ELECTROMAGNETIC MASS DIFFERENCES *

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In Cottingham's approach to electromagnetic masses (δm) one encounters the problem of unknown subtractions in the fixed mass dispersion relations for the virtual Compton scattering amplitude. We avoid this problem and express δm in terms of measurable quantities, assuming knowledge of light-cone singularities of the electromagnetic current commutator.

During the last decade many authors have tried to calculate the proton neutron mass difference following Cottingham's method [1], i.e., performing a Wick rotation in the Feynman integral for the electromagnetic mass δm and subsequently using the *fixed mass* dispersion relation (FMDR) for the virtual forward Compton scattering amplitude (FCA). Unfortunately such FMDR contains subtractions, which defy any clear control fom either the experimental or theoretical point of view. This difficulty makes the original Cottingham method hard to exploit practically***.

In this note we relate δm to measurable quantities using neither Wick rotation nor FMDR. Rather than FMDR two different kinds of dispersion relations will be used, which follow from causality of the electromagnetic current commutator†

$$\widetilde{C}(x) = \langle p|[j_\alpha(x),\ j_\alpha(0)]|p\rangle$$

$$C(q_0, q^2) = \frac{1}{2\pi}\int \mathrm{d}x\, \exp(\mathrm{i}qx)\,\widetilde{C}(x) = -3W_1 + \left(1 - \frac{q_0^2}{q^2}\right) W_2\,, \tag{1}$$

and for which the problem of subtractions is completely controlled by the light-cone behavior of $\widetilde{C}(x)$.

Let, for instance, $\widetilde{C}(x)$ possess a well defined equal time limit. Then the virtual FCA $T = g^{\alpha\beta}\, T_{\alpha\beta}$ can be expressed via the retarded product $2\mathrm{i}\theta(x_0)\,\widetilde{C}(x)$ and satisfies an *unsubtracted* dispersion relation

$$\mathrm{Re}\, T(q_0, q_0^2 - |\boldsymbol{q}|^2) = \frac{2}{\pi}\ \mathrm{P}\!\int_0^\infty \mathrm{d}q'_0\, \frac{q'_0}{q_0'^2 - q_0^2}\ C(q'_0, q_0'^2 - |\boldsymbol{q}|^2)\,, \tag{2}$$

where P denotes the principal value.

Analogously, provided one subtracts from the amplitude T its light-cone singularities, it is possible to write for such a truncated amplitude an *unsubtracted* dispersion relation in the variable $q_\perp = q_0 + q_3$ with remaining components of q fixed [3]. Equivalently, this can be written as the two-parametrical family of dispersion sum rules††

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*** For a review of this approach to δm see [2].
† States are normalized as $\langle p'|p\rangle = (2\pi)^3 E/M\, \delta^3(\boldsymbol{p}' - \boldsymbol{p})$ and we work in the rest frame $p = (1, 0, 0, 0)$. An average over spin states is understood.
†† These relations are valid when smeared over ξ with a test function from the class S. Consequently, for certain C_S, eq. (3) can break down at some isolated values of ξ, typically at $\xi = 0$. See refs. [4, 5].

$$\int_{-\infty}^{+\infty} \mathrm{d}q_o\, q_o^n\, [C(q_o, 2\xi q_o - \eta) - C_S(q_o, 2\xi q_o - \eta)] = 0\,,\quad \eta \gtrsim \xi^2\,, \tag{3}$$

where $n = 0$, 1 and C_S is any *causal* amplitude, for which the difference $\widetilde{C}(x) - \widetilde{C}_S(x)$ vanishes on the surface of the light-cone. A rigorous proof of the sum rules (3) can be found in ref. [4].

Eqs. (2) and (3) relate FCA containing time-like and space-like photons. We shall show that they can be used to eliminate all unmeasurable pieces in the expression for δm, provided one knows the light-cone singularities of $\widetilde{C}(x)$.

We suppose the light-cone behavior of $\widetilde{C}(x)$ to be determined in terms of Bjorken asymptotics of inelastic electron scattering cross-sections* as it is described in refs. [4, 7]. For simplicity we assume the vanishing of the longitudinal scaling function. Furthermore it will be assumed that the leading light-cone singularity of $\widetilde{C}(x)$ (i.e., $\partial_o[\epsilon(x_o)\delta(x^2)]$) is followed by non-leading singularities of canonical type (c.f. $\epsilon(x_o)\delta(x^2)$ and $\epsilon(x_o)\delta(x^2)$) with coefficients given by the first 3 terms in the asymptotic expansion of the transverse and longitudinal virtual photo-absorption cross-sections in the Bjorken limit. As a particular consequence of these hypothesis $\widetilde{C}(x)$ vanishes at $x_o = 0$.

