On the Validity of the Geometrical Theory of Diffraction by Star-Shaped Cylinders

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INTRODUCTION

Let \( U(X, X_0; k) \) be the (Green’s) function defined by the equations

\[
(\Delta + k^2) U = \delta(X, X_0), \quad X, X_0 \in \mathcal{E}; \\
U = 0, \quad X \in \mathcal{B}, \quad X_0 \in \mathcal{E};
\]

\[
\lim_{R \to \infty} \int_{|X| = R} |\partial U/\partial |X| - i k U|^2 \, dX = 0,
\]

where \( \mathcal{E} \) is the (2-dimensional) region exterior to a piecewise smooth star-shaped curve \( \mathcal{B} \). We obtain a rigorous asymptotic approximation of \( U(X, X_0; k) \) in the shadow \( S(X_0) \) of \( \mathcal{B} \) under the assumption that \( \mathcal{B} \) coincides with a circle \( \mathcal{B}_0 \) near the points of diffraction.

In Part I, using a priori estimates obtained by Morawetz and Ludwig [3], we establish that if \( \mathcal{B} \) coincides with \( \mathcal{B}_0 \) in the shadow, then

\[
U(X, X_0; k) = U_0(X, X_0; k) \left[ 1 + O(\exp(-k^{1/3} \sigma)) \right]
\]

as \( k \to \infty \), uniformly on every closed bounded subset of \( S(X_0) \). Here \( U_0(X, X_0; k) \) is Green’s function for the case \( \mathcal{B} = \mathcal{B}_0 \), and \( \sigma \) is a positive number independent of \( k \) and \( X \).

Using this result, we prove in Part II that if \( \mathcal{B} \) coincides with \( \mathcal{B}_0 \) only near the points of diffraction, then

\[
U(X, X_0; k) = U_0(X, X_0; k) \left[ 1 + O(\exp(-k^{1/3} \gamma)) \right]
\]

as \( k \to \infty \), uniformly on every closed bounded subset \( S(X_0) \) sufficiently far from \( \mathcal{B} \), where \( \gamma \) is a positive number independent of \( k \) and \( X \).

The asymptotic approximations obtained in this study are believed to be the first ones established for solutions of diffraction problems with nonconvex boundaries. They were announced in [1]. These results generalize those
obtained by Bloom and Matkowsky [2]; they considered diffraction by infinite cylinders of convex cross section. A reasonably complete account of other literature on this subject is given in [2].

**PART I**

Let $U_1(X, X_0; k)$ be the solution of the scattering problem $P_1$:

(i) $[\Delta + k^2] U = \delta(X, X_0), X, X_0 \in \mathcal{E}_1 (= \text{exterior of closed curve } \mathcal{B}_1);$  
(ii) $U = 0, X \in \mathcal{B}_1 (= \text{piecewise smooth and star-shaped});$

(iii) $\lim_{R \to \infty} \int_{|X| = R} |\partial U/\partial X| |X| - ikU|^2 \cdot |dX| = 0.$

Assume: (1) $\mathcal{B}_1$ is obtained by deforming the portion of the circle $\mathcal{B}_0 (\{X : |X| = a\})$ "illuminated" by the "source" $X_0$ into a piecewise smooth arc $A_1$ (see Fig. 1).

(2) $A_1$ has no points on the tangents to $\mathcal{B}_0$ that pass through $X_0$.

(3) $A_1$ cuts $\mathcal{B}_0$ at a finite number of points.

**THEOREM 1.** As $k \to \infty,$

$$U_1(X, X_0; k) = U_0(X, X_0; k) \cdot [1 + O(\exp\{-k^{1/\sigma}\})],$$

uniformly in $X$ for $X \in S_1 \prec (X_0)$.

Here $\sigma$ is a positive number independent of $X$ and $k$. The function $U_0(X, X_0; k)$ is the solution of the scattering problem $P_0$:

(i)' $[\Delta + k^2] U = \delta(X, X_0), |X|, |X_0| > a;$

(ii)' $U = 0; |X| = a;$

(iii)' $\lim_{R \to \infty} \int_{|X| = R} |\partial U/\partial X| |X| - ikU|^2 \cdot |dX| = 0.$
$S_{1}(X_0)$ is the "shadow" of $\mathcal{B}_1 : X \in S_{1}(X_0)$ if and only if $X \in \mathcal{B}_1 \cup \mathcal{B}_1$ and the straight line through $X$ and $X_0$ cuts $\mathcal{B}_1$ at 2 distinct points $S_{1}<(X_0)$ is any closed, bounded subset of $S_{1}(X_0)$ (see Fig. 2).

