$$

$$\theta(t)\frac{\varepsilon\alpha_2}{2\alpha_1} = \log|1 - Q\mathrm{e}^{2\varepsilon\alpha_3 t}| - \varepsilon\alpha_4 t + \theta_0 \tag{18}$$

where Q and θ_0 are the constants of the integration.

In summary, to solve the system (12) we have transformed the dependent variables (x_1, x_2, x_3) to (z_1, z_2, z_3) and to (ξ_1, ξ_2, ξ_3) and then to (r, θ, ξ_3). From (A.2) we obtain x_1, x_2 and x_3 in terms of r, θ and ξ_3 as

$$\begin{aligned}x_1(t) &= r\cos(t - \theta) + \xi_3\mathrm{e}^{-kt} + \varepsilon\left[-\frac{p_1 + p_2}{2}r_2 + \left(\frac{p_7}{2} - \frac{p_1}{3}\right)r^2\cos 2(\theta - t)\right. \\ &\quad \left. + \left(\frac{p_3}{2} + \frac{p_6}{3}\right)r^2\sin 2(t - \theta) + \frac{ar}{4}\{b_1\sin(t - \theta) - c_1\cos(t - \theta)\}\right]\end{aligned} \tag{19}$$

$$x_2(t) = \frac{a_{21}r(t)}{a_{21}{}^2 + 1}\{a_{21}\cos(t - \theta) + \sin(t - \theta)\} + \frac{a_{21}\xi_3(0)}{a_{21} - k}\mathrm{e}^{-kt} \tag{20}$$

$$x_3(t) = \frac{r(t)}{a_{13}}\{\sin(t - \theta) - a_{11}\cos(t - \theta)\} + \frac{k - a_{11}}{a_{13}}\xi_3(0)\mathrm{e}^{-kt} \tag{21}$$

We have expressed x_2 and x_3 only up to the order of unity in ε while x_1 was obtained up to the first order. Substituting (17) and (18) and the solution of (16) into (19), (20) and (21) we obtain the approximate solutions to the system. We can see from (19), (20) and (21) that the equilibrium $(x_1, x_2, x_3) = (0, 0, 0)$ corresponds with $r = 0$ since the terms proportional to $\xi_3(0)$ in (20) and (21) die out asymptotically.

Nonlinear stability in the small may be predicted by linear analysis, but this is not the case with a nonlinear system whose linearized system is neutral. The xenon-temperature controlled reactor whose linearized system is neutral can either be stable or unstable in the small, depending upon the steady state flux level ϕ_0. Then it is an interesting problem to consider what will happen if the operating point on the critical curve shifts to the linear stability or the linear instability regions. This would of course depend upon whether the neutral operating point is (nonlinearly) stable or not.

According to our calculation, (ASAHI, 1972) for any operating point (γ, ϕ_0) the parameter α_1 depends on γ_c and ϕ_0 and it is positive if $\phi_0 > 10^{10}\,\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$ and negative otherwise. Therefore we divide the $\gamma - \phi_0$ plane by a straight line $\phi_0 \sim 10^{-10}\,\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$. Figures 2–5 will help us to visualize the dynamic behavior of the system depending upon whether α_1 and α_3 are positive or negative. The four cases of Figs. 2–5 correspond with the four regions in Fig. 1. The dotted curve shows the curves of $\mathrm{d}r/\mathrm{d}t$ vs r for the corresponding reference points on the critical curve. Since α_3/a is always positive regardless of ϕ_0 and γ, α_3 is negative in the linear stability region

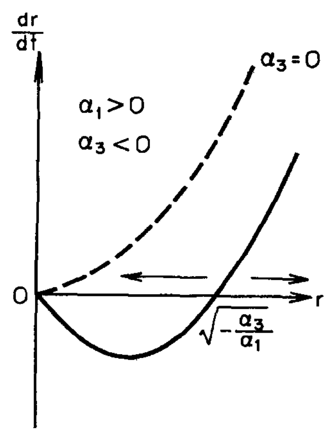

FIG. 2.—$\frac{\mathrm{d}r}{\mathrm{d}t}$ vs r in Case I (linear stability).

