FULL PROOFS for the MATCHUP PAPER

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Below are proofs, in full detail, for the two major theorems in Bean, Birge, Mittenthal and Noon [1986].

Proof of Theorem 1: McKenzie [Lemma 1, 1976] proves the result given (a). This condition does not cover scheduling problems with $x_0 \in \partial X_0$, the boundary of X_0. Also, the interiority of $X_t \cap Y_t$ may be difficult to verify. Conditions (b), (c) and (d) are reasonable assumptions that may be more readily verified. We show that each implies (i) and use (i) and the structure of f_t to show (ii).

Condition (b) implies that there is sufficient slack in the schedule that, at some future time, all resources will be utilized under capacity. We wish to show that $F^t(x_t)$ is subdifferentiable at x_t.

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Consider $x_t \in X_t \cap Y_t$. Let $\{x_\tau, \tau > t\}$ attain $F^t(x_t)$ so that

$$F^t(x_t) = \sum_{\tau=t}^{\infty} f_\tau(x_\tau, x_{\tau+1}).$$

Let $x'_t = x_t + \gamma$ where γ is a vector of perturbations such that $x'_t \in X_t \cap Y_t$. To show subdifferentiability, we show that there does not exist a γ such that the one sided directional derivative of F^t with respect to γ is $-\infty$ (Theorem 23.3, Rockafellar [1970]). For this, it is sufficient to show that $F^t(x'_t) \geq F^t(x_t) - K||x'_t - x_t||$, where K is a constant. If $F^t(x'_t) \geq F^t(x_t)$, the result is trivial.

If $F^t(x_t) \geq F^t(x'_t)$, construct a path, $\{x_\tau\}_{\tau \geq t}$, with a suitable bound. Assume $\{x'_\tau\}$ is an optimal path from x'_t. Let T be the next hypothesized slack time following t. In the following, we use the notation

$$1_{\{s=d(i,k)\}} = \begin{cases} 1, & \text{if } s = d(i,k) \text{ for any } k \\ 0, & \text{otherwise} \end{cases}.$$

For $\tau = t+1, \ldots, T$, let $x_\tau(i) = x_t(i) + \sum_{s=t}^{\tau}[x'_{s+1}(i) - x'_s(i) + 1_{\{s=d(i,k)\}}p(i,k)]$ for all $i \in I^1 \equiv \{i|\gamma(i) \geq 0\}$. For $i \notin I^1$, let $\underline{s}(i) = \inf\{s \geq t | x'_{s+1}(i) - x'_s(i) + 1_{\{s=d(i,k)\}}p(i,k) > 0\}$. Let $I^2 = \{i|\underline{s}(i) < T, i \notin I^1\}$. For $i \in I^2$, let

$$x_\tau(i) = \begin{cases} x_t(i) - \sum_{s=t}^{\tau-1} 1_{\{s=d(i,k)\}}p(i,k), & \tau \leq \underline{s}(i) \\ x'_\tau(i), & \tau > \underline{s}(i) \end{cases}.$$

This is possible by scaling γ such that $\gamma(i) \leq x'_{\underline{s}(i)+1}(i) - x'_{\underline{s}}(i) + 1_{\{\underline{s}(i)=d(i,k)\}}p(i,k)$. For all i not in $I = I^1 \cup I^2$, $\tau \leq T$, let $x_\tau(i) = x_t(i) - \sum_{s=t}^{\tau-1} 1_{\{s=d(i,k)\}}p(i,k)$. Then we have $x_T(i) \leq x'_T(i) - \gamma(i)$ for all i.

Define $x_{T+1}(i) = x'_{T+1}(i)$ for all $i \in I^1$ by scaling γ until adequate resources are available for all $i \in I^1$. This is possible by the slackness hypothesis. We have defined a path from x_t to x_{T+1} such that $x_{T+1}(i) = x'_{T+1}(i)$ for all $i \in I$ and $x_{T+1}(i) = x'_{T+1}(i) - \gamma(i)$ for all $i \notin I$.

