

Approximation to a Parabolic System Modeling the Thermoelastic Contacts of Two Rods

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In this article, we study a sequence of finite difference approximate solutions to a parabolic system, which models two dissimilar rods that may come into contact as a result of thermoelastic expansion. We construct the approximate solutions based on a set of finite difference schemes to the system, and we will prove that the approximate solutions converge strongly to the exact solutions. Moreover, we obtain and prove rigorously the error bound, which measures the difference between the exact solutions and approximate solutions in a reasonable norm. © 1998 John Wiley & Sons, Inc. Numer Methods Partial Differential Eq **14**: 1–25, 1998

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I. INTRODUCTION

The purpose of this article is to prove convergence and derive the error bound for a system of finite difference approximations to a parabolic system modeling the thermoelastic contacts of two dissimilar rods. Problems involving thermoelastic contact arise naturally in many applied situations, particularly those involving industrial processing, where two or more materials may come into contact or may lose contact as a result of thermoelastic expansion or contraction. The model of this article is one such situation: the mechanical behavior of two dissimilar rods, which are each fixed at one end but which may come into contact at their free ends as a result of thermoelastic expansions within finite time. Our model of the mechanical behavior originally consisted of a system of energy and elastic equations involving the temperature and displacement of each rod. The problem was then reformulated and reduced to solving an initial-boundary value problem for a nonlinear parabolic system containing only the two temperatures (see [1]).

The physical setting consists of two thin rods, each of which is clamped at one end but which may come into contact at their free ends. We assume that the process is independent of all but the horizontal variable. We use $\theta(x, t)$ to represent the temperature of the left rod on $0 \leq x \leq l_1$ and $\psi(x, t)$ to represent the temperature of the right rod on $l_2 \leq x \leq 1, l_1 \leq l_2$ both in nondimensional units. The fixed ends occur at $x = 0$ and $x = 1$, while the ends $x = l_1, x = l_2$ are free to expand

or contact. We let $\tilde{g} = l_2 - l_1$ to denote the initial gap between the rods in the reference configuration and the physical setting is displayed in Figure 1.

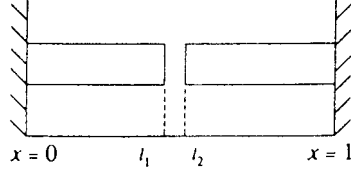


FIG. 1.

The complete parabolic system of equations is

$$(1 + a_1^2)\theta_t - \theta_{xx} = \alpha_1 g_t(\theta, \psi), 0 \leq x \leq l_1, t > 0, \quad (1.1)$$

$$(d + a_2^2)\psi_t - \psi_{xx} = \alpha_2 g_t(\theta, \psi), l_2 \leq x \leq 1, t > 0, \quad (1.2)$$

where

$$g(\theta, \psi)(t) = \max \left\{ a_1 \int_0^{l_1} \theta(x, t) dx + a_2 \int_{l_2}^1 \psi(x, t) dx - \tilde{g}; 0 \right\}, \quad (1.3)$$

$$\alpha_1 = \frac{a_1 \lambda}{l_1(1 + \lambda)}, \alpha_2 = \frac{a_2 \lambda}{(1 - l_2)(1 + \lambda)}, \tilde{g} = l_2 - l_1. \quad (1.4)$$

The initial condition is

$$\theta(x, 0) = \tilde{\theta}(x), 0 \leq x \leq l_1; \psi(x, 0) = \tilde{\psi}(x), l_2 \leq x \leq 1, \quad (1.5)$$

and boundary conditions are

$$\theta(0, t) = \psi(1, t) = 0, t \geq 0, \quad (1.6)$$

$$-k_1 \theta_x(l_1, t) = k(\theta(l_1, t) - \psi(l_2, t)), t \geq 0, \quad (1.7)$$

$$-k_2 \psi_x(l_2, t) = k(\theta(l_1, t) - \psi(l_2, t)), t \geq 0. \quad (1.8)$$

a_1, a_2 , and d are positive constants related to material properties of the rods; k_1, k_2 are the heat condition coefficients and k is the heat exchange coefficient; λ is the elastic modulus. For most materials that arise in practice, a_1 and a_2 are very small, therefore, we will assume that α_1, α_2 , which are defined in (1.4), are also small. The exact smallness conditions will be specified later. The current model is rather complicated in the sense that the two temperatures are coupled not only by a nonlinear system of equations, but also in the boundary conditions. The two source terms on the right hand side of (1.1), (1.2) are nonlocal as well as nonlinear. The system is also unusual in that each temperature function is defined on a different domain.

