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# Interpolation Problems in Function Spaces

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Let $D$ be a domain in the complex plane, let $\{z_n\}$ be a sequence of distinct points in $D$, and let $\{w_n\}$ be an arbitrary sequence of complex numbers. Given a space $E$ of functions on $D$, the problem arises to characterize the pairs of sequences $\{z_n\}$ and $\{w_n\}$ for which there is a function $f \in E$ with $f(z_n) = w_n$, $n = 1, 2,\ldots$. In the present paper, we solve a general interpolation problem of this type. We then apply the result to obtain criteria for interpolation by $H^p$ functions, $1 \leqslant p \leqslant \infty$, by harmonic functions of class $h^p$, and by functions belonging to certain Hilbert spaces. The main tool is a general theorem, closely related to the Hahn–Banach theorem, on the extension of functionals over normed linear spaces.

## 1. A GENERAL INTERPOLATION THEOREM

We shall make use of the following extension theorem, due essentially to F. Riesz and E. Helly (see Banach [2, p. 55]).

THEOREM A. *Let $\phi$ be a functional defined on a set $A$ in a complex Banach space $B$. Then, in order that there exist a continuous linear functional $\Phi \in B^*$ with $\| \Phi \| \leqslant M$ and $\Phi(x) = \phi(x)$ for all $x \in A$, it is necessary and sufficient that*

$$\left| \sum_{k=1}^{n} c_k \phi(x_k) \right| \leqslant M \left\| \sum_{k=1}^{n} c_k x_k \right\|$$

*for each $n$, for arbitrary elements $x_1 ,\ldots, x_n$ of $A$, and for arbitrary complex numbers $c_1 ,\ldots, c_n$ .*

From this result we shall deduce a general theorem which has direct applications to interpolation theory. Let $S$ be any set, and let $E$ be a complex Banach space of complex-valued functions defined on $S$. Addition and scalar multiplication in $E$ are understood to be the usual pointwise operations. For each $s \in S$, there is a linear functional $\lambda_s$ on $E$ defined by

$$\lambda_s(f) = f(s), \qquad f \in E.$$

We shall assume that each functional $\lambda_s$ is continuous. In other words, each $\lambda_s$ belongs to $E^*$, the dual space of $E$. In case $E$ is itself (isometrically isomorphic to) the dual space of a Banach space $E_0$, it may happen that the functionals $\lambda_s$ are weak-star continuous. In this case, each $\lambda_s$ can be identified with an element of $E_0$ under the canonical map of $E_0$ into $E^*$ [9, p. 112].

We are now prepared to state the general interpolation theorem.

THEOREM 1. *Let $E$ be a complex Banach space of functions over a set $S$. Suppose $E$ is dual to a Banach space $E_0$, and let each functional $\lambda_s$ ($s \in S$) be weak-star continuous. Let $g$ be a complex-valued function defined on a subset $U \subset S$. Then, in order that there exist a function $f \in E$ with $\|f\| \leqslant M$ and $f(s) = g(s)$ for all $s \in U$, it is necessary and sufficient that*

$$\left| \sum_{k=1}^{n} c_k g(s_k) \right| \leqslant M \left\| \sum_{k=1}^{n} c_k \lambda_{s_k} \right\| \tag{1}$$

*for each $n$, for arbitrary elements $s_1, \ldots, s_n$ of $U$, and for arbitrary complex numbers $c_1, \ldots, c_n$.*

*Proof.* The necessity of the condition (1) is clear. To prove the sufficiency, let $x_s \in E_0$ be the element which corresponds to $\lambda_s$, and define $\ell : S \to E_0$ by $\ell(s) = x_s$, $s \in S$. Let $\phi$ be the functional defined on $\ell(U)$ by

$$\phi(x_s) = g(s), \qquad s \in U.$$

Then, by Theorem A, this functional $\phi$ can be extended to a functional $\Phi \in E_0^*$ with $\| \Phi \| \leqslant M$, if and only if

$$\left| \sum_{k=1}^{n} c_k \phi(x_{s_k}) \right| \leqslant M \left\| \sum_{k=1}^{n} c_k x_{s_k} \right\| \tag{2}$$

for all $n$, for all $s_1, \ldots, s_n \in U$, and for all complex numbers $c_1, \ldots, c_n$.