Next let $J^1 = \{j|u(j) = 0, j \notin I\}$. Note that J^1 includes all setup states ($j > n$) not in I. According to the slackness hypothesis, for all $j \leq n$ and $j \in J^1$, we can increase production by $-\gamma(j)$, remain feasible and avoid earliness costs. Let $x''_{\tau+1}(j) =$

$x'_{\tau+1}(j) - \gamma(j)$ for all $\tau \geq T$, $j \leq n$, $j \in J^1$. For setup states ($j > n$), we have state constraints, $0 \leq x_\tau(j) \leq 1$, and dynamic constraints.

A feasible perturbation of $\{x'_\tau(j)\}$ for all $j > n$, $j \in J^1$, is obtained by defining $x''_{T+1}(j) = x'_{T+1}(j) - \gamma(j)$ for $j > n$, $j \in J^1$. For $\tau \geq T+2$, $j > n$, $j \in J^1$, define recursively,

$$x''_\tau(j) = \begin{cases} \max\{x'_\tau(j), x''_{\tau-1}(j)\}, & \text{if } x'_\tau(j) - x'_{\tau-1}(j) \geq 0 \\ x''_{\tau-1}(j) + (x'_\tau(j) - x'_{\tau-1}(j)) & \text{if } x'_\tau(j) - x'_{\tau-1}(j) < 0. \end{cases}$$

For all $j \notin J^1$, $\tau \geq T+1$, let $x''_\tau(j) = x'_\tau(j)$. This defines a feasible path since $x''_\tau(j) \geq x'_\tau(j)$, $x''_{\tau+1}(j) - x''_\tau(j) \leq x'_{\tau+1}(j) - x_\tau(j)$ and $x''_\tau(j) = x'_\tau(j)$ whenever $x'_\tau(j) \geq x'_{T+1}(j) - \gamma(j)$. Then $F^{T+1}(x''_{T+1}) \leq F^{T+1}(x'_{T+1})$ since $u(j) = 0$ for $j \in J^1$.

We now have $x_{T+1}(i) = x''_{T+1}(i)$ for all $i \in I \cup J^1$. Let $J^2 = \{j | j \notin I \cup J^1, \underline{s}(j) < \infty\}$, and let $\bar{T} = \max\{\underline{s}(j) | j \in J^2\}$. Define

$$x_\tau(i) = \begin{cases} x''_\tau(i), & i \in I \cup J^1, \tau = T+1, \ldots, \bar{T} \\ x_\tau(i) - \sum_{s=t}^{\tau-1} 1_{\{s=d(i,k)\}} p(i,k), & i \in J^2, \tau = T+1, \ldots, \underline{s}(j) - 1 \\ x_\tau(i) - \sum_{s=t}^{\tau-1} 1_{\{s=d(i,k)\}} p(i,k), & i \notin I \cup J^1 \cup J^2, \tau = T+1, \ldots, \bar{T} \\ x''_\tau(i), & i \in J^2, \tau = \underline{s}(j), \ldots, \bar{T}. \end{cases}$$

Again, this path is feasible since it requires no processing beyond x'_t.

For all $\tau \geq t$, $j \notin I \cup J^1 \cup J^2$, we have

$$x''_\tau(j) = x'_t(j) - \sum_{s=t+1}^{\tau} 1_{\{s=d(j,k)\}} p(j,k) < x_t(j) - \sum_{s=t+1}^{\tau} 1_{\{s=d(j,k)\}} p(j,k).$$

Let $T^j = \inf\{\tau | \sum_{s=t+1}^{\tau} 1_{\{s=d(j,k)\}} p(j,k) \geq x'_t(j) - \gamma(j)\}$. Let $J^3 = \{j \notin I \cup J^1 \cup J^2 | T^j < \infty\}$, the set of channels with no processing under $\{x'_t\}$ and with demand exceeding current inventory. Let $\bar{\bar{T}} = \min\{T \geq T^j | j \in J^3$, all resources slack for $j \in J^3\}$. Define

$$x'''_\tau(i) = \begin{cases} x''_\tau(i) - \gamma(i), & i \in J^3, \tau \geq \bar{\bar{T}} + 1 \\ x''_\tau(i), & \text{otherwise} \end{cases}$$