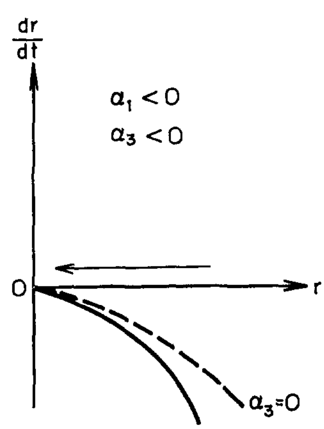

FIG. 3.—$\frac{\mathrm{d}r}{\mathrm{d}t}$ vs r in Case II (linear stability)

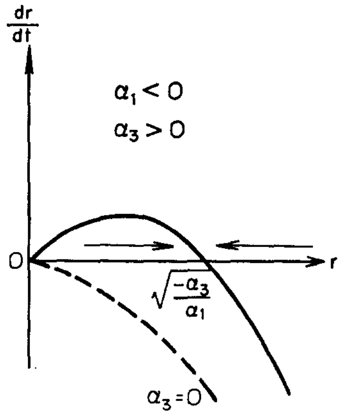

FIG. 4.—$\frac{dr}{dt}$ vs r in Case III (linear instability)

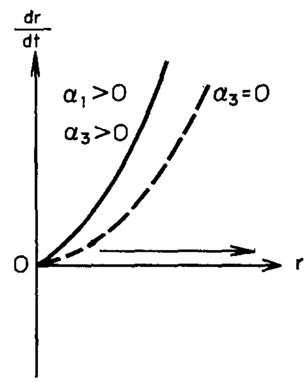

FIG. 5.—$\frac{dr}{dt}$ vs r in Case IV (linear instability)

and positive otherwise. Hence the shifts of the operating points from the critical curve into the linear stability region (Case I and Case II) lower the curve and hence stabilizes the system, while the shifts into the linear instability region (Case III and Case IV) raise the curve and hence destabilize the system. These figures show that our result is consistent with what the linear analysis predicts. Figures 2 and 5 show that the solutions are not bounded in Case I and Case IV so that the higher order calculation is necessary to obtain the boundedness of the solutions which we have shown.

Next, we will discuss those four cases separately. It would help us to get physical meaning of the discussion to refer to the corresponding figure whenever it is necessary.

Case I. If the initial value of $r(t)$ is smaller than $\sqrt{-\alpha_3/\alpha_1}$, then the system is asymptotically stable. To express the above condition in terms of $x_1(0)$, we find from (19), (20) and (21)

$$\begin{aligned}
x_1(0) &= r(0)\cos\theta(0) + \xi_3(0) \\
x_2(0) &= a_{21}\left\{\frac{a_{21}\cos\theta(0) - \sin\theta(0)}{a_{21}^2 + 1}\, r(0) + \frac{\xi_3(0)}{a_{21} - k}\right\} \\
x_3(0) &= \frac{1}{a_{13}}\{-r(0)(a_{11}\cos\theta(0) + \sin\theta(0)) + \xi_3(0)(k - a_{11})\}
\end{aligned} \tag{22}$$

up to the order of unity in ε. Since the iodine and xenon concentrations cannot physically be disturbed from outside the system, we assume that

$$x_2(0) = x_3(0) = 0 \tag{23}$$

It follows from (22) and (23) that

$$|x_1(0)| = c_0 r(0) \tag{24}$$

where

$$c_0 = \sqrt{\frac{k^2 + 1}{a_{33}{}^2 + 1}}$$

Hence if

$$\left| \frac{\phi(0) - \phi_0}{\phi_0} \right| < c_0 \sqrt{-\frac{\alpha_3}{\alpha_1}} \tag{25}$$

then the system is asymptotically stable. Since α_3 is proportional to $\gamma_c - \gamma$, the stability margin for the initial perturbation $|\phi(0) - \phi_0|$ increases as the operating point moves away from the critical curve into the linear stability region. Figure 2 also shows that there is an unstable limit cycle in the $\xi_1 - \xi_2$ plane and hence that if the condition (25) is not satisfied, the solutions are not bounded unless there is a stable limit cycle enclosing the unstable limit cycle in the $\xi_1 - \xi_2$ plane. We expect that indeed such a limit cycle with a large amplitude exists and could be obtained if we do higher order calculations. We suspect that nonlinear oscillations of linearly stable systems are characteristic not only of xenon-temperature controlled reactors, but also of any bounded nonlinear dynamical system whose corresponding neutral system is (nonlinearly) unstable.

