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Invariant Sets of Morphisms on Projective and Affine Number Spaces

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1. *Introduction*

Narkiewicz, in a series of papers [5–10], has discussed invariant sets of algebraic points for polynomial transformations (i.e., morphisms) on affine space and on the projective line. In particular, he has shown that under suitable hypotheses all such sets are finite. Kubota [1, 2] extended the problem and studied sets of algebraic points on the affine line which have the same image set under two morphisms. It is our purpose here to give a simple proof of a theorem which encompasses all these results. Our proof uses the concept of heights on projective space. A height function is implicit both in Narkiewicz's and in Kubota's work, but neither uses the most naive formulation of height. In the case of algebraic number fields the results on heights that we need are quite elementary and are easily derived. We do so here, for completeness sake. In the case of function fields, we use results on heights derived by Lang and Neron [3].

We denote m-dimensional affine number space by $\mathscr{A}^m$ and m-dimensional projective number space by $\mathscr{P}^m$. Thus, $\mathscr{A}^m$ consists of all ordered m-tuples from some sufficiently large algebraically closed field Ω, and $\mathscr{P}^m$ consists of all the classes of ordered nontrivial $(m + 1)$-tuples from Ω subject to the usual equivalence relation. Let K be a subfield of Ω. If $\mathscr{U}$ is some subset of $\mathscr{A}^m$, we denote by $\mathscr{U}_K$ those points of $\mathscr{U}$ with coordinates in K. Similarly, if $\mathscr{U}$ is a subset of $\mathscr{P}^m$, then $\mathscr{U}_K$ consists of those points in whose ratios lie in K.

A *rational mapping* **F** from $\mathscr{P}^m$ to $\mathscr{P}^m$ defined over K is of the form

$$\mathbf{x} \overset{\mathbf{F}}{\mapsto} (F_0(\mathbf{x}),\dots, F_m(\mathbf{x})), \tag{1}$$

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where the F_j are forms (homogeneous polynomials) over K of the same degree d. We will call d the *order* of the mapping. The mapping $\mathbf{F}$ is a *morphism* on $\mathscr{P}^m$ provided it is defined everywhere on $\mathscr{P}^m$; i.e., provided the forms F_j have no common nontrivial zero in Ω. Thus a necessary and sufficient condition that $\mathbf{F}$ be a morphism is that the resultant R of $F_0 ,..., F_m$ be different from zero.

A morphism from $\mathscr{P}^m$ to $\mathscr{P}^m$ possesses two special properties: (a) It is a pointwise surjective mapping, and (b) the cardinality of the inverse image of each point is finite.

To see that (a) is true, let $\mathbf{c}$ be an $(m+1)$-tuple over Ω representing a point in $\mathscr{P}^m$. Some coordinate of $\mathbf{c}$, say c_0, is nonzero, and we can find nontrivial $\mathbf{x}$ with coordinates in Ω satisfying

$$c_0F_j - c_jF_0 = 0, \qquad (j = 1,..., m). \tag{2}$$

If $F_0(\mathbf{x}) = 0$, then $\mathbf{x}$ would be a common nontrivial zero of the F_j, and hence the resultant $R = 0$, contrary to $\mathbf{F}$ being a morphism on $\mathscr{P}^m$. Hence, $F_0(x) \neq 0$ and $(F_0(\mathbf{x}),..., F_m(\mathbf{x}))$ and $\mathbf{c}$ represent the same point in $\mathscr{P}^m$.

To see that (b) is true, suppose $\mathbf{F}(\mathbf{x}) = \mathbf{c}$ for an infinity of classes $\mathbf{x}$. Some coordinate of $\mathbf{c}$, say c_0, is not zero and it follows that the variety defined by (2) contains a curve $\mathscr{C}$. But in $\mathscr{P}^m$ each curve meets every surface, and hence $\mathscr{C}$ meets the surface $F_0 = 0$ in at least one point. If $\mathbf{y}$ represents that point, then $\mathbf{y}$ is a common zero of the F_j and hence $R = 0$, contrary to $\mathbf{F}$ being a morphism on $\mathscr{P}^m$. Hence the inverse image of each point is a finite set. If the morphism is of order d, the cardinality of the inverse image of a point is d^m; i.e., the degree of the morphism is d^m.

A morphism $\mathbf{H}$ from $\mathscr{A}^m$ to $\mathscr{A}^m$ defined over K is of the form

$$\mathbf{x} \overset{\mathbf{H}}{\mapsto} (H_1(\mathbf{x}),..., H_m(\mathbf{x})), \tag{3}$$

where the H_j are polynomials, not necessarily homogeneous, over K. We call $d = \max(\deg H_1 ,..., \deg H_m)$ the *order* of $\mathbf{H}$. Clearly, a morphism of $\mathscr{A}^m$ to $\mathscr{A}^m$ need not be pointwise surjective—the simplest example being a projection of $\mathscr{A}^m$ onto a line in $\mathscr{A}^m$. While the cardinality of the inverse image of any generic point of a component of Im($\mathbf{H}$) is finite, this need not be the case for all points in Im($\mathbf{H}$) even when $\mathbf{H}$ is pointwise surjective. For example, the morphism

$$(x, y) \mapsto (x^2 + x, xy + x) \tag{4}$$

is pointwise surjective on $\mathscr{A}^2$ and takes the y-axis into the origin.

One can imbed $\mathscr{A}^m$ into $\mathscr{P}^m$ by the rule: $(x_1 ,..., x_m)$ in $\mathscr{A}^m \leftrightarrow$ class of $(1, x_1 ,..., x_m)$ in $\mathscr{P}^m$. We then speak of the points of $\mathscr{P}^m$ not in $\mathscr{A}^m$ as the points of $\mathscr{P}^m$ at infinity or as the points on the infinite hyperplane $X_0 = 0$.