The particular model has been studied previously in [1] and [2]. In [1], they proved that under appropriate assumptions of the data, the system has a unique strong solution, i.e., a solution where each temperature lies in a Sobolev space. In [2], they presented some numerical simulations to the system, but there was no report of the convergence of those simulations and there was no error bound analysis either. Our goal in this article is, therefore, to construct a sequence of approximate (numerical) solutions to the system and to understand the convergence properties

of those solutions in a rigorous mathematical way. More numerical works to related physical problems can be found in [2] and [3]. But in each case, the problem involves but a single temperature and a single displacement, so that decoupling leads to one nonlinear equation for a single temperature.

We construct the approximate solutions based on a set of finite difference schemes that are implicit, but linear. We show that the schemes can be solved easily for all time. We also establish *a priori* energy estimates to the approximate solutions; these are similar to those satisfied by the exact solutions, and are required to derive the weak truncation errors. We then show that the approximate solutions converge to the exact solutions in a stronger sense, i.e., stronger than the sense of distribution. Finally, we apply the weak truncation error estimates in obtaining the bound for the difference between the exact and the approximate solutions. The overall mesh requirement is essentially the standard stability requirement by a standard scheme to the parabolic equation.

We assume that $\tilde{\theta}(x), \tilde{\psi}(x)$ are smooth enough to satisfy the following conditions:

$$\tilde{C}_0(x) = \|\tilde{\theta}\|_{H^2(0,l_1)}^2 + \|\tilde{\psi}\|_{H^2(l_2,1)}^2 + (\tilde{\theta}(l_1) - \tilde{\psi}(l_2))^2 < \infty. \quad (1.9)$$

We now construct the schemes. Let Δt be the time increment, Δx be the space increment such that $1 = M\Delta x, l_1 = M_1\Delta x, l_2 = (M - M_2)\Delta x$, where M, M_1, M_2 are positive integers such that $M_1 \leq M_2 \leq M$. Throughout the article, we will use δ, δ^2 to represent the operators:

$$\delta w_j = \frac{w_{j+1/2} - w_{j-1/2}}{\Delta x}, \delta w_{j+1/2} = \frac{w_{j+1} - w_j}{\Delta x}, \delta^2 w_j = \frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2}, \quad (1.10)$$

and we often use the following summation by parts formulae:

$$\sum_{j=L_1}^{L_2} \delta w_j z_j \Delta x = - \sum_{j=L_1}^{L_2-1} w_{j+1/2} \delta z_{j+1/2} \Delta x + w_{L_2+1/2} z_{L_1} - w_{L_1-1/2} z_{L_1}, \quad (1.11)$$

$$\sum_{j=L_1}^{L_2} \delta w_{j+1/2} z_{j+1/2} \Delta x = - \sum_{j=L_1+1}^{L_2} w_j \delta z_j \Delta x + w_{L_2+1} z_{L_2+1/2} - w_{L_1+1/2} z_{L_1+1/2}, \quad (1.12)$$

where L_1, L_2 are integers. The two approximate temperatures $\{\theta_j^q\}, \{\psi_j^q\}$ are then computed from the following implicit, but linear schemes:

$$(1 + a_1^2) \frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} - \delta^2 \theta_j^{q+1} = \alpha_1 \frac{g(\theta^q, \psi^q) - g(\theta^{q-1}, \psi^{q-1})}{\Delta t}, 0 \leq j \leq M_1, \quad (1.13)$$

$$(d + a_2^2) \frac{\psi_j^{q+1} - \psi_j^q}{\Delta t} - \delta^2 \psi_j^{q+1} = \alpha_2 \frac{g(\theta^q, \psi^q) - g(\theta^{q-1}, \psi^{q-1})}{\Delta t}, M_2 \leq j \leq M, \quad (1.14)$$

$$g(\theta^q, \psi^q) = \max \left\{ a_1 \sum_{j=1}^{M_1} \theta_j^q \Delta x + a_2 \sum_{j=M_2}^M \psi_j^q \Delta x - \tilde{g}; 0 \right\}, q \geq 0. \quad (1.15)$$