![Figure 2](image)

Applying Green's second identity to $U_1(X', X_0; k)$ and $U_0(X, X'; k)$, and then integrating over the region

$$\mathcal{E}_1 \cap \mathcal{E}_0 \quad (\mathcal{E}_o = \{X : |X| > a\}),$$

we get the integral equation

$$U_1(X, X_0; k) = U_0(X, X_0; k) + I_1(X, X_0; k) - I_2(X, X_0; k),$$

where

$$I_1(X, X_0; k) = \int_{\mathcal{E}_1 \cap \mathcal{E}_0} U_0(X, X'; k) \cdot \frac{\partial U_1}{\partial n}(X', X_0; k) \, |dX'|,$$

and

$$I_2(X, X_0; k) = \int_{\mathcal{S}} \frac{\partial U_0}{\partial n}(X, X'; k) \cdot U_1(X', X_0; k) \, |dX'|.$$

In the last integral

$$\mathcal{S} = \mathcal{B}_0 \cap \mathcal{E}_1 - \mathcal{B}_0 \cap \mathcal{B}_1.$$

To prove Theorem 1 we show that as $k \to \infty$,

$$I_1(X, X_0; k) - I_2(X, X_0; k) = U_0(X, X_0; k) \cdot O(\exp\{- k^{1/\alpha}\}),$$

uniformly in $X$ for $X \in S_{1}<(X_0)$. Setting

$$U_1(X', X_0; k) = U_1^{(a)}(X', X_0; k) + \frac{i}{4} H_0^{(1)}(k |X' - X_0|),$$
\((H_0^{(1)}(z) = \text{Hankel function of first kind of order zero})\), we get the integral equation

\[
I_1(X, X_0; k) = I_{11}(X, X_0; k) + I_{12}(X, X_0; k),
\]

where

\[
I_{11}(X, X_0; k) = \frac{i}{4} \int_{\mathcal{A}_1 \cap \overline{\mathcal{A}_0}} U_0(X, X'; k) \cdot \frac{\partial H_0^{(1)}}{\partial n} (k | X' - X_0 |) \, dX',
\]

\[
I_{12}(X, X_0; k) = \int_{\mathcal{A}_1 \cap \overline{\mathcal{A}_0}} U_0(X, X'; k) \frac{\partial}{\partial n} U_1^{(1)}(X', X_0; k) \, dX'.
\]

and the integral equation

\[
I_2(X, X_0; k) = I_{21}(X, X_0; k) + I_{22}(X, X_0; k),
\]

where

\[
I_{21}(X, X_0; k) = \frac{i}{4} \int_{\mathcal{A}_1} \frac{\partial U_0}{\partial n} (X, X'; k) \cdot H_0^{(1)}(k | X' - X_0 |) \, dX',
\]

\[
I_{22}(X, X_0; k) = \int_{\mathcal{A}_1} \frac{\partial U_0}{\partial n} (X, X'; k) \cdot U_1^{(1)}(X', X_0; k) \, dX'.
\]

Using Schwarz' inequality, we derive the estimate

\[
I_1(X, X_0; k) \leq \max_{\mathcal{A}_1 \cap \overline{\mathcal{A}_0}} |U_0(X, X'; k)|
\]

\[
\cdot \left[ \mathbb{L}_1^{1/2} \left( \int_{\mathcal{A}_1} \left| \frac{\partial U_1^{(1)}}{\partial n} (X', X_0; k) \right|^2 \, dX' \right)^{1/2}
\]

\[
+ \int_{\mathcal{A}_1} \left| \frac{\partial H_0^{(1)}}{\partial n} (k | X' - X_0 |) \right| \, dX' \right],
\]

where \(\mathbb{L}_1\) is the length of \(\mathcal{A}_1\), and the estimate

\[
I_2(X, X_0; k) \leq (2\pi a)^{1/2} \max_{\mathcal{A}_1 \cap \overline{\mathcal{A}_0}} \left| \frac{\partial U_0}{\partial n} (X, X'; k) \right|
\]