and

$$x_\tau(i) = \begin{cases} x'''_\tau(i), & i \in I \cup J^1 \cup J^2, \tau \geq \bar{T} \\ x'''_\tau(i), & i \in J^3, \tau \geq \bar{\bar{T}} + 1 \\ x_t(i) - \sum_{s=t}^{\tau-1} 1_{\{s=d(i,k)\}} p(i,k), & i \in J^3, \tau = \bar{T}, \ldots, \bar{\bar{T}} \\ x_t(i) - \sum_{s=t}^{\tau-1} 1_{\{s=d(i,k)\}} p(i,k), & i \notin I \cup J^1 \cup J^2 \cup J^3, \tau \geq \bar{T}. \end{cases}$$

Note that

$$F^{\bar{\bar{T}}+1}(x'''_{\bar{\bar{T}}+1}) \leq F^{\bar{\bar{T}}+1}(x''_{\bar{\bar{T}}+1}) + \sum_{\tau \geq \bar{\bar{T}}+1, j \in J^3} w(j)\gamma(j).$$

Hence, $w(j)$ must be zero for $j \in J^3$ at state $x''_{\bar{\bar{T}}+1}$ on an optimal path from x'_t.

In the remaining case, $\sum_{s=t+1}^{\infty} 1_{\{s=d(j,k)\}} p(j,k) =_{\text{def}} P_j \leq x'_t(j) - \gamma(j) = x_t(j)$. Since we can scale down γ arbitrarily (maintaining $\gamma \neq 0$), we can choose $-\gamma(j) < P_j - x'_t(j)$ for any j such that $P_j > x'_t(j)$. For all such j, $T^j < \infty$. Hence, we can assume $x_t(j) - P_j = \delta(j) > 0$ for all $j \notin I \cup J^1 \cup J^2 \cup J^3$ and some $\delta(j) > 0$. We then have $x_\tau(j) \geq \delta(j) > 0$ for all $\tau \geq t$ and $j \notin I \cup J^1 \cup J^2 \cup J^3$. Note that $0 \leq x'''_\tau(j) \leq x_\tau(j) + \gamma(j) < x_t(j)$ for all $\tau \geq T^j$, $j \notin I \cup J^1 \cup J^2 \cup J^3$. Then $F^{\bar{\bar{T}}+1}(x_{\bar{\bar{T}}+1}) \geq F^{\bar{\bar{T}}+1}(x'''_{\bar{\bar{T}}+1}) + \sum_{\tau=\bar{\bar{T}}+1}^{\infty} \gamma(j)u(j) = \infty$, implying that $x_t \notin X_t$ for any $\gamma(j) > 0$ and $x'_t \in X_t$. Therefore, we have $I \cup J^1 \cup J^2 \cup J^3 = \{1, 2, \ldots, 2n\}$ and $F^t(x_t) \leq F^t(x'_t) + K||x'_t - x_t||$, for some $K \leq (\sum_{i=1}^{n} \max\{w_i, u_i\})(\bar{\bar{T}} - t)$.

A similar argument is used if (c) holds by noting that only a finite number of costs are reduced in optimal paths from x'_t and x_t. This again implies subdifferentiability.

Condition (d) can be interpreted as a generalization of (c), in which a finite cost path is eventually obtainable from every feasible path at decreasing cost. Note that $\sum_{\tau=t}^{T_K-1} f_\tau(x_\tau, x_{\tau+1}) + f_{T_K}(x_{T_K}, x'_{T_K+1})$ has a finite number of terms, each with bounded slope, and is hence subdifferentiable. Hence, there exists p_t^K and path $\{x''_{t+1}, x''_{t+2}, \ldots x''_{T_K}\}$ such that

$$\sum_{\tau=t}^{T_K-1} f_\tau(x_\tau, x_{\tau+1}) + f_{T_K}(x_{T_K}, x'_{T_K+1}) - p_t^K x_t$$