Each morphism $\mathbf{H}$ of $\mathscr{A}^m$ into $\mathscr{A}^m$ can be extended to a rational mapping of $\mathscr{P}^m$ into $\mathscr{P}^m$. For, if d is the order of the morphism $\mathbf{H}$, let

$$F_0 = X_0{}^d, F_1 = X_0{}^d H_1\left(\frac{X_1}{X_0},\dots,\frac{X_m}{X_0}\right),\dots, F_m = X_0{}^d H_m\left(\frac{X_1}{X_0},\dots,\frac{X_m}{X_0}\right).$$

Clearly the mapping

$$\mathbf{x} \overset{\mathbf{F}}{\mapsto} (F_0(\mathbf{x}),\dots, F_m(\mathbf{x}))$$

is a rational mapping of $\mathscr{P}^m$ into $\mathscr{P}^m$ which agrees with $\mathbf{H}$ on the points in $\mathscr{A}^m$. If the extended mapping $\mathbf{F}$ is a morphism on $\mathscr{P}^m$ we say $\mathbf{H}$ is an *extendable morphism.* Let

$$H_j = H_j' + H_j, \qquad (j = 1,\dots, m),$$

where H_j' is the sum of the monomials in H_j of degree d and H_j'' is the sum of the remaining terms of H_j. The H_j' are homogeneous and R, the resultant of $F_0,\dots, F_m$, is a power of R', the resultant of $H_1',\dots, H_m'$. Hence the extended mapping $\mathbf{F}$ is a morphism on $\mathscr{P}^m$ exactly when $R' \neq 0$. Thus $\mathbf{H}$ is an extendable morphism exactly when $R' \neq 0$.

Since $\mathbf{H}$ and its extended mapping $\mathbf{F}$ agree on $\mathscr{A}^m$, we see that if $\mathbf{H}$ is an extendable morphism, then (i) $\mathbf{H}$ is pointwise surjective on $\mathscr{A}^m$ and (ii) for each $\mathbf{x}$ in $\mathscr{A}^m$ the cardinality of $\mathbf{H}^{-1}(\mathbf{x})$ is finite. It should be remarked that properties (i) and (ii) do not characterize extendable morphisms. The morphism

$$(x, y) \mapsto (xy + x, xy + y) \tag{5}$$

has $R' = 0$ and possesses properties (i) and (ii).

The principal result of Narkiewicz [10] can be stated as follows: An extendable morphism on $\mathscr{A}^m$ of order d defined over an algebraic number field has an infinite invariant subset in $\mathscr{A}_K{}^m$ if and only if $d = 1$. Kubota [1] has gone on to show: Let K be a global field, i.e., an algebraic number field or function field in one variable over a finite field, and let F, G be polynomials over K of degrees f and g, respectively, with $f > g$. If $\mathscr{X}$ is a subset of K such that G is injective on $\mathscr{X}$ and if $F(\mathscr{X}) = G(\mathscr{X})$, then the cardinality of $\mathscr{X}$ is finite. Furthermore, Kubota showed, by examples, that the hypotheses that G be injective on $\mathscr{X}$ and that $f > g$ are essential.

We shall prove:

THEOREM. *Let K be a finitely generated field. Let $\mathbf{F}$, $\mathbf{G}$ be morphisms on $\mathscr{P}^m$ to $\mathscr{P}^m$, defined over K, of orders f, g respectively, with $f > g$. If $\mathscr{X}$ is a subset of $\mathscr{P}_K{}^m$ such that $\mathbf{G}$ is injective on $\mathscr{X}$ and $\mathbf{F}(\mathscr{X}) \supset \mathbf{G}(\mathscr{X})$, then the cardinality of $\mathscr{X}$ is finite.*

None of the hypotheses: (a) K finitely generated, (b) $f > g$, and (c) **G** injective on $\mathscr{X}$ can be omitted. That (b) and (c) are necessary follows from the examples of Kubota. We can see that (a) is necessary by observing that if K contains all the roots of unity, then the morphisms on $\mathscr{P}^1$ to $\mathscr{P}^1$:

$$(x, y) \overset{\mathbf{S}}{\mapsto} (x^2, y^2), \qquad (x, y) \overset{\mathbf{I}}{\mapsto} (x, y),$$

together with the infinite set

$$\mathscr{X} = \{(1, \zeta) \text{ all } \zeta, \text{ where } \zeta^p = 1, p \text{ an odd prime}\}$$

satisfy all the hypotheses of the theorem except (a) and $\mathscr{X}$ is an infinite set. Similarly, if K contains a^{2^n}, $n = 0, \pm 1, \pm 2,\ldots$, then **S**, **I** and the set $\mathscr{X}' = \{(1, a^{2^n}) \mid n \in \mathbf{Z}\}$ satisfy all the hypotheses except (a) and $\mathscr{X}'$ is infinite. Roughly speaking, for the theorem to be true for an infinite field K, K must be a long way from being algebraically closed.

One obtains an immediate corollary by substituting in the statement of the theorem "$\mathscr{A}^m$" for "$\mathscr{P}^m$" and "extendable morphism" for "morphism." As Narkiewicz [10] has shown, the corollary is not true if we replace "extendable morphism on $\mathscr{A}^m$" by "morphism on $\mathscr{A}^m$". The morphism on $\mathscr{A}^2$ given by (5) has an infinite invariant subset in $\mathscr{A}_K{}^2$, namely all the K points on the x-axis and on the y-axis.

The morphism given by (5) is a special case of the following more general phenomena. Suppose **H** is a morphism on $\mathscr{A}^m$ of order d, not extendable to a morphism on $\mathscr{P}^m$. Let

$$H_j = H_j^{(d)} + \cdots + H_j^{(1)} + H_j^{(0)}, \qquad (j = 1,\ldots, m),$$

where $H_j^{(k)}$ is a form of degree k in $X_1,\ldots, X_m$. Suppose the algebraic set $\mathscr{V}$ in $\mathscr{A}^m$ defined by the equations

$$H_1^{(d)} + \cdots + H_1^{(2)} = 0, \ldots, H_m^{(d)} + \cdots + H_m^{(2)} = 0$$

is of dimension at least one. Such would certainly be the case if there were a common nontrivial solution of the equations

$$H_1^{(d)} = H_1^{(d-1)} = \cdots = H_1^{(2)} = H_2^{(d)} = \cdots = H_m^{(2)} = 0.$$

Suppose further that $\mathscr{V}_K$ contains an infinite set of points $\mathscr{X}$ invariant under the linear morphism

$$\mathbf{H}^* = (H_1^{(1)} + H_1^{(0)},\ldots, H_m^{(1)} + H_m^{(0)}).$$

Then $\mathbf{H}$ has $\mathscr{X}$ as an invariant subset. We have been unable to decide if this situation is the only one yielding nonlinear morphisms on $\mathscr{A}^m$ defined over K having infinite invariant subsets of $\mathscr{A}_K{}^m$. To resolve this question it would appear we would need to know the behavior both of rational mappings on K points of Zariski closed subsets of $\mathscr{P}^m$ and of morphisms on K points of varieties of $\mathscr{A}^m$ in the form of something like our Lemma 2.