The asymptotic form of the approximate solution under the condition (25) can be obtained as follows. As time increases,

$$(r(t), \theta(t)) \to (Ae^{\varepsilon\alpha_3 t}, \theta_0 - \varepsilon\alpha_4 t) \tag{26}$$

Substituting (26) into (16) to obtain the asymptotic form of $\xi_3(t)$ and then substituting these into (19), we obtain

$$\begin{aligned} \frac{\phi(t)}{\phi_0} \to 1 &+ Ae^{\alpha_3 t}\{(1 + ap_{25}) \cos (\nu t - \theta_0) + ap_{26} \sin (\nu t - \theta_0)\} \\ &+ A^2 e^{2\alpha_3 t}\{p_{27} + p_{28} \cos 2(\nu t - \theta_0) + p_{29} \sin 2(\nu t - \theta_0)\} \\ &+ \text{higher order terms} \end{aligned} \tag{27}$$

where

$$\nu = 1 + \alpha_4$$

$$A = \sqrt{\frac{\alpha_3}{\alpha_1}}\, Q$$

Hence the flux asymptotically approaches to the steady state level as a decaying oscillatory function of time.

Case II. Figure 3 shows that in this case an operating point is asymptotically stable in the large at least within the validity of the second order approximation in ε. We note that although Regions I and II belong to the linear stability region, the dynamic behaviors are quite different from each other, depending upon the stability properties of the respective reference point. The asymptotic form of the approximate solution is also given by (27) without any restriction on $x_1(0)$ to ensure the asymptotic stability.

Case III. Figure 4 shows that there is a stable limit cycle of radius $\sqrt{|\alpha_3/\alpha_1|}$ on the $\xi_1 - \xi_2$ plane. The corresponding limit cycle of $\phi(t)$ can be obtained as follows. As t increases, (17) and (18) yield

$$[r(t), \theta(t)] \rightarrow (r_0, \theta_1 + \varepsilon\alpha t) \tag{28}$$

where

$$r_0 \equiv \sqrt{\left|\frac{\alpha_3}{\alpha_1}\right|} \quad \text{and} \quad \alpha = \frac{\alpha_2\alpha_3}{\alpha_1} - \alpha_4$$

and θ_1 is a constant. Substituting (28) into (16) to obtain the asymptotic form of $\xi_3(t)$ and then substituting them into (19), we obtain

$$\frac{\phi(t)}{\phi_0} = 1 + (1 + ap_{30})r_0\cos(\nu t - \theta_1) + ap_{31}r_0\sin(\nu t - \theta_1)$$
$$+ r_0^2[p_{32} + p_{33}\cos 2(\nu t - \theta_1) + p_{34}\sin 2(\nu t - \theta_1)] + \text{higher order terms} \tag{29}$$

where $\nu = 1 - \alpha$.

We observe that there is a shift in the mean value of the flux oscillation from the equilibrium level ϕ_0, by an amount of

$$\phi_0 r_0^2 p_{32} = -\phi_0 r_0^2 \frac{a_0 a_{11} a_{21}}{2k} \tag{30}$$

which is proportional to a_{11} and hence to γ_c. Therefore, if γ_c is positive, then the flux oscillates with a mean value smaller than the equilibrium level. Since we have restricted ourselves only to positive values of γ, our approximate solutions agrees qualitatively with the exact relationship between φ_{av} and $(\varphi^2)_{\text{av}}$.

Case IV. Figure 5 shows that any solution except for $r = 0$ is unbounded within the validity of the second order approximation. Hence it is necessary to do a higher order calculation in order to obtain the boundedness. The fact that the second order calculation fails to give the boundedness implies that the limit cycle in this case will contain many higher harmonics so that it will be very distorted from purely sinusoidal shape.

In these four cases we did not try to obtain explicit time dependence of $x_2(t)$ and $x_3(t)$ since the temporal behavior of iodine and xenon cannot be observed during reactor operation.

