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Note on the Modified Two-Stream Approximation of Sagan and Pollack

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The modified two-stream approximation is derived from the radiative transfer equation using a two-point gaussian quadrature and neglecting terms of order $l \geqslant 3$ in the Legendre expansion of the phase function. The results are compared with the Eddington approximation and with exact results for the special cases of perfect absorption, perfect scattering, and a semi-infinite layer with isotropic scattering.

The modified two-stream approximation developed by Sagan and Pollack (1967) is useful for problems involving the transmission of diffuse radiation through scattering and absorbing layers, or for evaluating the spherical albedo of such layers for direct radiation. It has been used, for example, by Rasool and Schneider (1971) in calculating the effect of aerosols on the global climate.

The purpose of this note is to clarify some points in the derivation of this approximation and to compare it with the Eddington approximation. The results of both approximations are then numerically compared with exact results in the three special cases of perfect absorption, perfect scattering, and a semi-infinite layer with isotropic scattering.

Sagan and Pollack's derivation begins with the transfer equation, which may be written as follows:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\tau} I(\tau,\mu) = -I(\tau,\mu) + \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')\, p(\mu,\mu')\,\mathrm{d}\mu' \quad (1)$$

where $I(\tau, \mu)$ is the specific intensity and $p(\mu, \mu')$ is the scattering phase function, both averaged over azimuthal angles. τ is the optical depth and μ is the cosine of the angle between the direction of propagation and the vertical.

From this equation, Sagan and Pollack immediately write down the two-stream equations as follows:

$$\frac{1}{\sqrt{3}}\frac{\mathrm{d}I_+}{\mathrm{d}\tau} = -I_+ + I_+\tilde{\omega}_0(1-\beta) + I_-\tilde{\omega}_0\beta$$

$$\frac{-1}{\sqrt{3}}\frac{\mathrm{d}I_-}{\mathrm{d}\tau} = -I_- + I_-\tilde{\omega}_0(1-\beta) + I_+\tilde{\omega}_0\beta \quad (2)$$

where $\tilde{\omega}_0$ is the single-scattering albedo and β is a measure of the fraction of radiation singly scattered into the backward hemisphere of the incident radiation. I_+ and I_- are identified as the average specific intensity in the positive and negative μ hemispheres, respectively. The factor $1/\sqrt{3}$ is explained as an appropriate average value of μ in the two-stream approximation (this value is derived from Gauss's quadrature formula, as shown by Chandrasekhar, 1960).

Using these equations with the boundary condition of zero radiation incident on the bottom of the layer, Sagan and Pollack derived the following expressions for the reflectivity, transmissivity, and absorptivity of the layer:

$$\mathscr{R} = \frac{(u+1)(u-1)[\exp(\tau_{\mathrm{eff}}) - \exp(-\tau_{\mathrm{eff}})]}{(u+1)^2\exp(\tau_{\mathrm{eff}}) - (u-1)^2\exp(-\tau_{\mathrm{eff}})} \quad (3)$$

$$\mathscr{T} = \frac{4u}{(u+1)^2\exp(\tau_{\mathrm{eff}}) - (u-1)^2\exp(-\tau_{\mathrm{eff}})} \quad (4)$$

$$\mathscr{A} = 1 - \mathscr{R} - \mathscr{T} \quad (5)$$

where

$$u^2 = \frac{1 - \tilde{\omega}_0 + 2\beta\tilde{\omega}_0}{1 - \tilde{\omega}_0} \quad (6)$$

$$\tau_{\mathrm{eff}} = [3(1 - \tilde{\omega}_0)(1 - \tilde{\omega}_0 + 2\beta\tilde{\omega}_0)]^{1/2}\tau_1. \quad (7)$$

By comparing these results with those of Piotrowski (1956) in the case $\tilde{\omega}_0 = 1$ and $\tau_1 \gg 1$, Sagan and Pollack arrived at the following expression for β:

$$\beta = \tfrac{1}{2}(1 - \tfrac{1}{3}\tilde{\omega}_1) \quad (8)$$

where $\tilde{\omega}_1$ is the first-order coefficient in the Legendre expansion of the phase function.