It would be natural to inquire into the possible relations existing between morphisms $\mathbf{F}$, $\mathbf{G}$ of the same order defined over K such that they have the same image on some infinite subset of K points. For example, the morphisms on $\mathscr{A}^1$:

$$x \mapsto x^3, \qquad x \mapsto 7 - x^3$$

agree on the set $\mathscr{X}$ of rational points on the affine line where $\mathscr{X}$ consists of all the coordinates of the infinity of rational points on the elliptic curve $X^3 + Y^3 = 7$. The relation one might expect to obtain would be: If $\mathbf{F}$ is the composition of morphisms, say $\mathbf{F} = \mathbf{F}_1 \circ \mathbf{F}_2 \circ \cdots \circ \mathbf{F}_s$, then

$$\mathbf{G} = \mathbf{L}_0 \circ \mathbf{F}_1 \circ \mathbf{L}_1 \circ \mathbf{F}_2 \circ \mathbf{L}_2 \circ \cdots \circ \mathbf{L}_{s-1} \circ \mathbf{F}_s \circ \mathbf{L}_s,$$

where the $\mathbf{L}_j$ are linear morphisms. We leave discussion of this problem to another time.

Part I: The Case where K is an Algebraic Number Field

2. *Properties of Naive Height Function*

Let K be a given fixed algebraic number field and let $\mathfrak{S}$ denote the set of all equivalence classes of valuations on K. Then, with a suitable normalization of these valuations, we have for all nonzero a in K:

$$|a|_{\mathfrak{p}} = 1 \text{ for all but finitely many } \mathfrak{p} \text{ in } \mathfrak{S}, \tag{6}$$

$$\prod_{\mathfrak{p}\in\mathfrak{S}} |a|_{\mathfrak{p}} = 1, \tag{7}$$

and

$$\prod_{\mathfrak{p}\ \text{archimedean}} |a|_{\mathfrak{p}} = |N_{K/\mathbf{Q}}(a)|. \tag{8}$$

If $x_0, \ldots, x_m$ are in K, not all 0, define

$$h(x_0, \ldots, x_m) = \prod_{\mathfrak{p}\in\mathfrak{S}} \max\{|x_0|_{\mathfrak{p}}, \ldots, |x_m|_{\mathfrak{p}}\}. \tag{9}$$

This product converges because of property (6). Also, because of property (7), for $a \neq 0$, we have

$$h(ax_0, \ldots, ax_m) = h(x_0, \ldots, x_m).$$

Hence, we can define the height of a point P in $\mathscr{P}_K{}^m$ to be $h(x_0, \ldots, x_m)$, where $(x_0, \ldots, x_m)$ is any representation of P. The height function we have defined depends on K and strictly speaking we should write h_K. If we used $h_K(P)^{1/[K:\mathbf{Q}]}$, we would have a height function defined for all points P in $\mathscr{P}_A{}^m$, where $\mathbf{A}$ is the field of algebraic numbers. But we are concerned with points in $\mathscr{P}_K{}^m$ and so the simpler function will suffice.

Since each point P in $\mathscr{P}_K{}^m$ has a representation in which 1 appears as a coordinate, we see that

$$h(P) \geqslant 1$$

for all P in $\mathscr{P}_K{}^m$.

For each point P in $\mathscr{P}_K{}^m$ there are representations where the coordinates are integers in K and where the ideal generated by the coordinates is one of a finite set of integral ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_h$, one from each of the ideal classes of K. We call such a representation a *reduced representation*. When we use a reduced representation for P, we see that

$$\begin{aligned} h(P) &= \prod_{\mathfrak{p}\text{ non-arch}} \max_j\{|x_j|_{\mathfrak{p}}\} \cdot \prod_{\mathfrak{q}\text{ arch}} \max_j\{|x_j|_{\mathfrak{q}}\} \\ &= N(x_0, \ldots, x_m)^{-1} \prod_{\mathfrak{q}\text{ arch}} \max_j\{|x_j|_{\mathfrak{q}}\} \\ &\geqslant N(x_0, \ldots, x_m)^{-1} \max_j\{|N(x_j)|\}, \end{aligned} \tag{11}$$

where N is the norm from the ideals of K to $\underset{\sim}{Q}$. Hence

$$h(P) \geqslant \max_j\{|N(x_j)|\} \cdot [\max_i N\mathfrak{A}_i]^{-1}. \tag{12}$$

Since there are only finitely many integers of K with norm less than a given bound we can conclude

LEMMA 1. *If K is an algebraic number field, there are only finitely many points in $\mathscr{P}_K{}^m$ of height less than a given bound.*

LEMMA 2. *Let $\mathbf{F}$ be a morphism on $\mathscr{P}^m$, of order d, defined over an algebraic number field K, then there exists constants C_1, C_2 such that*

$$C_1 h^d(P) \leqslant h(\mathbf{F}(P)) \leqslant C_2 h^d(P) \tag{13}$$

for all P in $\mathscr{P}_K{}^m$. The constants depend on $\mathbf{F}$ and K, but not on P.

Proof. The morphism $\mathbf{F}$ is given by forms $F_0, \dots, F_m$ of degree d defined over K which have a nonzero resultant R. We can further assume that the coefficients of the F_j are integers of K which generate one of the finitely many ideals $\mathfrak{A}_1, \dots, \mathfrak{A}_h$.