COMPARISON OF OUR ANALYSIS WITH COMPUTER CALCULATION

We performed computer calculations to compare with our analytical result. In Case I our analysis predicts relaxation oscillations if the initial perturbation to the flux is sufficiently large so that (25) does not hold. A computer calculation was done for

$$\phi_0 = 10^{11}\,\text{cm}^{-2}\,\text{sec}^{-1} \quad \text{and} \quad \frac{\gamma - \gamma_c}{\gamma_c} = 0{\cdot}5 \times 10^{-2}.$$

The condition (25) then becomes

$$\left|\frac{\phi(0) - \phi_0}{\phi_0}\right| = |x_1(0)| < 1{\cdot}02.$$

The computer calculation showed that the solutions are asymptotically stable for $x_1(0) = 1{\cdot}5$ and $2{\cdot}0$, and a relaxation oscillation for $x_1(0) = 5{\cdot}0$ and $10{\cdot}0$. Hence our analysis gives a conservative estimate for the initial value $x_1(0)$ to ensure the asymptotic stability. One reason for this numerical discrepancy may be that we considered only the terms of order of unity in ε to obtain the condition (25). Another reason may be that we did not calculate in (22), the terms of higher orders in ε whose contribution should be negative and prevails for large r and hence gives a larger value for r_0 and a larger threshold value for $|x_1(0)|$. When a long time has passed after a sufficiently small pertubation at $t = 0$, $\phi(t)$ can be approximated by the first two terms in (27),

$$x_1(t) = (a\ \text{constant}) \times e^{\alpha_3 t} \cos(\nu t - \theta')$$

Table 2 shows the period and the ratio of the successive peak values of $x_1(0)$ for the analytical and computer results. In the computer calculation we used the Runge–Kutta method in which the time step is controlled in such a way that if the largest of the relative increment

$$\left|\frac{\Delta x_i}{x_i}\right| \qquad i = 1, 2 \text{ and } 3$$

in the last time step exceeds a given value e, the time step will be decreased to the half and the calculation will be repeated again. Table 2 also shows the dependence of the computer result on e. To analyze the solutions in Case II we can follow the same procedure as that for the asymptotically stable solutions in Case I.

In Case III a computer calculation was performed for

$$\phi_0 = 2 \times 10^{9}\,\text{cm}^{-2}\,\text{sec}^{-1}, \qquad \frac{\gamma - \gamma_c}{\gamma} = -0{\cdot}165 \quad \text{and} \quad e = 10^{-3}.$$

According to our analysis the period of the limit cycle turned out to be $1{\cdot}11\ T_c$ while the computer calculation gave $1{\cdot}09\ T_c$ where $T_c = 2\pi/\omega_c = 48{\cdot}9$ hr. Figure 6 shows the comparison of the shapes of the limit cycle obtained by these two methods. The agreement would be improved if a smaller value for e were used.

CONCLUSIONS

This work has been concerned with dynamics of the xenon-temperature controlled nuclear reactor which is a special case of rate equations with quadratic nonlinearities. It has been shown under the physically least restrictive initial conditions that the solutions are nonnegative and that, if the flux coefficient is positive (or stabilizing), the solutions are bounded from above and hence there should be isolated periodic or

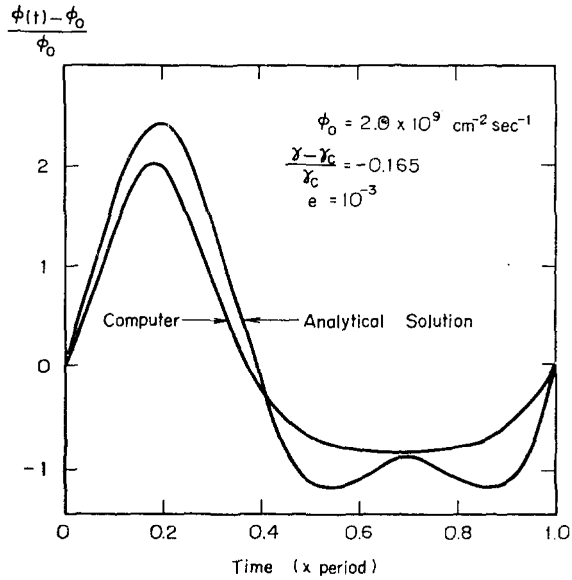

FIG. 6.—A comparison of computer and analytical results for the limit cycle in the linear instability region (Case III).

quasi-periodic solutions. An exact formula between the mean and mean square value of the incremental flux was obtained for any periodic solution. It was then found that the mean of the periodic solution for the flux is always less than the equilibrium level.

