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On the Number of Unique Subgraphs*

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Entringer and Erdös introduced the concept of a unique subgraph of a given graph G and obtained a lower bound for the maximum number of unique subgraphs in any n-point graph, which we now improve.

1. A BOUND ON THE MAXIMUM NUMBER OF UNIQUE SUBGRAPHS

We say a spanning subgraph H of a graph G with n points is *unique* if H is not isomorphic to any other subgraph of G. In other words, if α is an isomorphism mapping H onto a second subgraph H', then $H' = H$ and α is in fact an automorphism of H. Following [2], we write K_n and $\overline{K}_n$ for the complete n-point graph and its complement. Obviously G itself is a unique subgraph, and so is $\overline{K}_n$. Let $f(G)$ be the number of unique spanning subgraphs in G, and let

$$f(n) = \max\{f(G) : G \text{ has } n \text{ points}\}.$$

Entringer and Erdös [1] used a clever device to construct an n-point graph G for each $c > 3\sqrt{2}/2$ having

$$f(G) \geqslant 2^{n^2/2 - cn^{3/2}}. \tag{1}$$

This provides a lower bound for $f(n)$. By a slight modification of their construction and by means of a related lemma, we are able to improve this bound.

THEOREM. *For each $c > 2$, there exists an $N = N(c)$ such that for all $n \geqslant N$,*

$$2^{n^2/2} n^{-cn} < f(n) \leqslant 2^{n^2/2 - n/2}. \tag{2}$$

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Proof. The upper bound for $f(n)$ is trivial since it is simply the total number of possible labeled graphs on n points, as observed in [2, p. 178]. The lower bound is the more difficult part and, for clarity, we first examine a very special case.

Special Case.

$$n = \binom{\{ck\}}{k} + \{ck\} \qquad \text{with} \qquad k \geqslant 3.$$

For convenience, we write

$$m = \binom{\{ck\}}{k}.$$

We proceed to construct a graph G which will have many unique subgraphs. Let A be a $\{ck\}$-point identity graph having maximum degree 3. Such graphs are known to exist for $\{ck\} \geqslant 6$, see [2, p. 161]. Let B be the complete graph K_m. We form the n-point graph G by setting up a one-to-one correspondence between each point of B and each k-subset of $V(A)$, the point set of A. We then join each point of B to its k corresponding points in A to obtain G.

Now each B-point of G has degree $m - 1 + k$, but each A-point of G has degree at most $3 + km/\{ck\} \leqslant 3 + m/c$. Now suppose H is a subgraph of G obtained by removing the lines of a subgraph J of B having maximum degree at most $m(c-1)/c$. We claim H is a unique subgraph, for, if not, then there is an isomorphism α mapping H onto $H' \neq H$. Now all the B-points in H have degree greater than that of the A-points in G, and so α must permute the B-points. Similarly, α must permute the A-points. But A is an identity graph and all its lines are in H. Therefore, α restricted to A is the identity. Moreover, each B-point is joined to a unique k-subset of $V(A)$, and so each B-point is also fixed by α. Thus α is the identity automorphism, and H is an identity subgraph of G. It remains to estimate how many subgraphs of K_m have maximum degree at most $m(c-1)/c$. Call this number $g(k, c)$.

LEMMA. *For every* $\epsilon > 0$ and $c > 2$, there exists a $k_0 = k_0(c, \epsilon)$ such that, for $k \geqslant k_0$,

$$2^{\binom{m}{2}}(1-\epsilon) < g(k, c) \leqslant 2^{\binom{m}{2}}. \tag{4}$$

Proof of lemma. Once again, the upper bound is trivially the number of all m-point labeled graphs. To obtain the lower bound, we use a crude but adequate estimate on $g(k, c)$, the number of labeled graphs having maximum degree at most $\Delta = [m(c-1)/c]$. Only those graphs having larger maximum degree fail to contribute to $g(k, c)$. There are m ways to select a point which is to have degree larger than Δ, then there are $2^{(m-1)(m-2)/2}$ ways to add lines among the remaining $m-1$ points, and finally, $\binom{m-1}{i}$ ways for that point to have degree $i > \Delta$. We have shown that

$$\begin{aligned} g(k, c) &\geqslant 2^{\binom{m}{2}} - m\, 2^{\binom{m-1}{2}} \sum_{i=\Delta+1}^{m-1} \binom{m-1}{i} \\ &\geqslant 2^{\binom{m}{2}} \left\{1 - m\, 2^{1-m} \sum_{i=\Delta+1}^{m-1} \binom{m-1}{i}\right\}. \end{aligned} \tag{5}$$