\[
\cdot \left[ (2\pi a)^{1/2} \max_{\mathcal{A}_1 \cap \overline{\mathcal{A}_0}} \left| U_1^{(1)}(X', X_0; k) \right|
\]

\[
+ \int_{\mathcal{A}_0} \left| H_0^{(1)}(k | X' - X_0 |) \right|^2 \, dX' \right]^{1/2}.
\]
Morawetz and Ludwig [3] have obtained the following a priori estimates, as $k \rightarrow \infty$:

$$\max_{\mathcal{S}_0 \cap \mathcal{S}_1 - \mathcal{S}_1 \cap \mathcal{S}_0} \left| \frac{\partial U_1^{(\omega)}}{\partial n} (X', X_0; k) \right| \left[ \int_{\mathcal{S}_1} \left| T \cdot \nabla H_0^{(1)} (k \mid X' - X_0 \mid)^2 \right| dX' \right]^{1/2}$$

$$= \mathcal{O} \left( \left\{ \int_{\mathcal{S}_1} \left| T \cdot \nabla H_0^{(1)} (k \mid X' - X_0 \mid)^2 \right| dX' \right\}^{1/2} \right)$$

uniformly in $X_0$, $X_0 \in \text{any closed subset of } \mathcal{S}_1$ where $T$ is the unit tangent to $\mathcal{S}_1$ at $X'$.

(ii) \[
\int_{\mathcal{S}_1} \left| \frac{\partial H_0^{(1)}}{\partial n} (k \mid X' - X_0 \mid) \right| dX' = \mathcal{O}(k^{1/2} / |X_0|^{1/2}),
\]

and

(iii) \[
\int_{\mathcal{S}_0} \left| H_0^{(1)} (k \mid X' - X_0 \mid)^2 \right| dX' = \mathcal{O} \left( \left\{ \frac{\ln |k/X_0|}{k} \right\}^{1/2} \right),
\]

uniformly in $X_0 (|X_0| \geq \rho > 0)$.

Consequently, as $k \rightarrow \infty$,

$$I_1(X, X_0; k) = \max_{\mathcal{S}_1 \cap \mathcal{S}_0} \left| U_0(X, X'; k) \right| \cdot \mathcal{O}(k^{1/2}),$$

$$I_2(X, X_0; k) = \max_{\mathcal{S}_0 \cap \mathcal{S}_1 - \mathcal{S}_1 \cap \mathcal{S}_0} \left| \frac{\partial U_0}{\partial n} (X, X'; k) \right| \cdot \mathcal{O} \left( \frac{\ln |k/X_0|}{k} \right)^{1/2},$$

uniformly in $X_0$ for $X_0 \in \text{any closed subset of } \mathcal{S}_1$.

As $k \rightarrow \infty$,

$$U_0^{-1}(X, X_0; k) = \mathcal{O}(k^{1/2} \cdot \exp\{k^{1/3}(\Im \tau_1) a^{-2/3} \lambda_m(X, X_0)\}),$$

uniformly in $X$ for $X \in S_0 \subset (X_0) (= S_1 \subset (X_0))$;

$$U_0(X, X'; k) = \sum_{1}^{1} \mathcal{O}(k^{1/2} \cdot \exp\{- k^{1/3}(\Im \tau_1) a^{-2/3} \lambda_m(X, X')\}),$$

$$\frac{\partial U_0}{\partial n} (X, X'; k) = \sum_{1}^{1} \mathcal{O}(k^{1/2} \cdot \exp\{- k^{1/3}(\Im \tau_1) a^{-2/3} \lambda_m(X, X')\}),$$
uniformly in $X'$ and $X$ for $X' \in \mathcal{B}_1 \cap \mathcal{B}_0 \cap \mathcal{B}_0 \cap \mathcal{B}_1$ and $X \in S_1^{<}(X') (\subseteq S_1^{<}(X_0))$.