$$\leq \sum_{\tau=t}^{T_K-1} f_\tau(x''_\tau, x''_{\tau+1}) + f_{T_K}(x''_{T_K}, x'_{T_K+1}) - p_t^K x''_t, \tag{9}$$

for all $x''_t \in X_t$. Rewrite (9) as

$$p_t^K(x''_t - x_t) \leq \sum_{\tau=t}^{T_K-1} f_\tau(x''_\tau, x''_{\tau+1}) - \sum_{\tau=t}^{T_K-1} f_\tau(x_\tau, x_{\tau+1}) +$$

$$f_{T_K}(x''_{T_K}, x'_{T_K+1}) - f_{T_K}(x_{T_K}, x'_{T_K+1}). \tag{10}$$

Note that $|F^t(x_t)| < \infty$ and $|F^t(x''_t)| < \infty$. Hence p_t^K has a limit point, p_t, and by (d)

$$p_t(x''_t - x_t) \leq F^t(x''_t) - F^t(x_t), \tag{11}$$

for all $x_t'' \in X_t$. Hence, F^t is subdifferentiable at x_t.

To show conclusion (ii), we must show that $(p_{t-1}^*, -p_t^*) \in \partial f_{t-1}(x_{t-1}^*, x_t^*)$, where $p_{t-1}^* \in \partial F^{t-1}(x_{t-1}^*)$ and $p_t^* \in \partial F^t(x_t^*)$. From (i), $F^{t-1}(x_{t-1}^*) - p_{t-1}^* x_{t-1}^* \leq F^{t-1}(x_{t-1}) - p_{t-1}^* x_{t-1}$ for all x_{t-1}. Hence,

$$f_{t-1}(x_{t-1}^*, x_t^*) + F^t(x_t^*) - p_{t-1}^* x_{t-1}^* \leq f_{t-1}(x_{t-1}, x_t) + F^t(x_t) - p_{t-1}^* x_{t-1}, \tag{12}$$

for all (x_{t-1}, x_t). For $g^t(x_{t-1}, x_t) = f_{t-1}(x_{t-1}, x_t) + F^t(x_t)$, (12) implies that $(p_{t-1}^*, 0) \in \partial g^t(x_{t-1}^*, x_t^*)$. Note that f_{t-1} is polyhedral. Also, note that for any $x_t \in ri(X_t)$, the point (x_t, x_t) belongs to $ri(dom\ \bar{F}^t)$, where $\bar{F}^t(y, x_t) = F^t(x_t)$ when $y \in \Re^{2n}$. Hence, $dom\ f_{t-1} \cap ri(dom\ \bar{F}^t) \neq \emptyset$. The subgradients of g^t are then the sums of subgradients of f_{t-1} and $\bar{F}^t$ (Theorem 23.8, Rockafellar [1970]), that is,

$$(p_{t-1}^*, 0) = (q_{t-1}, q_t) + (0, p_t^*), \tag{13}$$

where $(q_{t-1}, q_t) \in \partial f_{t-1}(x_{t-1}^*, x_t^*)$ and $(0, p_t^*) \in \partial \bar{F}^t(x_{t-1}^*, x_t^*)$, or, $p_t^* \in \partial F^t(x_t^*)$. This completes the result.∎

Proof of Theorem 2: We want to show

$$\inf_{(z_t, z_{t+1}) \in Z_t^*} ||(x_t', x_{t+1}') - (z_t, z_{t+1})|| < \epsilon. \tag{14}$$

Let x' have supporting prices p' as in (ii). Let $v_t(z_t^*) = (p_t^* - p_t')(x_t' - z_t^*)$. By summing inequalities (ii), for all T,

$$p_{T+1}^*(x_{T+1}' - x_{T+1}^*) \geq \sum_{t=0}^{T} (f_t(x_t^*, x_{t+1}^*) - f_t(x_t', x_{t+1}')) - p_0^*(x_0 - x_0'), \tag{15}$$

and

$$p_{T+1}'(x_{T+1}^* - x_{T+1}') \geq \sum_{t=0}^{T} (f_t(x_t', x_{t+1}') - f_t(x_t^*, x_{t+1}^*)) - p_0'(x_0' - x_0). \tag{16}$$

From the finiteness of $F^0(x_0')$ and $F^0(x_0^*)$, both right hand sides in (15) and (16) are uniformly bounded from below for all T. Hence, we know

$$v_t(x_t^*) \geq K > -\infty, \tag{17}$$

for all t and x_t^*.