The Bogoliubov's method of averaging was applied to obtain the approximate solutions which, in turn, were used to investigate the stability of the system. In Case I the mechanism of the relaxation oscillation in the linear stability region following sufficiently large initial disturbances was analyzed and found due to the fact that the corresponding neutral system is (nonlinearly) unstable. An estimate of the upper bound for the initial disturbances to ensure the asymptotic stability also was obtained. In Case II the system is asymptotically stable in the large at least within the validity of the second order calculation. In Cases I and II the asymptotically stable solutions were obtained analytically. In Case III there exists a limit cycle which was also obtained analytically. Computer calculations were performed and it was found that they were in good agreement with our analytical results.

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APPENDIX

Method of averaging (CASE, 1963, unpublished paper; NOBLE, 1965)

A system of equations is "in Standard Form" if the equations are in the following vector form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}(t) = \varepsilon\mathbf{y}^{(1)}[\mathbf{z}(t), t] + \varepsilon^2\mathbf{y}^{(2)}[\mathbf{z}(t), t] + \dots + \dots + \varepsilon^n\mathbf{y}^{(n)}[\mathbf{z}(t), t] \tag{A.1}$$

where $\mathbf{z}$ is a vector whose elements are the dependent variables of the system, ε is the small parameter, and $\mathbf{y}^{(k)}$ are vectors which may depend on the dependent variables $\mathbf{z}$ as well as the independent variable t. The solution of (A.1) is formally constructed by means of the method of averaging in the following manner: Let

$$\mathbf{z}(t) = \boldsymbol{\xi}(t) + \varepsilon\mathbf{F}^{(1)}(\boldsymbol{\xi}, t) + \varepsilon^2\mathbf{F}^{(2)}(\boldsymbol{\xi}, t) + \dots \tag{A.2}$$

where $\mathbf{F}^{(k)}(\boldsymbol{\xi}, t)$ depends upon time as well as $\boldsymbol{\xi}$, and, in a manner to be described below, we try to choose $\boldsymbol{\xi}$ in such a way that $\mathbf{z}(t)$ does not become unbounded due to the explicit time dependent part of $\mathbf{F}^{(n)}$ and that the long time behavior of the solution is reflected in $\boldsymbol{\xi}$. Thus let

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}(t) = \epsilon\mathbf{G}^{(1)}(\boldsymbol{\xi}, t) + \epsilon^2\mathbf{G}^{(2)}(\boldsymbol{\xi}, t) + \dots \tag{A.3}$$

where $\mathbf{G}^{(n)}$ can hopefully be chosen in such a manner that the qualitative long time behavior of $\mathbf{z}(t)$ can be described by (A.3). Noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}(t) = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} + \sum_j \epsilon^j \frac{\partial}{\partial t}\mathbf{F}^{(j)} + \sum_j \epsilon^j\left(\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}\cdot\nabla_{\xi}\right)\mathbf{F}^{(j)} \tag{A.4}$$

and that the functions $\mathbf{y}^{(k)}$ in (A.1) can be Taylor expanded as

$$\begin{aligned}\mathbf{y}^{(k)}(\mathbf{z}, t) &= \mathbf{y}^{(k)}(\boldsymbol{\xi}, t) + [(\mathbf{z} - \boldsymbol{\xi})\cdot\nabla_{\xi}]\mathbf{y}^{(k)}(\boldsymbol{\xi}, t) \\ &\quad + \tfrac{1}{2}(\mathbf{z} - \boldsymbol{\xi})(\mathbf{z} - \boldsymbol{\xi}) : \nabla_{\xi}\cdot\nabla_{\xi}\mathbf{y}^{(k)}(\boldsymbol{\xi}, t) \\ &\quad + \dots\end{aligned} \tag{A.5}$$

we obtain the following relations for each like power in ε upon substituting the above relations into (A.1).

