On Microwave Bremsstrahlung from a Cool Plasma

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I. Introduction

In a relatively cool plasma, the major source of Bremsstrahlung is expected to be electron encounters with neutrals, the low degree of ionization compensating for the disparity in cross section between this and electron-ion Bremsstrahlung. Since the nuclear charge is screened by atomic electrons, a major problem in obtaining the cross section or radiated power is determining the effective value of Z, the atomic number, for the neutrals. Likewise, Debye shielding is operative with the ions. In plasmas with low degree of ionization, the Debye length is sufficiently great that such shielding may be neglected for the neutrals, of course, whose own screened fields drop off much more rapidly.

Some previous work along these lines exists. For example, Breen and Nardone (Ref. 1), working with O atoms at 8000^{o}K, find the free-free absorption cross section to be given by the classical Kramers' law, with an effective $Z = 0.31$ at 10,000 Å and $Z = 0.27$ at 20,000 Å. Their matrix element, involving only S and P partial waves, is the result of machine computation. Now Kramers' law is the high-frequency limit of the exact Sommerfeld cross section for a pure Coulomb potential and, as can be shown by putting in a few numbers, would in the absence of screening describe the contribution from most of the

velocity distribution at the temperatures and frequencies they use. However, for microwaves the cross section is not necessarily of the same form, and since the importance of screening is a function of electron velocity for a given radiation frequency, the effective Z for microwave radiation is not necessarily the same as theirs, and is moreover different for different parts of the electron velocity distribution, over which they have averaged.

Rather than using machine wave functions, we base our work on a screened Coulomb potential, with the screening radius taken as the Debye length for ion Bremsstrahlung and obtained from a Thomas-Fermi model for neutrals. Since the Schrödinger equation cannot be solved exactly for this potential, approximate solutions must be used. The range of validity of these approximations depends on the microwave frequency, Debye shielding length, and electron velocity. Thus, for the sake of concreteness, we specify these by choosing ν, electron density, and T

$$\nu = 50 \text{ KMc}$$

$$n_e = 10^{13}/\text{cm}^3$$

$$T = 5 \times 10^3 \ {}^\circ\text{C} .$$

II. Electron-Ion Bremsstrahlung from the Faster Electrons

In this section we refer extensively to the work of Dewitt (Refs. 2, 3), who has applied the Born approximation to the Debye-screened Coulomb potential to treat Bremsstrahlung in a fully-ionized gas. He has investigated the validity of this approximation in detail. Let us refer to incident and scattered quantities by the subscripts 1 and 2. It is useful to talk in terms of $n_{1,2} = \frac{Ze^2}{\hbar v_{1,2}}$. For the pure Coulomb field the Sommerfeld exact Bremsstrahlung cross section may be expanded in powers of $n_2 - n_1$; the first term of this expansion is the Born approximation result. Thus for the pure Coulomb field, the Born approximation describes well the situation of low-frequency radiation ($v_1 \sim v_2$) from not-too-slow electrons. That is, $n_1 >> 1$ is permissable as long as $n_2 - n_1 << 1$. For a screened potential, although there is no exact solution to compare it with, the Born approximation is shown by Dewitt to be even better for the low-frequency part of the spectrum, i.e. it is valid for smaller v_1. Since we note that for $v_1 = \sqrt{\frac{kT}{m}}$ and $\nu = 50$ KMc, $n_1 \sim 8$, and $n_2 - n_1 \sim 10^{-4}$, we shall use it down to this value of v_1. For faster electrons it is of course even better, but we restrict ourselves to non-relativistic electrons, naturally.

In what follows we use the following notation:

P = momentum in energy units = mvc

K = photon energy = $h\nu$

$\mu = mc^2$

$r_o = e^2/mc^2$

$$\alpha = e^2/\hbar c^2$$

$$\sigma_0 = \frac{16}{3}\alpha Z^2 r_o^2 \mu^2 / P_1^2$$

$$\gamma = \hbar c/\lambda$$

$$\lambda, \text{ the Debye length } = \sqrt{\frac{kT}{4\pi n_e e^2}}$$

In terms of these, the Born differential Bremsstrahlung cross section may be written

$$d\sigma(K, P_1) = \sigma_0 \frac{dK}{K}\left\{\frac{1}{2}\ln\frac{(P_1+P_2)^2+\gamma^2}{(P_1-P_2)^2+\gamma^2} - \frac{2\gamma^2 P_1 P_2}{[(P_1+P_2)^2+\gamma^2][(P_1-P_2)^2+\gamma^2]}\right\} \tag{2.1}$$

in which, of course, P_2 is to be eliminated by conservation of energy,

$$P_1^2 - P_2^2 = 2K\mu \tag{2.2}$$

Since the power is obtained by multiplying the cross section by the incident flux and photon energy, and integrating over the electron velocity distribution, the contribution from this velocity range to the power/ unit volume/ circular frequency interval may be written, where $Z = 1$ for the ions always,

$$\underline{P}\, d\omega = n_e^2 \frac{16}{3} Z^2 e^6 \frac{4\pi}{m^4 c^5}\left(\frac{m}{2\pi kT}\right)^{3/2} d\omega \int_{c\sqrt{kTm}}^{\infty} dP_1 \quad P_1 e^{-P_1^2/2\mu kT}$$

$$\left\{\frac{1}{2}\ln\frac{(P_1+P_2)^2+\gamma^2}{(P_1-P_2)^2+\gamma^2} - \frac{2\gamma^2 P_1 P_2}{[(P_1+P_2)^2+\gamma^2][(P_1-P_2)^2+\gamma^2]}\right\} \tag{2.3}$$

