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EXCITATION OF AN ELASTIC HALF-SPACE BY A TIME-DEPENDENT DIPOLE—II. THE VERTICAL SURFACE DISPLACEMENTS DUE TO A BURIED DIPOLE

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Abstract—The surface, vertical displacements generated by a buried vertical dipole in an elastic half-space are given. The dipole is taken to have a ramp time-dependence and the results are in the form of integrals. These integrals are evaluated as functions of time for several epicentral distances.

1. INTRODUCTION

IN A companion paper [1], closed form expressions were given for the surface displacements generated by an arbitrarily oriented, surface dipole with a ramp time-dependence. However, in common with point load and point torque problems (see Refs. [10–12] of [1]), it appears that closed-form expressions for the surface displacements cannot be found for the case of a buried dipole. But, using the method given in [1], results in terms of integrals can be found. In this paper, the surface vertical displacements due to a buried, vertical dipole of unit strength with a ramp time-dependence are given in terms of integrals. These integrals are evaluated numerically for several epicentral distances and the results are discussed.

2. DEVELOPMENT OF THE FORMAL SOLUTIONS

Cylindrical coordinates r, ϕ and z are used, with the origin being at the dipole, which is a depth d below the free surface and the z-axis pointed vertically downwards. The dipole is taken to be vertical with its normal along the x-axis (whose orientation is, of course, arbitrary), that is,

$$f_1 = 0, \qquad f_2 = 0, \qquad f_3 = 1$$

$$n_1 = 1, \qquad n_2 = 0, \qquad n_3 = 0$$

where the f's and n's are the Cartesian components of unit vectors parallel and perpendicular to the plane of the dipole, respectively.

For algebraic convenience, attention is restricted to solids for which the Lamé constants λ and μ are equal. Also, the source time-dependence $h(t)$ is taken as that of a ramp function i.e.

$$h(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t}{q}, & 0 < t < q \\ 1, & t > q. \end{cases}$$

Then, following the procedure given in [1], it can be shown that the Laplace-transformed, surface, vertical displacements $\bar{u}_z$ are given by

$$4\pi\rho c_s^3 QR\bar{u}_z = -2(\bar{I}_1 - \bar{I}_2)\cos\phi \tag{1}$$

where

$$\bar{I}_1 = \int_0^\infty x^2 J_1\left(\frac{p}{c_s}xr\right)N_1(x)\exp\left[-\frac{pd}{c_s}\left(x^2+\frac{1}{3}\right)^{\frac{1}{2}}\right]\mathrm{d}x \tag{2}$$

$$\bar{I}_2 = \int_0^\infty x^2 J_1\left(\frac{p}{c_s}xr\right)N_2(x)\exp\left[-\frac{pd}{c_s}(x^2+1)^{\frac{1}{2}}\right]\mathrm{d}x \tag{3}$$

$$m_1(x) = (1+2x^2)^2 - 4x^2(x^2+1)^{\frac{1}{2}}(x^2+\tfrac{1}{3})^{\frac{1}{2}}$$

$$m_1(x)N_1(x) = (1+2x^2)(x^2+\tfrac{1}{3})^{\frac{1}{2}}, m_1(x)N_2(x) = 2x^2(x^2+\tfrac{1}{3})^{\frac{1}{2}}$$

$$rQ = c_s q, R = \sqrt{(z^2+r^2)}$$

and J designates a Bessel function of the first kind, ρ is the density, $c_s = \sqrt{(\mu/\rho)}$ is the shear wave speed and the bar denotes the Laplace transform, parameter p. Also, the branches chosen for the square roots are such that their real parts are positive in the right-half plane.

In [1], it was shown that if

$$\bar{S}_n = \int_0^\infty x^{n+1} J_n\left(\frac{p}{c_s}xr\right)F_1(x)\exp\left[-\frac{pd}{c_s}\left(x^2+\frac{1}{3}\right)^{\frac{1}{2}}\right]\mathrm{d}x \tag{4}$$

where n is an integer and $F_1(x)$ is a function of x which is free of exponentials, then

$$S_n = \frac{2c_s}{\pi R}\int_{l/\sqrt{3}}^{\tau} I\frac{v^{n+1}(w)\Theta_3 F_1[iv(w)]\cosh\Theta_1}{[(\frac{1}{3}-w^2)(\tau-w)]^{\frac{1}{2}}\Theta_2}\mathrm{d}w$$