For each P in $\mathscr{P}_K{}^m$ we choose a reduced representation $\mathbf{x}$. Clearly, for such a representation the $F_j(\mathbf{x})$ are integers of K, and hence

$$\prod_{\mathfrak{p}\text{ non-arch}} \max_j\{|F_j(\mathbf{x})|_{\mathfrak{p}}\} \leqslant 1. \tag{14}$$

For each archimedean valuation $\mathfrak{q}$ on K we have

$$|F_j(\mathbf{x})|_{\mathfrak{q}} \leqslant C_{j\mathfrak{q}}[\max_j\{|x_j|_{\mathfrak{q}}\}]^d,$$

where $C_{j\mathfrak{q}}$ is a constant determined by the form F_j and by $\mathfrak{q}$. It then follows (see (8)) that

$$|N(F_j(\mathbf{x}))| \leqslant C'\left[\prod_{\mathfrak{q}\text{ arch}} \max_j\{|x_j|_{\mathfrak{q}}\}\right]^d, \qquad (j = 0, \dots, m), \tag{15}$$

where C' is a constant depending on $\mathbf{F}$ and K. Combining (11) and (15) we obtain

$$|N(F_j(\mathbf{x}))| \leqslant C'M^d h^d(P) = C_2 h^d(P), \tag{16}$$

where $M = \max_j\{|N\mathfrak{A}_j|\}$. Finally combining (14) and (16) we obtain

$$h(\mathbf{F}(\mathbf{x})) \leqslant C_2 h^d(P). \tag{17}$$

From elimination theory (see [12]) we know that there exist forms $A_{ij}(X_0, \dots, X_m)$ of degree $\leqslant d(m-1)+1$ with coefficients which are polynomials over $\mathbf{Z}$ in the coefficients of the F_ν, and hence are integers of K, such that

$$A_{i0}F_0 + \cdots + A_{im}F_m = RX_i^{dm+1}, \qquad (i = 0, \dots, m). \tag{18}$$

It follows from (18) that the ideal $\mathfrak{B}_P = (F_0(\mathbf{x}), \dots, F_m(\mathbf{x}))$ divides one of the ideals $(R)\,\mathfrak{A}_i^{dm+1}$. Hence, for all P in $\mathscr{P}_K{}^m$, $N\mathfrak{B}_P$ is bounded, say by B. Also, for each archimedean valuation $\mathfrak{q}$ we have

$$|A_{ij}(\mathbf{x})|_{\mathfrak{q}} \leqslant C''[\max_j\{|x_j|_{\mathfrak{q}}\}]^{d(m-1)+1},$$

where C'' is a constant depending on K and on the A_{ij} and so on $\mathbf{F}$ and K. It then follows from (18) that for all j,

$$\max_j\{|F_j(x)|_{\mathfrak{q}}\} \geqslant C'''[\max_j\{|x_j|_{\mathfrak{q}}\}]^d,$$

where C''' depends on C'', R and K and so on $\mathbf{F}$ and K. Hence, (see (11)) we have

$$h(\mathbf{F}(P)) \geqslant B^{-1}C''' \prod_{\mathfrak{q}\text{ arch}} [\max_j\{|x_j|_{\mathfrak{q}}\}]^d$$

$$\geqslant C_1 h^d(P).$$

Combining this last inequality with (17) gives (13) and completes the proof of the lemma.

3. *Proof of the Theorem when K is an Algebraic Number Field*

Let $\mathbf{F}, \mathbf{G}$ be morphisms on $\mathscr{P}^m$ to $\mathscr{P}^m$ of orders f, g, respectively, with $f > g$. Then, by Lemma 2, there exists a positive real number H such that for P, Q in $\mathscr{P}_K{}^m$,

(α) If $h(P) > H$, then $h(\mathbf{F}(P)) > h(\mathbf{G}(P))$, and

(β) If $h(P) > H$, and $\mathbf{F}(P) = \mathbf{G}(Q)$, then $h(Q) > h(P)$.

Let $\mathscr{Y} = \{P \in \mathscr{X} \mid h(P) \leqslant H\}$ and $\mathscr{Z} = \{P \in \mathscr{X} \mid h(P) > H\}$. By Lemma 1, $\mathscr{Y}$ is a finite set. Since $\mathbf{F}(\mathscr{X}) \supset \mathbf{G}(\mathscr{X})$, if $Q \in \mathscr{Y}$ there exists a point B in $\mathscr{X}$ such that $\mathbf{F}(B) = \mathbf{G}(Q)$. It follows from ($\beta$) that B is in $\mathscr{Y}$. Hence, $\mathbf{F}(\mathscr{Y}) \supset \mathbf{G}(\mathscr{Y})$. Since $\mathbf{G}$ is injective on $\mathscr{X}$, $\mathbf{G}$ is injective on $\mathscr{Y}$, and hence

$$\text{card}\,\mathscr{Y} \geqslant \text{card}\,\mathbf{F}(\mathscr{Y}) \geqslant \text{card}\,\mathbf{G}(\mathscr{Y}) = \text{card}\,\mathscr{Y}.$$

Thus

$$\mathbf{F}(\mathscr{Y}) = \mathbf{G}(\mathscr{Y}),$$

and hence

$$\mathbf{F}(\mathscr{Z}) \supset \mathbf{G}(\mathscr{Z}).$$

It follows from Lemmas 1 and 2 that there are only finitely many points in $\mathscr{P}_K{}^m$ such that $h(\mathbf{G}(P))$ is less than a given bound. Hence, the set of real numbers $\{h(\mathbf{G}(P)) \mid P \in \mathscr{Z}\}$ is discrete and if $\mathscr{Z}$ is not empty, there exists a point Q in $\mathscr{Z}$ such that

$$h(\mathbf{G}(Q)) = \min_{P \in \mathscr{Z}} h(\mathbf{G}(P)). \tag{19}$$

Since $\mathbf{F}(\mathscr{Z}) \supset \mathbf{G}(\mathscr{Z})$, there is a point P_0 in $\mathscr{Z}$ such that $\mathbf{F}(P_0) = \mathbf{G}(Q)$. But then ($\alpha$) implies $h(\mathbf{F}(P_0)) > h(\mathbf{G}(P_0))$ and we have $h(\mathbf{G}(Q)) = h(\mathbf{F}(P_0)) > h(\mathbf{G}(P_0))$, contrary to (19). Hence $\mathscr{Z}$ must be the empty set and $\mathscr{X} = \mathscr{Y}$ is a finite set.

The same proof would apply for a function field in one variable over a finite field of constants.