The subgradient set of f_t at (x_t^*, x_{t+1}^*) is

$$\partial f_t(x_t^*, x_{t+1}^*) = co\{\begin{pmatrix} v \\ 0 \end{pmatrix} | v_i = -w_i, x_t^*(i) \geq 0, \text{ or } v_i = u_i, x_t^*(i) \leq 0\} \\ + N(dom\ f_t; (x_t^*, x_{t+1}^*)), \tag{18}$$

where co denotes the convex hull and, for any convex set, S, and point, x, $N(S;x) = \{v | v^T(y-x) \leq 0, \forall y \in S\}$, the normal cone to S at x and, for two sets A and B, $A+B = \{x | x = a+b, a \in A, b \in B\}$. Equation (18) follows from noting that the subgradient set of any proper, closed, convex function is the convex hull of all limits of neighboring gradients plus the normal cone to the effective domain of the function (Rockafellar, Theorem 25.6). We use (18) to show that

$$Z_t^* = \bar{Z}_t =_{\text{def}} \{(z_t, z_{t+1}) | z_t(i) = \lambda_i x_t^*(i), \lambda_i \geq 0, \text{ if } x_t^*(i) \neq 0, u_i \neq -w_i; (z_t, z_{t+1}) \in dom\ f_t\}. \tag{19}$$

First consider any vector $(z_t, z_{t+1}) \in Z_t^*$. There exists some $(p_t^*, -p_{t+1}^*)^T \in \partial f_t(x_t^*, x_{t+1}^*)$ such that

$$f_t(x_t^*, x_{t+1}^*) - p_t^* x_t^* + p_{t+1}^* x_{t+1}^* = f_t(z_t, z_{t+1}) - p_t^* z_t + p_{t+1}^* z_{t+1}. \tag{20}$$

Let $(p_t^*, -p_{t+1}^*)^T = (v, 0)^T + (n_1, n_2)^T$ as in (18) where $(v, 0)^T$ is in the convex hull of neighboring gradients and $(n_1, n_2)^T$ is in the normal cone. Write (20) as

$$f_t(z_t, z_{t+1}) = f_t(x_t^*, x_{t+1}^*) + v^T(z_t - x_t^*) + n_1^T(z_t - x_t^*) + n_2^T(z_{t+1} - x_{t+1}^*). \tag{21}$$

Note that $(z_t, z_{t+1}) \in dom\ f_t$ since $(x_t^*, x_{t+1}^*) \in dom\ f_t$. Since $\lambda(n_1, n_2)^T \in N(dom\ f_t; (x_t^*, x_{t+1}^*))$ for all $\lambda \geq 0$, we have $n_1^T(z_t - x_t^*) + n_2^T(z_{t+1} - x_{t+1}^*) \leq 0$. If $n_1^T(z_t - x_t^*) + n_2^T(z_{t+1} - x_{t+1}^*) < 0$, then $f_t(z_t, z_{t+1}) < f_t(x_t^*, x_{t+1}^*) + p_t^*(z_t - x_t^*) - p_{t+1}^*(z_{t+1} - x_{t+1}^*)$ for some $(p_t^*, -p_{t+1}^*) \in \partial f_t(x_t^*, x_{t+1}^*)$, violating convexity of f_t. Hence, $n_1^T(z_t - x_t^*) + n_2^T(z_{t+1} - x_{t+1}^*) = 0$. From the definition of $f_t(z_t, z_{t+1})$ and equation (21),

$$\sum_{i=1}^{2n} (-w_i z_t(i)^- + u_i z_t(i)^+) = f_t(z_t, z_{t+1})$$

$$= f_t(x_t^*, x_{t+1}^*) + \sum_{i=1}^{2n} v_i(z_t(i) - x_t^*(i))$$

$$= \sum_{i=1}^{2n}(-w_i x_t^*(i)^- + u_i x_t^*(i)^+) + \sum_{i=1}^{2n} v_i(z_t(i) - x_t^*(i))$$

$$= \sum_{i=1}^{2n} v_i z_t(i), \tag{22}$$

where

$$\begin{cases} v_i = -w_i, & x_t^*(i) < 0 \\ v_i = u_i, & x_t^*(i) > 0 \\ -w_i \le v_i \le u_i & x_t^*(i) = 0. \end{cases}$$