We take the following initial data

$$\theta_j^0 = \tilde{\theta}(x_j), 0 \leq j \leq M_1; \psi_j^0 = \tilde{\psi}(x_j), M_2 \leq j \leq M, \quad (1.16)$$

and the boundary conditions:

$$\theta_0^q = 0; -k_1 \delta \theta_{M_1+1/2}^q = -k_1 \frac{\theta_{M_1+1}^q - \theta_{M_1}^q}{\Delta x} = k(\theta_{M_1}^q - \psi_{M_2}^{q-1}), \quad (1.17)$$

$$\psi_M^q = 0; -k_2 \delta \psi_{M_2-1/2}^q = -k_2 \frac{\psi_{M_2}^q - \psi_{M_2-1}^q}{\Delta x} = k(\theta_{M_1}^q - \psi_{M_2}^q). \quad (1.18)$$

In order to make the schemes complete, we also extend the schemes to

$$\theta_j^{-1} = \theta_j^0, 0 \leq j \leq M_1; \psi_j^{-1} = \psi_j^0, M_2 \leq j \leq M, \quad (1.19)$$

and

$$\theta_{-1}^q = -\theta_1^q, \psi_{M+1}^q = -\psi_{M-1}^q, 0 \leq q \leq N. \quad (1.20)$$

We can prove that

$$\begin{aligned} C_0 = & \sum_{j=0}^{M_1} ((\theta_j^0)^2 + (\delta^2 \theta_j^0)^2) \Delta x + \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^0)^2 \Delta x + \sum_{j=M_2}^M ((\psi_j^0)^2 + (\delta^2 \psi_j^0)^2) \Delta x \\ & + \sum_{j=M_2}^{M-1} (\delta \psi_{j+1/2}^0)^2 \Delta x + (\theta_{M_1}^0 - \psi_{M_2}^0)^2 \leq 5\tilde{C}_0 < \infty. \end{aligned} \quad (1.21)$$

Since the source terms are nonlocal as well as nonlinear, no standard results in the parabolic theory can be quoted here. Instead, we derive *a priori* energy estimates peculiar to this particular problem, and use the estimates to prove convergence and derive error bound. Comparable results can also be found in [4]–[8], where finite difference approximate solutions to other nonlinear parabolic system equations were studied. But those schemes are all implicit and nonlinear, which means a system of nonlinear equations must be solved at each time-step. Moreover, the initial data to those systems were required to be sufficiently small. By contrast, our schemes are implicit, but linear, which is quite easy to implement, and there is no smallness requirement to the initial data. It is also amenable to a complete rigorous convergence and error bound analysis.

The plan of the article is as follows. In Section II, we derive *a priori* energy estimates to the solutions $\{\theta_j^q\}, \{\psi_j^q\}$ of the schemes, and prove that the schemes are solvable. Then, in Section III, we construct the approximate solutions $\{\theta^h\}, \{\psi^h\}$, and we apply the estimates obtained in Section II to derive bounds for the weak truncation errors, which measure the extent to which the approximation solutions fail to be exact solutions. In Section IV, we prove that the approximate solutions converge and we apply the weak truncation error estimates in deriving the error bound. Finally, we provide some computational results in Section V.

Most of the estimates occurring in the following sections are long and technical; many of them are symmetric between θ and ψ . We have, therefore, omitted most of the repeating details.

II. SOLVABILITY AND A PRIORI ENERGY ESTIMATES

In this section, we will prove solvability and derive various energy estimates of the schemes.

