Abstract. We show that for any group \( H \) (finite or infinite) there exists an independence structure with automorphism group isomorphic to \( H \). The proof is by construction and shows that for any \( H \) there is a geometric lattice with automorphism group isomorphic to \( H \).

§ 1: Introduction

We show that for any group \( H \) (finite or infinite) there exists an independence structure with automorphism group isomorphic to \( H \). The proof is by construction and shows that for any \( H \) there is a geometric lattice with automorphism group isomorphic to \( H \).

An independence structure on a set \( S \) is a family \( \mathcal{I} \) of subsets of \( S \) such that:

(i) \( \emptyset \in \mathcal{I} \).

(ii) If \( A \in \mathcal{I} \) and \( B \subseteq A \), then \( B \in \mathcal{I} \).

(iii) If \( A \in \mathcal{I} \) and \( B \in \mathcal{I} \) with \( |A| = |B| + 1 \), there exists \( a \in A - B \) such that \( B \cup \{a\} \in \mathcal{I} \).

(iv) \( \mathcal{I} \) has finite character, that is if \( X \) is an infinite subset of \( S \) and every finite subset \( Y \subseteq X \) also belongs to \( \mathcal{I} \), then \( X \in \mathcal{I} \).

When \( S \) is finite, the independence structure is a matroid \( M \) on \( S \). The members of \( \mathcal{I} \) are called independent sets. A base is a maximal independent set. Another matroidal terminology is that of \([4]\) and apart from the existence of a dual it carries over to independence structures in the obvious way.

* Revised version received 21 April 1971; final version received 16 August 1971.
A permutation \( \phi \) of \( S \) preserves incidence provided \( X \in I \) if and only if \( \phi(X) \in I \). The automorphism group of an independence structure \( I \) is the collection of permutations of \( S \) which preserve independence, and will be denoted by \( A(I) \), or by \( A(M) \) when the structure is a matroid \( M \).

The graph theory terminology is fairly standard, see [3].

**Theorem 1.** Given any group \( H \) there exists an independence structure \( I \) such that the automorphism group of \( I \) is isomorphic to \( H \).

From the proof, it will be clear that when the group \( H \) is finite, we can find a matroid \( M \) such that the automorphism group of \( M \) is isomorphic to \( H \).

§ 2. Proof of the main theorem

To prove Theorem 1, we need the following lemmas about infinite graphs.

A graph \( G \) consists of a set \( V = V(G) \) (possibly infinite) of vertices and a subset \( E = E(G) \) of edges, that is, of unordered pairs \( \{u, v\} \) of distinct vertices. A cycle in \( G \) is a finite sequence of edges \( \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\} \), where \( v_i \) (\( 1 < i < n \)) are distinct elements of \( V \). If we let \( X \subseteq E \) be a member of \( M(G) \) if and only if \( X \) does not contain a cycle, then it is easily seen (Piff [6]) that \( M(G) \) is an independence structure on \( E \). In the case where \( G \) is finite, \( M(G) \) is just the cycle matroid of \( G \); see [3]. Let \( A_p(G) \) be the point automorphism group of \( G \), and let \( A_c(G) \) be the cycle automorphism group of \( G \). That is, a permutation \( \pi \) of \( V \) is a member of \( A_p(G) \) provided \( \{v_1, v_2\} \in E(G) \) if and only if \( \{\pi(v_1), \pi(v_2)\} \in E(G) \). A permutation \( \phi \) of \( E \) is a member of \( A_c(G) \) if a subset \( X \) is a cycle of \( G \) if and only if \( \phi(X) \) is also a cycle of \( G \).

**Lemma 1.** \( A_c(G) = A(M(G)) \).

We omit the proof, which is obvious from the definition.
Lemma 2. If G is 3-connected, then $A_p(G) \cong A_c(G)$.

Proof. In the finite case, the theorem is essentially proved by Whitney [9]; see [5, Theorem 15.4.4.].