The two-stream equations (2) and the correct expression for β follow directly from the transfer equation if I_+ and I_- are identified instead as the values of the specific intensity at

$$\mu = \pm\frac{1}{\sqrt{3}}$$

(the first-order gaussian quadrature points, or the zeros of P_2). Substituting these values of μ into the transfer equation, and setting

$$I_\pm(\tau) \equiv I\left(\tau, \pm\frac{1}{\sqrt{3}}\right) \quad (9)$$

we obtain

$$\frac{1}{\sqrt{3}}\frac{\mathrm{d}I_+}{\mathrm{d}\tau} = -I_+ + \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')\,p\left(\frac{1}{\sqrt{3}},\mu'\right)\mathrm{d}\mu' \quad (10)$$

$$\frac{-1}{\sqrt{3}}\frac{\mathrm{d}I_-}{\mathrm{d}\tau} = -I_- + \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')\,p\left(\frac{-1}{\sqrt{3}},\mu'\right)\mathrm{d}\mu'. \quad (11)$$

Using the two-point gaussian quadrature formula (Chandrasekhar, 1960, p. 61) these integrals may be written

$$\int_{-1}^{1} I(\tau,\mu')\,p\left(\frac{\pm 1}{\sqrt{3}},\mu'\right)\mathrm{d}\mu' \cong I_+(\tau)\,p\left(\frac{\pm 1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + I_-(\tau)\,p\left(\frac{\pm 1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right). \quad (12)$$

If the phase function is expanded in a series of Legendre polynomials with coefficients $\tilde{\omega}_l$, we have from the addition theorem of spherical harmonics (Chandrasekhar, 1960, p. 150)

$$p(\mu,\mu') = \sum_{l=0}^{\infty} \tilde{\omega}_l\,P_l(\mu)\,P_l(\mu'). \quad (13)$$

Dropping terms of order $l \geqslant 3$, we then have

$$p\left(\pm\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) = \tilde{\omega}_0 \pm \tfrac{1}{3}\tilde{\omega}_1 \quad (14)$$

$$p\left(\pm\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) = \tilde{\omega}_0 \mp \tfrac{1}{3}\tilde{\omega}_1. \quad (15)$$

The two-stream equations (2) then follow directly from (10), (11), and (12), with

$$\beta = \frac{1}{2}\left(1 - \frac{1}{3}\frac{\tilde{\omega}_1}{\tilde{\omega}_0}\right) \quad (16)$$

which is the correct expression in the case $\tilde{\omega}_0 \leqslant 1$.

To reiterate, the important points in the derivation of the two-stream equations are (1) the use of the two-point gaussian quadrature for the evaluation of the integral term in the transfer equation, and (2) the neglect of the terms of order $l \geqslant 3$ in the Legendre expansion of the phase function.

In deriving the expressions for the transmissivity and reflectivity from the two-stream equations, the further assumption is made that the hemispheric flux in the upward and downward directions is proportional to the intensity at

$$\mu = \pm\frac{1}{\sqrt{3}}.$$

This is essentially the application of the two-point gaussian quadrature to the flux integral.

In the Eddington approximation, the specific intensity is expanded in a series of Legendre polynomials which is truncated after the first two terms. That is,

$$I(\tau,\mu) = I_0(\tau) + I_1(\tau)\,\mu. \quad (17)$$

The equations which result from the substitution of this expression into the transfer equation (Shettle and Weinman,

1970) are equivalent to the two-stream equations under the change of variable

$$I_{\pm}(\tau) = I_0(\tau) \pm \frac{1}{\sqrt{3}} I_1(\tau). \tag{18}$$

These equations may be solved, with the boundary condition of zero upward flux at the lower boundary, to yield the reflectivity and transmissivity as in the case of the two-stream approximation. The results of the Eddington approximation may be written in the same form as the two-stream approximation, except with

$$u^2 = \frac{3}{4} \frac{1 - \tilde{\omega}_0 + 2\beta\tilde{\omega}_0}{1 - \tilde{\omega}_0}. \tag{19}$$

These results are plotted and compared with exact results in Figs. 1–3 for the three special cases $\tilde{\omega}_0 = 0$, $\tilde{\omega}_0 = 1$, and $\tau_1 = \infty$.