Theorem 2.1. *Assume that schemes (1.13)–(1.20) have been solved to t^n ($t^n \geq \Delta t$), then the schemes can be advanced from t^n to t^{n+1} if Δx is chosen to satisfy*

$$\Delta x \leq \min \left\{ \frac{2k_1}{k}, \frac{2k_2}{k} \right\}. \quad (2.1)$$

Proof. We will only prove the solvability for $\{\theta_j^n\}$; the proof for $\{\psi_j^n\}$ is similar and will be omitted. To advance $\{\theta_j^n\}$ to $\{\theta_j^{n+1}\}$, we need to solve the following system of linear equations:

$$A\theta_j^{n+1} = B(\theta_j^n, \theta_j^{n-1}), \quad (2.2)$$

where

$$\theta_j^{n+1} = (\theta_0^{n+1}, \theta_1^{n+1}, \dots, \theta_{M_1+1}^{n+1})^T \quad (2.3)$$

is an $M_1 + 2$ dimensional vector and A is an $(M_1 + 2) \times (M_1 + 2)$ tridiagonal matrix with entries

$$a_{0,0} = 1; a_{M_1+2, M_1+2} = -k_1; a_{j,j} = 1 + a_1^2 + 2 \frac{\Delta t}{\Delta x^2}, j = 1, 2, \dots, M_1; \quad (2.4)$$

$$a_{j,j+1} = -\frac{\Delta t}{\Delta x^2}, j = 0, 1, \dots, M_1; a_{j+1,j} = -\frac{\Delta t}{\Delta x^2}, j = 0, 1, \dots, M_1 - 1; \quad (2.5)$$

$$a_{M_1, M_1+1} = k_1 - k\Delta x. \quad (2.6)$$

B is an $M_1 + 2$ dimensional vector with entries

$$B_0 = 0, B_{M_1+1} = -k\Delta x \psi_{M_2}^{n-1}, \quad (2.7)$$

$$B_j = (1 + a_1^2)\theta_j^n + \alpha_1(g(\theta^n, \psi^n) - g(\theta^{n-1}, \psi^{n-1})), j = 1, 2, \dots, M_1. \quad (2.8)$$

By (2.1), A is a diagonally dominant matrix and so is invertible by the Gerschgorin Theorem. Therefore, $\{\theta_j^n\}$ can be advanced to $\{\theta_j^{n+1}\}$. ■

We now give the energy estimates in the next two theorems.

Theorem 2.2. Assume that schemes (1.13)–(1.20) have been solved to t^n ($t^n \geq \Delta t$). If $a_1, a_2, \alpha_1, \alpha_2$ are small enough, $\Delta x, \Delta t$ are chosen such that they satisfy (2.1) and

$$\Delta t \leq \Delta x^{1+p}, \text{ for some constant } p, 0 \leq p \leq 1, \quad (2.9)$$

$$k\Delta x^p + 2(a_1 + a_2) \left(\frac{k_1 \alpha_1^2}{1 + a_1^2} + \frac{k_2 \alpha_2^2}{d + a_2^2} \right) \leq \frac{1}{2} \min\{k_1(1 + a_1^2), k_2(d + a_2^2)\}, \quad (2.10)$$

then there exists a constant C , which satisfies

$$C \leq \frac{\max\{k_1, k_2, k\}}{\min\left\{\frac{1}{2}k_1(1 + a_1^2), \frac{1}{2}k_2(d + a_2^2), \frac{k_1}{1 + a_1^2}, \frac{k_2}{d + a_2^2}, k_1, k_2, k\right\}}, \quad (2.11)$$

such that

$$\begin{aligned} & \sum_{j=0}^{M_1-1} (\delta\theta_{j+1/2}^n)^2 \Delta x + \sum_{j=M_2}^{M-1} (\delta\psi_{j+1/2}^n)^2 \Delta x + \sum_{q=0}^n \sum_{j=0}^{M_1} (\delta^2\theta_j^q)^2 \Delta x \Delta t \\ & + \sum_{q=0}^n \sum_{j=M_2}^M (\delta^2\psi_j^q)^2 \Delta x \Delta t + \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t \left(\frac{\psi_j^{q-1} - \psi_j^q}{\Delta t} \right)^2 \Delta x \Delta t \\ & + \sum_{q=0}^{n-1} \sum_{j=M_2}^M + (\theta_{M_1}^n - \psi_{M_2}^n)^2 \leq CC_0. \end{aligned} \quad (2.12)$$

Remark 2.1. (2.10) is not the CFL condition, it is weaker than the CFL condition. The optimal error bound, which will be given in Theorem 4.4., will depend on p .