It is not possible to introduce approximations to the bracketed term valid over the whole range of integration. $P_1 >> \gamma$ is always valid, and thus $(P_1+P_2) >> \gamma$. However, $P_1 - P_2 = \gamma$ at about $P_{1c} = 15c\sqrt{kTm}$, and since $P_1 - P_2 \sim K\mu/P_1$,

$P_1 - P_2 < \gamma$ for greater P_1. Since most of the contribution to this integral comes from $P_1 < P_{1c}$, a fair approximation to the bracket is, if one is required,

$\ln\left(\frac{2P_1^2}{K\mu}\right) - \frac{\gamma^2 P_1^2}{2(K\mu)^2}$.

Now for the very slowest electrons with $P_1^2 > 2K\mu$, the microwave radiation is the high-frequency limit of their spectrum. The Born approximation fails here for the pure Coulomb field, but the Sommerfeld solution is reproduced excellently when the Born approximation is modified by the Elwert factor. Although its use cannot clearly be justified for the screened potential, we might hope the screening is weak enough to approximate the Coulomb case, and that it would be correct order-of-magnitude.

The expression is

$$d\sigma_{B-E}(K, P_1) = \sigma_0 \frac{dK}{K} \frac{P_1}{P_2} \frac{1 - e^{-2\pi n_1}}{1 - e^{-2\pi n_2}}$$

$$\left\{ \frac{1}{2} \ln \frac{(P_1+P_2)^2+\gamma^2}{(P_1-P_2)^2+\gamma^2} - \frac{2\gamma^2 P_1 P_2}{\left[(P_1+P_2)^2+\gamma^2\right]\left[(P_1-P_2)^2+\gamma^2\right]} \right\}. \tag{2.4}$$

The contribution from the very lowest velocities is described by the limit here $P_2/P_1 \longrightarrow 0$, $P_1 >> \gamma$, or

$$d\sigma_{B-E}(K, P_1) \longrightarrow 2\sigma_0 \frac{dK}{K} , \tag{2.5}$$

which differs from the Kramers' result in having a factor of 2 rather than $\frac{\pi}{\sqrt{3}}$.

These slowest electrons then contribute (again $Z = 1$ for 0^+.)

$$\underline{P} d\omega = n_e^2 \frac{32}{3} Z^2 e^6 \frac{4\pi}{m^4 c^5} \left(\frac{m}{2\pi kT}\right)^{3/2} d\omega \int P\, dP\, e^{-P^2/2\mu kT} . \tag{2.6}$$

This is valid for only a very small range, since for P_1 = even $2c\sqrt{2mK}$, $P_2 / P_1 = \sqrt{3/2}$, which is certainly not the high-frequency limit.

We would like now a cross section valid for electron velocities between the very lowest and the thermal range. The classical impulse approximation will be used to furnish such a result.

III. The Impulse Approximation

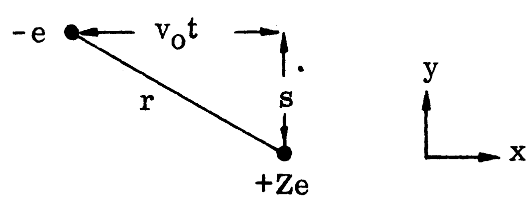
Ideally, an exact classical trajectory treatment should be used to fill in the gap here. However, this cannot be carried out, and we are forced to resort to the impulse approximation, which has been used elsewhere (Roberts, Ref. 4) for Bremsstrahlung in a pure Coulomb field. Since a screened field causes less acceleration at large distances, the impulse approximation should be better for a given electron velocity here than for the Coulomb case.

Now in a classical treatment we obtain a particle trajectory which is a function of the impact parameter s and the initial velocity v_o. For fixed values of these, the radiated power from one electron in $d\omega$ is (Ref. 5)

$$P_{s, v_o} d\omega = \frac{8\pi}{3} \frac{e^2}{c^3} \left|\vec{a}(\omega)\right|^2 d\omega \quad , \tag{3.1}$$

where $\vec{a}(\omega)$ is the Fourier transform of the vector acceleration, and $\left|\vec{a}(\omega)\right|^2 = \left[a_x(\omega)\right]^2 + \left[a_y(\omega)\right]^2$. This is then multiplied by the flux $N_n d_n(v_o) v$, and averaged over annuli of radius s. Finally, of course, we average over that part of the electron velocity spectrum for which the expression is valid.