where

$$\Theta_1(w) = n\cosh^{-1}\left[1+\frac{\tau-w}{(1-l^2)^{\frac{1}{2}}v(w)}\right]$$

$$\Theta_2(w) = [\tau+w-2l^2w-2l(1-l^2)^{\frac{1}{2}}(\tfrac{1}{3}-w^2)^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$\Theta_3(w) = lw+(1-l^2)^{\frac{1}{2}}(\tfrac{1}{3}-w^2)^{\frac{1}{2}}$$

$$w = l(\tfrac{1}{3}-v^2)^{\frac{1}{2}}+v(1-l^2)^{\frac{1}{2}}, \qquad v = w(1-l^2)^{\frac{1}{2}}-l(\tfrac{1}{3}-w^2)^{\frac{1}{2}}$$

$$Rl = d, r\tau = c_s t$$

t being time and I denoting imaginary part. Using this, and analogous results when $(x^2+1)^{\frac{1}{2}}$ arises in the exponential in (4), (1) can be written

$$4\pi\rho c_s^3 Q R^2 u_z = -2\{T_1(\tau)-T_2(\tau)-H(\tau-Q)[T_1(\tau-Q)-T_2(\tau-Q)]\}\cos\phi \quad (5)$$

where

$$T_1(\tau) = -\frac{2c_s}{\pi}\int_{l/\sqrt{3}}^{\tau} I\frac{\Theta_3(w_1)\Theta_4(w_1)N_1[iv_1(w_1)]}{[(\frac{1}{3}-w_1^2)(\tau-w_1)]^{\frac{1}{2}}\Theta_2(w_1)}\mathrm{d}w_1,\ l\neq 1 \quad (6)$$

$$T_2(\tau) = -\frac{2c_s}{\pi}\int_{l/\sqrt{3}}^{\tau} I\frac{\Theta_5(w_2)\Theta_6(w_2)N_2[iv_2(w_2)]}{[(\frac{1}{3}-w_2^2)(\tau-w_2)]^{\frac{1}{2}}\Theta_7(w_2)}\mathrm{d}w_2,\ l\neq 1 \quad (7)$$

$$\Theta_4(w_1) = [\tau(1-l^2)^{\frac{1}{2}}-l(\tfrac{1}{3}-w_1^2)^{\frac{1}{2}}][w_1(1-l^2)^{\frac{1}{2}}-l(\tfrac{1}{3}-w_1^2)^{\frac{1}{2}}]$$

$$\Theta_5(w_2) = lw_2+(1-l^2)^{\frac{1}{2}}(1-w_2^2)^{\frac{1}{2}}$$

$$\Theta_6(w_2) = [\tau(1-l^2)^{\frac{1}{2}}-l(1-w_2^2)^{\frac{1}{2}}][w_2(1-l^2)^{\frac{1}{2}}-l(1-w_2^2)^{\frac{1}{2}}]$$

$$\Theta_7(w_2) = [\tau+w_2-2l^2w_2-2l(1-l^2)^{\frac{1}{2}}(1-w_2^2)^{\frac{1}{2}}]^{\frac{1}{2}}.$$

In equation (5), H stands for Heaviside unit step function. The restriction $l\neq 1$ stems from the fact that some of the contour integration methods used in [1] do not hold for $r=0$. An interesting overall feature of the results is that u_z decays like $1/R^2$, whereas it decays like $1/R$ in the case of a point source.

3. NUMERICAL RESULTS

Equation (5) is still quite complicated in that the upper limit is variable and singularities appear in the integrand. However it appears that simpler expressions cannot be found and (5) must be evaluated numerically. The biggest problem as regards this is the presence of singular integrands. In all cases singularities occur at $w=\tau$. Two subcases arise depending on whether $(\frac{1}{3}-w^2)$ or $(1-w^2)$ arises in the denominator. In the former case, no singularity arises since it turns out that the imaginary parts of the integrands are zero for $w_{1,2}<1/\sqrt{3}$. However in the latter case a singularity does appear at $w=1$.