Therefore, we have

$$\begin{aligned} 0 = & \sum_{\{i|z_t(i)\le 0, x_t^*(i)<0\}} 0 + \sum_{\{i|z_t(i)\ge 0, x_t^*(i)>0\}} 0 \\ & + \sum_{\{i|z_t(i)>0, x_t^*(i)<0\}} (u_i + w_i)z_t(i) + \sum_{\{i|z_t(i)<0, x_t^*(i)>0\}} (u_i + w_i)(-z_t(i)) \\ & + \sum_{\{i|z_t(i)\ge 0, x_t^*(i)=0\}} (u_i - w_i)z_t(i) + \sum_{\{i|z_t(i)<0, x_t^*(i)=0\}} (u_i + w_i)(-z_t(i)). \end{aligned} \tag{23}$$

Every term in the sum in (23) must be zero since each has nonnegative components. Hence, $Z_t^* \subset \bar{Z}_t$. Note also that if $(z_t, z_{t+1}) \in \bar{Z}_t$, then we can choose $v_i = u_i$ if $z_t(i) > 0$ and $v_i = -w_i$ if $z_t(i) < 0$ for all i such that $x_t^*(i) = 0$. In this case, we have constructed $(p_t^*, -p_{t+1}^*) \in \partial f_t(x_t^*, x_{t+1}^*)$ to satisfy (20). This proves that $Z_t^* = \bar{Z}_t$.

Suppose (14) does not hold. There exists $\epsilon > 0$ and a sequence of times, $\{t_j\} \to \infty$ such that

$$\inf_{(z_{t_j}, z_{t_j+1}) \in Z_{t_j}^*} ||(x'_{t_j}, x'_{t_j+1}) - (z_t, z_{t+1})|| \ge \epsilon. \tag{24}$$

Suppose there exists a subsequence, $\{(x'_{t_{j_k}}, x'_{t_{j_k}+1})\}$, of the sequence, $\{(x'_{t_j}, x'_{t_j+1})\}$, such that

$$\inf_{(z_{t_{j_k}}, z_{t_{j_k}+1}) \in Z_{t_{j_k}}^*} ||z_{t_{j_k}} - x'_{t_{j_k}}|| \to 0 \tag{25}$$

as $t_{j_k} \to \infty$. Then, for any $\delta > 0$, there exists some $\bar{t}, x'_{\bar{t}}, (z_{\bar{t}}, z_{\bar{t}+1}) \in Z_{\bar{t}}^*$ such that $||z_{\bar{t}} - x'_{\bar{t}}|| \le \delta$ but $||z - x'_{\bar{t}+1}|| \ge \epsilon - \delta$ for all z such that $(z_{\bar{t}}, z)$ is feasible. However, by construction of f_t, for any x_t, if $x_t = x_t + \rho$ for a perturbation vector, ρ, with $||\rho|| \le \delta$, then, in the worst case, we have changed by ρ, the level of some channel i that bounds production for all outputs. In this case, for any $\bar{x}_{t+1}$, feasible from $\bar{x}_t$, there exists some x_{t+1}, feasible from x_t, such that $||\bar{x}_{t+1} - x_{t+1}|| \leq 2n|\rho|$. In particular, this is true for $t = \bar{t}$ and $\bar{x} = x'$, hence, $2n\delta \geq ||z - x'_{\bar{t}+1}|| \geq \epsilon - \delta$ for all δ. Hence, (24) and (25) are inconsistent.