Proof. We square both sides of (1.13) to get

$$\begin{aligned} (1 + a_1^2)^2 \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 + (\delta^2 \theta_j^{q+1})^2 - 2(1 + a_1^2) \delta^2 \theta_j^{q+1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right) \\ = \alpha_1^2 \left(\frac{g(\theta^q, \psi^q) - g(\theta^{q-1}, \psi^{q-1})}{\Delta t} \right)^2. \end{aligned} \quad (2.13)$$

We then multiply both sides of (2.13) by $\frac{k_1}{1+a_1^2} \Delta x \Delta t$, sum over q, j to get

$$\begin{aligned} k_1(1 + a_1^2) \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t + \frac{k_1}{1 + a_1^2} \sum_{q=1}^n \sum_{j=0}^{M_1} (\delta^2 \theta_j^{q+1})^2 \Delta x \Delta t \\ - 2k_1 \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right) (\delta^2 \theta_j^{q+1}) \Delta x \Delta t \\ = \frac{k_1 \alpha_1^2}{1 + a_1^2} \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{g(\theta^q, \psi^q) - g(\theta^{q-1}, \psi^{q-1})}{\Delta t} \right)^2 \Delta x \Delta t. \end{aligned} \quad (2.14)$$

We then apply the summation by parts formula to the third term on the left-hand side of (2.14) to get

$$\begin{aligned} -2k_1 \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right) (\delta^2 \theta_j^{q+1}) \Delta x \Delta t \\ = 2k_1 \sum_{q=0}^{n-1} \sum_{j=0}^{M_1-1} \frac{\delta \theta_{j+1/2}^{q+1} - \delta \theta_{j+1/2}^q}{\Delta t} (\delta \theta_{j+1/2}^{q+1}) \Delta x \Delta t \\ - 2k_1 \sum_{q=0}^{n-1} \frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \delta \theta_{M_1+1/2}^{q+1} \Delta t + 2k_1 \sum_{q=0}^{n-1} \frac{\theta_0^{q+1} - \theta_0^q}{\Delta t} \delta \theta_{-1/2}^{q+1} \Delta t. \end{aligned} \quad (2.15)$$

The last term of (2.15) is zero by the first part of the boundary condition (1.17). We use the second part of (1.17) to the middle term and simplify the first term on the right-hand side of (2.15) to get

$$\begin{aligned} -2k_1 \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} (\delta^2 \theta_j^{q+1}) \Delta x \Delta t \geq k_1 \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^n)^2 \Delta x \\ - k_1 \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^0)^2 \Delta x - 2k_1 \sum_{q=0}^{n-1} \frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} (\theta_{M_1}^{q+1} - \psi_{M_2}^q) \Delta t. \end{aligned} \quad (2.16)$$

To estimate the term on the right-hand side of (2.14), we use the fact that, for any number r ,

$$\max\{r, 0\} = \frac{1}{2}r + \frac{1}{2}|r|, \quad (2.17)$$

by using the definition of $g(\theta, \psi)$ and (2.17), we get

$$\begin{aligned}
 & \frac{g(\theta_j^q, \psi_j^q) - g(\theta_j^{q-1}, \psi_j^{q-1})}{\Delta t} \\
 &= \left(\max \left\{ a_1 \sum_{j=0}^{M_1} \theta_j^q \Delta x + a_2 \sum_{j=M_2}^M \psi_j^q - \tilde{g}; 0 \right\} \right. \\
 & \quad \left. - \max \left\{ a_1 \sum_{j=0}^{M_1} \theta_j^{q-1} \Delta x + a_2 \sum_{j=M_2}^M \psi_j^{q-1} \Delta x - \tilde{g}; 0 \right\} \right) / \Delta t \\
 &= \frac{a_1}{2} \sum_{j=0}^{M_1} \frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \Delta x + \frac{a_2}{2} \sum_{j=M_2}^M \frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \Delta x \\
 & \quad + \frac{1}{2} \left(\left| a_1 \sum_{j=0}^{M_1} \theta_j^q \Delta x + a_2 \sum_{j=M_2}^M \psi_j^q \Delta x - \tilde{g} \right| \right. \\
 & \quad \left. - \left| a_1 \sum_{j=0}^{M_1} \theta_j^{q-1} \Delta x + a_2 \sum_{j=M_2}^M \psi_j^{q-1} \Delta x - \tilde{g} \right| \right) / \Delta t, \tag{2.18}
 \end{aligned}$$