The impulse approximation consists in taking the acceleration which would be associated with an undeviated straight-line trajectory, i.e. acceleration but not displacement results from the presence of the scattering center, which is like the effect of an impulse. The geometry for the calculation is given by the following sketch:



Then

$$\left.\begin{aligned} r^2 &= s^2 + v_o^2 t^2 & x &= v_o t \quad , \quad y = s \\ \vec{a} &= -\frac{1}{m}\nabla V(r) & V(r) &= -Ze^2 \frac{e^{-r/\lambda}}{r} \equiv -Ze^2 W \\ a_x &= -\frac{Zq^2}{m}\frac{x}{r}\frac{\partial W}{\partial r} & a_y &= -\frac{Zq^2}{m}\frac{y}{r}\frac{\partial W}{\partial r} \end{aligned}\right\} \qquad (3.2)$$

and, where the bar here indicates a Fourier transform on a, or $a(\omega)$,

$$\bar{a}_x = -\frac{Ze^2 v_o}{2\pi m}\int_{-\infty}^{\infty}\frac{t e^{i\omega t}}{r}\frac{\partial W}{\partial r}\,dt\ , \qquad \bar{a}_y = -\frac{Ze^2 s}{2\pi m}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{r}\frac{\partial W}{\partial r}\,dt\ . \qquad (3.3)$$

Since $g(r) \equiv -\frac{\partial W}{\partial r}$ is even in t, we have

$$\bar{a}_x = \frac{Ze^2 v_o i}{\pi m}\int_0^{\infty} t \sin \omega t\, g(r)\frac{dt}{r}\ ,$$

$$\bar{a}_y = \frac{Ze^2 s}{\pi m}\int_0^{\infty}\cos \omega t\, g(r)\frac{dt}{r}\ . \qquad (3.4)$$

Consider the integral in $\bar{a}_x$, which we will call $I(\omega)$. We change the integration variable to r from (3.2), and integrate once by parts, obtaining

$$I(\omega) = \frac{\omega}{v_o^3}\int_0^{\infty}\frac{\cos \frac{\omega}{v_o}\sqrt{r^2-s^2}\, e^{-r/\lambda}}{\sqrt{r^2 - s^2}}\,dr \qquad (3.5)$$

and with $r = s\cosh t$, we have finally

$$I(\omega) = \frac{\omega}{v_o^3}\int_0^{\infty}\cos\left[\frac{\omega s}{v_o}\sinh t\right] e^{-s/\lambda \cosh t}\,dt$$

$$= \frac{\omega}{v_o^3} K_o\left[\sqrt{\left(\frac{s}{\lambda}\right)^2 + \left(\frac{s\omega}{v_o}\right)^2}\right]\ , \qquad (3.6)$$

in which K is a modified Hankel function. Then

$$\bar{a}_x = \frac{i Z e^2 \omega}{\pi m v_o^2} K_o \left[\sqrt{\left(\frac{s}{\lambda}\right)^2 + \left(\frac{s\omega}{v_o}\right)^2} \right] . \tag{3.7}$$

But

$$\frac{\partial \bar{a}_y}{\partial \omega} = - \frac{Z e^2 s}{\pi m} I(\omega) , \tag{3.8}$$

so

$$\bar{a}_y = \frac{Z e^2 s}{\pi m} \left[- \int_0^\omega I(x)\, dx + \int_0^\infty g(r) \frac{dt}{r} \right] . \tag{3.9}$$

Now consider

$$J(\omega) = \int_0^\omega I(x)\, dx = \frac{1}{v_o^3} \int_0^\omega x K_o \left[\sqrt{a^2 + b^2 x^2} \right] dx , \tag{3.10}$$

$$\text{where} \quad a = \frac{s}{\lambda} , \quad b = \frac{s}{v_o}$$

$$= \frac{1}{b^2 v_o^3} \int_a^{\sqrt{a^2 + b^2 \omega^2}} u K_o(u)\, du$$

$$= \frac{1}{b^2 v_o^3} \left[a K_1(a) - \sqrt{a^2 + b^2 \omega^2}\, K_1 \left(\sqrt{a^2 + b^2 \omega^2} \right) \right] . \tag{3.11}$$

We now need only

$$F = \int_0^\infty g(r) \frac{dt}{r} = \frac{1}{v_o} \int_s^\infty \frac{g(r)\, dr}{\sqrt{r^2 - s^2}} ,$$

which with the substitution $r = s \cosh t$ becomes

$$F = \frac{1}{v_o s^2} \int_0^\infty e^{-a \cosh t} \left[\frac{1}{\cosh^2 t} + \frac{a}{\cosh t} \right] dt . \tag{3.12}$$