Part II: The General Case

4. A Height Function for Points Defined over Function Fields

Lemma 3. *Let K be a function field of transcendence degree 1 over a field k which is algebraically closed in K. Then there exists a nonsingular complete projective curve $\mathscr{T}$ such that $\mathscr{T}$ is the projective closure of an affine curve $\mathscr{T}'$, K is the quotient field of $k[t_1, ..., t_m]$ where $(t_1, ..., t_m)$ is a generic point of $\mathscr{T}'$ and $t_2, ..., t_m$ are each integral over $k[t_1]$.*

Proof. It is well known, for example, see [4, p. 406] that given K/k there exists a complete nonsingular projective curve $\mathscr{C} \subset \mathscr{P}^m$, for some m, defined over k such that if $\mathscr{H}$ is any hyperplane not containing $\mathscr{C}$, then the affine curve $\mathscr{C}' = \mathscr{C} \cap \mathscr{H}$ has K as the quotient field of its coordinate ring. Let $M = k[x_1, ..., x_m]$ be the coordinate ring for C', so that $(x_1, ..., x_m)$ is a generic point of $\mathscr{C}'$. The curve $\mathscr{C}$ is the unique projective closure (relative to $\mathscr{H}$) of $\mathscr{C}'$, see [11, p. 14]. By Noether's normalization theorem, see [4, p. 4], there exists a t in M such that M is integrally dependent on $k[t]$. Let $\mathbf{t} = (t, x_1, ..., x_m)$ and let $\mathscr{T}' = \text{loc}_k\, \mathbf{t}$. Then $M = k[t, x_1, ..., x_m]$ and M is the coordinate ring for $\mathscr{T}'$. Let $\mathscr{T}$ be the projective closure of $\mathscr{T}'$ (relative to $\mathscr{H}$). Clearly there is a projection of $\mathscr{T}'$ onto $\mathscr{C}'$, and hence, see [11, p. 18], there is a projection of $\mathscr{T}$ onto $\mathscr{C}$. Since $\mathscr{C}$ is nonsingular so is $\mathscr{T}$. The curve $\mathscr{T}$ has the properties required for the lemma to hold.

We now seek to define a height function on the points $\mathscr{P}_K{}^m$, where K is a function field of transcendence degree 1 over k. While it is not necessary, it is most convenient to do so by defining a height function on $\mathscr{P}_L{}^m$, where $L = Kk^c$ and k^c is the algebraic closure of k. We can then express L as

$$L = k^c(t_1, ..., t_s), \tag{20}$$

where the t_j are transcendental over k; $t_2, ..., t_s$ are integral over $k^c[t_1]$, and $(t_1, ..., t_s)$ are the ratios of a generic point of a nonsingular projective curve $\mathscr{T}$ defined over k^c. To simplify notation, in this section we shall assume $k = k^c$ and we let $t = t_1$.

We now recall the definition of height on points of $\mathscr{P}_L{}^m$ given by Lang and Neron [3]. The elements of L can be viewed as functions on $\mathscr{T}$ and, as is customary, we let (x) denote the divisor on $\mathscr{T}$ associated with the element x from L; i.e.,

$$(x) = \sum_{\mathfrak{p}} \nu_{\mathfrak{p}}(x)\mathfrak{p},$$

where the sum is over all places on L trivial on k and $\nu_{\mathfrak{p}}(x)$ denotes the order of x at $\mathfrak{p}$. If $x_0, ..., x_m$ are from L, define

$$h(\mathbf{x}) = h(x_0, ..., x_m) = -\deg\{\inf_{x_i \neq 0}(x_i)\}. \tag{21}$$

Since the degree of a principal divisor (x) is 0 and since $(xy) = (x) + (y)$, whenever $xy \neq 0$, we see that

$$h(xx_0, \ldots, xx_m) = h(x_0, \ldots, x_m).$$

We now define the height of a point P in $\mathscr{P}_L^m$, by

$$h(P) = h(\mathbf{x}),$$

where $\mathbf{x}$ is any representation for P. Clearly $h(P)$ is a rational integer, and since each P has a representation in which 1 appears as a coordinate we see that $h(P)$ is always a nonnegative integer. Furthermore, since nonconstant functions always have poles, we see that

$$h(P) = 0 \qquad \text{if and only if} \quad P \in \mathscr{P}_k^m.$$

Each point P in $\mathscr{P}_L^m$, not in $\mathscr{P}_k^m$, determines a curve $\mathscr{Z}(P)$ in $\mathscr{P}^m$ defined over k; namely,

$$\mathscr{Z}(P) = \operatorname{loc}_k P = \operatorname{loc}_k \mathbf{x},$$

where $\mathbf{x}$ is any representation of P. Also, there is a rational mapping

$$\mathscr{T} \overset{f}{\mapsto} \mathscr{P}^m \tag{22}$$

with

$$f(\mathbf{t}) = P.$$

Hence f is a rational surjective mapping of $\mathscr{T}$ onto $\mathscr{Z}(P)$. As is shown in [3],

$$h(P) = (\deg f)(\deg \mathscr{Z}(P)), \tag{23}$$

where $\deg \mathscr{Z}(P)$ is the projective degree of the curve $\mathscr{Z}(P)$. It follows that if $h(P)$ is bounded then either $P \in \mathscr{P}_k^m$ or $\deg \mathscr{Z}(P)$ is bounded.

It should be noted that our height function is relative to the field L and more appropriately we should write h_L. Also, the height function we have defined relative to function fields corresponds to the logarithm of the height function we defined relative to algebraic number fields. In the case of function fields, the logarithmic height function is more convenient and so we use it.

We shall call a place on $\mathscr{T}$ finite (infinite) if it corresponds to a specialization $\mathbf{t} \mapsto \mathbf{a}$ extending $t \to a$, where a is in k (where $a = \infty$). We let

$$(x)_F = \sum_{\mathfrak{p}\ \text{finite}} \nu_{\mathfrak{p}}(x)\mathfrak{p},$$

$$(x)_I = \sum_{\mathfrak{q}\ \text{infinite}} \nu_{\mathfrak{q}}(x)\mathfrak{q}.$$

Clearly we can assume that a point P in $\mathscr{P}_L{}^m$ has a representation $\mathbf{x}$ whose coordinates are polynomials over k in $t, t_2, \ldots, t_s$ and so are integral over $k[t]$. Let

$$D = \sum_{\mathfrak{p}\text{ finite}} \inf_{x_j \neq 0} \{\nu_{\mathfrak{p}}(x_j)\}\mathfrak{p} = \inf_{x_j \neq 0} (x_j)_F .$$

