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SUPERSONIC FLOW PAST A DELTA WING AT ANGLES OF ATTACK AND YAW

BY

R. C. F. BARTELS

O. LAPORTE

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1. Introduction

A general method for treating the linearized equations for the flow past a conical body placed in a uniform stream with supersonic speed is formulated in a previous report[1)]. It is shown there that a conical flow past the body can be completely described in terms of an analytic function of a single complex variable and that this function is determined by boundary conditions corresponding to those to which the flow must conform at points on the surface of the body and at points on the Mach cone enveloping the body. This method is employed in the present report to determine the flow past a delta wing at angles of attack and yaw. The wing is restricted to lie entirely within its Mach cone. Also, as has been implied in the general theory of the previous report, it is supposed that the velocity of the flow is continuous at all points of space exterior to the wing surface. Therefore no provision is made in this report for the

1) O. Laporte and R. C. F. Bartels, An investigation of the exact solutions of the linearized equations for the flow past conical bodies. Bumble Bee Report No. 75, 1948. This report will be referred to simply as BB. Report No. 75.

presence of a vortex sheet behind the wing in the case when one of its edges becomes a trailing edge[2].

As in the case of the delta wing at zero angles of yaw treated in the previous report, it is shown here that also in the case of the wing at arbitrary angles of attack and yaw the boundary conditions determined by the body surface and the Mach cone are not sufficient to determine uniquely the conical flow past the wing. However, it is shown that when these conditions are supplemented by the additional condition requiring the normal force coefficient of the wing to be finite, the flow is then completely determined for all angles of attack and yaw. The corresponding flow is such that the velocity along either edge is always infinite, except when the angle of yaw is equal to half the flare angle of the wing.

2. Formulation of the Problem for the Delta Wing with Angle of Attack and Yaw

Let the delta wing be formed by a sector of a plane inclined at an angle α with respect to the uniform flow. The angle of flare of the wing; i.e., the angle of the sector, will be taken as 2γ and the angle of yaw as ψ (see Figure 1). If the origin of the X, Y, and Z coordinate system is placed at the vertex of the wing with the Z-axis in the direction of the uniform flow, the equation of the plane containing the wing sector may be taken as[3]

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2) The case of flows with surfaces along which the velocity is discontinuous will be treated further in a subsequent report. Such a surface of discontinuity is admitted by Busemann in the treatment of the triangular wing with trailing edge; see A. Busemann, Infinitesimale kegelige Uberschallstromung, Deutschen Akademie fur Luftfahrtforschung, Vol. 7B (1943), pp. 105-122. A vortex sheet is also admitted by Hayes in his treatment of the same problem. In this work the flow is determined by assuming a condition at the trailing edge of the wing analogous to the condition of Kutta. See W. D. Hayes, Linearized Supersonic Flow, Thesis, Cal. Inst. Tech., June, 1947.

3) It should be noted that the orientation of the axes are such that a positive angle of α will produce a force of lift in the direction of the negative Y-axis.

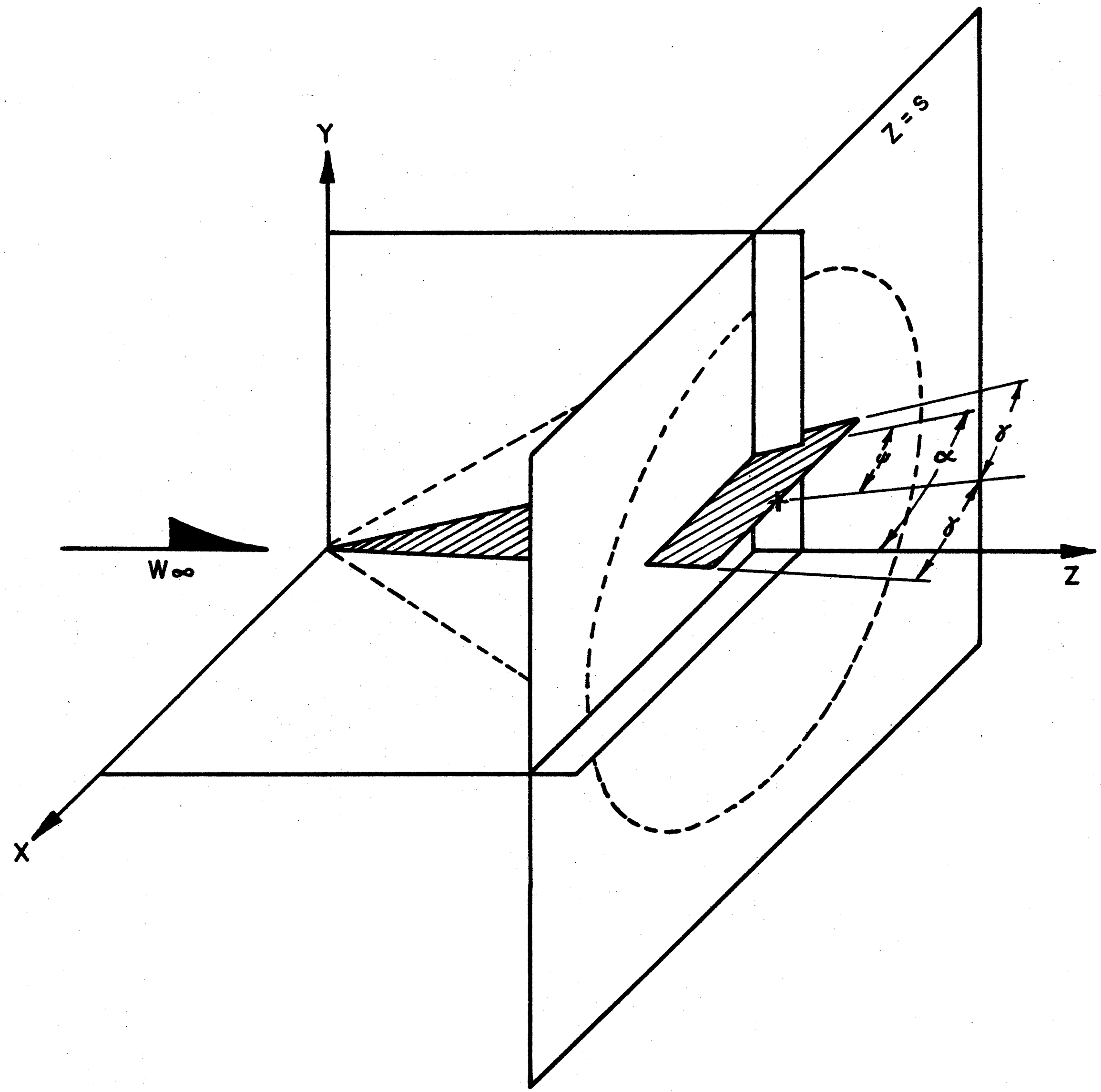

Figure 1. The Delta Wing at an Angle of Attack and Yaw.

$$Y = Z \tan \alpha \, . \tag{1}$$

It will be assumed throughout that the angles α, 2γ, and ψ of attack, flare, and yaw, respectively, are so restricted that the wing is always contained within the Mach cone whose vertex is at the vertex of the wing sector; i.e., the cone whose equation is

$$Z^2 - \beta^2 (X^2 + Y^2) = 0, \tag{2}$$

where

$$\beta = \cot \mu \tag{3}$$

with μ equal to the Mach angle of the uniform flow.

The problem is to determine the conical flow field representing a solution of the linearized equations for a supersonic, irrotational flow for which (a) the delta wing forms a stream surface, and (b) the transition from the constant state of the uniform flow ahead of the wing to the disturbed start of the flow that envelops the wing takes place across the Mach cone of equation (2). Let w_∞ represent the velocity of the uniform flow, and let u, v, and w represent the components of the additional velocity parallel to the X, Y, and Z axes, respectively, so that the components of the total velocity at any point are, respectively, u, v, and $w + w_\infty$. Then by virtue of equation (1) the condition (a) implies that

$$v = (w + w_\infty) \tan \alpha \tag{4}$$

at points on the wing surface, and the condition (b) implies that

$$u = v = w = 0 \tag{5}$$

on the Mach cone.