But because the canonical embedding of $E_0$ in $E^* = E_0^{**}$ is an isometry, the condition (2) is equivalent to (1). Furthermore, since $E_0{}^* = E$, the functional $\Phi \in E_0{}^*$ can be identified with a function $f \in E$, and we have $\|f\| = \|\Phi\|$ and

$$f(s) = \lambda_s(f) = \Phi(x_s) = \phi(x_s) = g(s), \qquad s \in U.$$

Thus $f$ is the desired extension of $g$.

It has been pointed out to us by Y. Katznelson that weak-star continuity of the functionals $\lambda_s$ is essential and cannot be relaxed to continuity in the norm topology of $E$. For example, consider the space $A$ of absolutely convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \|f\| = \sum_{n=0}^{\infty} |a_n| < \infty.$$

Now $A$ is dual to the space $A_0$ of power series $g(z) = \sum b_n z^n$ where $b_n \to 0$, and the dual of $A$ is the space of power series $g(z) = \sum b_n z^n$ where $\{b_n\}$ is bounded, the pairing in both dualities being $(f, g) = \sum a_n b_n$. Regarding $A$ as a Banach function space on $|z| \leqslant 1$, it is easily seen that point evaluation at $\zeta$ is weak-star continuous for $|\zeta| < 1$, while for $|\zeta| = 1$ it is continuous but not weak-star continuous. In the notation of Theorem 1, suppose $U = \{s_0, s_1, \ldots\}$, where $|s_0| = 1$, $|s_n| < 1$ for $n \geqslant 1$, and $s_n \to s_0$. Since for $n \geqslant 1$ the functionals $\lambda_{s_n}$ are weak-star continuous, they belong to a norm-closed subspace of $A^*$ (the image of $A_0$ under the canonical embedding) which does not include $\lambda_{s_0}$. By the Hahn-Banach theorem there is a functional $\psi \in A^{**}$ such that $\psi(\lambda_{s_0}) = 1$ and $\psi(\lambda_{s_n}) = 0$ for $n \geqslant 1$. Define $g(s_n) = \psi(\lambda_{s_n})$, $n \geqslant 0$. Then (1) is satisfied with $M = \|\psi\|$. However, there is no $f \in A$ such that $f(s_0) = g(s_0) = 1$ and $f(s_n) = g(s_n) = 0$ for $n \geqslant 1$, since each $f \in A$ is continuous in the closed disk.

## 2. Applications to Hilbert Spaces

Consider now the case in which $E$ is a Hilbert space $H$ of functions over $S$, and let ( , ) denote the inner product. Each of the "point evaluation" functionals $\lambda_s$ is assumed to be continuous (and therefore, of course, weak-star continuous). Thus by the Riesz representation theorem, to each $t \in S$ there corresponds an element $k_t \in H$ such that $\|k_t\| = \|\lambda_t\|$ and

$$f(t) = \lambda_t(f) = (f, k_t), \qquad f \in H.$$

The function $K(s, t) = k_t(s)$ is known as the *reproducing kernel* of $H$. Since

$$(k_t, k_s) = k_t(s) = K(s, t),$$

it follows that $K(t, s) = \overline{K(s, t)}$. For a general discussion of reproducing kernels see [1].

A function $F(s, t)$ is said to be *positive semidefinite* on $S \times S$ if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j F(s_i, s_j) \geqslant 0$$

for all $n$, for all $s_1, ..., s_n \in S$, and for all complex numbers $c_1, ..., c_n$.

THEOREM 2. *Let $H$ be a Hilbert space of complex-valued functions on a set $S$, with reproducing kernel $K(s, t)$. Let $g(s)$ be a complex-valued function defined on a subset $U \subset S$. Then in order that there exist $f \in H$ with $\|f\| \leqslant M$ and $f(s) = g(s)$ for all $s \in U$, it is necessary and sufficient that $[M^2K(s, t) - g(s)\overline{g(t)}]$ be positive semidefinite on $U \times U$.*

*Proof.* By Theorem 1, the condition (1) is necessary and sufficient for the existence of the extension $f$. But this condition is equivalent to

$$\left| \sum_{i=1}^{n} c_i g(s_i) \right|^2 \leqslant M^2 \left( \sum_{i=1}^{n} \bar{c}_i k_{s_i}, \sum_{j=1}^{n} \bar{c}_j k_{s_j} \right),$$

or

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j g(s_i) \overline{g(s_j)} \leqslant M^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{c}_i c_j (k_{s_i}, k_{s_j}),$$

or

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j [M^2 K(s_i, s_j) - g(s_i) \overline{g(s_j)}] \geqslant 0,$$

as asserted.