Consider the integral here, which is a function of a only; call it $f(a)$. Then

$$f'(a) = - a \int_0^\infty e^{-a \cosh t}\, dt = - a K_o(a) \tag{3.13}$$

so that

$$f(a) = -\int_{\infty}^{a} x K_o(x)\, dx - \lim_{x\to\infty} xK_1(x) = aK_1(a) \tag{3.14}$$

and

$$F = \frac{aK_1(a)}{v_o s^2} \quad . \tag{3.15}$$

Then

$$\bar{a}_y = \frac{Ze^2}{\pi m v_o} \sqrt{(\frac{1}{\lambda})^2 + (\frac{\omega}{v_o})^2} \quad K_1\left[s\sqrt{(\frac{1}{\lambda})^2 + (\frac{\omega}{v_o})^2}\right] \tag{3.16}$$

$$\bar{a}_x = \frac{i\, Ze^2 \omega}{\pi m v_o^2} \; K_o \left[s \sqrt{(\frac{1}{\lambda})^2 + (\frac{\omega}{v_o})^2}\right] \quad . \tag{3.17}$$

It should be noted that for $\lambda \rightarrow \infty$ (no shielding), these agree with the corresponding quantities deduced by Roberts (Ref. 5). Then we have for the quantity $|\vec{a}(\omega)|^2$, which we may call A^2,

$$A^2 = C^2 \left[\delta^2 K_1^2(\delta s) - \frac{\omega^2}{v_o^2} \; K_o^2(\delta s)\right] \tag{3.18}$$

with

$$C = \frac{Ze^2}{\pi m v_o} \quad , \qquad \delta^2 = (\frac{1}{\lambda})^2 + (\frac{\omega}{v_o})^2 \quad .$$

Next we want to average over impact parameter s. We have no improvement to suggest over the usual procedure of taking the lower limit at $s_{min} = \lambda_{DeBroglie}$, the distance within which the electron cannot be localized, so that it makes no sense to talk about closer approaches to the nucleus. Then we must evaluate the integral

$$B = 2\pi \int_{\epsilon}^{\infty} s\, ds\ C^2 \left[\delta^2 K_1^2(\delta s) - \frac{\omega^2}{v_o^2} K_o^2(\delta s) \right]$$

$$= 2\pi C^2 \left[\int_x^{\infty} + K_1^2(t)\, dt - \frac{\omega^2}{\delta^2 v_o^2} \int_x^{\infty} t\, K_o^2(t)\, dt \right] \tag{3.19}$$

in which we have used the symbol ϵ for $\lambda_{DeBroglie}$ to eliminate confusion with the screening radius, and $x = \delta\epsilon$. Let the first of these integrals be designated F(x), the second G(x). Now an integration by parts shows that

$$F(x) = x K_o(x) K_1(x) - G(x) \quad , \tag{3.20}$$

so we can concentrate on the second integral. As may be verified by differentiation, this is simply

$$G(x) = \frac{x^2}{2} \left[K_1^2(x) - K_o^2(x) \right] + \lim_{x \to \infty} \frac{x^2}{2} \left[K_o^2(x) - K_1^2(x) \right] = \frac{x^2}{2} \left[K_1^2(x) - K_o^2(x) \right] . \tag{3.21}$$

Then

$$B = 2\pi C^2 \left[\delta\epsilon K_o(\delta\epsilon) K_1(\delta\epsilon) - \frac{(\delta\epsilon)^2}{2} \left\{ K_1^2(\delta\epsilon) - K_o^2(\delta\epsilon) \right\} \right.$$

$$\left. - \frac{\omega^2}{2 v_o^2} \epsilon^2 \left\{ K_1^2(\delta\epsilon) - K_o^2(\delta\epsilon) \right\} \right] \tag{3.22}$$

and

$$\frac{dP_{v_o}(\omega)}{d\omega} = \frac{8\pi}{3} \frac{e^2}{c^3} N_n v_o dn_e(v_o) B = \frac{16}{3} \frac{Z^2 e^6}{m^2 v_o c^3} N_n dn_e(v_o) \tag{3.23}$$

$$\left[\eta K_o(\eta) K_1(\eta) - \frac{\eta^2}{2} \left\{ K_1^2(\eta) - K_o^2(\eta) \right\} - \frac{1}{2} \left(\frac{\omega \lambda_{DeB}}{v_o} \right)^2 \left\{ K_1^2(\eta) - K_o^2(\eta) \right\} \right] ,$$

in which

$$\eta = \sqrt{\left(\frac{\lambda_{DeB}}{\lambda}\right)^2 + \left(\frac{\omega \lambda_{DeB}}{v_0}\right)^2} \quad \text{and} \quad \lambda_{DeBroglie} = \frac{\hbar}{m v_0} \quad .$$

In order to apply this result, we need a criterion for the validity of the classical description. This can be obtained in rough form by following Bohm (Ref. 6). We require that the size of a wave packet representing the electron be $\lesssim$ the impact parameter, and that the momentum uncertainty involved in forming this packet be much smaller than that transferred during the collision. The impulse approximation should be better for the shielded than the pure Coulomb potential, so we will get a criterion no worse than that usually employed, which is

$$\frac{2s^2}{\hbar v} \int_{-\infty}^{\infty} F(r) \frac{dx}{r} >> 1 \quad , \tag{3.24}$$

which leads to an integral previously evaluated, and yields ($Z = 1$ for us)

$$\frac{4 Ze^2 s}{\hbar v \lambda} K_1 \left(\frac{s}{\lambda}\right) >> 1 \quad . \tag{3.25}$$