Let $G$ be 3-connected and infinite. As in the finite case, call a subset $X$ of $E(G)$ a cocycle of $G$ if it is a minimal disconnecting set of edges of $G$. In general, independence structures do not have duals in the same way as matroids. However, analogous to the finite case, it is true that $X$ is a cocycle of $G$ if and only if $X$ has a non-null intersection with every spanning tree of $G$, that is, with every base of $M(G)$, and $X$ is minimal with respect to this property. Alternatively, we can prove that a cocycle of $G$ is the complement in $E(G)$ of some hyperplane (maximal proper closed set) of $M(G)$. Now if $\phi \in A_c(G)$, then $\phi \in A(M(G))$, and hence $\phi$ preserves hyperplanes of $M(G)$. Thus $\phi$ preserves cocycles of $G$.

If $v \in V(G)$, let $st(v)$, the star at $v$, be the set of edges of $G$ incident with $v$. Then $st(v)$ is a cocycle of $G$ since $G$ is 3-connected. If $e_1, e_2 \in E - st(v)$, since $G$ is 3-connected, $G - st(v)$ is at least 2-connected and hence there exists a cycle $C \subset E(G) - st(v)$ such that $\{e_1, e_2\} \subset C$.

Thus

$$\phi(\{e_1, e_2\}) \subset \phi(C) \subset E - \phi(st(v)).$$

Thus the removal of $\phi(st(v))$ from $E(G)$ results in a graph $G'$ in which any pair of edges is contained in a cycle and therefore, since also $\phi(st(v))$ is a cocycle, this can only happen if there exists some vertex $v' \in V(G)$ such that $\phi(st(v)) = st(v').$

Define the map $\Psi: V(G) \rightarrow V(G)$ by $\Psi(v) = v'$. This is clearly a bijection, and if there exists an edge $[v_1, v_2] \in E(G)$, then

$$\phi(st(v_1)) \cap \phi(st(v_2)) = \emptyset,$$

hence $[\Psi(v_1), \Psi(v_2)] \in E(G)$ and $\Psi \in A_p(G)$. It is trivially verified that the map $f: A_c(G) \rightarrow A_p(G)$ defined by $f(\phi) = \Psi$ is a group isomorphism, thus completing the proof of Lemma 2.

In [8], Sabidussi proves:

Lemma 3. Given a group $H$, there exists a graph $G$ such that $H$ is isomorphic to $A_p(G)$.
The graph so constructed in [8] is however only 1-connected and \( A_p(G) \neq A_c(G) \).

**Lemma 4.** For any positive integer \( p \) and finite group \( H \), there exists a finite \( p \)-connected graph \( G \) with \( A_p(G) \cong H \).

This is proved in Sabidussi [7].

For the proof of Theorem 1 and Corollary 1 we need a 3-connected graph with given group \( H \).

**Lemma 5.** For any positive integer \( p \) and group \( H \), there exists an infinite \( p \)-connected graph \( G \) with \( A_p(G) \cong H \).

**Proof.** By [8], there exist non-isomorphic 1-connected graphs \( G_1, G_2, \ldots, G_p \) such that \( A_p(G_1) \cong H \) and \( A_p(G_2) = \ldots = A_p(G_p) = 1 \). Since the \( G_i \)'s are 1-connected, they are prime relative to cartesian multiplication. Hence by [7, Lemma 2.3], \( G = G_1 \times G_2 \times \ldots \times G_p \) is \( p \)-connected and by [7, Lemma 2.10].

\[
A_p(G) \cong A_p(G_1) \times A_p(G_2) \times \ldots \times A_p(G_p) \cong H.
\]

This is really all that is needed for the proof of Theorem 1!

We can now complete the proof of Theorem 1. By Lemma 5, for any group \( H \) there is a 3-connected graph \( G \) such that \( A_p(G) \cong H \). By Lemma 2, \( A_p(G) \cong A_c(G) \), and by Lemma 1, \( M(G) \) has automorphism group isomorphic to \( H \).

**Corollary 1.** Given any group \( H \) there exists an independence structure \( I \) such that all the following properties hold:

(i) \( A(1) \cong H \),

(ii) \( I \) is a geometry,

(iii) \( I \) is graphic.