It is shown in the previous report[4] that when the equations of flow are transformed by making the substitutions

$$X = R\frac{\xi}{\beta\lambda}\ , \quad Y = R\frac{\eta}{\beta\lambda}\ , \quad Z = R\frac{1-\lambda}{\lambda}\ , \tag{6}$$

where

$$\lambda = \frac{1}{2}(1-\xi^2-\eta^2)\ , \qquad R = \sqrt{Z^2-\beta^2(X^2+Y^2)}\ , \tag{6a}$$

then the real parts of the following three analytic functions of the complex variable $\zeta = \xi + i\eta$:

$$\begin{aligned} U(\zeta) &= \frac{\beta}{2}\int (1+\zeta^2)\,F(\zeta)\,d\zeta\,, \\ V(\zeta) &= i\frac{\beta}{2}\int (1-\zeta^2)\,F(\zeta)\,d\zeta\,, \\ W(\zeta) &= -\int \zeta\,F(\zeta)\,d\zeta\,, \end{aligned} \tag{7}$$

define, respectively, the additional velocity components u, v, and w of an irrotational, conical flow field past a conical obstacle, provided that $F(\zeta)$ is analytic in the appropriate region of the ζ-plane, and that conditions of the type (a) and (b) of the last paragraph are satisfied on the surface of the obstacle and the Mach cone. By means of equations (6), the points of the Mach cone (2) correspond to the unit circle $|\zeta| = 1$ in the ζ-plane, and the points of the plane (1) containing the wing sector correspond to the circle (see Figure 2b) with the equation

$$\xi^2 + \eta^2 - 2\frac{\cot\alpha}{\beta}\eta + 1 = 0\,. \tag{8}$$

4) cf. BB. Report No. 75, Section 1.2.

Consequently, the region of disturbed flow within the Mach cone and surrounding the delta wing corresponds to the double-connected region of the ζ-plane between the unit circle and an arc of the circle (8). The wing edges correspond to the end points (ξ_E^+, η_E^+) and (ξ_E^-, η_E^-) of this arc. Therefore, the function $F(\zeta)$ which defines the appropriate conical flow past the delta wing is analytic in the double-connected region of the ζ-plane and such that the real parts u, v, and w of the three functions in equations (7) are single-valued in this region and, in accordance with equations (4) and (5), satisfy the following boundary conditions:

$$\begin{aligned} &(a') \text{ On the circular arc:} \quad v - w\tan\alpha = w_\infty \tan\alpha\,. \\ &(b') \text{ On the unit circle } |\zeta| = 1: \quad u = v = w = 0. \end{aligned} \tag{9}$$

As a consequence of the second of the boundary conditions (9) and the last of equations (7) it can be shown[5] that the function $F(\zeta)$ is necessarily continuous on the unit circle $|\zeta| = 1$.

Let the problem formulated above in terms of the function $F(\zeta)$ be modified by the introduction of the function

$$H(\zeta) = V(\zeta) - W(\zeta)\tan\alpha\,, \tag{10}$$

so that

$$H(\zeta) = \int L(\zeta)\,F(\zeta)\,d\zeta\,, \tag{11}$$

where

$$L(\zeta) = -\frac{i}{2}\left\{\beta(\zeta^2-1) + 2i\,\zeta\tan\alpha\right\}. \tag{12}$$

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5) From general theorems on the theory of functions it follows that: if the real part of a function of a complex variable is constant along an analytic arc of the boundary of a region in which it is analytic, the function is itself analytic on this arc. Thus the continuity of $F(\zeta)$ follows from the analyticity of $W(\zeta)$ on the circle $|\zeta| = 1$.

Then, for the appropriate function $F(\zeta)$, $H(\zeta)$ is analytic in the double-connected region of the ζ-plane, its real part is single-valued within this region and, by (9), constant on the interior and exterior boundaries of the region.

Let the problem be further modified by mapping the doubly-connected region of the ζ-plane conformally onto the annular region $r_0 \leqq |z| \leqq 1$ in the plane of the complex variable $z = x + iy$ (see Figure 2d) by means of the relation

$$\zeta = \omega(z) \tag{13}$$

in such a manner that the points of the unit circle $|\zeta| = 1$ correspond to the points of the circle $|z| = 1$ forming the interior boundary of the annulus, and the points of the (two-sided) circular arc with end points $(\xi_\varepsilon^+, \eta_\varepsilon^+)$ and $(\xi_\varepsilon^-, \eta_\varepsilon^-)$ correspond to the circle $|z| = r_0$ forming the interior boundary. Then the functions $U(z)$, $V(z)$, $W(z)$, and $h(z)$ obtained from $U(\zeta)$, $V(\zeta)$, $W(\zeta)$, and $H(\zeta)$, respectively, by performing the change of variable in (13) are analytic within the annulus. In particular,

$$\begin{aligned} U(z) &= \frac{\beta}{2} \int [1 + \omega^2(z)]\, f(z)\, dz, \\ V(z) &= i\frac{\beta}{2} \int [1 - \omega^2(z)]\, f(z)\, dz, \\ W(z) &= -\int \omega(z)\, f(z)\, dz, \end{aligned} \tag{14}$$

and, if $\ell(z)$ represents the function obtained from $L(\zeta)$ in (12) by making the substitution (13),

$$h(z) = \int \ell(z) f(z)\, dz \ , \tag{15}$$

where in these expressions f(z) is an analytic function of z in the annulus and is related to the function $F(\zeta)$ of equations (7) by the equation

$$F(\zeta) = \frac{f(z)}{\omega'(z)} . \tag{16}$$

Moreover, the real part of the function h(z) is single-valued in the annulus and on its boundaries satisfies the conditions corresponding to (9), namely[6)]

$$\begin{aligned} &(a'')\ \text{On the circle } |z| = r_o \quad : \mathcal{R}\{h(z)\} = v_\infty \tan\alpha . \\ &(b'')\ \text{On the circle } |z| = 1 \quad : \mathcal{R}\{h(z)\} = 0. \end{aligned} \tag{17}$$

The result of the foregoing modifications of the problem of the flow past the delta wing as originally formulated may be summarized as follows: <u>The problem of the conical flow past the delta wing consists of determining the function f(z) which is analytic in the annular region $r_o < |z| < 1$ and such that the real part of the function h(z) defined by equation (15) is single-valued in this region and satisfies the conditions (17) on its boundaries</u>. The functions U(z), V(z), and W(z) representing the components of the "complex velocity" of the flow are obtained from the solution of this problem by means of equations (14). It is necessary, of course, to select the appropriate constants of integration in the integral formulae of equations (14) and (15) in order that the components u, v, and w of the real velocity vanish on the circle $|z| = 1$.

It is important to observe that the function f(z) is necessarily continuous at points of the circle $|z| = 1$. In view of equation (16), this property follows from the fact that $F(\zeta)$ is continuous on the corresponding

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6) The symbol $\mathcal{R}\{\ \}$ denotes the real part of the complex quantity within the brace.

circle $|\zeta| = 1$ in the ζ-plane, and the fact that the mapping function $\omega(z)$ and its derivative $\omega'(z)$ are continuous everywhere on the boundaries of the annulus, except at the points of the interior boundary $|z| = r_o$ which correspond to the wing edges. It also is important to observe that the boundary conditions on the function $H(\zeta)$ are such as to insure the continuity of the function $f(z)$ at all points on the circle $|z| = r_o$, except possibly at the points corresponding to the wing edges. Since the real part of $H(\zeta)$ is constant at points of the circular arc in the ζ-plane which corresponds to the wing sector, it follows[7] that $H(\zeta)$ and its derivatives are continuous at all points of the arc, with the possible exception of the end points $(\xi_\epsilon^{\pm}, \eta_\epsilon^{\pm})$ corresponding to the wing edges. Since the zeros of the quadratic function $L(\zeta)$ are shown later to be on the circle $|\zeta| = 1$, the continuity of the function $F(\zeta)$ at points of the arc follows from the continuity of the derivative $H'(\zeta)$ by virtue of the relation (11). Therefore, by (16), the function $f(z)$ is continuous on $|z| = r_o$, save possibly at the points corresponding to the wing edges.

3. Mapping of the Doubly-Connected Region onto the Annulus

It is convenient to regard the conformal mapping of the doubly-connected region of the ζ-plane onto the annulus in the z-plane as effected in a series of three steps (see Figures 2b, 2c, 2d). First, let

$$\zeta = \frac{\zeta_1 + ib}{1 - ib\zeta_1}, \tag{18}$$

where $\underline{b}$ is the intercept of the circular arc in the ζ-plane with the η-axis

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7) See the general theorem of function theory referred to in footnote ~~4~~5.