We start by rewriting the usual second order perturbation theory expression for δm in a form, which contains only positive frequency parts of the FCA and of the photon propagator, c.f.

$$\delta m = \frac{\alpha}{\pi \mathrm{i}} \int \mathrm{d}x \, \frac{\theta(x_o)}{x^2 - \mathrm{i}\epsilon x_o} \, \langle p | j_\alpha(x)\, j^\alpha(0) | p \rangle\,. \tag{4}$$

In momentum space this can be expressed as

$$\delta m = \frac{\alpha}{\pi} \int_0^\infty \mathrm{d}q_o \int_0^\infty \mathrm{d}|\boldsymbol{q}|\, \frac{|\boldsymbol{q}|}{|\boldsymbol{q}| + q_o}\, C(q_o, q_o^2 - |\boldsymbol{q}|^2)\,, \tag{5a}$$

where we have used the fact that the Fourier transform of $\langle p|j_\alpha(x)\, j^\alpha(0)|p\rangle$ vanishes for $q_o < 0$ and coincides with $C(q)$ for $q_o > 0$. It is interesting to note that eq. (5a) is form-invariant with respect to the Fourier transform: In the coordinate space one has

$$\delta m = 2\mathrm{i}\alpha \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}r\, \frac{r}{r+t}\, \widetilde{C}(t, t^2 - r^2)\,, \tag{5b}$$

which can be directly obtained from eq. (4), by expressing the current product as dispersion integral in time over the commutator $\widetilde{C}(x)$.

We isolate in the formulae (5a) and (5b) the part of current commutator singular on the light-cone writing

$$\widetilde{C}(x) = \widetilde{C}_S(x) + \widetilde{C}_R(x)\,, \tag{6}$$

* This includes the hypothesis that the amplitude C does not contain polynomials of the form

$$\epsilon(\nu) \sum_k \nu^k \rho_k(q^2)\, \theta(q^2 - s_o) \qquad \text{with} \qquad \int \mathrm{d}s\, s^n \rho_k(s) \neq 0 \text{ for } n = 0\,, 1\, \ldots, k+1\,.$$

Such terms are causal and they would contribute to the light cone singularities of C, although they cannot be detected in inelastic electron scattering experiment [6]. An incomplete test of this hypothesis has been performed for the amplitude $q^{-2}\, W^2$ in refs. [3], evaluating the corresponding sum rule (3) for $\xi = 0.1$ and $\eta = 1.1\ \mathrm{GeV}^2$.

where i) both $\tilde{C}_S$ and $\tilde{C}_R$ are causal, Lorentz invariant and odd, ii) $\tilde{C}_R$ vanishes on the surface of the light-cone and iii) $C_R(q)$ does not contain any singularity† at $q^2 = 0$. We denote by δm_S and δm_R the contributions of C_S and C_R to the formula (5a) or (5b).

The above three conditions leave the choice of C_S rather arbitrary. Later on we shall further restrict this choice and discuss one particularly suitable way how to provide the splitting (6).

Let us briefly comment about the divergences of δm. The formula (5b) could contain 'short range divergences' coming from the end point $t = r = 0$ and 'long range divergences', which might arise from the asymptotic tail of the integral (5b). Since the short range divergences are induced by the light-cone singularities of $\tilde{C}(x)$, they will be contained in δm_S. Putting the explicit form of these singularities [4, 7] into the integral (5b), one reproduces the well known logarithmic divergence†† which is absent provided the leading terms of the transverse and longitudinal photo-absorption cross-sections in the Bjorken limit satisfy the sum rule given in the refs. [4, 7]. We assume that this short range divergences does not contribute to the mass differences of hadrons from the same isomultiplet.

Next, we discuss δm_R, i.e., the contribution of $C_R(q_o, q^2)$ to the integral (5a), which should be finite. Splitting the integrand in the corresponding formula (5a) into odd and even parts with respect to $|\boldsymbol{q}| \to -|\boldsymbol{q}|$ we rewrite δm_R in the form†††

$$\delta m_R = \alpha \int_0^\infty \mathrm{d}|\boldsymbol{q}|\, |\boldsymbol{q}| \frac{1}{\pi}\, \mathrm{P}\int_0^\infty \mathrm{d}q_o \frac{q_o}{q_o^2 - |\boldsymbol{q}|^2}\, C_R(q_o, q_o^2 - |\boldsymbol{q}|^2) + \frac{\alpha}{2\pi} \int_0^\infty \mathrm{d}q_o\, \mathrm{P}\int_{-\infty}^{+\infty} \mathrm{d}|\boldsymbol{q}|\, \frac{|\boldsymbol{q}|}{|\boldsymbol{q}| + q_o}\, C_R(q_o, q_o^2 - |\boldsymbol{q}|^2). \tag{7}$$