Thus, a limiting impact parameter is determined as a function of n_1 or v_1. For $v_{threshold} = \sqrt{\frac{2K}{m}}$, (3.25) is satisfied out to $s/\lambda = 5$ or 6, which should include most of the effect of the potential, while for $v = \sqrt{\frac{kT}{m}}$, $s/\lambda \sim 2.5$ is the limit. However, computation in both cases shows that integrating out to $s = \infty$ is justified because of the rapid decrease of the K functions. Incidentally, another requirement that the classical description be valid for this potential is that the relative variation of the potential over the size of the equivalent wave packet be small. That is,

$$\lambda_{\text{DeBroglie}} \frac{\frac{\partial V}{\partial r}}{V} < 1 , \qquad (3.26)$$

or

$$\frac{\hbar}{m v_o} \left(\frac{1}{r} + \frac{1}{\lambda}\right) < 1 . \qquad (3.27)$$

Since $r \geqslant s \geqslant \lambda_{\text{DeBroglie}}$ always in this description, and $\frac{\lambda_{\text{De Broglie}}}{\lambda} < .01$, this will be satisfactory in general.

The contribution from the range $\sqrt{\frac{2K}{m}} < v \leqslant \sqrt{\frac{kT}{m}}$ should then be given by

$$\underline{P}\, d\omega = \frac{16}{3} n_e^2 \frac{Z^2 e^6 d\omega}{m^4 c^5} 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_{\sqrt{\frac{2K}{m}}}^{\sqrt{\frac{kT}{m}}} P\, dP\, e^{-P^2/2\mu kT} \qquad (3.28)$$

$$\left\{ \sqrt{\left(\frac{\lambda_D}{\lambda}\right)^2 + \left(\frac{\mu \omega \lambda_D}{P}\right)^2}\, K_o(\sqrt{\ }) K_1(\sqrt{\ }) - \frac{1}{2} \left[\left(\frac{\lambda_D}{\lambda}\right)^2 + 2\left(\frac{\mu \omega \lambda_D}{P}\right)^2\right] \left[K_1^2(\sqrt{\ }) - K_o^2(\sqrt{\ })\right] \right\}.$$

It may be noted that the ratio of the impulse approximation to the Kramers' result at the lower limit is approximately 1/7. At the upper limit, the agreement with the Born approximation result is much better, the ratio being ~0.83. Since the faster electron is deviated less from its trajectory, the superior agreement at the upper end may be interpreted as resulting from better validity of the impulse approximation, and is thus in agreement with expectation. It may also be noted that Dewitt gives a "classical low-frequency" expression (eq. 18 of Ref. 3) valid for weak shielding. This is applicable to only a very narrow velocity range, just about $\sqrt{\frac{kT}{m}}$, for our ω. At that limit, the ratio of the impulse cross section to his result is 1.31, which is quite reasonable agreement.

IV. Electron-Neutral Bremsstrahlung

As stated in the introduction, we base our calculations here on a screening radius derived from the Thomas-Fermi atom. While this is admittedly not very good for Z as small as 8, it is hoped that the model is still more physical than that of Nedelsky's (Ref. 7) frequently-quoted paper, which uses the potential

$$V(r) = \begin{cases} \dfrac{Ze^2}{a} - \dfrac{Ze^2}{r}, & r < a \\ = 0 & r > a \end{cases} \tag{4.1}$$

and must determine Z and a by recourse to experiment.