If $\deg D > 2g$, where g is the genus of the curve $\mathscr{T}$, then by the Riemann–Roch theorem there exists an element y in L, not in k, such that

$$(y) \geqslant -D.$$

Thus $(y)_F \geqslant -D = -\inf_{x_j \neq 0} (x_j)_F$ and $(y)_I \geqslant 0$ whence $\deg(y)_F < 0$. Clearly $(yx_0, \ldots, yx_m)$ is also a representation of P and

$$D_y = \inf_{x_j \neq 0} (yx_j)_F = D + (y)_F \geqslant 0,$$

whence the yx_j are integral over $k[t]$. Also

$$0 \leqslant \deg D_y < \deg D.$$

Now, from among all such y, choose one such that $\deg(y)_F$ is minimal. Then, for such y, $0 \leqslant D_y \leqslant 2g$; for otherwise we could repeat the technique to obtain a y' such that $\deg(yy')_F < \deg(y)_F$ and $\deg D_{yy'} < \deg D_y$. Thus we can assume that each point P in $\mathscr{P}_L{}^m$ has a representation $\mathbf{x}$ where

(i) x_j are integral over $k[t]$, and

(ii) $\inf_{x_j \neq 0} (x_j)_F$ has bounded degree ($\leqslant 2g$).

We shall call such a representation a *reduced representation.*

LEMMA 4. *Let L be a function field in one variable over an algebraically closed field k. Let* $\mathbf{F}$ *be a morphism on* $\mathscr{P}^m$ *of order d defined over L. Then there exist constants* $C_1, \ldots, C_5$ *such that*

$$C_1 + C_2 dh(P) \leqslant h(\mathbf{F}(P)) \leqslant C_3 + C_4 d + C_5 dh(P) \tag{24}$$

for all P in $\mathscr{P}_L{}^m$. *The constants depend on L and* $\mathbf{F}$, *but not on P.*

Proof. We can assume that $\mathbf{F}$ is given by forms $F_0, \ldots, F_m$ of degree d having no common zero and having coefficients which are polynomials over k in $t, t_2, \ldots, t_s$ and hence the coefficients are integral over $k[t]$. As we have seen earlier, there exist forms A_{ij} over L such that the identity

$$A_{i0}F_0 + \cdots + A_{im}F_m = RX_i^{dm+1} \tag{25}$$

holds. Here R is the resultant of the F_j, hence is a polynomial over k in $t, t_2, \ldots, t_s$. The coefficients of the A_{ij} are from the ring generated by the coefficients of the F_j and hence lie in $k[t, t_2, \ldots, t_s]$.

Let $\mathfrak{p}$ be a fixed place on $\mathscr{T}$. From the identity (25) we obtain

$$\nu_{\mathfrak{p}}(Rx_i^{dm+1}) = \nu_{\mathfrak{p}}\left(\sum_j A_{ij}(\mathbf{x})\, F_j(\mathbf{x})\right), \qquad (i = 0, \ldots, m),$$

and hence

$$\begin{aligned}\nu_{\mathfrak{p}}(R) + (dm+1)\,\nu_{\mathfrak{p}}(x_i) &\geqslant \inf \nu_{\mathfrak{p}}(A_{ij}(\mathbf{x})) + \inf \nu_{\mathfrak{p}}(F_j(\mathbf{x}))\\ &\geqslant \inf \nu_{\mathfrak{p}}(b) + [d(m-1)+1] \inf \nu_{\mathfrak{p}}(x_j)\\ &\quad + \inf \nu_{\mathfrak{p}}(F_j(\mathbf{x})), \qquad (i = 0, \ldots, m),\end{aligned}$$

where the first inf is over all coefficients b of the A_{ij}. Therefore,

$$\begin{aligned}&\nu_{\mathfrak{p}}(R) + (dm+1) \inf \nu_{\mathfrak{p}}(x_i)\\ &\qquad \geqslant \inf_b \nu_{\mathfrak{p}}(b) + [d(m-1)+1] \inf \nu_{\mathfrak{p}}(x_j) + \inf_j \nu_{\mathfrak{p}}(F_j(\mathbf{x})),\end{aligned}$$

or

$$\nu_{\mathfrak{p}}(R) + d \inf \nu_{\mathfrak{p}}(x_j) \geqslant \inf_b \nu_{\mathfrak{p}}(b) + \inf \nu_{\mathfrak{p}}(F_j(\mathbf{x})).$$

Consequently,

$$-(R)\, d[-\inf\, (x_j)] \leqslant -\inf_b\, (b) - \inf\, (F_j(\mathbf{x})),$$

and on taking degrees we obtain

$$C_1 + dh(P) \leqslant h(\mathbf{F}(P)), \tag{26}$$

where $C_1 = \deg\, (\inf_b\, (b))$.

Let $\mathbf{x}$ be a reduced representation of the point P in $\mathscr{P}_L{}^m$. Since the x_j are integral over $k[t]$, we have

$$\inf_{x_j \neq 0} (x_j) = \inf_{x_j \neq 0} (x_j)_F - \sup_{x_j \neq 0} (x_j^{-1})_I\,. \tag{27}$$

Similarly, since the coefficients of the F_j are integral over $k[t]$, we have

$$\inf\, (F_j(\mathbf{x})) = \inf\, (F_j(\mathbf{x}))_F - \sup_{F_j(\mathbf{x}) \neq 0} (1/F_j(\mathbf{x})). \tag{28}$$

If $z \neq 0$ and is integral over $k[t]$, then

$$\deg(z^{-1})_I = -\sum_{\mathfrak{q}\, \inf} \nu_{\mathfrak{q}}(z) = \deg_t\{N_{L/k(t)}(z)\}.$$