In the case $\tilde{\omega}_0 = 0$, the two-stream and Eddington approximations give nearly indistinguishable values of the transmissivity, although the expressions are different. In this case the transfer equation can be solved exactly (the integral term vanishes), with the result

$$\mathscr{T} = 2E_3(\tau_1), \tag{20}$$

which is also plotted in Fig. 1.

For $\tilde{\omega}_0 = 1$, the two-stream approximation gives slightly greater values of the reflectivity than the Eddington approximation. The former is in somewhat better agreement with the exact results computed by Hansen (1969) for small τ_1, while the opposite is true for large τ_1.

For a semi-infinite layer with isotropic scattering ($\tau_1 = \infty$, $\tilde{\omega}_1 = 0$) the two-stream approximation gives decidedly better results than the Eddington approximation, as compared with the exact results computed by Chandrasekhar (1960, p. 125). The Eddington approximation, in fact,

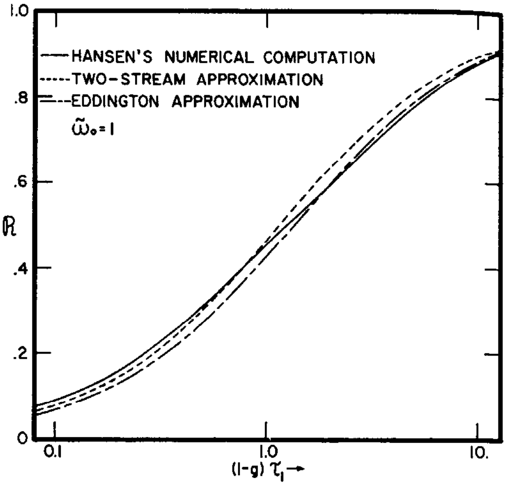

FIG. 2. Reflectivity of a perfectly scattering layer of optical thickness τ_1, with $\tilde{\omega}_1 = 3\tilde{\omega}_0 g$.

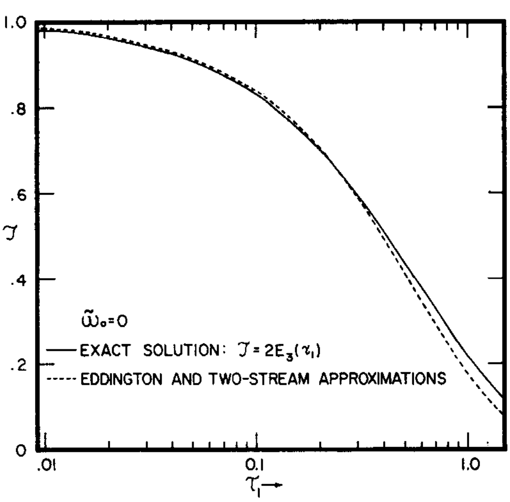

FIG. 1. Transmissivity of a perfectly absorbing layer of optical thickness τ_1.

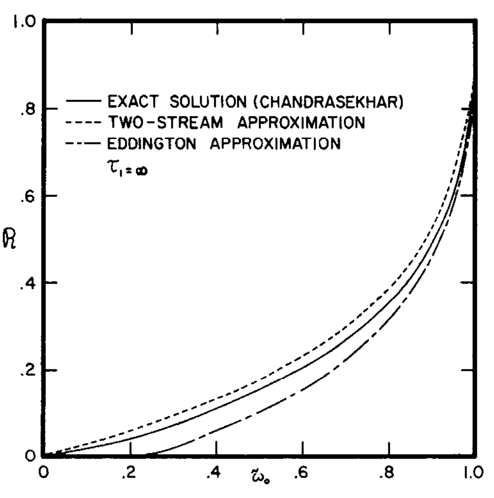

FIG. 3. Reflectivity of a semi-infinite scattering and absorbing layer, with isotropic scattering ($\tilde{\omega}_1 = 0$).

yields a negative reflectivity when $u < 1$, or

$$\tilde{\omega}_0 < \frac{1}{4}(1 + \tilde{\omega}_1). \tag{21}$$

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