Here $2^{1/3} e^{\pi \sqrt{3/3}}$ is that zero of the Airy function closest to zero, and

$$\lambda_<(X, X_0) = \min_{m=1,2} \lambda_m(X, X_0),$$

$$\lambda_1(X, X') = a[\theta - \theta' - \arccos[a/X] - \arccos[a/X']],$$

$$\lambda_2(X, X') = a[2\pi - \theta - \theta' - \arccos[a/X] - \arccos[a/X']],$$

$$\theta = \arg X, \quad \text{and} \quad \theta' = \arg X'.$$

Let $\hat{X}_m$ be the point on

$$\mathcal{F}_1 = \mathcal{B}_1 \cap \mathcal{B}_0 \cap \mathcal{B}_0 \cap \mathcal{B}_1$$

such that

$$\lambda_m(X, \hat{X}_m) = \min_{X' \in \mathcal{F}_1} \lambda_m(X, X'), \quad X \in S_1^{<}(X_0).$$

If $0 < \theta < \pi$, and $0 < \theta - \hat{\theta}_1$, then

(i) $\lambda_<(X, X_0) = \lambda_1(X, X_0)$, $\theta_0 = 0$,

(ii) $\lambda_1(X, \hat{X}_1) = \lambda_1(X, X_0) + \mu_1(X)$, $\mu_1(X) > 0$, and

(iii) $\mu_1(X) \geq \min_{S_1^{<}(X_0)} \mu_1(X') \geq \Delta > 0$.

Statements (i)–(iii) imply the inequality

$$-\lambda_1(X, \hat{X}_1) \leq -\lambda_<(X, X_0) - \Delta \quad \text{(see Fig. 3).}$$

![Figure 3](image-url)
If $0 \leq \theta \leq \pi$ and $0 \geq \theta - \hat{\theta}_1$, then

(i) $\lambda_<(X, X_0) = \lambda_1(X, X_0)$, $\theta_0 = 0$,
(ii) $\lambda_2(X, X_0) \geq \lambda_1(X, X_0)$, $\theta_0 = 0$,
(iii) $\lambda_2(X, X_0)_{\hat{\theta}_0=0} = \lambda_1(X, X_0)_{\hat{\theta}_0=2\pi}$,
(iv) $\lambda_1(X, X_1) = \lambda_1(X, X_0)_{\hat{\theta}_0=2\pi} + \mu_2(X)$, $\mu_2(X) > 0$,
(v) $\mu_2(X) \geq \min_{S_1 < (X_0)} \mu_2(X') \geq \Delta > 0$.

Statements (i)-(v) imply the inequality

$$-\lambda_1(X, \hat{X}_1) < -\lambda_<(X, X_0) - \Delta$$

(see Fig. 4).

FIGURE 4

If $\pi \leq \theta \leq 2\pi$ and $0 \geq \theta - \hat{\theta}_1$, then

(i) $\lambda_<(X, X_0) = \lambda_2(X, X_0)$, $\theta_0 = 0$,
(ii) $\lambda_2(X, X_0)_{\hat{\theta}_0=0} = \lambda_1(X, X_0)_{\hat{\theta}_0=2\pi}$,
(iii) $\lambda_1(X, \hat{X}_1) = \lambda_1(X, X_0)_{\hat{\theta}_0=2\pi} + \mu_3(X)$, $\mu_3(X) > 0$,
(iv) $\mu_3(X) \geq \min_{S_1 < (X_0)} \mu_3(X') \geq \Delta > 0$.

Statements (i)-(iv) imply the inequality

$$-\lambda_1(X, \hat{X}_1) < -\lambda_<(X, X_0) - \Delta$$

(see Fig. 5).
Statements (i)–(iv) imply the inequality

$$-\lambda_1(X, \hat{X}_1) \leq -\lambda_<(X, X_0) - \Delta$$  \hspace{1em} (see Fig. 6).

It follows from the above series of inequalities that for all \( X \in S_1 <(X_0) \)

$$-\lambda_1(X, \hat{X}_1) \leq -\lambda_<(X, X_0) - \Delta,$$

where \( \Delta \) is positive and independent of \( X \).

Similarly,

$$-\lambda_2(X, \hat{X}_2) \leq -\lambda_<(X, X_0) - \Delta$$

for all \( X \in S_1 <(X_0) \).
Since
\[ -\min_{\mathcal{A}_1 \cap \mathcal{A}_0} \lambda_m(X, X') - \min_{\mathcal{A}_0 \cap \mathcal{A}_1 - \mathcal{A}_0 \cap \mathcal{A}_1} \lambda_m(X, X') \leq -\lambda_m(X, X_m), \]