From the dispersion relation (2) it is seen that the first term is nothing but an integral over the real part of

$$f_1^R(\nu) = f_1(\nu) - f_1^S(\nu)\,, \tag{8}$$

where $f_1(\nu)$ stands for the *real* ($q^2 = 0$) spin non-flip FCA and $f_1^S(\nu)$ denotes the contribution of C_S to this amplitude. As for the second term of eq. (7), the change of the integration variable $|\boldsymbol{q}| = -q_o - \xi$, leads to the expression

$$\frac{\alpha}{2\pi}\, \mathrm{P}\int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{\xi} \int_0^\infty \mathrm{d}q_o\, (\xi + q_o)\, C_R(q_o, -2\xi q_o - \xi^2)\,, \tag{9}$$

which contains the contribution from space-like photons ($\xi > 0$ or $0 > -2q_o > \xi$) as well as from time-like photons ($0 > \xi > -2q_o$). The latter can be eliminated using the sum rules (3) for the amplitude C_R with $\eta = \xi^2$, i.e.*

$$\int_0^\infty \mathrm{d}q_o q_o^n C_R(q_o, 2\xi q_o - \xi^2) = (-1)^n \int_0^\infty \mathrm{d}q_o q_o^n C_R(q_o, -2\xi q_o - \xi^2), \qquad n = 0, 1\,. \tag{10}$$

Combining eqs. (5a), (2), (7), (10) and using the notation (8), we obtain the following expression for the electromagnetic mass δm

† Because of this condition, $\tilde{C}_S(x)$ cannot be identified with the leading light-cone singularities of $\tilde{C}(x)$.

†† The quadratic divergence is absent since we have assumed $F_S(\xi) = 0$.

††† The origin of this formula can also be seen by taking the real part of the standard expression for δm:

$$\mathrm{Re}\, \mathrm{i}\alpha \int \mathrm{d}^4 q\, T(q)\, (q^2 + \mathrm{i}\epsilon)^{-1} = \alpha \int \mathrm{d}^4 q\, [\pi\delta(q^2)\, \mathrm{Re}\, T(q) - q^{-2}\, \mathrm{Im}\, T(q)]\,.$$

* Note that $C_R(-q_o, q^2) = -C_R(q_o, q^2)$.

$$\delta m = \frac{\alpha}{\pi}\int_0^\infty \mathrm{d}q_\mathrm{o}\int_0^\infty \mathrm{d}|\boldsymbol{q}|\,\frac{|\boldsymbol{q}|}{|\boldsymbol{q}|+q_\mathrm{o}}\,C_\mathrm{S}(q_\mathrm{o},q_\mathrm{o}^2-|\boldsymbol{q}|^2)-\frac{1}{\pi}\int_0^\infty \mathrm{d}\nu\nu\,\mathrm{Re}\,f_1^\mathrm{R}(\nu)+\frac{\alpha}{\pi}\int_0^1\frac{\mathrm{d}\xi}{\xi}\int_0^\infty \mathrm{d}q_\mathrm{o}\,(\xi+q_\mathrm{o})\,C_\mathrm{R}(q_\mathrm{o},-2\xi q_\mathrm{o}-\xi^2) \tag{I}$$

where the support properties of $C_\mathrm{R}(q_\mathrm{o}, -2\xi q_\mathrm{o} - \xi^2)$ have been used in the last integral ($C_\mathrm{R}(q_\mathrm{o}, -2\xi q_\mathrm{o} - \xi^2) = 0$ for $\xi > 1$ and $q_\mathrm{o} > 0$).

Obviously, the formal manipulations leading to the formula (I) cannot be used for each choice of C_S: The conditions i)–iii) imposed on the amplitude C_S assure the validity of eqs. (3) and (10) for almost all ξ, but they do not guarantee the local validity of eq. (10) at $\xi = 0$††. Furthermore, although the whole contribution δm_R is convergent*, the two terms in eq. (7) and also the two last integrals in (I) might diverge separately for certain C_S. Consequently, in order that (I) makes sense, it is necessary to choose C_S so that the sum rule (10) be locally valid at $\xi = 0$ and that the ξ-integrals in (9) and in the third term of (I) converge at this point. For such choice of C_S the second integral in eq. (I) should converge automatically.