Suppose $z = \sum_1^n p_i(t)\, w_i$, where $w_1, \ldots, w_n$ is a basis for L over $k(t)$ and the w_i are integral over $k[t]$. Then $N_{L/k(t)}(z)$ is a form J of degree n over $k[t]$ in the $p_i(t)$ which vanishes only if all $p_i(t)$ are 0. It follows that

$$n \max_i \deg_t p_i(t) \leqslant \deg_t \{N_{L/k(t)}(z)\} \leqslant m + n \max_i \deg_t p_i(t),$$

where m depends on the coefficients of J and hence on the field L. As a consequence, if z_1, z_2 are of this form, then

$$\deg_t N_{L/k(t)}(z_1 + z_2) \leqslant m + \max_i \deg_t N_{L/k(t)}(z_i).$$

Let $\mathfrak{q}_0$ be a fixed infinite place on $\mathscr{T}$. We have

$$\begin{aligned} -\nu_{\mathfrak{q}_0}(F_j(\mathbf{x})) &\leqslant \deg_t N_{L/k(t)}(F_j(\mathbf{x})) \\ &\leqslant m + \max_{\mathfrak{m}} N_{L/k(t)}(\mathfrak{m}), \end{aligned}$$

where $\mathfrak{m}$ is a monomial of $F_j(\mathbf{x})$. Then

$$\begin{aligned} -\nu_{\mathfrak{q}_0}(F_j(x)) &\leqslant M + dm + d \sup_i \deg_t N_{L/k(t)}(x_i) \\ &\leqslant M + dm + d \sup_{x_i \neq 0} \deg(\bar{x}_i^1\ )_I \\ &\leqslant M + dm + d \deg \sup_{x_i \neq 0} (\bar{x}_i^1\ )_I, \end{aligned}$$

where M depends on the coefficients of the F_j. By (27) this gives

$$-\nu_{\mathfrak{q}_0}(F_j(\mathbf{x})) \leqslant M + dm + d[h(P) + \deg \inf(x_i)_F].$$

But then

$$\sum_{\mathfrak{q} \text{ infinite}} \sup_{F_j(\mathbf{x}) \neq 0} (\nu_{\mathfrak{q}}(1/F_j(\mathbf{x})) \leqslant n[M + dm + d\{h(P) + 2g\}],$$

since $\mathbf{x}$ is a reduced representation. Hence

$$\begin{aligned} h(\mathbf{F}(P)) &= -\deg \inf(F_j(\mathbf{x}))_F + \deg \sup_{F_j(\mathbf{x}) \neq 0} (1/F_j(\mathbf{x}))_I \\ &\leqslant \deg \sup_{F_j(\mathbf{x}) \neq 0} (1/F_j(\mathbf{x}))_I \\ &\leqslant n[M + dm + 2gd + dh(P)]. \end{aligned} \tag{29}$$

The inequalities (26) and (29) give (24). This completes the proof of the lemma.

5. *Proof of the General Theorem*

We now consider the general case of a finitely generated field K. The proof will be by induction on the transcendence degree of K over its prime field. We shall denote that degree by $r(K)$. The theorem is trivial true if K is a finite field since then card $\mathscr{P}_K{}^m$ is finite. Also, as proved in Section 3, the theorem holds true for algebraic number fields. Hence the theorem is true for finitely generated fields K with $r(K) = 0$.

We now assume the theorem holds true for finitely generated fields of transcendency less than or equal to r and prove it for a finitely generated field K with $r(K) = r + 1$. Then $K = k_0(\theta_1, ..., \theta_n)$, where k_0 is the prime field of K (the field generated by 1). Let $K_0 = k_0$, $K_1 = k_0(\theta_1), ..., K_n = k_0(\theta_1, ..., \theta_n) = K$. Then $r(K_j) \leq r + 1$ for all j and there is a first integer v such that $r(K_{v+1}) = r + 1$. Let k be the field obtained by adjoining to K_v all the θ_i, with $i > v$, which are algebraic over K_v. Then k is finitely generated, $r(k) = r$, and K is a function field in one variable over k. But then we can write

$$K = k(t_1, t_2, ..., t_s),$$

where each t_i is transcendental over k, $t_2, ..., t_s$ are integral ovre $K[t_1]$ and the t_v are ratios of a generic point of a nonsingular projective curve $\mathscr{T}$.

Let $\mathbf{F}$ be a morphism from $\mathscr{P}^m$ to $\mathscr{P}^m$ of order d defined over K. We can suppose the coefficients of the forms $F_j(\mathbf{X})$ are polynomials in $t_1, ..., t_s$ with coefficients in k. Hence the resultant of the F_j is nonzero and is a polynomial in $t_1, ..., t_s$ over k. Let $R(\mathbf{T})$ be a polynomial in $T_1, ..., T_s$ such that $R(\mathbf{t}) = R$. Since $R(\mathbf{t}) = R \neq 0$, we see that the surface $\mathscr{R}$ defined by $R(\mathbf{T}) = 0$ does not contain the generic point $\mathbf{t}$ of the curve $\mathscr{T}$ and hence $\mathscr{T}$ and $\mathscr{R}$ meet in only finitely many points. Hence if $\mathbf{a}$ is a finite point on $\mathscr{T}_{k^c}$ not on the surface $\mathscr{R}$, then the specialization $\mathbf{t} \underset{k}{\mapsto} \mathbf{a}$ induces a morphism $\mathbf{F}_\mathbf{a}$ on $\mathscr{P}^m$ of order d defined over $k(\mathbf{a})$. The coefficients of the forms defining $\mathbf{F}_\mathbf{a}$ being the images under the specialization $\mathbf{t} \mapsto \mathbf{a}$ of the corresponding coefficients of the forms defining $\mathbf{F}$.

LEMMA 5. *Suppose the theorem holds for finitely generated fields k of transcendency at most r. Let $K = k(t_1, ..., t_s)$ where $r(K) = 1 + r(k)$ and $r(k) = r$. Let* $\mathbf{F}$, $\mathbf{G}$ *be morphisms on $\mathscr{P}^m$ defined over K of orders f, g, respectively, with $f > g$. Let $\mathscr{X}$ be a subset of $\mathscr{P}_K{}^m$ such that* (i) $\mathbf{G}$ *is injective on $\mathscr{X}$,* (ii) $\mathbf{F}(\mathscr{X}) \supset \mathbf{G}(\mathscr{X})$, *and* (iii) *the points of $\mathscr{X}$ are of bounded height. Then $\mathscr{X}$ is a finite set.*

Proof. As we have just seen, there exist finitely many points $\mathscr{B}$ on $\mathscr{T}_{k^c}$ such that if $\mathbf{a}$ is on $\mathscr{T}_{k^c}$ and not in $\mathscr{B}$, then $\mathbf{F}_\mathbf{a}$ and $\mathbf{G}_\mathbf{a}$ are morphisms on $\mathscr{P}^m$ defined over $k(\mathbf{a})$ of orders f, g, respectively, with $f > g$.