it follows from the above estimates for \( U_0(X, X'; k), \partial U_0(X, X'; k)/\partial n, \) and \( U_0^{-1}(X, X_0; k), \) that as \( K \to \infty, \)
\[
\max_{\mathcal{A}_1 \cap \mathcal{A}_0} \left| U_0(X, X'; k) \right| = | U_0(X, X_0; k) | \cdot O(\exp(-K^{1/3}(\text{Im } \tau_1) a^{-2/3} \Delta)),
\]
and
\[
\max_{\mathcal{A}_0 \cap \mathcal{A}_1 - \mathcal{A}_0 \cap \mathcal{A}_1} \left| \frac{\partial U_0}{\partial n} (X, X'; k) \right| = | U_0(X, X_0; k) | \cdot O(k^{1/2} \exp(-k^{1/3}(\text{Im } \tau_1) a^{-2/3} \Delta)),
\]
uniformly in \( X \) for \( X \in S_1 <(X_0). \)

We therefore conclude that as \( K \to \infty, \)
\[
\begin{align*}
I_1(X, X_0; k) &= \max_{\mathcal{A}_1 \cap \mathcal{A}_0} \left| U_0(X, X'; k) \right| \cdot O(k^{1/2}) \\
&= | U_0(X, X_0; k) | \cdot O(\exp(-k^{1/3} a)),
\end{align*}
\]
and
\[
\begin{align*}
I_2(X, X_0; k) &= \max_{\mathcal{A}_0 \cap \mathcal{A}_1 - \mathcal{A}_0 \cap \mathcal{A}_1} \left| \frac{\partial U_0}{\partial n} (X, X'; k) \right| \cdot O((\ln[ka/\rho])^{1/2}) \\
&= | U_0(X, X_0; k) | \cdot O(\exp(-k^{1/3} a)),
\end{align*}
\]
uniformly in \( X \) for \( X \in S_1 <(X_0). \)

**PART II**

Let \( U_2(X, X_0; k) \) be the solution of the scattering problem \( P_2: \)

(i) \( [\Delta + k^2] U = \delta(X, X_0), \) \( X, X_0 \in \mathcal{B}_2 (= \text{exterior of closed curve } \mathcal{B}_2); \)

(ii) \( U = 0, \) \( X \in \mathcal{B}_2 (= \text{star-shaped deformation of } \mathcal{A}_1); \)

(iii) \( \lim_{R \to \infty} \int_{|X| = R} |\partial U/\partial X| - i k U |^2 \cdot |dX| = 0. \)

Assume: (1) \( \mathcal{B}_2 \) is obtained by deforming the “dark” portion of \( \mathcal{B}_1 \) into a piecewise smooth arc \( A_2; \) see Fig. 7. (2) \( A_2 \) cuts \( \mathcal{B}_1 \) at a finite number of points.
THEOREM 2. If \( B_2 \) is obtained by deforming the “dark” portion of the boundary of Theorem I into a piecewise smooth arc \( A_2 \), then as \( k \to \infty \)

\[
U(X, X_0; k) = U_0(X, X_0; k) \cdot [1 + O(\exp(- k^{1/3}))],
\]

uniformly in \( X \), for \( X \in S_2^\circ(X_0) - B \).

Here \( S_2^\circ(X_0) \) is any bounded closed subset of the shadow \( S_2(X_0) \) of \( B_2 \), and \( B \) is the “region of influence” of \( A_2 \) defined as follows. Let \( e \) be the end of \( A_2 \) closest to the illuminated side of \( B_2 \), and let \( f \) be a point on \( B_2 \) between \( e \) and the shadow boundary at \( g \). Let \( f' \) be a point on \( B_2 \) between the other end of \( A_2 \) and the shadow boundary, that is as far from the illuminated side of \( B_2 \) as \( f \). Assume \( f \) is so located that the tangents to \( B_2 \) at \( f \) and \( f' \) have no points on \( A_2 \). \( B \) is that part of \( S_2(X_0) \) bounded by \( B_2 \), and the tangents from \( f \) and \( f' \) (see Fig. 8).