Therefore, if (24) holds, we must have some $\delta > 0$ such that

$$\inf_{(z_{t_j}, z_{t_j+1}) \in Z^*_{t_j}} ||x'_{t_j} - z_{t_j}|| \geq \delta, \tag{26}$$

for all $\{t_j\}$. Note that (19) implies that the set of $z_{t_j} \in Z^*_{t_j}$, for some feasible z_{t_j+1}, defines a cone, $C^*_{t_j}$, corresponding to the orthant that contains $x^*_{t_j}(i)$ for $u_i \neq -w_i$ plus all coordinate directions, i, such that $x^*_t(i) = 0$ or $u_i = -w_i$. Consider $z^*_{t_j}$ defined by

$$z^*_{t_j}(i) = \begin{cases} 0 & \text{if } i \in A =_{\text{def}} \{i | x^*_{t_j}(i) \neq \lambda x'_{t_j}(i) \text{ for any } \lambda \geq 0 \text{ and } u_i \neq -w_i\} \\ x'_{t_j}(i) & \text{otherwise.} \end{cases} \tag{27}$$

The vector $x'_{t_j} - z^*_{t_j}$ is then normal to $C^*_{t_j}$ at $z^*_{t_j}$, so it achieves the infimum in (26). We then have

$$\inf_{(z_{t_j}, z_{t_j+1}) \in Z^*_{t_j}} ||x'_{t_j} - z_{t_j}|| = \sum_{i \in A} |x'_{t_j}(i)| \geq \delta. \tag{28}$$

Note that (28) implies that there exists some i such that $u_i \neq -w_i$. Let $\nu = \min\{u_i + w_i | u_i + w_i \neq 0\} > 0$. We then have, for any $(p^*_{t_j}, -p^*_{t_j+1}) \in \partial f_t(x^*_{t_j}, x^*_{t_j+1})$,

$$\begin{aligned} & f_{t_j}(x'_{t_j}, x'_{t_j+1}) - p^*_{t_j}(x'_{t_j} - z^*_{t_j}) + p^*_{t_j+1}(x'_{t_j+1} - z^*_{t_j+1}) \\ =& \sum_{i=1}^{2n} (-w_i x'_{t_j}(i)^- + u_i x'_{t_j}(i)^+) + \sum_{\{i | z^*_{t_j} = x'_{t_j}(i)\}} 0 - \sum_{\{i | z^*_{t_j} \neq x'_{t_j}(i)\}} v_i x'_{t_j}(i) \\ =& \sum_{\{i | z^*_{t_j} = x'_{t_j}(i)\}} (-w_i x'_{t_j}(i)^- + u_i x'_{t_j}(i)^+) + \sum_{\{i | z^*_{t_j} \neq x'_{t_j}(i)\}} (u_i + w_i)|x'_{t_j}(i)| \\ \geq& f_{t_j}(z^*_{t_j}, z^*_{t_j+1}) + \nu\delta. \end{aligned} \tag{29}$$

By definition of Z^*_t, for any (x^*_t, x^*_{t+1})

$$f_t(x^*_t, x^*_{t+1}) - p^*_t x^*_t + p^*_{t+1} x^*_{t+1} = f_t(z^*_t, z^*_{t+1}) - p^*_t z^*_t + p^*_{t+1} z^*_{t+1}. \tag{30}$$

From (29) and (30), we have for any T,

$$\sum_{t=0}^{T} f_t(x_t^*, x_{t+1}^*) - \sum_{t=0}^{T} f_t(x_t', x_{t+1}')$$

$$\leq (p_0^* - p_0')(x_0' - x_0^*) - (p_T^* - p_T')(x_T' - x_T^*) - \sum_{\{j | t_j \leq T\}} \nu ||(z_{t_j}^*, z_{t_j+1}^*) - (x_{t_j}', x_{t_j+1}')||. \quad (31)$$

By (17), if (14) does not hold, then the right-hand side of (31) approaches $-\infty$ as T approaches ∞ since the normed term exceeds δ infinitely often. This contradicts the finiteness of $F^0(x_0')$.∎

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