(see Figure 2b). This relation maps the region of the ζ-plane bounded by the unit circle $|\zeta| = 1$ and containing the circular arc corresponding to the wing sector onto the region bounded by the unit circle $|\zeta_1| = 1$ in the plane of the complex variable $\zeta_1 = \xi_1 + i\eta_1$ in such a manner that the points of the circular arc are mapped onto a segment of the real axis in the ζ_1-plane. The end points (ξ_E^+, η_E^+) and (ξ_E^-, η_E^-) of the circular arc are thereby mapped onto the end points $(\xi_{1E}^+, 0)$ and $(\xi_{1E}^-, 0)$, respectively, of the segment of the ξ_1-axis.

Next, let

$$\zeta_1 = \frac{\zeta_2 + a}{1 + a\zeta_2}, \tag{19}$$

where $\underline{a}$ is a real constant that will be specified later. This relation maps the region of the ζ_1-plane bounded by the unit circle $|\zeta_1| = 1$ onto the region bounded by the unit circle $|\zeta_2| = 1$ in the plane of the complex variable $\zeta_2 = \xi_2 + i\eta_2$ in such a manner that the real axis in the ζ_1-plane corresponds to the real axis in the ζ_2-plane. With an appropriate choice of the real constant $\underline{a}$, the segment of the real axis in the ζ_1-plane between the points $(\xi_{1E}^{\pm}, 0)$ is mapped onto a segment with the symmetrically located end points $(\pm\xi_{2E}, 0)$ (see Figure 2d).

It has been indicated in the previous report[8] that when the constants $\underline{b}$ and $\underline{a}$ of equations (18) and (19) are defined as follows:

$$b = \tanh \frac{\tilde{\alpha}}{2} \tag{20}$$

and

$$a = \tanh \frac{\tilde{\psi}}{2}, \tag{21}$$

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8) cf. BB. Report No. 75, Section 1.3.

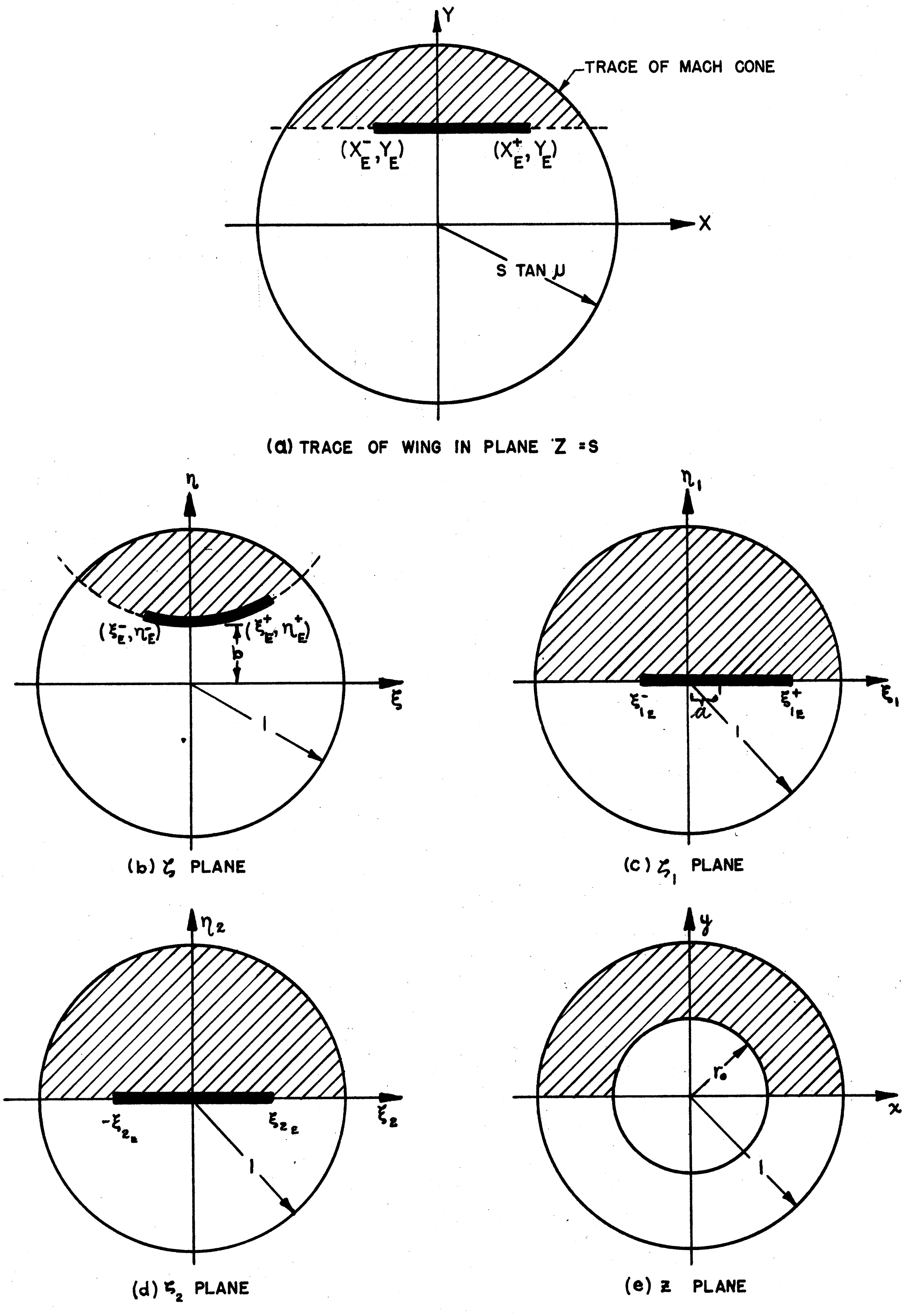

Figure 2. Correspondence between the XYZ-Space and the ζ, ζ_1, ζ_2, and z Planes

the transformations (18) and (19) are equivalent to non-Euclidean rotations about the X-axis and Y-axis, respectively, in pitch and yaw of magnitudes specified by the quantities $\tilde{\alpha}$ and $\tilde{\psi}$ that correspond to the Euclidean angles of attack α and yaw ψ, respectively. Since $\underline{b}$ is the intercept of the circle of equation (8) with the η-axis, it is evident that α and $\tilde{\alpha}$ are related as follows:

$$\tanh \tilde{\alpha} = \beta \tan \alpha . \tag{22}$$

In order to derive an expression for $\tilde{\psi}$ in terms of the Euclidean angles describing the shape and position of the delta wing, let it be observed that under the transformation (6) the polar angle ϕ of a point (X,Y) in the plane Z = constant of the (X,Y,Z)-space is equal to the polar angle ϕ of the corresponding point (ξ,η) in the ζ-plane. Taking $(X_\varepsilon^+, Y_\varepsilon)$ and $(X_\varepsilon^-, Y_\varepsilon)$ as the coordinates of the points on the wing edges in the plane Z = constant (see Figure 2a), then

$$X_\varepsilon^\pm : Y_\varepsilon = \xi_\varepsilon^\pm : \eta_\varepsilon^\pm . \tag{23}$$

But (see Figure 1)

$$\frac{X_\varepsilon^\pm}{Y_\varepsilon} = \tan(\psi \pm \gamma) \csc \alpha . \tag{24}$$

Also by equation (18)

$$\xi + i\eta = (1 - b^2)\frac{\xi_1}{1 + b^2 \xi_1^2} + i b \frac{1 + \xi_1^2}{1 + b^2 \xi_1^2} ,$$

where b is defined in (20), therefore

$$\frac{\xi_\varepsilon^\pm}{\eta_\varepsilon^\pm} = \frac{2\,\xi_{1\varepsilon}^\pm}{1 + \xi_{1\varepsilon}^{\pm 2}} \operatorname{csch} \tilde{\alpha} . \tag{25}$$

Hence, if $\tilde{\gamma}$ is defined in terms of the coordinates of the end points $(\pm \xi_{2_\varepsilon}, 0)$ of the symmetrically-located segment in the ζ_2-plane by the equation

$$\xi_{2_\varepsilon} = \tanh \frac{\tilde{\gamma}}{2} , \tag{26}$$

then, by equations (19), (21) and (25),

$$\frac{\xi_\varepsilon^\pm}{\eta_\varepsilon^\pm} = \tanh(\tilde{\psi} \pm \tilde{\gamma}) \cosh\tilde{\alpha} ,$$

and this, together with equations (23) and (24) gives the two equations

$$\tanh(\tilde{\psi} \pm \tilde{\gamma}) \cosh\tilde{\alpha} = \tan(\psi \pm \gamma) \csc\alpha . \tag{27}$$

Equation (22) and the two equations (27) serve to determine the three quantities $\tilde{\alpha}, \tilde{\gamma}$, and $\tilde{\psi}$ that correspond, respectively, to the Euclidean angles α, γ, and ψ determining the shape and orientation of the delta wing in the flow space. In this connection, it should be remarked that any arbitrary choice of values for the triplet $\tilde{\alpha}, \tilde{\gamma}, \tilde{\psi}$ will yield a triplet α, γ, ψ consistent with the requirement that the wing be contained entirely within the Mach cone.