$$\begin{aligned}&\mathbf{G}^{(1)}(\boldsymbol{\xi}, t) + \frac{\partial}{\partial t}\mathbf{F}^{(k)}(\boldsymbol{\xi}, t) = \mathbf{y}^{(1)}(\boldsymbol{\xi}, t) \\ &\mathbf{G}^{(2)}(\boldsymbol{\xi}, t) + \frac{\partial}{\partial t}\mathbf{F}^{(2)}(\boldsymbol{\xi}, t) = \mathbf{y}^{(2)}(\boldsymbol{\xi}, t) - (\mathbf{G}^{(1)}\cdot\nabla_{\xi})\mathbf{F}^{(1)}(\boldsymbol{\xi}, t) + (\mathbf{F}^{(1)}\cdot\nabla_{\xi})\mathbf{y}^{(1)}(\xi, t) \\ &\qquad\qquad\qquad\qquad\quad \equiv \mathbf{R}^{(2)}(\boldsymbol{\xi}, t) \\ &\quad\vdots \\ &\mathbf{G}^{(n)}(\boldsymbol{\xi}, t) + \frac{\partial}{\partial t}\mathbf{F}^{(n)}(\boldsymbol{\xi}, t) = \mathbf{R}^{(n)}(\boldsymbol{\xi}, t) \\ &\quad\vdots\end{aligned} \tag{A.6}$$

where $\mathbf{R}^{(n)}(\boldsymbol{\xi}, t)$ are known functions of $\boldsymbol{\xi}$ and t since they involve solutions of previous equations in the hierarchy and the known functions $\mathbf{y}^{(k)}(\boldsymbol{\xi}, t)$ of $\boldsymbol{\xi}$ and t. The left hand side of the equation in (A.6) contains the two unknown functions $\mathbf{G}^{(n)}$ and $\mathbf{F}^{(n)}$. We will choose these two functions in such a manner that the equations (A.6) are satisfied and that $\mathbf{F}^{(n)}$ do not become unbounded through their explicit time dependence. $\mathbf{F}^{(n)}$ may become unbounded for large times, but it is desired that they do so only through their dependence on $\boldsymbol{\xi}$.

In most applications of the method of averaging which are described in the literature, $\mathbf{R}^{(n)}$ are functions which have a time average when $\boldsymbol{\xi}(t)$ is treated as a constant. However, this is not the case with the kinetic equations in question. We consider the following type of equation

$$\frac{dx}{dt} = A\mathbf{x} + \sum_{k=1}^{n} \epsilon^k \mathbf{f}^{(k)}(\mathbf{x}, t) \tag{A.7}$$

Equation (A.7) can be cast into standard form if A is a constant matrix with distinct eigenvalues. If any of the eigenvalues of A has a nonzero real part, as is the case with our problem, the time average of $\mathbf{R}^{(n)}$ will, in general, not exist. A straight forward generalization to the case where $\mathbf{R}^{(n)}$ does not have the time average has been presented by CASE (1963). Actually, in the case treated here, the extension is particularly simple. The idea is based upon the fact that the separation of $\mathbf{R}^{(n)}$ into $G^{(n)}$ and $\partial/\partial t\, \mathbf{F}^{(n)}$ should preserve the property

$$M_t \frac{\partial}{\partial t} \mathbf{F}^{(n)} = 0$$

where the operator M_t is defined as

$$M_t \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t.$$

suppose that

$$\mathbf{R}^{(n)}(\boldsymbol{\xi}, t) = \mathbf{R}_1^{(n)}(\boldsymbol{\xi}, t) + \mathbf{R}_2^{(n)}(\boldsymbol{\xi}, t)$$

where $\mathbf{R}_2^{(n)}$ is unbounded as $t \to \infty$. Then we choose

$$\mathbf{G}^{(n)} = M_t \mathbf{R}_1^{(n)} + \mathbf{R}_2^{(n)} \tag{A.9}$$

and

$$\frac{\partial \mathbf{F}^{(n)}}{\partial t} = (1 - M_t)\mathbf{R}_1^{(n)} \tag{A.10}$$

We note that (A.10) indeed preserves the property (A.8). The constant of integration for $\mathbf{F}^{(n)}$ is determined by the further requirement that

$$M_t \mathbf{F}^{(n)} = 0$$

The solution for $\mathbf{z}$ is then obtained by substituting (A.9) into (A.3), solving for $\boldsymbol{\xi}(t)$ and by integrating (A.10) with respect to t (holding $\boldsymbol{\xi}$ fixed).