(iv) \( I \) is non-separable,

(v) \( I \) is a non-transversal structure.

**Proof.** The only statement not immediately obvious from the proof of Theorem 1 is (v). Every 3-connected graph without parallel edges con-
§ 3. The infinite version of Lemma 5.

contains \(K_4\) as a homeomorph (perhaps this is a well-known result). For suppose \(G\) is 3-connected and \([v_1, v_2]\) is an edge of \(G\). Let \(p_1, p_2\) be paths joining \(v_1\) and \(v_2\) in \(G\) disjoint from \([v_1, v_2]\), and suppose \(v_3\) and \(v_4\) are intermediate nodes on \(p_1\) and \(p_2\), respectively. Then, since \(G\) is 3-connected, there exists a path \(p_3\) from \(v_3\) to \(v_4\) which does not pass through \(v_1\) or \(v_2\). Let \(v_5\) be the last vertex of \(p_1 \cap p_3\) and \(v_6\) the following vertex of \(p_2 \cap p_3\). Put \(p_4\) equal to the section of \(p_3\) joining \(v_5\) and \(v_6\). The subgraph consisting of \([v_1, v_2]\), \(p_1, p_2\) and \(p_4\) is a homeomorph of \(K_4\).

Taking a 3-connected graph \(G\) with \(A_p(G) \cong H\), we see that \(G\) contains \(K_4\) as a homeomorph, hence, by a recent result of Bondy [2], \(M(G)\) is not transversal.

From (ii) and the correspondence between geometries and geometric lattices (semimodular, relatively complemented, atomic, and of finite length), we have:

**Corollary 2.** Given any group \(H\), there exists a geometric lattice with automorphism group isomorphic to \(H\).

This result neither implies nor is implied by the theorem of Birkhoff [1] who shows the existence of a distributive lattice with arbitrary automorphism group.

§ 3. The infinite version of Lemma 5

The strong infinite version of Lemma 5 is of independent interest and is now demonstrated.

**Theorem 2.** For any cardinal \(\rho > 0\) and infinite group \(H\), there exists an infinite \(\rho\)-connected graph \(G\) with \(A_p(G) \cong H\).

**Proof.** Let \(G_1\) and \(G_2\) be edge- and vertex-disjoint, 1-connected graphs such that \(A_p(G_2) \cong H\) and \(A_p(G_1)\) is the identity group, and such that \(|V(G_1)| > \rho\), \(|V(G_1)|\) is infinite. Let \(V(G_1) = V_1 = \{v_i : i \in I\}\) and \(V(G_2) = V_2 = \{v_j : j \in J\}\). Let \(V = \{v_{ij} : i \in I, j \in J\}\) be a set disjoint from \(V_1\) and \(V_2\). Let \(G'\) be a graph with vertex set \(V_1 \cup V_2 \cup V\), such that \(G'-(V_1 \cup V) = G_2\), \(G'-(V_2 \cup V) = G_1\). Also for each \(i \in I, j \in J\), let
there be edges in \( G' \) joining \( v_j \) to \( v_i \) and \( v_j \). Thus in \( G' \), if \( v \in V_1 \cup V_2 \), then \( v \) has degree strictly greater than \( \rho \), and if \( v \in V \), then \( v \) has degree 2.

Let \( u, v \) denote two arbitrary vertices of \( G' \). If \( \{u, v\} \subset V_1 \) or if \( \{u, v\} \subset V_2 \), there exist at least \( \rho \) vertex-disjoint paths from \( u \) to \( v \) in \( G' \). If \( u \in V_2 \), \( v \in V_1 \), we find \( \rho \) vertex-disjoint paths in \( G' \) from \( u \) to \( v \) as follows. There is one path of length 2. There are \( \rho - 1 \) paths of length 2 from \( u \) into \( V_1 - v \), followed by \( \rho - 1 \) disjoint paths returning from \( V_1 - v \) to \( V_2 - u \), and finally \( \rho - 1 \) paths from \( V_2 - u \) to \( v \).