Applying Green’s second identity to \( U_0(X, X'; k) \) and \( U_1(X', X_0; k) \), and then integrating over the region \( \bar{S}_2 \cap \bar{S}_1 \), we get the integral equation

\[
U_2(X, X_0; k) = U_1(X, X_0; k) - I_3(X, X_0; k) + I_4(X, X_0; k),
\]

where

\[
I_3(X, X_0; k) = \int_{S_2 \cap \bar{S}_1} U_1(X', X_0; k) \cdot \frac{\partial U_2}{\partial n} (X, X'; k) \cdot |dX'|,
\]

and

\[
I_4(X, X_0; k) = \int_{S_2 \cap \bar{S}_2 \cup \bar{S}_1 \cap \bar{S}_2} \frac{\partial U_1}{\partial n} (X', X_0; k) \cdot U_2(X, X'; k) \cdot |dX'|.
\]

To prove Theorem 2 we show that as \( k \to \infty \),

\[
I_3(X, X_0; k) - I_4(X, X_0; k) = U_1(X, X_0; k) \cdot O(\exp(- k^{1/3}))
\]
uniformly in \( X \) for \( X \in S_2 \prec (X_0) - \mathcal{R} \). Set

\[
U_d(X, X'; k) = U_2^{(2)}(X, X'; k) + \frac{i}{4} H_0^{(1)}(k | X - X'|)
\]

in the above integrals and use Schwarz' inequality to make the estimates

\[
|I_d(X, X_0; k)| \leq \max_{\mathcal{B}_2 \cap \mathcal{S}_2} \left| U_1(X', X_0; k) \right|
\]

\[
\cdot \left[ L_2^{1/2} \left\{ \int_{\mathcal{B}_2} \left| \frac{\partial U_2^{(2)}}{\partial n} (X, X'; k) \right|^2 |dX'| \right\}^{1/2} \right]
\]

\[
+ \int_{\mathcal{B}_2} \left| \frac{\partial H_0^{(1)}}{\partial n} (k | X - X'|) \right| |dX'|
\]

and

\[
|I_d(X, X_0; k)| \leq L_1^{1/2} \cdot \max_{\mathcal{B}_1 \cap \mathcal{S}_2 - \mathcal{B}_1 \cap \mathcal{B}_2} \left| \frac{\partial U_1}{\partial n} (X', X_0; k) \right|
\]

\[
\cdot \left[ L_1^{1/2} \cdot \max_{\mathcal{B}_1 \cap \mathcal{S}_2 - \mathcal{B}_1 \cap \mathcal{B}_2} \left| U_2^{(2)}(X, X'; k) \right| \right]
\]

\[
+ \left\{ \int_{\mathcal{B}_1} \left| H_0^{(1)}(k | X - X'|) \right|^2 |dX'| \right\},
\]

where \( L_2 \) is the length of \( \mathcal{B}_2 \).
By essentially the same argument used to prove Theorem 1 we get the result that as $k \to \infty$

$$
\frac{\partial U_1}{\partial n} (X', X_0; k) = \frac{\partial U_0}{\partial n} (X', X_0; k) \cdot [1 + O(\exp(- k^{1/3}))] ,
$$
uniformly in $X'$ for $X' \in S_2^<(X_0) (\subseteq S_1^<(X_0))$.

It follows from this result, Theorem 1, the estimates of Morawetz and Ludwig [3], and the above inequalities that, as $k \to \infty$,

$$
I_3(X, X_0; k) = \max_{\mathscr{A}_2 \cap \mathfrak{F}_1} |U_0(X', X_0; k)| \cdot O(k^{1/3}),
$$

$$
I_4(X, X_0; k) = \max_{\mathscr{A}_1 \cap \mathfrak{F}_2 = \mathfrak{F}_1} \left| \frac{\partial U_0}{\partial n} (X', X_0; k) \right| \cdot O((\ln |h\rho|)^{1/2}),
$$

uniformly in $X$ for $X \in S_2^<(X_0) - \mathscr{R}$.

It follows from Theorem 1 that, as $k \to \infty$,

$$
U_1^{-1}(X, X_0; k) = O(k^{1/3} \exp( - k^{1/3}(\text{Im } \tau_1) a^{2/3} \lambda_<(X, X_0)))
$$

uniformly in $X$ for $X \in S_2^<(X_0) - \mathscr{R} (\subseteq S_1^<(X_0))$.