It is easy to see the meaning of this *additional constraint on* C_S: C_S should not only represent the asymptotic behavior of C in the Bjorken limit (what is already guaranteed by the conditions i)–iii)), but also iv) the contribution of C_S to the *real* ($q^2 = 0$) FCA should reproduce its high energy behavior so that $q_\mathrm{o}C_\mathrm{R}(q_\mathrm{o},0)$ become superconvergent.

The formula (I) can be written using any C_S which satisfies i)–iii) and the supplementary condition iv). All divergences of δm are contained in the first term (i.e., in δm_S) and presumably, they are exclusively of the 'short range' type as described before**. The formula (I) contains only measurable quantities. In order to construct C_S one has to know, besides the light-cone singularities of C, the high energy behavior of the forward scattering of *real* photons, which is experimentally accessible.

In the remaining we shall briefly discuss one particular choice of C_S, for which the formula (I) simplifies and which might be suitable for an approximate evaluation of δm. For this purpose we refer to the important feature of inelastic e–p scattering data discovered by Bloom and Gilman [9] and further investigated by Rittenberg and Rubinstein [10]. These authors show that an appropriate modification of the scaling variable $\omega = -2q_\mathrm{o}/q^2$ in the scaling laws for $q_\mathrm{o}W_2$ (W_1) leads to an amplitude, which interpolates quasi-locally the low energy (and even low q^2 [10]) behavior of $q_\mathrm{o}W_2$ (W_1) in the vicinity of each resonance peak. One might expect that this phenomenon, reflecting the 'dual character' of the scaling laws, has a larger scope of validity than the analysis of existing data is able to show [11]. In particular there should exist an amplitude $C_\mathrm{int}(q_\mathrm{o}, q^2)$, which would not only account for the behavior of C in the deep inelastic region, but would also interpolate C in the resonance region analogously, as do Bloom and Gilman's 'modified scaling laws'. Provided such C_int is *causal*, it satisfies the three conditions i)–iii) we have imposed on the amplitude C_S. In addition, the sum rules (3) for the difference $C-C_\mathrm{int}$ will now be satisfied as finite energy sum rules with the integration range extending over a vicinity of each resonance peak crossed by the integration line. As a particular consequence the sum rules (3) will split into two 'semi-finite energy sum rules'

$$\int_0^\infty \mathrm{d}q_\mathrm{o}q_\mathrm{o}^n\,[C(q_\mathrm{o},\pm 2\xi q_\mathrm{o}-\xi^2)-C_\mathrm{int}(q_\mathrm{o},\pm 2\xi q_\mathrm{o}-\xi^2)]=0\,,\quad n=0,1 \tag{11}$$

which will now be locally valid for each value of ξ.

Hence, making a particular choice $C_\mathrm{S} = C_\mathrm{int}$ in the general formula (I), both sides of eq. (10) and also the third term in the formula (I) identically vanish. (As pointed out before, this is sufficient to guarantee the convergence

†† See footnote on p. 105.

* This can be seen by writing the integral (5a) for δm_R in terms of variables q_o and $\xi = q_\mathrm{o} + |\boldsymbol{q}|$ and using the sum rules (3) for $\eta = \xi^2 > 1$.

** We have computed δm_S using a model representation for C_S constructed similarly as in ref. [3]. We have found no long range divergences in δm_S.

of the second term in eq. (I).) We thus arrive at the special case of the formula (I), which reads

$$\delta m = \frac{\alpha}{\pi} \int_0^\infty \mathrm{d}q_0 \int_0^\infty \mathrm{d}|\boldsymbol{q}| \frac{|\boldsymbol{q}|}{|\boldsymbol{q}| + q_0} C_{\mathrm{int}}(q_0, q_0^2 - |\boldsymbol{q}|^2) - \frac{1}{\pi} \int_0^\infty \mathrm{d}\nu\nu \,\mathrm{Re}[f_1(\nu) - f_1^{\mathrm{int}}(\nu)] \,, \tag{II}$$

where f_1^{int} denotes the contribution of C_{int} to the real spin non-flip FCA.

Unfortunately, the explicit interpolating amplitudes given by the authors of refs. [9, 10] are not causal [11] and consequently, they cannot be used directly to evaluate formula (II). Furthermore, the existing data are presumably not sufficient to extract a causal C_{int} both for the proton and for the neutron. We thus conclude that more experimental and phenomenological work is needed before formula (II) (or the more general formula (I)) will allow a model independent evaluation of the proton neutron mass difference.

A more detailed account of this work will be given elsewhere.

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