By way of illustration, let us choose $S$ to be the unit disk $|z| < 1$ in the complex plane. Then we have the following examples:

(i) For the space $H^2$ of analytic functions $f(z) = \sum a_n z^n$ in $|z| < 1$ such that

$$\|f\|^2 = \frac{1}{2\pi} \lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

the reproducing kernel is $K(z, \zeta) = (1 - z\bar{\zeta})^{-1}$.

(ii) For the Bergman space of analytic functions $f(z)$ such that

$$\|f\|^2 = \frac{1}{\pi} \iint_{|z|<1} |f(z)|^2 \, dx \, dy = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty,$$

the reproducing kernel is $K(z, \zeta) = (1 - z\bar{\zeta})^{-2}$.

(iii) For the space of analytic functions $f(z)$ with $f(0) = 0$ and finite Dirichlet integral

$$\|f\|^2 = \frac{1}{\pi} \iint_{|z|<1} |f'(z)|^2 \, dx \, dy = \sum_{n=1}^{\infty} n \, |a_n|^2 < \infty,$$

the kernel function is $K(z, \zeta) = \log((1 - z\bar{\zeta})^{-1})$.

More generally, for the Bergman space over an arbitrary simply connected domain $D$ with at least two boundary points, the reproducing kernel is $K(z, \zeta) = \psi'(z)\overline{\psi'(\zeta)}$, where $\psi$ maps $D$ conformally onto the unit disk and $\psi(\zeta) = 0$. (See, for example [6, p. 253].)

## 3. Applications to $H^p$ Spaces

A function $f(z)$ analytic in $|z| < 1$ is said to be of class $H^p$ $(0 < p < \infty)$ if the integral means

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p}$$

remain bounded as $r \to 1$. If $p \geqslant 1$, $H^p$ is a Banach space under the norm

$$\|f\|_p = \lim_{r \to 1} M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right\}^{1/p},$$

where $f(e^{i\theta})$ is the radial limit of $f(z)$, defined almost everywhere. The space $H^\infty$ of bounded analytic functions is a Banach space under the norm

$$\|f\|_\infty = \sup_{|z|<1} |f(z)|.$$

An account of the theory of $H^p$ spaces may be found in [4].

We can now apply Theorem 1 to obtain the following criterion for interpolation by $H^p$ functions.

THEOREM 3. *Let $\{z_n\}$ be a sequence of distinct points in the open unit disk, and let $\{w_n\}$ be a sequence of complex numbers. Suppose $1 \leqslant p \leqslant \infty$. Then in order that there exist $f \in H^p$ with $\|f\|_p \leqslant M$ and $f(z_n) = w_n$, $n = 1, 2, \ldots$, it is necessary and sufficient that*

$$\left| \sum_{i=1}^{n} c_i w_i \right| \leqslant M \max_{g \in H^p, \|g\|_p \leqslant 1} \left| \sum_{i=1}^{n} c_i g(z_i) \right| \tag{3}$$

*for each $n$ and for all complex numbers $c_1, \ldots, c_n$.*

*Proof.* The condition (3) is equivalent to (1). Thus, the theorem will follow from Theorem 1 if we can show that $H^p$ is a dual space and that point evaluation is weak-star continuous. For $1 < p < \infty$, there is no difficulty, since $H^p$ is reflexive. Hence, we need to discuss only $H^1$ and $H^\infty$.

To show that $H^1$ is a dual space, we identify each $f \in H^1$ with the complex-valued measure $f(e^{i\theta})\, d\theta$ on the unit circle $\Gamma$. This identification embeds $H^1$ in the space $M(\Gamma)$ of finite regular Borel measures on $\Gamma$. Furthermore, $M(\Gamma)$ is the dual of the space $C(\Gamma)$ of continuous functions on $\Gamma$, and by the theorem of F. and M. Riesz (see [4], p. 41), $H^1$ is a weak-star closed subspace of $M(\Gamma)$. Thus [4, p. 111], $H^1$ is dual to $C(\Gamma)/A_0$, where $A_0$ is the subspace of all $g \in C(\Gamma)$ such that

$$\int_0^{2\pi} g(e^{i\theta}) f(e^{i\theta})\, d\theta = 0 \qquad \text{for all} \quad f \in H^1.$$