Since \( A_\rho(G_2) \cong H \) and \( H \) is infinite, \( |V_2| = \infty \) and since \( V_1 \) is infinite and \( > \rho \), we can partition \( V_1 \) into disjoint \((\rho - 1)\)-subsets, say \((A_k : k \in K)\). For each \( k \in K \) and \( v_{i_0} \in V_2 \), if \( A_k = \{v_i : i \in I_k\} \), we form the complete graph with vertex set \( \{v_{i_0} : i \in I_k\} \). Denote this graph by \( G(k, i_0) \). Now let \( G = G' \cup \bigcup_{k, i_0} G(k, i_0) \). We claim that \( G \) is \( \rho \)-connected and \( A_\rho(G) \cong H \). There are several cases to consider. We list in each case \( \rho \) disjoint paths connecting two arbitrary vertices \( u \) and \( v \), and throughout this listing will often refer to "paths" when we mean "collection of vertex-disjoint paths".

**Case 1.** There exist \( k, j_0 \) such that \( \{u, v\} \subset G(k, j_0) \). It is trivial that \( \rho \) disjoint paths exist.

**Case 2.** \( u \in G(k_1, j_0), v \in G(k_2, j_0) \), where \( k_1 \neq k_2 \). There is a path \( \{u, v_{j_0}, v\} \) and \( \rho - 1 \) additional paths obtained by connecting \( \rho - 1 \) paths from \( u \) to \( A_{k_1} \) to the \( \rho - 1 \) paths of length 4 from \( A_{k_1} \) to \( A_{k_2} \) and then connecting the \( \rho - 1 \) paths from \( A_{k_2} \) to \( v \).

**Case 3.** \( u \in G(k, j_0), v \in G(k, j_1) \). There are \( \rho - 1 \) paths from \( u \) to \( A_k \) and \( \rho - 1 \) paths from \( A_k \) to \( v \). There is a further disjoint path from \( u \) to \( v \) through \( v_{j_0} \) and \( v_{j_1} \).

**Case 4.** \( u \in G(k_1, j_1), v \in G(k_2, j_2) \), where \( k_1 \neq k_2, j_1 \neq j_2 \). There is one path \( P \) from \( A_{k_1} \) to \( A_{k_2} \) which only intersects \( A_{k_1} \) and \( A_{k_2} \) at its endpoints and otherwise lies in \( G_2 \). Letting \( u_1, v_1 \) be the endpoints of \( P \), we have a path \( u_1, P, v_1, v \). Now consider the \( \rho - 2 \) paths of length 4 from \( A_{k_1} - u_1 \) to \( A_{k_2} - v_1 \) which pass through \( V_2 - v_j - v_j \). These can
be extended to paths from $u$ to $v$. Finally, it is easy to see that there is a path $Q$ of length 4 from $v_{j_1}$ to $v_{j_2}$ such that $Q \cap P = Q \cap A_{k_1} = Q \cap A_{k_2} = \emptyset$. The path $u v_{j_1} Q v_{j_2} u$ is a path disjoint from the $\rho - 1$ previously constructed.

Case 5. $u \in G(k, j_0)$, $v = v_{j_0}$. In this case, $\rho - 1$ disjoint paths are immediately obvious. The last is obtained by taking the edge from $u$ to $A_k$ followed by a path in $G_1$ to a vertex outside $A_k$ and then a path of length 2 to $v_{j_0} = v$.

Case 6. $u \in G(k, j_0)$, $v = v_{j_1}, j_1 \neq j_0$. There exist $\rho - 1$ disjoint paths from $u$ through $G(k, j_0)$ to $A_k$. To these connect the $\rho - 1$ disjoint paths from $A_k$ through $G(k, j_1)$ to $v_{j_1} = v$. Also there is a path disjoint from these of the form $uv_{j_0}Q$ where $Q$ is a path from $v_{j_0}$ to $v_{j_1}$ contained in $G_2$.