We therefore take for the potential seen by an incident electron

$$V(r) = -\frac{Ze^2}{r}\,\phi(r) \;, \tag{4.2}$$

with ϕ a solution of

$$x^{1/2}\,\frac{d^2\phi}{dx^2} = \phi^{3/2} \;, \tag{4.3}$$

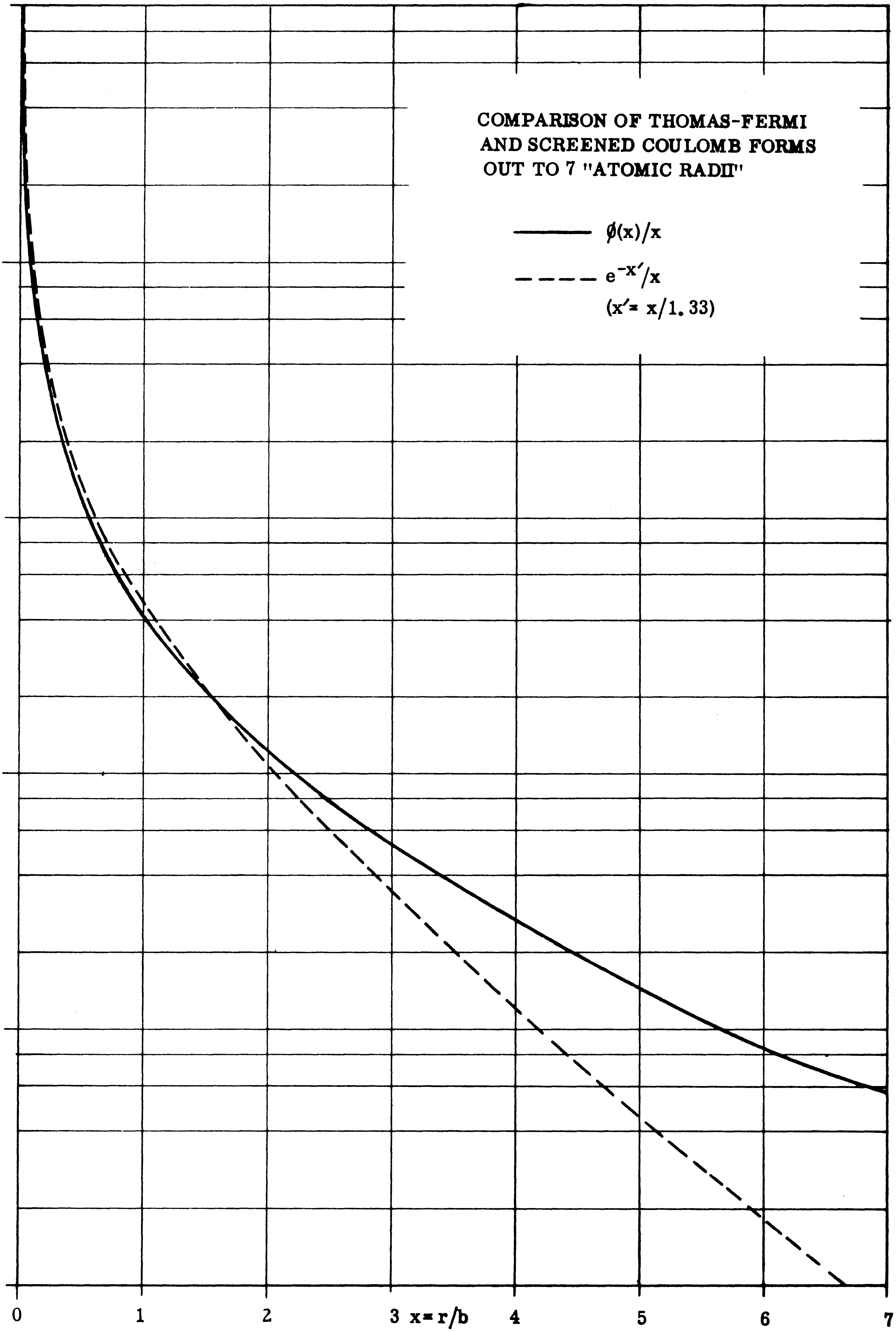
in which $x = r/b$, $b = \dfrac{0.885\,a_0}{Z^{1/3}}$, and a_0 is the Bohr radius, $\hbar^2/me^2$.

$\phi(x)$ has been tabulated by Bush and Caldwell (Ref. 8). It may be fitted quite well by $e^{-r/\lambda}$, with $\lambda = 1.33\,b = 3.13 \times 10^{-9}$ cm for O. We give here a plot of $V(x) = -\dfrac{e^{-bx/\lambda}}{x}$ and $\dfrac{1}{x}\,\phi(x)$.

Again, the Born approximation cross section may be used for $P > c\sqrt{mkT}$, but different approximations are permitted with the stronger screening here. We write, then, for this part of the electron velocity distribution,

$$P_1 + P_2 \sim 2P_1 \qquad \text{and} \qquad P_1 - P_2 = \frac{P_1^2 - P_2^2}{P_1 + P_2} \sim \frac{\mu K}{P_1} \tag{4.4}$$

so that


COMPARISON OF THOMAS-FERMI
AND SCREENED COULOMB FORMS
OUT TO 7 "ATOMIC RADII"
ø(x)/x
e^-x'/x
(x'= x/1.33)
0
1
2
3 x=r/b
4
5
6
7

$$\underline{P}\, d\omega = N_n n_e \frac{16}{3} e^6 Z^2 \frac{4\pi}{m^4 c^5} \left(\frac{m}{2\pi kT}\right)^{3/2} d\omega \int_{C\sqrt{mkT}}^{\infty} dP\, P\, e^{-P^2/2\mu kT}$$

$$\left\{\frac{1}{2} \ln\left(1 + \frac{4P^2}{\gamma^2}\right) - \frac{2P^2}{\gamma^2 (1 + 4\, P^2/\gamma^2)}\right\} . \tag{4.5}$$

Although, because the strong screening makes this case much different from the pure Coulomb field, we cannot justify the Born-Elwert result, we may use it to get some indication of the contribution of the slowest electrons. We need now the limit $P_2 / P_1 \rightarrow 0$, $P_1 << \gamma$, and find

$$d\sigma(K) = 2\sigma_0 \frac{dK}{K}\left(\frac{P_1}{\gamma}\right)^4 , \tag{4.6}$$

which because of the shielding is quite a small result compared to the contribution from the thermal range.