(By the F. and M. Riesz theorem, $H^1$ is the annihilator of $A_0$ in $M(\Gamma)$.) The Cauchy representation [4, p. 40]

$$f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta} - \zeta} f(e^{i\theta})\, d\theta, \qquad |\zeta| < 1, \tag{4}$$

with $e^{i\theta}(e^{i\theta} - \zeta)^{-1} \in C(\Gamma)$, shows that each functional $\lambda_\zeta(f) = f(\zeta)$ is weak-star continuous.

To show that $H^\infty$ is a dual space, we regard $H^\infty$ as a subspace of $L^\infty$, the space of bounded measurable functions on $\Gamma$. Then $L^\infty = (L^1)^*$, and $H^\infty$ is the annihilator of $H_0^1$, the subspace of functions $f \in L^1$ such that

$$\int_0^{2\pi} e^{in\theta} f(e^{i\theta})\, d\theta = 0, \qquad n = 0, 1, 2, \ldots.$$

Thus, $H^\infty = (L^1/H_0^1)^*$, and the Cauchy representation (4) again shows that point evaluation is weak-star continuous. This concludes the proof of Theorem 3.

COROLLARY. *Suppose* $1 \leqslant p \leqslant \infty$ *and* $1/p + 1/q = 1$. *Then there exists* $f \in H^p$ *such that* $\|f\|_p \leqslant M$ *and* $f(z_n) = w_n$, $n = 1, 2, \ldots$, *if and only if*

$$\left| \sum_{i=1}^{n} c_i w_i \right| \leqslant M \min_{g \in H^q} \| k - g \|_q \tag{5}$$

*for each n and for all complex numbers* $c_1, \ldots, c_n$, *where*

$$k(z) = \sum_{i=1}^{n} \frac{c_i}{z - z_i}.$$

*Proof.* The kernel $k(z)$ generates the linear functional

$$\phi(f) = \sum_{i=1}^{n} c_i f(z_i) = \frac{1}{2\pi i} \int_{|z|=1} k(z) f(z)\, dz.$$

By the duality relation for extremal problems in $H^p$ spaces [4, p. 130],

$$\| \phi \| = \max_{\|f\|_p \leqslant 1} | \phi(f) | = \min_{g \in H^q} \| k - g \|_q ,$$

where "max" and "min" indicate that the extrema are attained.

In the case $p = q = 2$, it is known [4, p. 142] that the "natural kernel" $k$ is always extremal; so we have $\| \phi \| = \| k \|_2$. But a straightforward calculation gives

$$\| k \|_2^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{c_i \bar{c}_j}{1 - z_i \bar{z}_j}.$$

Thus, Theorem 3 gives the following result, which is actually a special case of Theorem 2.

THEOREM 4. *In order that there exist* $f \in H^2$ *with* $\|f\|_2 \leqslant M$ *and* $f(z_n) = w_n$ *for* $n = 1, 2, \ldots$, *it is necessary and sufficient that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j \left[ \frac{M^2}{1 - z_i \bar{z}_j} - w_i \bar{w}_j \right] \geqslant 0$$

*for each n and for all complex numbers* $c_1, \ldots, c_n$.

This result should be compared with the following theorem of R. Nevanlinna and G. Pick, characterizing the pairs of sequences $\{z_n\}$ and $\{w_n\}$ which admit an $H^\infty$ interpolation.

THEOREM B. *In order that there exist $f \in H^\infty$ with $\|f\|_\infty \leqslant M$ and $f(z_n) = w_n$ for $n = 1, 2,\ldots$, it is necessary and sufficient that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j \frac{M^2 - w_i \overline{w_j}}{1 - z_i \bar{z}_j} \geqslant 0$$

*for each $n$ and all complex $c_i$.*

This theorem may also be expressed in terms of functions $f(z)$ with positive real part. For a proof and further references, see [8]. We have not been able to deduce the Nevanlinna–Pick theorem from Theorem 3.