Finally let

$$\zeta_2 = \frac{cn(z_1;k) + i k' sn(z_1;k)}{dn(z_1;k)} , \tag{28}$$

where

$$z_1 = \frac{2K}{\pi i} \log z , \tag{29}$$

k is the modulus of the Jacobi elliptic functions, k' the complementary modulus, and K the value of the complete elliptic integral of the first kind having the modulus k. This relation maps the circular region $|\zeta_2| \leq 1$ in the ζ_2-plane with the symmetrically-placed slit on the real axis onto the annular region $r_0 \leq |z| \leq 1$ in the plane of the complex variable $z = x + iy$ (see Figure 2e) in such a manner that the points of the unit circle $|\zeta_2| = 1$

correspond to the points of the unit circle $|z| = 1$, and the points of the segment $-\xi_{2_E} \leq \xi_2 \leq \xi_{2_E}$ of the real axis in the ζ_2-plane, where

$$\xi_{2_E} = \left(\frac{1-k'}{1+k'}\right)^{1/2}, \tag{30}$$

correspond to the points of the circle $|z| = r_0$ in the z-plane whose radius is given by

$$r_0 = e^{-\frac{\pi K'}{2K}}, \tag{31}$$

where K' is the value of the complete elliptic integral of the first kind having the modulus k'. This is made evident first by setting $z = e^{i\theta}$. In this case (28) becomes[9]

$$\zeta_2 = \frac{cn\frac{2K}{\pi}\theta + ik'sn\frac{2K}{\pi}\theta}{dn\frac{2K}{\pi}\theta} = \frac{dn\frac{2K}{\pi}\theta}{cn\frac{2K}{\pi}\theta - ik'sn\frac{2K}{\pi}\theta} = \frac{1}{\bar{\zeta}_2},$$

where $\bar{\zeta}_2$ represents the conjugate of the complex number ζ_2. Therefore, for points $z = e^{i\theta}$ on the unit circle about the origin in the z-plane,

$$|\zeta_2|^2 = \zeta_2\bar{\zeta}_2 = 1,$$

whence the corresponding points in the ζ_2-plane also lie on the unit circle about the origin. Next, setting $z = r_0e^{i\theta}$, ~~then~~ equation (28) becomes[9]

$$\zeta_2 = \frac{dn\frac{2K}{\pi} - k'}{k\,cn\frac{2K}{\pi}\theta} = \frac{k\,cn\frac{2K}{\pi}\theta}{k' + dn\frac{2K}{\pi}\theta},$$

whence ζ_2 is real for all values of θ (real) and is confined to the interval

9) cf. E. T. Whittaker and G. H. Watson, Modern Analysis, 4th Ed., London, 1935, pp. 493, et seq. Here and throughout the remainder of the report the modulus of the Jacobi elliptic functions is understood to be k.

of values $-\xi_{2_E} \leq \xi_2 \leq \xi_{2_E}$ defined by (30). Hence the points of the circle $|z| = r_0$ forming the interior boundary of the annulus in the z-plane correspond to the points of the segment of the real axis in the ζ_2-plane. In particular, the end points $(\pm \xi_{2_E}, 0)$ of the segment which correspond to the wing edges are mapped into the points $z = \pm r_0$, respectively.

The mapping defined by equation (28) is determined by the single parameter k. According to equation~~s~~ ~~(29)~~ (30) this parameter depends on the width of the slit on the real axis of the ζ_2-plane. But, by equation (26), the width of the slit is in turn determined by the quantity $\tilde{\gamma}$ which corresponds to the semi-vertex angle γ of the delta wing. Therefore, by equations (26) and (29), it follows that the mapping function (28) is completely defined in terms of the geometry of the delta wing by means of the relation

$$k = \tanh \tilde{\gamma} \; . \tag{32}$$

The function $\omega(z)$ in equation (13) which accomplishes the desired mapping of the doubly-connected region of the ζ-plane onto the annulus in the z-plane is therefore obtained by combining equations (18), (19), and (28).

4. Formulation of the Properties of the Function f(z)

Precisely as in the earlier report on the conical flow over a delta wing at zero yaw[10], it can be shown that the function f(z), representing the solution of the problem formulated in Section 2 for the general case of the wing at an angle of yaw different from zero, is expressible in terms of

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10) cf. BB. Report No. 75, Section 2.3. Here the function f(z) corresponds to the function G(z) in the earlier report.

elliptic functions. Here again, it is shown that the boundary conditions (4) and (5) are not sufficient to specify completely the function $f(z)$, but must be supplemented by an additional condition such as the finiteness of the total normal force coefficient of the wing. The arguments are essentially the same as in the earlier report, so that a brief outline of the steps will suffice here.

Let the function $h_1(z)$ be defined by the equation

$$h_1(z) = z\,h'(z)\,, \tag{33}$$

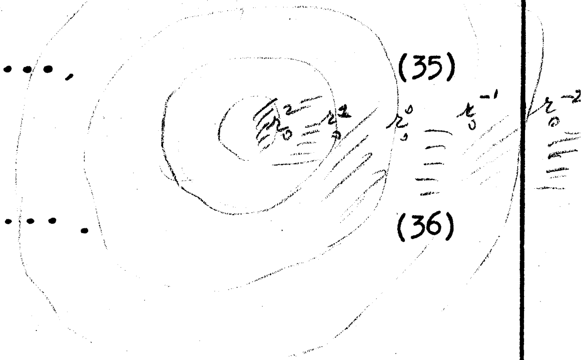
or by the equation

$$h(z) = \int_1^z h_1(\nu)\frac{d\nu}{\nu}\,, \tag{34}$$

where the lower limit of integration conforms with the second of the boundary conditions (17). Since $h(z)$ is analytic within the annular region $r_0 < |z| < 1$ and is continuously differentiable on the circular boundaries of the region (except possibly at the points corresponding to the wing edges; namely, $z = \pm r_0$), it follows that the function $h_1(z)$ is analytic within the annulus and continuous along its boundaries (except possibly at the edge points, $z = \pm r_0$). Moreover, as a consequence of the conditions (17), which require the real part of $h(z)$ to be constant along the circular boundaries of the annulus, it immediately follows that the imaginary part of the function $h_1(z)$ vanishes at all points of these circles, and therefore that the values of $h_1(z)$ on the circular boundaries of the annulus are real.

Let the definition of the function $h_1(z)$ be continued analytically across the circular boundaries of the annulus so as to cover the entire z-plane. As a consequence of the properties of $h_1(z)$ on the annulus, it follows from the principle of reflection[11] that its analytic extension is analytic in the whole of the z-plane (except possibly at the points $z = \pm r_0$ and at congruent points obtained from these by successive reflections with respect to the circles bounding the annulus) and is such that[12]

$$h_1(z) = h_1(z r_0^{-2n}) , \quad n = 0, \pm 1, \pm 2, \cdots , \tag{35}$$



and

$$h_1(z) = h_1(z e^{2m\pi i}) , \quad m = 0, \pm 1, \pm 2, \cdots . \tag{36}$$

Moreover,[13]

$$\overline{h_1\left(\frac{1}{\bar{z}}\right)} = h_1\left(\frac{1}{z}\right) = \overline{h_1(z)} \; . \tag{37}$$

//// defined by (35)

≡ " " (37)

As in equation (29), set

$$z_1 = \frac{2K}{\pi i} \log z ,$$

and define the function $\wp(z_1)$ of the complex variable $z_1 = x_1 + iy_1$ as follows:

$$\wp(z_1) = \frac{\pi}{2K} h_1(z) \, . \tag{38}$$

This function is analytic throughout the entire z_1-plane, except possibly at the points in the z_1-plane corresponding to the edge points $z = \pm r_0$ and the

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11) See, for example, E. J. Townsend, Functions of a Complex Variable, Holt, New York, 1915, p. 255.

12) cf. BB. Report No. 75, equations (2.28) and (2.29), respectively.