The points in the set $\mathscr{X}$ have representations with coordinates which are polynomials over k in $t_1, \ldots, t_s$. We let $\mathscr{X}_{\mathbf{a}}$ be the set of points in $\mathscr{P}^m$ which have a representation $\mathbf{y}$ which is the image under the specialization $\mathbf{t} \underset{k}{\mapsto} \mathbf{a}$ of a representation of some point P in $\mathscr{X}$. If $\mathbf{x}$ is a representation of P, we denote the point represented by $\mathbf{y}$ by $P_{\mathbf{a}}$. Clearly $\mathscr{X}_{\mathbf{a}}$ is a subset of $\mathscr{P}_{k(\mathbf{a})}$ and $\mathbf{F}_{\mathbf{a}}(\mathscr{X}_{\mathbf{a}}) \supset \mathbf{G}(\mathscr{X}_{\mathbf{a}})$.

We next show that we can choose a point $\mathbf{a}$ on $\mathscr{T}_{k^c}$ not in $\mathscr{B}$ such that (I) the mapping $P \to P_{\mathbf{a}}$ is injective on $\mathscr{X}$, and (II) $\mathbf{G}_{\mathbf{a}}$ is injective on $\mathscr{X}_{\mathbf{a}}$.

Since the points P in $\mathscr{X}$ are of bounded height we know that the curves $\mathscr{Z}(P) = \operatorname{loc}_{k^c} P$ are of bounded degree. The degree of a curve is the number of intersections it has with a generic hyperplane. Let

$$\mathscr{H} : \mu_0 X_0 + \cdots + \mu_m X_m = 0$$

be such a hyperplane. Let $\mathbf{x}$ be a reduced representation of a point P in $\mathscr{X}$, then $\deg \mathscr{Z}(P)$ is the number of specializations $\mathbf{t} \underset{k}{\mapsto} \mathbf{a}$ which takes

$$z = \mu_0 x_0 + \cdots + \mu_m x_m$$

into 0 and such that some x_j does not specialize to 0. Since x_i are integral over $k^c[t_1]$, $(z)_0 = (z)_F$, and

$$\deg (z)_F = \deg \mathscr{Z}(P) + \deg \inf_j (x_j)_F .$$

We are given that $\deg \mathscr{Z}(P)$ is bounded and, since $\mathbf{x}$ is a reduced representation for P, $\deg \inf_j (x_j)_F$ is also bounded. Hence $\deg (z)_F$ is bounded. But

$$\deg (z)_F = \deg (z^{-1})_I = \deg_{t_1} N_{K/k(t_1)}(z),$$

and hence $\deg_{t_1} N_{K/k(t_1)}(z)$ is bounded. Let $w_1, \ldots, w_n$ be an integral basis for K over $k(t_1)$. Then

$$x_j = \sum \alpha_{j\rho}(t_1)\, w_\rho , \tag{30}$$

where $\alpha_{j\rho}(T)$ are polynomials in T, over k. Let $e = \max \deg_{t_1} \alpha_{j\rho}(t_1)$. Then

$$\deg_{t_1} N_{K/k(t_1)}(z) \geqslant e[K : k(t_1)].$$

Hence we can conclude that if $\mathbf{x}$ is a reduced representation of a point in $\mathscr{X}$ (a set of points in $\mathscr{P}_K{}^m$ of bounded height), then the coordinates of $\mathbf{x}$ are of the form (30) where e is bounded.

Let $\mathscr{X} = \{P^{(\sigma)} \mid \sigma \text{ runs over an index set } \Lambda\}$, and let $\mathbf{x}^{(\sigma)}$ be a reduced representation for $P^{(\sigma)}$. Consider the set of polynomials

$$G_{i,j,\sigma,\tau} = N_{K/k(t)}[G_i(\mathbf{x}^{(\sigma)})\, G_j(\mathbf{x}^{(\tau)}) - G_j(\mathbf{x}^{(\sigma)})\, G_i(\mathbf{x}^{(\tau)})],$$

where $0 \leqslant i < j \leqslant m$, and $\sigma \neq \tau$ run over Λ. Since the $\mathbf{x}^{(\sigma)}$, $\mathbf{x}^{(\tau)}$ are reduced representations of points of bounded height, the polynomials $G_{i,j,\sigma,\tau}$ have bounded degree. Since $\mathbf{G}$ is injective on $\mathscr{X}$, for each $\sigma \neq \tau$ there exist u, v with $0 \leqslant u < v \leqslant m$ such that $G_{u,v,\sigma,\tau}$ is not the zero polynomial. Let $\mathbf{a}$ be a point on $\mathscr{T}_{k'}$ having the degree of a_1 over k very large. Then $\mathbf{a}$ is not in the set $\mathscr{B}$ and a_1 is not a zero of $G_{u,v,\sigma,\tau}$. It follows that $\mathbf{x}_{\mathbf{a}}^{(\sigma)} \neq \mathbf{x}_{\mathbf{a}}^{(\tau)}$ and that $\mathbf{G}_{\mathbf{a}}$ is injective on $\mathscr{X}_{\mathbf{a}}$. This proves (I) and (II).

By hypothesis the theorem holds true for fields of transcendency degree r and so holds true for $k(\mathbf{a})$. Our choice of $\mathbf{a}$ is such that $\mathbf{F}_{\mathbf{a}}$, $\mathbf{G}_{\mathbf{a}}$, $\mathscr{X}_{\mathbf{a}}$, $k(\mathbf{a})$ satisfy the hypotheses of the theorem and $r(k(\mathbf{a})) = r$. We can therefore conclude that $\mathscr{X}_{\mathbf{a}}$ is a finite set. Since $P \to P_{\mathbf{a}}$ is an injective mapping on $\mathscr{X}$, it follows that $\mathscr{X}$ is a finite set. This completes the proof of the lemma.

We can now complete the proof of the theorem for finitely generated K along the same lines as employed in Section 3. By Lemma 4, there exists an H such that properties (α) and (β) of Section 3 hold. Define $\mathscr{Y}$ and $\mathscr{Z}$ as in Section 3 and show that $\mathbf{F}(\mathscr{Y}) \supset \mathbf{G}(\mathscr{Y})$. One can then apply Lemma 5 to conclude that $\mathscr{Y}$ is a finite set and $\mathbf{F}(\mathscr{Y}) = \mathbf{G}(\mathscr{Y})$. One then argues exactly as in Section 3 that $\mathscr{Z}$ is the empty set, and hence, $\mathscr{X} = \mathscr{Y}$ is a finite set.

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