The impulse approximation is found not to be valid for neutrals, and no method has been found for treating the contribution of the intermediate part of the electron velocity distribution. However, there is no physical reason to expect any special phenomena to characterize this range, so that we still expect the contribution to the total power to be given essentially by (4.5); the power rising with velocity from that given by (4.6) to this value. In fact, up to velocities of the order of $\sqrt{\frac{kT}{m}}$ the extreme screening limit still applies, and the integrands for power have the same form. Thus the ratio of power/ cm^3 in equal frequency ranges from thermal electrons to that from slow ones is thus essentially

$$\frac{P_{therm}}{P_{slow}} \sim \frac{v_{th}}{v_s}\, e^{-\frac{m v_{th}^2}{2kT}} \left(\frac{v_{th}}{v_s}\right)^4 \sim 10^7 \tag{4.7}$$

so the slowest electrons can be neglected with respect to those at $v = \sqrt{\frac{kT}{m}}$. Likewise, the high-energy tail $P_1 \gtrsim \gamma$ contributes little, and in fact, the integrand of (4.5) peaks near $v_M = \sqrt{\frac{kT}{m}}$, dropping off faster for $v > v_M$ than for $v < v_M$, but such that (4.5) does give the essential contribution.

V. Evaluation and Discussion of Results

The integration indicated in (4. 5) for the Bremsstrahlung power from neutrals may be carried out, leading to a closed form. Consider

$$I_1 = \int_{\sqrt{\mu kT}}^{\infty} P\,dP\, e^{-P^2/2\mu kT} \left\{ \frac{1}{2} \ln\left(1 + 4\frac{P^2}{\gamma^2}\right) - \frac{2P^2}{\gamma^2(1+4P^2/\gamma^2)} \right\} \tag{5.1}$$

With $y = P^2/2\mu kT$, $\beta = 8\mu kT/\gamma^2$, we find

$$I_1 = \frac{1}{2}\mu kT \int_{Y_2}^{\infty} e^{-y} \left\{ \ln(1+\beta y) - \frac{\beta y}{1+\beta y} \right\} dy, \tag{5.2}$$

and two integrations by parts yield finally

$$I_1 = \frac{1}{2}\mu kT \left\{ e^{-1/2}\left[\ln(1+\beta/2) - 1\right] - e^{1/\beta}(1+\beta)\,\mathrm{Ei}\left(-\frac{1}{2} - \frac{1}{\beta}\right) \right\} \tag{5.3}$$

Here Ei is the exponential integral, defined by

$$-\mathrm{Ei}(-x) = \int_x^{\infty} \frac{e^{-t}}{t}\,dt \tag{5.4}$$

Numerically,

$$I_1 = 4.65 \times 10^{-22}\ \mathrm{erg}^2 \tag{5.5}$$

and (4. 5) becomes, for the power/cm^3 in $d\omega$ from neutrals at 50 KMc,

$$\underline{P}\,d\omega = N_n n_e \frac{16}{3} e^6 Z^2 \frac{4\pi}{m^4 c^5} \left(\frac{m}{2\pi kT}\right)^{3/2} \times 4.65 \times 10^{-22}\, d\omega\ \mathrm{ergs/cm^3\ sec}$$

$$= N_n\, d\omega\ 4.45 \times 10^{-29}\ \mathrm{ergs/sec\ cm^3} \tag{5.6}$$

Incidentally, we might ask what "effective Z" this corresponds to, i.e., "If Kramers' law gave the correct cross section throughout the whole velocity distribution, what value of Z in it would reproduce this power ?" Now Kramers' law integrated from $P = \sqrt{2k\mu}$ to ∞ yields

$$\underline{P}\, d\omega = N_n n_e \frac{16}{3} e^6 Z^2 \frac{4\pi}{m^4 c^5} \left(\frac{m}{2\pi kT}\right)^{3/2} d\omega \frac{\pi}{3} \mu kT e^{-K/kT} \tag{5.7}$$

so we find

$$Z_{eff} = 8 \left[\frac{\text{"5.6"}}{\text{"5.7"}}\right]^{1/2} = 0.17 \tag{5.8}$$

This is in reasonable agreement with the results of Breen and Nardone (Ref. 1), since our longer wavelength radiation may come from more distant collisions, classically speaking, and greater shielding.