Case 7. $u \in G(k, j_0)$, $v \in A_k$. There is a path of length at most 2 from $u$ to $v$. There is a second from $u$ to $v_{j_0}$ to $v_{j_1}$ ($j_1 \neq j_0$) and then two further edges across to $v$. The remaining $\rho - 2$ paths are obtained by travelling along $\rho - 2$ paths through $G(k, j_0)$ to $A_k$, then across to $\rho - 2$ distinct points in $G_1 - v_{j_0}v_{j_1}$ followed by $\rho - 2$ paths to $v$.

Case 8. $u \in G(k, j_0)$, $v \in G_2 - A_k$. This is very similar to case 7 and clear from a diagram.

As these eight cases are exhaustive, this completes the proof that $G$ is $\rho$-connected. Now consider any $\phi \in A_p(G)$. Clearly $\phi$ must map $V$ onto itself from vertex degree arguments. Consider $G - V$ and $\phi' = \phi$ restricted to $G - V$. Then $\phi'$ must be an automorphism of $G - V$. Since $G_1$ and $G_2$ are connected components of $G - V$, we have $\phi'(V_1) = V_1$ and $\phi'(V_2) = V_2$. Hence since $A_p(G_1)$ is the identity group, $\phi'$ and thus $\phi$ restricted to $G_1$ must be the identity; therefore $\phi$ is uniquely determined by its effect on $G_2$. That is, $A_p(G) \cong A_p(G_2) \cong H$, which proves the theorem.
§ 4. Further results

A trivial but useful result is the following:

**Theorem 3.** For any matroid \( \mathcal{M} \) on \( S \) and its dual \( \mathcal{M}^* \), \( A(\mathcal{M}^*) = A(\mathcal{M}) \).

**Proof.** Obvious since the dual is unique.

**Theorem 4.** A matroid \( \mathcal{M} \) on a set of cardinality \( n \) has \( S_n \) as its automorphism group if and only if \( \mathcal{M} \) has as bases every \( k \)-subset of \( S \) for some \( k \), \( 1 \leq k \leq n \), i.e., is \( k \)-uniform.

**Proof.** Let \( A(\mathcal{M}) = S_n \). Let \( \mathcal{M} \) have rank \( r \). Take any base \( B \). For any \( r \)-set \( X \subseteq S \), there exists \( \pi \in A(\mathcal{M}) \) such that \( \pi(B) = X \). Hence every \( r \)-set in \( S \) must be independent.

**Theorem 5.** An independence structure 1 on an infinite set \( S \) has the full permutation group as its automorphism group if and only if either \( (a) \) it is \( k \)-uniform for some finite \( k \) or \( (b) \) it is the trivial structure in which every subset of \( S \) is independent.

The proof is very similar to that of Theorem 4 and will be omitted.

**Theorem 6.** There is no matroid on a set of \( n \) elements with automorphism group equal to the alternating group \( A_n \) for any \( n \geq 3 \).

**Proof.** Let \( n \geq 3 \). It is well-known that \( A_n \) is \((n-2)\)-ply transitive. Hence suppose \( A(\mathcal{M}) = A_n \) and \( \mathcal{M} \) has rank \( r < n - 2 \). Then every \( r \)-subset of \( S \) is independent and hence \( A(\mathcal{M}) = S_n \). If \( \mathcal{M} \) has rank \( r > n - 2 \), then \( \mathcal{M}^* \) has rank \( \leq n - 2 \) and so \( A(\mathcal{M}^*) = S_n \), whence by Theorem 3, \( A(\mathcal{M}) = S_n \).

An interesting application of the above theory is as follows. Consider the 3-dimensional Desargues configuration. Regarded as a matroid with independence induced by projective independence, it has rank 4 on a set of 10 elements and is easily seen to be the same matroid as the cycle matroid of \( K_5 \). Hence its automorphism group is the same as the automorphism group of \( M(K_5) \). Also since \( K_5 \) is 3-connected,
A(M(K₅)) \cong A_p(K₅) = S₅, and we see that the Desargues configuration has automorphism group isomorphic to S₅.

References