It is interesting to observe that if $f \in H^p$, $1 \leqslant p \leqslant \infty$, then the condition (5) with $c_1 = \cdots = c_{n-1} = 0$ and $c_n = 1$ gives

$$|f(z_n)| \leqslant M \| k \|_q = O((1 - | z_n |)^{-1/p}).$$

(See [4, lemma on p. 65].) This is close to the best possible result

$$f(z) = o((1 - | z |)^{-1/p}), \qquad | z | \rightarrow 1,$$

which can be proved (for $0 < p < \infty$) by other methods (see [4, p. 84]).

## 4. HARMONIC INTERPOLATION

Theorem 1 may also be applied to interpolation problems for harmonic functions. Although we shall discuss only the case of the unit disk, the methods extend easily to harmonic functions over more general domains. For $0 < p < \infty$, let $h^p$ be the space of complex-valued functions $u(z)$ harmonic in $| z | < 1$, such that $M_p(r, u)$ is bounded for $0 \leqslant r < 1$. Let $h^\infty$ be the space of bounded harmonic functions in $| z | < 1$. Then $h^p$ is a Banach space if $1 \leqslant p \leqslant \infty$. If $1 < p \leqslant \infty$, $h^p$ is isometrically isomorphic to the space $L^p$ over the unit circle $\Gamma$; while $h^1$ may be identified with the space $M(\Gamma)$ of finite regular Borel measures on $\Gamma$. (See [4, Chapter 1].) In particular, $h^p$ is reflexive if $1 < p < \infty$. The space $h^\infty$ may be regarded as the dual of $L^1$, and the Poisson formula shows that each point evaluation functional $\lambda_\zeta(u) = u(\zeta)$ $(| \zeta | < 1)$ is weak-star continuous on $h^\infty$. Finally, $h^1$ may be viewed as the dual of $C(\Gamma)$, the space of continuous functions on $\Gamma$; and the Poisson–Stieltjes representation shows that evaluation at each point $\zeta$ $(| \zeta | < 1)$ is weak-star continuous on $h^1$. Thus, Theorem 1 applies to $h^p$, and we obtain the following analogue of Theorem 3.

THEOREM 5. *Let $\{z_n\}$ be a sequence of distinct points in the open unit disk, and let $\{w_n\}$ be a sequence of complex numbers. Then in order that there exist $u \in h^p$ with $\| u \|_p \leqslant M$ and $u(z_n) = w_n$, $n = 1, 2, \ldots$, it is necessary and sufficient that*

$$\left| \sum_{i=1}^{n} c_i w_i \right| \leqslant M \sup_{v \in h^p, \|v\|_p \leqslant 1} \left| \sum_{i=1}^{n} c_i v(z_i) \right| \tag{6}$$

*for each $n$ and for all complex numbers $c_1, \ldots, c_n$.*

This theorem may also be expressed in a dual form involving the Poisson kernel

$$P_z(e^{it}) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}, \qquad z = re^{i\theta}.$$

COROLLARY. *If $1 \leqslant p \leqslant \infty$, a necessary and sufficient condition for the existence of $u \in h^p$ with $\| u \|_p \leqslant M$ and $u(z_n) = w_n$, $n = 1, 2, \ldots$, is that*

$$\left| \sum_{i=1}^{n} c_i w_i \right| \leqslant M \left\| \sum_{i=1}^{n} c_i P_{z_i} \right\|_q$$

*for each $n$ and for all complex numbers $c_1, \ldots, c_n$, where $q = p/(p - 1)$ is the conjugate index.*

*Proof.* This follows at once from the Poisson representation of $v \in h^p$ for $1 < p \leqslant \infty$, and from the Poisson–Stieltjes representation of $v \in h^1$.

## 5. UNIVERSAL INTERPOLATION SEQUENCES

A sequence $\{z_n\}$ of distinct points in $|z| < 1$ is called a *universal interpolation sequence* if for each bounded sequence $\{w_n\}$ there is a function $f \in H^\infty$ with $f(z_n) = w_n$, $n = 1, 2, \ldots$. The complete description of the universal interpolation sequences was given by L. Carleson [3]. Call a sequence $\{z_n\}$ *uniformly separated* if there is a $\delta > 0$ such that

$$\prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| \geqslant \delta, \qquad n = 1, 2, \ldots.$$

Then Carleson's result is as follows.

THEOREM C. *$\{z_n\}$ is a universal interpolation sequence if and only if it is uniformly separated.*

Shapiro and Shields [7] have generalized this theorem to include "weighted" interpolation in $H^p$ spaces, $1 \leqslant p \leqslant \infty$. An account of this theory may be found in [4, Chap. 9].