To handle the singularity at $w=\tau$, the range of integration was split into $l/\sqrt{3}$ to $\tau-\varepsilon$, and $\tau-\varepsilon$ to τ, where ε is some small, positive number. For the first range, numerical integration was carried out using eight-point, Gaussian quadrature. For $\tau-\varepsilon$ to τ, a technique given in Isaacson and Keller [2, pp. 346–350] was adopted, in which the non-singular part of the integrand is written as a Taylor series about the singularity. For integrals of the type

$$I = \int_a^b \frac{g(x)}{(x-a)^\theta}\mathrm{d}x, \qquad 0<\theta<1$$

the method gives, on retaining only two terms in the series,

$$I \simeq \left[\frac{g(a)}{1-\theta}+\frac{\varepsilon}{2-\theta}g'(a)\right]\varepsilon^{1-\theta}\int_{a+\varepsilon}^{b}\frac{g(x)}{(x-a)^\theta}\mathrm{d}x$$

where the prime denotes differentiation. In the present work, the first term was calculated analytically, but the second term was evaluated using numerical differentiation, that is, expressions of the type

$$\frac{1}{2\varepsilon}[g(\tau+\varepsilon)-g(\tau-\varepsilon)]$$

were evaluated numerically.

When a singularity arose at $w = 1$, the range of integration was divided into $l/\sqrt{3}$ to $1-\varepsilon_1$, $1-\varepsilon_1$ to $1+\varepsilon_1$, $1+\varepsilon_1$ to $\tau-\varepsilon$ and $\tau-\varepsilon$ to τ. Eight-point, Gaussian quadrature was used for the first and third ranges of integration, whereas the Isaacson–Keller scheme was used for the second and fourth ranges.

In arriving at the numerical results, it was found that a major consideration was the singularity at $w = \tau$. Consequently the choice of ε played a significant role. As an aid in the choice, the imaginary parts of the integrands were plotted as functions of w_1 (or w_2) for several values of τ. These plots revealed that the choice become more critical as τ increased and l decreased. To get some confidence in the results, two values of ε were chosen for the most extreme case that arose, namely, $l = 0{\cdot}005$ and $\tau = 1{\cdot}3$. Initially ε was taken as 0·0001

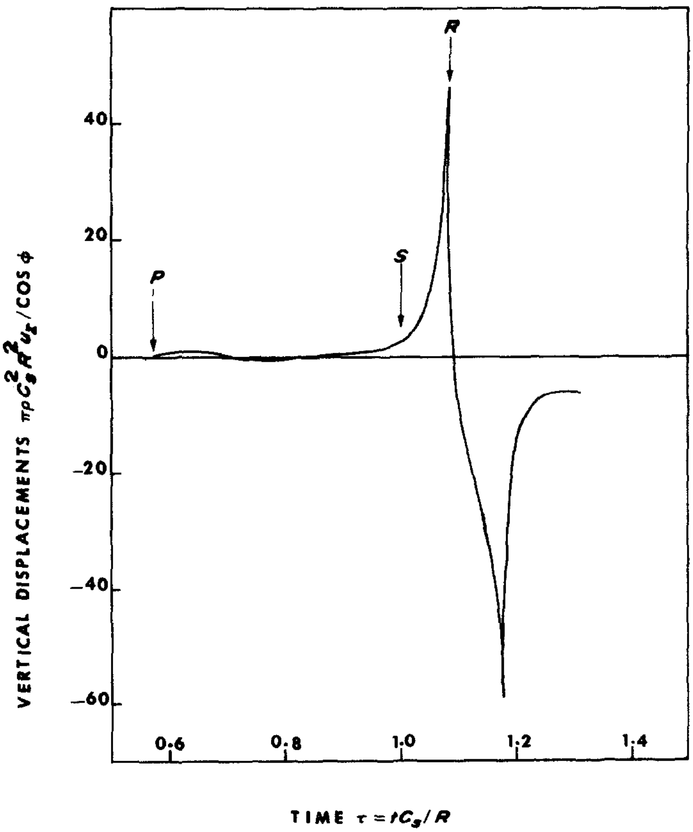

FIG. 1. The vertical displacements $\pi\rho c_s^2 R^2 u_z/\cos\phi$ as a function of time $\tau = tc_s/R$. $Q = 0{\cdot}1$ and $l = 0{\cdot}005$.