13) cf. BB. Report No. 75, equations (2.27).

points congruent to these with respect to successive reflections in the circles bounding the annulus. Moreover, it follows from equations (35), (36), (37), and (38) that

$$\wp(z_1 + 2niK') = \wp(z_1), \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{39}$$

$$\wp(z_1 + 4mK) = \wp(z_1), \qquad m = 0, \pm 1, \pm 2, \cdots, \tag{40}$$

and

$$\wp(\bar{z}_1) = \overline{\wp(z_1)}, \tag{41}$$

or, what is the same thing, $\wp(z_1)$ is real for real values of z_1. Therefore, according to the first two of these relations, $\wp(z_1)$ is a doubly-periodic function of z_1 with the periods 4K and 2iK', whence its values at all points of the z_1-plane are periodic repetitions of its values in the period rectangle:

$$0 \leq x_1 \leq 4K, \qquad -K' \leq y_1 \leq K' \tag{42}$$

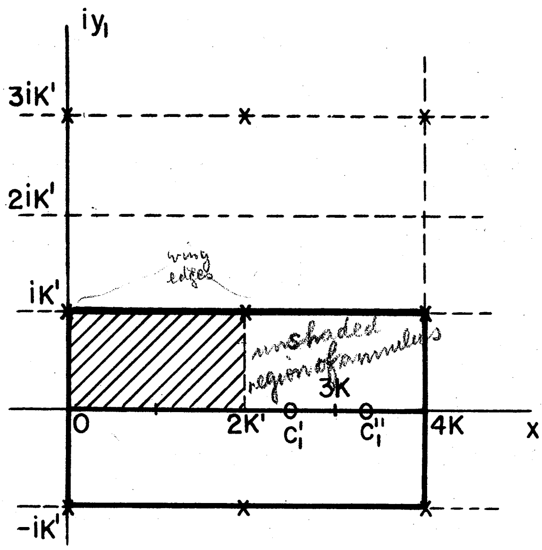

Fig. 3 Z_1–PLANE

(See Figure 3.) However, as a consequence, of the property expressed by equation (41), the values of $\wp(z_1)$ in the lower half of this rectangle are completely defined in terms of those in the upper half. In this connection, it should be observed that the upper half of the rectangle is mapped onto the annulus $r_0 \leq |z| \leq 1$ in the z-plane by the relation (29) in such a manner that the Mach circle $|z| = 1$ corresponds to the real axis $y_1 = 0$, and the circle $|z| = r_0$ corresponds to the side $y_1 = K'$. (The shaded region in Figure 3 corresponds to the shaded regions of Figure 2); in particular, the wing-edge points $z = +r_0$ and $z = -r_0$ are mapped into the points $z_1 = iK'$ and $z_1 = 2K + iK'$, respectively. Hence, the function $\wp(z_1)$ is completely defined at all points of the z_1-plane in terms of its values in the half of the period parallelogram which is mapped into the annulus in the z-plane.

As in the previous report[14], it follows that the doubly-periodic function $\wp(z_1)$ is an elliptic function of the variables z_1 with zeros and poles in each period parallelogram. The locations of the poles of this elliptic function are essentially determined by the conditions of the problem requiring that the velocity be continuous at each point within the flow and at each regular point[15] of the surface of the wing. For it is seen in Section 2, page 9, that these conditions require that h(z) and its derivative h'(z) are continuous within and on the boundaries of the annulus, with the possible exception of the points corresponding to the wing edges. Hence, by equations (33) and (38), the poles of the function $\wp(z_1)$, if they exist, are restricted to lie at the points of the half-period rectangle corresponding to these edge points; that is, at the points $z_1 = iK'$ and $z_1 = 2K + iK'$. Moreover, it will be seen later that the boundary conditions essentially determine the locations of two points, say $z_1 = c'$ and $z_1 = c''$, on the

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14) cf. BB. Report No. 75, page 39.

15) By a regular point of a surface is meant a point at which the surface has a continuous normal vector. See footnote 5), page 6, of BB. Report No. 75.

real axis of the z_1-plane at which $\wp(z_1)$ must vanish; by equations (15), (23), and (38), or the equation

$$\wp(z_1) = \frac{\pi}{2K} z f(z) \ell(z) \;, \tag{43}$$

it follows that the zeros of $\wp(z_1)$ correspond to the zeros of the function $L(\varsigma)$ in equation (12). However, neither the orders of these zeros, nor the orders of the poles of $\wp(z_1)$ are prescribed by the boundary conditions. Nor, for that matter, are the disposition and orders of other possible zeros of $\wp(z_1)$ prescribed by these conditions. By an argument similar to that employed in the previous report[16], it can be shown that, as a consequence of the boundary conditions, the residues of $\wp(z_1)$ at each of its poles are necessarily equal to zero, but, other than this, no further limitations are placed on the orders of the poles by these conditions. Conversely, let the elliptic function $\wp(z_1)$ have the following properties: (a) periods $4K$ and $2iK'$, (b) poles with zero residues at $z_1 = iK'$ and $z_1 = 2K + iK'$, (c) zeros in at least the two points $z_1 = c\,'$ and $z_1 = c\,''$, and finally (d) real for real values of z_1. Then there is determined a function $h(z)$ which satisfies the boundary conditions (17) and which in turn determines a conical flow past the wing. Since the complete specification of an elliptic function requires not only the locations of the zeros and the poles, but also their respective orders[17], it follows that there are infinitely many functions $\wp(z_1)$ possessing the foregoing properties. Consequently, the

16) cf. BB. Report No. 75, page 41.

17) The orders of the zeros and poles in a period parallelogram and their locations are not entirely independent. See for examples the Theorems of Liouville and Abel, Whittaker and Watson, loc. cit., page 432 et seq.

conical flow past the delta wing is not completely determined by the boundary conditions formulated in equations (4) and (5). On the other hand, as in the previous report, it is shown in Section 7 that the normal force exerted on a finite portion of the wing is finite only if the order of the poles of $\wp(z_1)$ is at most two. Since the residue of the function at each pole is zero, the order of the poles of $\wp(z_1)$ for the case of a flow yielding a finite normal force coefficient is exactly two. As a consequence, it will be seen that the elliptic function $\wp(z_1)$, as well as the function f(z) representing the solution of the flow problem, are uniquely determined by the addition of this supplementary condition. Hence, among all the conical flows past the delta wing satisfying the boundary conditions in equations (4) and (5), there is a unique one for which the normal force coefficient of the wing is finite. The particular functions $\wp(z_1)$ and f(z) which determine the flow with finite lift coefficient are given in the following section.

5. Determination of the Function $\wp(z_1)$

In the preceding section it is shown that the function f(z) is expressible in terms of an elliptic function $\wp(z_1)$ of the variable $z_1 = \frac{2K}{\pi i} \log z$ having the following properties:

(a) Periods 4K and 2iK'.

(b) Poles with zero residue at z_1 = iK' and z_1 = 2K + iK'.

(c) Real for real values of z_1.

(d) Zeros at points $z_1 = c'$ and $z_1 = c''$ on the real axis of the z_1-plane (see Figure 3).

Functions possessing these properties are readily constructed with the aid of the Jacobi elliptic functions. In this section the function f(z) corresponding to the particular elliptic function $\wp(z_1)$ having poles of at most the second order is derived. For, as will be shown, the lift coefficient of the wing is not finite for a conical flow corresponding to a function with poles of order greater than two at the points corresponding to the wing edges.

Consider the elliptic functions:

$$\begin{aligned} \wp_1(z_1) &= k\,\mathrm{sn}^2 z_1 - \mathrm{cn}\, z_1\, \mathrm{dn}\, z_1 \,, \\ \wp_2(z_1) &= k\,\mathrm{sn}^2 z_1 + \mathrm{cn}\, z_1\, \mathrm{dn}\, z_1 \,. \end{aligned} \tag{44}$$

These functions have the primitive periods 4K and 2iK', they are real for real values of z_1, and are such that:

(i) $\wp_1(z_1)$ has a single pole of the second order at $z_1 = iK'$ in the period parallelogram (42);

(ii) $\wp_2(z_1)$ has a single pole of the second order at $z_1 = 2K + iK'$ in the period parallelogram (42).

Since these functions have one pole in each period parallelogram, their residues at the poles must be equal to zero[18]. By the general theorems on elliptic functions[19], it follows that any elliptic function having periods 4K and 2iK', and a single pole of the second order at $z_1 = iK'$ in the period parallelogram (42) can differ from a real multiple of the function $\wp_1(z_1)$ by at most an additive real constant. On the other hand, if it has a single pole of second order at $z_1 = 2K + iK'$ in the period parallelogram (42), it can differ from a real multiple of the function $\wp_2(z_1)$ by at most an additive real constant. Finally, if it has second order poles at both $z_1 = iK'$ and $z_1 = 2K + iK'$, and if its residue is zero at each of these poles, then

18) The sum of the residues of an elliptic function at the poles in any period parallelogram is zero. cf. Whittaker and Watson, loc. cit., page 431.