We now proceed to simplify (5) by further estimation of these bounds. First, we recognize 2^{1-m} times the sum in (5) as the probability that a random variable x from a binomial distribution of $m-1$ trials exceeds Δ (see Lindgren and McElrath [3, p. 61]). Writing this as $P(x > \Delta)$, we obtain

$$g(k, c) \geqslant 2^{\binom{m}{2}}\{1 - mP(x > \Delta)\}. \tag{6}$$

But for large m this probability is accurately estimated by a normal distribution with mean $(m-1)/2$ and standard deviation $\sqrt{m-1}/2$, which can then be expressed in terms of the normal distribution

$$\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

(see [3, p. 99]) to yield

$$P(x > \Delta) = P(x \leqslant m - \Delta - 2) = \Phi\left(\frac{m/2 - \Delta - 1}{\sqrt{m-1}/2}\right). \tag{7}$$

But now

$$m/2 - (\Delta + 1) \leqslant m/2 - m(c-1)/c = m(2-c)/2c, \tag{8}$$

so since Φ is monotonic we can replace the lower bound in (6) by

$$g(k, c) \geqslant 2^{\binom{m}{2}}\{1 - m\Phi(m(2-c)/c\sqrt{m-1})\}. \tag{9}$$

Now, since $e^{-t^2/2} < 3/t^4$ for all $t < 0$, we may bound $\Phi(x)$ for $x < 0$ by

$$\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2}\, dt < \int_{-\infty}^{x} (3/t^4)\, dt = -1/x^3, \tag{10}$$

so that we may replace Φ in (9) by the larger quantity

$$-(c\sqrt{m-1}/m(2-c))^3$$

to obtain

$$g(k, c) \geqslant 2^{\binom{m}{2}}\{1 + mc^3(m-1)^{3/2}/(2-c)^3 m^3\}. \tag{11}$$

Recalling that $c > 2$, we find

$$g(k, c) \geqslant 2^{\binom{m}{2}}\{1 - c^3(m-1)^{3/2}/(c-2)^3 m^2\}. \tag{12}$$

Consequently, given any $\epsilon > 0$, we may choose k_0 so that m is large enough to assure that

$$g(k, c) > 2^{\binom{m}{2}}(1-\epsilon) \tag{13}$$

for $k \geqslant k_0$, as required.

We now return to the proof of the theorem. By our construction,

$$f(G) \geqslant g(k, c) > 2^{\binom{m}{2}}(1-\epsilon). \tag{14}$$

Recall that

$$m = n - \{ck\} \qquad \text{and} \qquad m = \binom{\{ck\}}{k} > c^k$$

so $\log m > k \log c$, and hence

$$\begin{aligned} m &\geqslant n - 1 - ck \\ &> n - 1 - c \log m/\log c \\ &\dot{>} n - 1 - c \log n/\log c. \end{aligned} \tag{15}$$

Thus, defining δ by the equation $2^\delta = 1 - \epsilon$, we have

$$\begin{aligned} f(G) \geqslant 2^{\binom{m}{2}}(1-\epsilon) &= 2^{m(m-1)/2+\delta} \\ &> 2^{(n-1-c\log n/\log c)(n-2-c\log n/\log c)/2+\delta} \\ &> 2^{n^2/2-cn\log n/\log c+c_1 n+c_2\log n+c_3+\delta} \end{aligned} \tag{16}$$

for appropriate constants c_1, c_2, and c_3. Now $\log c > \log 2$, so we observe that

$$2^{n^2/2-cn\log n/\log c+c_1 n+c_2\log n+c_3+\delta} > 2^{n^2/2-cn\log n/\log 2}. \tag{17}$$

Consequently, we have the result of the theorem

$$f(G) > 2^{n^2/2-cn\log n/\log 2} = 2^{n^2/2-cn\log_2 n} = 2^{n^2/2-n/2}. \tag{18}$$

It is important to keep in mind that this was accomplished under the assumption that $n = \binom{\{ck\}}{c} + \{ck\}$. To handle the general case of other values of n, we choose k so that

$$\binom{\{c(k-1)\}}{k-1} + \{c(k-1)\} < n \leqslant \binom{\{ck\}}{k} + \{ck\}. \tag{19}$$

We now proceed to define G in a similar manner, only now just some of the $\binom{\{ck\}}{k}$ subsets of A of size k are used. The bounds that arise are considerably more intricate, but the same sorts of crude estimates again suffice, and, omitting these tedious details, we again obtain equation (18) for all $n > N$ as specified in the theorem.

References

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3. B. W. Lindgren and G. W. McElrath, "Introduction to Probability and Statistics," Macmillan, New York, 1959.