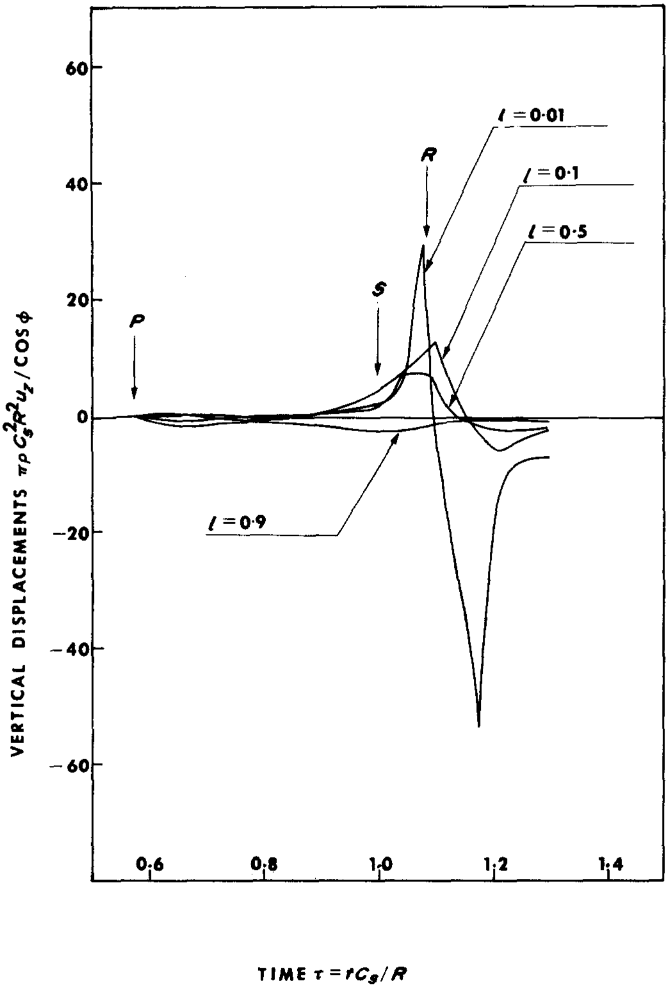

FIG. 2. The vertical displacements $\pi\rho c_s^2 R^2 u_z/\cos\phi$ as a function of time $\tau = tc_s/R$. $Q = 0{\cdot}1$ and $l = 0{\cdot}01$, $0{\cdot}1$, $0{\cdot}5$ and $0{\cdot}9$.

and then it was selected as 0·00001. It was found that the results differed by at most 4 per cent, which was felt to be acceptable.

Shown in Figs. 1 and 2 are the vertical displacements as functions of time for $Q = 0{\cdot}1$ and $l = 0{\cdot}005$, $0{\cdot}01$, $0{\cdot}1$, $0{\cdot}5$ and $0{\cdot}9$. The arrows designate the arrival of various events, P denoting the pressure wave, S the shear wave and R the Rayleigh wave. The computations, which proved to be quite expensive, were terminated soon after the arrival of the delayed Rayleigh wave, since experience would indicate that no significant event would occur after that time.

The most interesting feature that emerges is the evolution of the Rayleigh pulse. As l gets smaller, that is, as the distance from the source gets bigger, the larger the amplitude of the Rayleigh wave gets. This observation has also been made by Pekeris and Longman [3] in their work on a buried, torque pulse with a Heaviside step time-dependence. Pekeris and Lifson [4] in their work on a buried normal stress discontinuity with a Heaviside step time-dependence noted a similar feature. It required epicentral distances about five times the source depth before distinct Rayleigh events were visible. Finally, Nakano (see Ewing *et al.* [5, pp. 64–66]), who treated a line source with a harmonic time-dependence, showed, using the method of steepest descents, that Rayleigh waves do not arise when the epicentral distance is smaller than $c_R d/\sqrt{(c_D^2 - c_R^2)}$, c_R and c_D being the Rayleigh and dilatational wave speeds, respectively.

The curves also show that for $l = 0{\cdot}005$ and $0{\cdot}01$, the maximum magnitude occurs at the delayed Rayleigh arrival, whereas for $l = 0{\cdot}1$ and $0{\cdot}5$, the peak occurs at the Rayleigh arrival. For $l = 0{\cdot}9$, the maximum magnitude occurs between the S and R events. Finally it should be noted that for $l = 0{\cdot}005$, $0{\cdot}01$ and $0{\cdot}1$, the displacements are initially positive, whereas for $l = 0{\cdot}5$ and $0{\cdot}9$, the initial displacements are negative. In all cases considered, the displacements are negative after the arrival of the Rayleigh pulse.

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Абстракт—Определяются поверхностные вертикальные перемещения, вызванные погруженным, упруьим диполем в упругом полупространстве. Предполагается, что диполь облагает зависимостью наклона во времени. Результаты представляются в форме интегралов, Эти интегралы решаются как функции от времени для некоторых расстояний эпицентра.