19) cf. Whittaker and Watson, loc. cit., page 429 et seq.; also K. Knapp, Theory of Functions, Part II, Dover, 1947, page 73 et seq.

it can differ from a linear combination of the functions $\wp_1(z_1)$ and $\wp_2(z_1)$ with real coefficients by at most an additive real constant. Hence, the function $\wp(z_1)$ having poles of second order, and possessing the properties (a), (b), and (c) is of the form

$$\wp(z_1) = B' \left\{ \lambda_1 \wp_1(z_1) + \lambda_2 \wp_2(z_1) - 1 \right\} , \tag{45}$$

where B', λ_1, and λ_2 are real constants. Moreover, the coefficients λ_1 and λ_2 are uniquely determined by the position of the zeros $z_1 = c'$ and $z_1 = c''$ specified in the property (d), provided that

$$\begin{vmatrix} \wp_1(c_1') & \wp_2(c_1') \\ \wp_1(c_1'') & \wp_2(c_1'') \end{vmatrix} \neq 0 . \tag{46}$$

In other words, the elliptic function $\wp(z_1)$ is determined completely, with the exception of the real multiplier B', by the properties (a) to (d), provided that the zeros specified for $\wp(z_1)$ are such that the inequality (46) holds.

The real constant B' is determined by the first of the boundary conditions (17). By equations (29), (34), and (38),

$$h(z) = i \int_0^{z_1} \wp(\nu_1)\, d\nu_1 , \tag{47}$$

and therefore

$$h(z) = -i B' \left\{ \frac{\lambda_1 + \lambda_2}{k} \left[E(z_1) - z_1 \right] + (\lambda_1 - \lambda_2) \operatorname{sn} z_1 + z_1 \right\} , \tag{48}$$

where $E(z_1)$ is the fundamental elliptic function of the second kind[20]. For $z = r_0 e^{i\theta}$ and, therefore, $z_1 = x_1 + iK'$, ($x_1 = \frac{2K}{\pi}\theta$), it follows that

20) cf. Whittaker and Watson, loc. cit., page 517.

$$h(r_0 e^{i\theta}) = iB'\left\{\frac{\lambda_1+\lambda_2}{k}\left[E(x_1)+\frac{\mathrm{cn}\,x_1\,\mathrm{dn}\,x_1}{\mathrm{sn}\,x_1}\right]+\frac{\lambda_1-\lambda_2}{k\,\mathrm{sn}\,x_1}-\frac{\lambda_1+\lambda_2-k}{k}x_1\right\}$$

$$+\,B'\,\frac{kK'-(\lambda_1+\lambda_2)E'}{k},$$

and for all real values of θ

$$\mathcal{R}\left\{h(r_0 e^{i\theta})\right\} = B'\,\frac{kK'-(\lambda_1+\lambda_2)E'}{k}.$$

By comparison with the first of the conditions (17) it follows that the constant B' has the value given by the relation:

$$B' = \frac{k}{kK'-(\lambda_1+\lambda_2)E'}\,w_\infty \tan\alpha. \tag{49}$$

According to equation (43) the zeros of the function $\wp(z_1)$ are the points of the z_1-plane corresponding to the zeros of the quadratic function $L(\zeta)$ in equation (12). The two zeros of this function are

$$\zeta = \pm\sqrt{1-\frac{\tan^2\alpha}{\beta^2}} - i\,\frac{\tan\alpha}{\beta}. \tag{50}$$

These points are evidently distinct and lie on the unit circle $|\zeta| = 1$, provided that

$$\tan\alpha < \beta = \tan\left(\frac{\pi}{2}-\mu\right),$$

or

$$\alpha < \frac{\pi}{2}-\mu. \tag{51}$$

Moreover, for positive angles of attack α[21], these zeros lie on the portion

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21) The argument is readily modified for negative angles of attack.

of the circle below the real axis in the ζ-plane. Hence, by virtue of the properties of the transformations (18), (19), and (28), the corresponding points $z_1 = c_1'$ and $z_1 = c_1''$ in the z_1-plane at which the function $\wp(z_1)$ vanishes lie on the real axis and are such that

$$2K < c_1', c_1'' < 4K \tag{52}$$

(see Figure 3). This is true independent of the non-Euclidean angles $\tilde{\alpha}$, $\tilde{\gamma}$, and $\tilde{\psi}$ of attack, semi-flare, and yaw, respectively.

The precise location of the zeros of the function $\wp(z_1)$ can, of course, be obtained by solving the equations (18), (19), and (28) for z_1 in terms of the values of ζ given in (50). However, it is sufficient for the purpose of determining the constants λ_1 and λ_2 in (45) to obtain expression in the variable z_1 corresponding to the function $L(\zeta)$ in equation (12). To this end, consider the function $L(\zeta)/\zeta$. Making use of equations (18) and (19), this function, when expressed in terms of the variable ζ_2 becomes

$$\frac{L(\zeta)}{\zeta} = 2 \csc 2\alpha \, \frac{(1+i\tau_1)\zeta_2^2 + 2i\tau_2\zeta_2 - (1-i\tau_1)}{(1+i\sigma_1)\zeta_2^2 + 2i\sigma_2\zeta_2 - (1-i\sigma_1)}, \tag{53}$$

where

$$\tau_1 = \tau \sinh\tilde{\psi}, \quad \tau_2 = \tau\cosh\tilde{\psi}, \quad \tau = \frac{\sin 2\alpha}{\sin 2\mu}, \tag{54}$$

$$\sigma_1 = \sigma\sinh\tilde{\psi}, \quad \sigma_2 = \sigma\cosh\tilde{\psi}, \quad \sigma = \frac{\tan\mu}{\tan\alpha} = \coth\tilde{\alpha}, \tag{55}$$

and where μ is the Mach angle defined by equation (3). Next, introducing the variable z_1 by making the substitution defined in (28), equation (53) can be written in the form:

$$\frac{L(\zeta)}{\zeta} = 2\cos 2\alpha \, \frac{k' \operatorname{sn} z_1 + \tau_1 \operatorname{cn} z_1 + \tau_2 \operatorname{dn} z_1}{k' \operatorname{sn} z_1 + \sigma_1 \operatorname{cn} z_1 + \sigma_2 \operatorname{dn} z_1}. \tag{56}$$

Hence, the zeros of the function $\wp(z_1)$ denoted by $z_1 = c_1'$ and $z_1 = c_1''$ are the roots of the equation

$$k' \operatorname{sn} z_1 + \tau_1 \operatorname{cn} z_1 + \tau_2 \operatorname{dn} z_1 = 0. \tag{57}$$

From equation (57) it follows that for either one of the zeros of $\wp(z_1)$, which for convenience is denoted simply by $z_1 = c_1$,

$$(k'^2 + k^2\tau_2^2 + \tau_1^2)\operatorname{sn}^2 c_1 = \tau_1^2 + \tau_2^2 + 2\tau_1\tau_2 \operatorname{cn} c_1 \operatorname{dn} c_1 . \tag{58}$$

Therefore, by substituting for $\operatorname{sn}^2 c_1$ in equations (44), the values of $\wp_1(z_1)$ and $\wp_2(z_1)$ at either of the zeros of $\wp(z_1)$ are obtained in the form

$$\wp_1(z_1) = \frac{k(\tau_1^2+\tau_2^2)}{k'^2 + k^2\tau_2^2 + \tau_1^2}\left\{1 - \frac{k'^2 + (k\tau_2 - \tau_1)^2}{k(\tau_1^2+\tau_2^2)} \operatorname{cn} c_1 \operatorname{dn} c_1\right\},$$

$$\wp_2(z_1) = \frac{k(\tau_1^2+\tau_2^2)}{k'^2 + k^2\tau_2^2 + \tau_1^2}\left\{1 + \frac{k'^2 + (k\tau_2 + \tau_1)^2}{k(\tau_1^2+\tau_2^2)} \operatorname{cn} c_1 \operatorname{dn} c_1\right\}.$$