A sequence $\{z_n\}$ of distinct points in $|z| < 1$ will be called an *$(H^p, \ell^q)$ interpolation sequence* if for each $\{w_n\} \in \ell^q$ there is a function $f \in H^p$ such that $f(z_n) = w_n$, $n = 1, 2, \ldots$. In other words, the requirement is that

$$\{\{f(z_n)\} : f \in H^p\} \supset \ell^q.$$

Thus the $(H^\infty, \ell^\infty)$ interpolation sequences are the universal interpolation sequences. The next theorem describes the $(H^2, \ell^q)$ interpolation sequences. Similar results can be obtained for other spaces of functions.

THEOREM 6. *Suppose $1 \leqslant q \leqslant \infty$, and let $q' = q/(q-1)$ be the conjugate index. Then $\{z_n\}$ is an $(H^2, \ell^q)$ interpolation sequence if and only if there exists a constant $A > 0$ such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{c_i \bar{c}_j}{1 - z_i \bar{z}_j} \geqslant A \left\{ \sum_{i=1}^{n} |c_i|^{q'} \right\}^{2/q'} \tag{7}$$

*for each $n$ and for all complex numbers $c_1, \ldots, c_n$. If $q' = \infty$, the right-hand side of* (7) *should be interpreted as*

$$A \max_{1 \leqslant i \leqslant n} |c_i|^2.$$

*Proof.* Suppose (7) holds. Then for each $\{w_n\} \in \ell^q$ with $\|\{w_n\}\|_q \leqslant 1$, we have by Hölder's inequality

$$\left| \sum_{i=1}^{n} c_i w_i \right|^2 \leqslant \left\{ \sum_{i=1}^{n} |c_i|^{q'} \right\}^{2/q'} \leqslant \frac{1}{A} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{c_i \bar{c}_j}{1 - z_i \bar{z}_j}.$$

Therefore, by Theorem 4, there exists $f \in H^2$ with $f(z_n) = w_n$, $n = 1, 2, \ldots$.

Conversely, suppose $\{z_n\}$ is an $(H^2, \ell^q)$ interpolation sequence. This says that to each $w = \{w_n\} \in \ell^q$, there corresponds $f \in H^2$ with $f(z_n) = w_n$, $n = 1, 2, \ldots$. This correspondence induces a one-one linear mapping $T$ from $\ell^q$ into $H^2/N$, where

$$N = \{f \in H^2 : f(z_n) = 0,\ n = 1, 2, \ldots\}.$$

It is easy to verify that $T$ is a closed operator. Hence by the closed graph theorem, $T$ is bounded. In other words, there is a constant $M$ such that to each $w = \{w_n\} \in \ell^q$ with $\| w \|_q \leqslant 1$, there corresponds $f \in H^2$ with $\|f\|_2 \leqslant M$ and $f(z_n) = w_n$, $n = 1, 2, \ldots$ . Thus by Theorem 4,

$$\left| \sum_{i=1}^{n} c_i w_i \right|^2 \leqslant M^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{c_i \bar{c}_j}{1 - z_i \bar{z}_j}$$

for each $n$ and all complex $c_i$. Taking the maximum of the left-hand side over all $w \in \ell^q$ with $\| w \|_q \leqslant 1$, we obtain (7). This completes the proof.

Corollary 1. *If $\{z_n\}$ is uniformly separated, then*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{c_i \bar{c}_j}{1 - z_i \bar{z}_j} \geqslant A \left\{ \sum_{i=1}^{n} | c_i | \right\}^2$$

*for each $n$ and all complex $c_i$.*

*Proof.* This is an immediate consequence of Theorem C and Theorem 6.

Corollary 2.

$$\{\{f(z_n)\} : f \in H^2\} \supset \ell^2$$

*if and only if the eigenvalues of the Hermitian matrices $[(1 - z_i \bar{z}_j)^{-1}]$ $(i, j = 1, \ldots, n)$ have a positive lower bound, independent of $n$.*

*Note.* Professor A. K. Snyder informs us (private communication) that he has proved the existence of an $(H^2, \ell^\infty)$ interpolation sequence which is not uniformly separated. More recently, Duren and Shapiro [5] have constructed $(H^p, \ell^\infty)$ interpolation sequences (for each $p < \infty$) which are not uniformly separated.

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