Furthermore, as $k \to \infty$

$$
\max_{\mathfrak{F}_2} |U_0(X', X_0; k)|
$$

$$
= \sum_{m=1}^{2} O(k^{-1/3} \exp( - k^{1/3}(\text{Im } \tau_1) a^{2/3} \lambda_m(\vec{X}_m, X_0))),
$$

and

$$
\max_{\mathfrak{F}_2} \left| \frac{\partial U_0}{\partial n} (X', X_0; k) \right|
$$

$$
= \sum_{m=1}^{2} O(k^{1/2} \exp( - k^{1/3}(\text{Im } \tau_1) a^{2/3} \lambda_m(\vec{X}_m, X_0))),
$$

where

$$
\lambda_m(\vec{X}_m, X_0) = \min_{X \in \mathcal{T}_2} \lambda_m(X', X_0),
$$

with

$$
\mathcal{T}_2 = [\mathfrak{F}_2 \cap \mathfrak{F}_1] \cup [\mathfrak{F}_1 \cap \mathfrak{F}_2 - \mathfrak{F}_1 \cap \mathfrak{F}_2].
$$

To complete the proof of Theorem 2 we show that if $X \in S_2^<(X_0) - \mathscr{R}$ then

$$
- \lambda_m(\vec{X}_m, X_0) \leq - \lambda_<(X, X_0) - \Lambda,
$$
where $A$ is positive and independent of $X$. It then follows from the above estimates that, as $k \to \infty$,

$$I_0(X, X_0; k) = \max_{\mathcal{B}_1 \cap \mathcal{A}} | U_0(X', X_0; k)| \cdot O(k^{1/2})$$

$$= | U_1(X, X_0; k)| \cdot O(\exp(-k^{1/3} \gamma)),$$

$$I_0(X, X_0; k) = \max_{\mathcal{B}_1 \cap \mathcal{B}_2} \left| \frac{\partial U_0}{\partial n} (X', X_0; k) \right| \cdot O(\ln (ka/p)^{1/2})$$

$$= | U_1(X, X_0; k)| \cdot O(\exp(-k^{1/3} \gamma)),$$

uniformly in $X, X \in S_2 \subset (X_0) - \mathcal{R}$.

Now, if $0 \leq \theta \leq \pi$, and $X \in S_2 \subset (X_0) - \mathcal{R}$, then

(i) $\lambda_\prec (X, X_0) = \lambda_\prec (X, X_0)$, $\theta_0 = 0$,

(ii) $\lambda_\prec (X, X_0) = \lambda_\prec (X, X_0) + \nu_1(X)$, $\nu_1(X) > 0$,

(iii) $\nu_1(X) \geq \min_{S_2 \subset (X_0) - \mathcal{R}} \nu_1(X') \geq A > 0$.

Statements (i)–(iii) imply the inequality

$$-\lambda_\prec (X, X_0) \leq -\lambda_\prec (X, X_0) - A$$

(see Fig. 9).

If $\pi \leq \theta \leq 2\pi$, and $X \in S_2 \subset (X_0) - \mathcal{R}$, then

(i) $\lambda_\prec (X, X_0) = \lambda_\prec (X, X_0)$, $\theta_0 = 0$,

(ii) $\lambda_\prec (X^*, X_0)|_{\theta_0=0} = \lambda_\prec (X^*, X_0)|_{\theta_0=2\pi}$,

(iii) $\lambda_\prec (X^*, X_0) = \lambda_\prec (X^*, X_0)|_{\theta_0=0} + \nu_2$, $\nu_2 > 0$,

(iv) $\lambda_\prec (X^*, X_0)|_{\theta_0=2\pi} \geq \lambda_\prec (X, X_0)|_{\theta_0=0}$.

FIGURE 9
Statements (i)-(iv) imply the inequality
\[-\lambda_1(\vec{x}_1, X_0) \leq -\lambda_<(X, X_0) - A \] (see Fig. 10).

It follows from the above series of inequalities that, for all \(X \in S_k^<(X_0) - \mathcal{R}\),
\[-\lambda_1(\vec{x}_1, X_0) \leq -\lambda_<(X, X_0) - A,\]
and similarly that
\[-\lambda_2(\vec{x}_2, X_0) \leq -\lambda_<(X, X_0) - A.\]

**CONCLUSION**

In Theorems 1 and 2 we could just as well let \(\mathcal{R}_0\) be any convex curve such that \(U_0(X, X_0; k)\) behaves asymptotically as predicted by the geometrical theory of diffraction.

Furthermore, asymptotic approximations similar to the above can be obtained in 3-dimensions for perturbed spheres or for star-shaped perturbations of any convex surface \(S\) for which the geometrical theory of diffraction is known to be valid.

**REFERENCES**