The four values $\wp_1(c_1')$, $\wp_1(c_1'')$, $\wp_2(c_1')$, and $\wp_2(c_1'')$ are obtained from these expressions by replacing c_1 by c_1' and c_1'', respectively. Substituting these values in the determinant on the left of (46), gives

$$\begin{vmatrix} \wp_1(c_1') & \wp_2(c_1') \\ \wp_1(c_1'') & \wp_2(c_1'') \end{vmatrix} = \frac{k(\tau_1^2+\tau_2^2)}{k'^2 + k^2\tau_2^2 + \tau_1^2}\left\{\operatorname{cn} c_1'' \operatorname{dn} c_1'' - \operatorname{cn} c_1' \operatorname{dn} c_1'\right\}$$

$$= \frac{k(\tau_1^2+\tau_2^2)}{k'^2 + k^2\tau_2^2 + \tau_1^2}\left[\frac{d}{dz_1}\operatorname{sn} z_1\right]_{c_1'}^{c_1''}, \tag{59}$$

where the symbol $\left[\ \right]_{c_1'}^{c_1''}$ denotes the difference between the values of the function within the bracket for $z_1 = c_1''$ and $z_1 = c_1'$. Since the derivative of the function sn x_1 is monotone in the interval $2K \leqq x_1 \leqq 4K$ on the real axis, it follows that the determinant is different from zero for the real values $z_1 = c_1'$ and $z_1 = c_1''$ which satisfy (52). Hence, the inequality (46) holds, and as a consequence, it follows that the constants λ_1 and λ_2 in (45) are uniquely determined. In fact, with the aid of (58) and (59), it follows that $\wp(z_1)$ vanishes for $z_1 = c_1'$ and c_1'' if and only if

$$\lambda_1 = \frac{k'^2 + (k\tau_2 + \tau_1)^2}{2k(\tau_1^2 + \tau_2^2)} \quad , \qquad \lambda_2 = \frac{k'^2 + (k\tau_2 - \tau_1)^2}{2k(\tau_1^2 + \tau_2^2)} \, .$$

As a result of these values and the value of B' in equation (49), $\wp(z_1)$ becomes

$$\wp(z_1) = k^2 B \left\{ (k' \operatorname{sn} z_1 + \tau_1 \operatorname{cn} z_1 + \tau_2 \operatorname{dn} z_1)(k' \operatorname{sn} z_1 - \tau_1 \operatorname{cn} z_1 - \tau_2 \operatorname{dn} z_1) \right\} , \tag{60}$$

where

$$B = \frac{w_\infty \tan \alpha}{k^2(\tau_1^2 + \tau_2^2)K' - (k'^2 + k^2\tau_2^2 + \tau_1^2)} \, . \tag{61}$$

It should be noted that for the case of zero yaw, $\tilde{\psi} = 0$, it follows from (54) and (55) that

$$\tau_1 = \sigma_1 = 0 \quad , \quad \tau_2 = \tau \, , \quad \sigma_2 = \sigma \, .$$

Consequently, in this case, $\wp(z_1)$ becomes simply[22)]

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22) The expression given here for $\wp(z_1)$ is much simpler than that given for the corresponding function $h_1(z_1)$ in the earlier report; see equations (2.33), (2.34), (2.36), and (2.39) of BB. Report No. 75. The simplification is the result of selecting a more appropriate modulus for the Jacobi elliptic functions employed in the expression which establishes the conformal mapping of the circular region with the symmetric slit onto the annulus. The change in the modulus is effected by means of Gauss' transformation of the Jacobi elliptic functions. See footnotes on pages 43 and 52 of BB. Report No. 75.

$$\not p(z_1) = B_0 \left(sn^2 z_1 - A_0 \right),$$

where

$$B_0 = \frac{k^2 (k'^2 + k^2 \tau^2)}{k^2 \tau^2 K' - (k'^2 + k^2 \tau_2^2 + \tau_1^2) E'} \, w_\infty \tan\alpha ,$$

$$A_0 = \frac{\tau^2}{k'^2 + \tau^2 k^2} .$$

6. Determination of the Function W(z) and the Values of the Velocity Component w Along the Wing Surface

According to the last of equations (14), the component W of the complex velocity is given by the integral formula

$$W = - \int_1^z \omega(\nu) f(\nu) \, d\nu ,$$

where the choice of the lower limit conforms with the condition that real part of the function W(z) vanish along the circle $|z| = 1$. Making use of equations (29), (38), and (43) this equation can be written as

$$W = -i \int_0^{z_1} \not p(\nu_1) \frac{\omega_1(\nu_1)}{\ell_1(\nu_1)} \, d\nu_1 , \tag{62}$$

where $z_1 = (2K/\pi i) \log z$, and $\omega_1(z_1)$ and $\ell_1(z_1)$ denote the values of the functions $\omega(z)$ and $\ell(z)$ when expressed in terms of z_1. The ratio $\dfrac{\ell_1(z_1)}{\omega_1(z_1)}$ is given by the right hand member of equation (56). Therefore, by equations (56) and (60),

$$\not p(z_1) \frac{\omega_1(z_1)}{\ell_1(z_1)} = \frac{1}{2} k^2 B \sin 2\alpha \left\{ (k' sn\, z_1 + \sigma_1 cn\, z_1 + \sigma_2 dn\, z_1)(k' sn\, z_1 - \tau_1 cn\, z_1 - \tau_2 dn\, z_1) \right\} .$$

Hence

$$W = \frac{i}{2} B \sin 2\alpha \Big\{ (k'^2 + k^2\sigma_2\tau_2 + \sigma_1\tau_1) E(z_1) + k'(\sigma_1 - \tau_1) \,\mathrm{dn}\, z_1$$

$$+ k'k^2(\sigma_2 - \tau_2)\,\mathrm{cn}\, z_1 + k^2(\sigma_1\tau_2 + \sigma_2\tau_1)\,\mathrm{sn}\, z_1 \tag{62a}$$

$$- k'^2(1 + \sigma_1\tau_1) z_1 - k'\big[(\sigma_1 - \tau_1) + k^2(\sigma_2 - \tau_2)\big] \Big\}$$

The expression for the component w of the additional velocity at points on the wing surface is particularly simple. Thus, setting $z_1 = x_1 + iK'$ and taking the real part of the function W, one obtains

$$w = \frac{1}{2} B \sin 2\alpha \left\{ k' \, \frac{k(\sigma_2 - \tau_2)\,\mathrm{dn}\, x_1 + (\sigma_1 - \tau_1)\,\mathrm{cn}\, x_1}{\mathrm{sn}\, x_1} - A \right\}, \tag{63}$$

where

$$A = k^2(\sigma_1\tau_1 + \sigma_2\tau_2)K' - (k'^2 + k^2\sigma_2\tau_2 + \sigma_1\tau_1)E' . \tag{64}$$

The relation between the coordinate x_1 of a point on the line $y_1 = K'$ in the z_1-plane and the coordinate ξ_2 of a point on the slit along the real axis in the ζ_2-plane (see Figure 2) is readily obtained from equation (28). In fact, it follows from (28) that

$$\begin{aligned} \mathrm{dn}\, x_1 &= k' \, \frac{1 + \xi_2^2}{1 - \xi_2^2} , \\ \mathrm{cn}\, x_1 &= \frac{2k'\xi_2}{k(1 - \xi_2^2)} , \\ \mathrm{sn}\, x_1 &= \pm \frac{\sqrt{k^2(1 + \xi_2^2)^2 - 4\xi_2^2}}{k(1 - \xi_2^2)} , \end{aligned} \tag{65}$$

where the sign of the function sn x_1 is determined as follows:

+ for points on the upper side of the segment $(-\xi_{2_E} \leq \xi_2 \leq \xi_{2_E})$.
(66)
- for points on the lower side of the segment $(-\xi_{2_E} \leq \xi_2 \leq \xi_{2_E})$.

It is convenient at this point to introduce the real variable

$$\rho = \frac{2\xi_2}{k(1+\xi_2^2)} \; . \tag{67}$$

Then, as ξ_2 ranges over the segment $-\xi_{2_E} \leq \xi_2 \leq \xi_{2_E}$, the variable ρ ranges over the interval $-1 \leq \rho \leq 1$. The relation between the X-coordinate of a point on the surface of the wing along the line of intersection with the plane Z = constant = s (see Figure 4) and the variable ρ is readily obtained from the equations of transformation. From equations (6), (18), (20), (21), and (67) it readily follows that

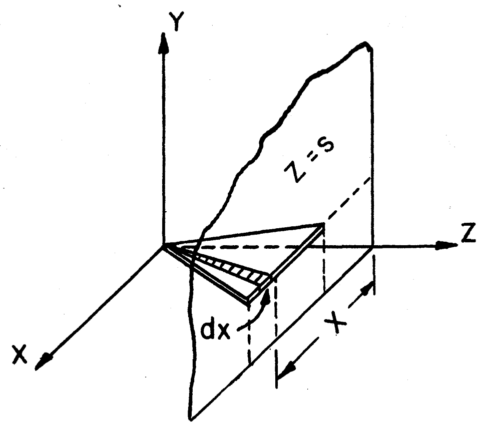

Fig. 4

$$X = s \tan\alpha \, \operatorname{csch}\tilde{\alpha} \, \frac{k\rho + \tanh\tilde{\psi}}{1 + k\rho \tanh\tilde{\psi}} \; . \tag{68}$$

It should be noted that in the special case of zero yaw ($\tilde{\psi} = 0$), the quantity ρ represents the fractional part of the semi-span of the wing measured along the line Z = constant.