For the ion Bremsstrahlung, the higher velocity electrons yielded the power expression

$$\underline{P} d\omega = n_e^2 \frac{16}{3} Z^2 e^6 \frac{4\pi}{m^4 c^5} \left(\frac{m}{2\pi kT}\right)^{3/2} d\omega$$

$$\int_{\sqrt{\mu kT}}^{\infty} P\, dP\, e^{-P^2/2\mu kT} \left\{ \ln\left(\frac{2P^2}{k\mu}\right) - \frac{\gamma^2 P^2}{2(K\mu)^2} \right\}$$

The integral I_2 appearing here may also be evaluated in closed form, and yields, with $\delta = (4kT/K)$ and $\epsilon = \dfrac{\gamma^2 kT}{K^2 \mu}$,

$$I_2 = \mu kT \left[e^{-1/2} \left\{ \ln \delta/2 - 3\frac{\epsilon}{2} \right\} - Ei\left(-\frac{1}{2}\right) \right] = 3.01 \times 10^{-18} \text{ erg}^2 , \tag{5.10}$$

so that

$$\underline{P}\, d\omega = 4.5 \times 10^{-14} \, d\omega \text{ erg/cm}^3 \text{ sec} \qquad (5.11)$$

The integral arising from the impulse approximation must be evaluated numerically. We find for it

$$I_3 = 1.24 \times 10^{-18} \text{ erg}^2 , \qquad (5.12)$$

so that the slower electrons contribute to the ion Bremsstrahlung

$$\underline{P}\, d\omega = 1.83 \times 10^{-14} \, d\omega \text{ erg/cm}^3 \text{ sec.} \qquad (5.13)$$

Finally, now that the power integrals have been investigated in detail, it is possible to go back and ask whether the replacement of the Thomas-Fermi potential by the screened Coulomb form is consistent. The question arises since they agree well only for small x, differing by more than an order of magnitude for $r > 7b$, when the exponential begins to drop off more rapidly. However, it is clear that the range of r for which the potential must be given accurately is determined by the way in which it enters into the Born approximation cross-section. This in turn need only be given well for that range of momentum transfer $|\vec{P}_1 - \vec{P}_2|$ from which a significant contribution to the power arises in the integral over electron velocity distribution.

Since the potential V(r) is spherically symmetric, it enters the Born approximation cross section only in the radial integral

$$I = \int_0^{\infty} dr\ r\ V(r) \sin \left|\vec{P_1} - \vec{P_2}\right|\ r/\hbar c \qquad (5.15)$$

Further, integrating I_1 only up to $P = \sqrt{10\mu kT}$ yields 90% of the total power. We must then investigate the integrand of (5.15) for $P < \sqrt{10\,\mu kT}$. The minimum period of the sine is clearly $\hbar c / \left|\vec{P_1} - \vec{P_2}\right|_{max} \sim \hbar c \cdot 2\pi/2P_{1max}$, which is 37.2b. Therefore the range of x within which the potential must be given well is that within which $\int_0 \phi(x) \sin\left(\frac{2\pi x}{37.2}\right) dx$ yields its essential contribution. For much larger periods, the integrand behaves like $x\phi(x)$, the integral of which must be examined. In the case of the minimum period, we find it satisfactory to fit the potential for smaller x, say $x < 8$. Unfortunately, for larger periods (i.e., smaller-angle scattering), we need the potential much farther out, where we have fitted it poorly with the exponential. This should lead to an underestimate of the scattered power from neutrals. However, since, as stated, we would expect a smaller effective Z for microwave than for infrared radiation, and yet a factor of only roughly 2 exists between our result and that of Ref. 1 for Z effective, it would seem that the underestimate has not been a gross one, so that the power given by equation (5.6) should be valid within an order of magnitude. This may perhaps be due to the fact that for large periods T, $\sin 2\pi \frac{x}{T}$ behaves like $2\pi x/T$ for small x, where the potential is appreciable, and the integral (5.15) is thus much smaller than it is for larger-angle scattering. That is, the potential drops off so rapidly that distant small-angle scattering does not contribute greatly to the cross-section, as it would for an unshielded Coulomb potential.

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REFERENCES

1. Breen, R. G. and Nardone, M. C., "The Free-Free Continuum of Oxygen," Space Sciences Laboratory, Missile and Space Vehicle Dept., General Electric Co., Philadelphia.

2. Dewitt, Hugh, "Scattering and Bremsstrahlung Energy Transfer in a Fully Ionized Gas Using the Debye Potential," UCRL, Livermore, Theo. PNU-56.

3. Dewitt, Hugh, "The Free-Free Absorption Coefficient in Ionized Gases," UCRL 5377.

4. Roberts, C. A., "Radio-Frequency Radiation from Hypersonic Plasmas with Impressed Oscillating Electric Fields," Douglas Aircraft Co., Engineering Paper No. 847, 20 Nov., 1959.

5. Landau and Lifshitz, "The Classical Theory of Fields," Addison-Wesley, 1951.

6. Bohm, D., "Quantum Theory," Prentice-Hall, 1951.

7. Nedelsky, L., "Radiation from Slow Electrons," P.R. 42, 641 (1932).

8. Bush, V. and Coldwell, S., "Thomas-Fermi Equation Solution by the Differential Analyzer," P.R. 38, 1898 (1931).