The relation between the variable ρ and the coordinate x_1 of a point on the line $y_1 = K'$ in the z_1-plane can be obtained from equations (65) and (67). Thus, one obtains

$$\frac{dn\,x_1}{sn\,x_1} = \pm \frac{k'}{\sqrt{1-\rho^2}} ,$$

$$\frac{cn\,x_1}{sn\,x_1} = \pm \frac{k'\rho}{\sqrt{1-\rho^2}} , \qquad (69)$$

$$\frac{cn\,x_1}{dn\,x_1} = \rho ,$$

where the signs of the first two are determined as in (66).

As a consequence of (63) and (69) the expression for the component w of the additional velocity at points along the wing surface, when written in terms of the variable ρ, takes the form

$$[w]_+ = \frac{1}{2} B \sin 2\alpha \left[k'^2(\sigma_2 - \tau_2) \frac{k + \rho \tanh \tilde{\psi}}{\sqrt{1-\rho^2}} - A \right] ,$$

$$(70)$$

$$[w]_- = \frac{1}{2} B \sin 2\alpha \left[-k'^2(\sigma_2 - \tau_2) \frac{k + \rho \tanh \tilde{\psi}}{\sqrt{1-\rho^2}} - A \right] ,$$

where the constants B and A are defined in equations (61) and (64), respectively. The first of these expressions represents the values of w along the positive (or "upper") face of the wing[23], and the second, the values of w on the negative (or "lower") face.

Since the wing edges correspond to $\rho = \pm 1$, it follows from equations (70) that the velocity component w is always infinite at both edges whatever the angle of yaw may be, except at an edge for which

$$k \pm \tanh \tilde{\psi} = 0 .$$

23) See footnote 3). The orientation of the axis in this report are such that the "upper" surface of the wing is the high pressure side.

Since $k = \tanh\tilde{\gamma}$, the exceptional cases correspond to

$$\tilde{\psi} = \pm\,\tilde{\gamma}\,.$$

Hence, it follows from equation (27) that the exceptions correspond to the cases in which an edge lies in the YZ-plane.

7. Calculation of the Normal Force Coefficient

The total normal force applied to the triangular tip of the delta wing ahead of the plane Z = s (see Figures 1 and 4) is given, within the limits of accuracy attained by the linear theory, by the integral

$$-\frac{1}{2}\rho_\infty w_\infty\, s\sec\alpha \int_{s\sec\alpha\tan(\psi+\gamma)}^{s\sec\alpha\tan(\psi+\gamma)} \Delta w\, dX\,,$$

where ρ_∞ is the density of the gas in the undisturbed region of the flow ahead of the wing and

$$\Delta w = [w]_+ - [w]_- \tag{71}$$

is the difference between the values of the velocity component w at adjacent points on the positive and negative faces of the wing. The non-dimensional normal force coefficient C_n obtained by dividing the value of this integral by the stagnation pressure $1/2\ \rho_\infty w_\infty^2$ and the area,

$$\frac{1}{2}s^2\sec^2\alpha\left[\tan(\psi+\gamma)-\tan(\psi-\gamma)\right] = s^2\sec\alpha\,\frac{\operatorname{sech}\tilde{\alpha}\,\tanh\tilde{\gamma}\,\operatorname{sech}^2\tilde{\psi}}{\beta\left[1-\tanh^2\tilde{\gamma}\,\tanh^2\tilde{\psi}\right]}\,,$$

of the portion of the wing under consideration can be written in the form

$$C_n = -\frac{\beta\left[1-\tanh^2\tilde{\gamma}\tanh^2\tilde{\psi}\right]}{s\,w_\infty \operatorname{sech}\tilde{\alpha}\tanh\tilde{\gamma}\operatorname{sech}^2\tilde{\psi}}\int_{s\sec\alpha\tan(\psi-\gamma)}^{s\sec\alpha\tan(\psi+\gamma)} \Delta w \, dX \, .$$

This can be expressed in terms of the variable ρ by making the substitution (68), whence

$$C_n = -\frac{1-\tanh^2\tilde{\gamma}\tanh^2\tilde{\psi}}{w_\infty}\int_{-1}^{+1}\Delta w \frac{d\rho}{\left[1+k\rho\tanh\tilde{\psi}\right]^2} \, . \tag{72}$$

On the other hand, making use of relations (69) and the definition of Δw in (71), the normal force coefficient can be written as an integral along the side y_1 = K' of the period rectangle (42) in the z_1-plane (see Figure 3) as follows:

$$C_n = -k'^2\,\frac{1-\tanh^2\tilde{\gamma}\tanh^2\tilde{\psi}}{w_\infty}\int_0^{4K}\frac{w\,\mathrm{sn}\,x_1}{(\mathrm{dn}\,x_1 + k\tanh\tilde{\psi}\,\mathrm{cn}\,x_1)^2}\,dx_1 \tag{73}$$

It was stated in Section 4 that normal force coefficient C_n is finite only if the order of the poles of the elliptic function $\wp(z_1)$ at the points z_1 = iK' and z_1 = 2K + iK' is at most two. This is easily seen with the aid of the integral formula (73). Since the residue of $\wp(z_1)$ at each of its poles is zero, it follows from (62) that the complex function W also has poles at the same points. The orders of these poles are, however, less by one than the corresponding poles of $\wp(z_1)$. Therefore, the real part, w, of W which appears in the integrand of the integral in (73) becomes infinite at the points on the path of integration where x_1 = 0 and x_1 = 2K. However, the Jacobi elliptic function sn x_1 has simple zeros at the same points. Consequently the integral in (73) is finite if and only if the poles of W are of at most the first order. In other words, the normal force coefficient C_n for the delta wing at an arbitrary angle of attack and yaw is finite if and only if the poles of the elliptic function $\wp(z_1)$, which occur at the points corresponding to the wing edges, are at most of the second order. The only function which satisfies this condition is shown in Section 5 to be given by equation (60), and the corresponding component W of the complex velocity having simple poles is given by equation (62a).

The value of the normal force coefficient corresponding to the function W defined by equation (62a) is readily obtained by means of the integral formula (72). From equations (70) it follows that the increment Δw in the velocity across the wing surface is given by the equation

$$\Delta w = k'^2 B(\sigma-\tau) \frac{1+\delta\rho}{\sqrt{1-\rho^2}} \sin 2\alpha \tanh\tilde{\gamma} \cosh\tilde{\psi} , \qquad (74)$$

where

$$\delta = \frac{1}{k}\tanh\tilde{\psi} = \frac{\tanh\tilde{\psi}}{\tanh\tilde{\gamma}} . \qquad (75)$$

Therefore the integral (72) can be written

$$C_n = -k'^2 \frac{B(\sigma-\tau)}{w_\infty} \sin 2\alpha \tanh\tilde{\gamma} \cosh\tilde{\psi}\left(1-\tanh^2\tilde{\gamma}\tanh^2\tilde{\psi}\right)\int_{-1}^{+1} \frac{(1+\delta\rho)\,d\rho}{(1+k^2\delta\rho)^2\sqrt{1-\rho^2}} .$$

Since

$$\int_{-1}^{+1} \frac{(1+\delta\rho)}{(1+k^2\delta\rho)^2\sqrt{1-\rho^2}}\,d\rho = \pi\,\frac{1-\delta^2k^2}{(1-\delta^2k^4)^{3/2}} = \pi\,\frac{\operatorname{sech}^2\tilde{\psi}}{(1-\tanh^2\tilde{\gamma}\tanh^2\tilde{\psi})^{3/2}} ,$$

the normal force coefficient can be written in the form

$$C_n = \pi C \sin 2\alpha , \tag{76}$$

where

$$C = k'^2 \frac{(\sigma - \tau) \tan \alpha \tanh \tilde{\gamma}}{(k'^2 + k^2 \tau_2^2 + \tau_1^2) E' - k^2 (\tau_1^2 + \tau_2^2) K'} \cdot \frac{\operatorname{sech} \tilde{\psi}}{\sqrt{1 - \tanh^2 \tilde{\gamma} \tanh^2 \tilde{\psi}}}$$