we then use a basic inequality $||a| - |b|| \leq |a - b|$ to (2.18) and get

$$\begin{aligned}
 & \left| \frac{g(\theta_j^q, \psi_j^q) - g(\theta_j^{q-1}, \psi_j^{q-1})}{\Delta t} \right| \\
 & \leq \frac{a_1}{2} \sum_{j=0}^{M_1} \left| \frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \right| \Delta x + \frac{a_2}{2} \sum_{j=M_2}^M \left| \frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \right| \Delta x \\
 & \quad + \frac{1}{2} \left| a_1 \sum_{j=0}^{M_1} \frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \Delta x + a_2 \sum_{j=M_2}^M \frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \Delta x \right| \\
 & \leq a_1 \sum_{j=0}^{M_1} \left| \frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \right| \Delta x + a_2 \sum_{j=M_2}^M \left| \frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \right| \Delta x. \tag{2.19}
 \end{aligned}$$

So,

$$\begin{aligned}
 & \left(\frac{g(\theta_j^q, \psi_j^q) - g(\theta_j^{q-1}, \psi_j^{q-1})}{\Delta t} \right)^2 \\
 & \leq 2a_1^2 \left(\sum_{j=0}^{M_1} \left| \frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \right| \Delta x \right)^2 + 2a_2^2 \left(\sum_{j=M_2}^M \left| \frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \right| \Delta x \right)^2 \\
 & \leq 2a_1^2 \left(\sum_{j=0}^{M_1} \Delta x \right) \left(\sum_{j=0}^{M_1} \left(\frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \right)^2 \Delta x \right)
 \end{aligned}$$

$$\begin{aligned}
& + 2a_2^2 \left(\sum_{j=M_2}^M \Delta x \right) \left(\sum_{j=M_2}^M \left(\frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \right)^2 \Delta x \right) \\
& = 2l_1 a_1^2 \left(\sum_{j=0}^{M_1} \left(\frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \right)^2 \Delta x \right) + 2l_2 a_2^2 \left(\sum_{j=M_2}^M \left(\frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \right)^2 \Delta x \right). \quad (2.20)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{k_1 \alpha_1^2}{1 + a_1^2} \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{g(\theta_j^q, \psi_j^q) - g(\theta_j^{q-1}, \psi_j^{q-1})}{\Delta t} \right)^2 \Delta x \Delta t \\
& \leq \frac{k_1 \alpha_1^2}{1 + a_1^2} \left(2l_1^2 a_1^2 \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \right)^2 \Delta x \Delta t \right. \\
& \quad \left. + 2l_2^2 a_2^2 \sum_{q=0}^{n-1} \sum_{j=M_2}^M \left(\frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \right)^2 \Delta x \Delta t \right) \\
& \leq \frac{2k_1 \alpha_1^2 (a_1^2 + a_2^2)}{1 + a_1^2} \left(\sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^q - \theta_j^{q-1}}{\Delta t} \right)^2 \Delta x \Delta t \right. \\
& \quad \left. + \sum_{q=0}^{n-1} \sum_{j=M_2}^M \left(\frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} \right)^2 \Delta x \Delta t \right). \quad (2.21)
\end{aligned}$$

If we put (2.16) and (2.21) back to (2.14), we then get

$$\begin{aligned}
& k_1 (1 + a_1^2) \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t + \frac{k_1}{1 + a_1^2} \sum_{q=1}^n \sum_{j=0}^{M_1} (\delta^2 \theta_j^q)^2 \Delta x \Delta t \\
& + k_1 \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^n)^2 \Delta x - 2k \sum_{q=0}^{n-1} \left(\frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \right) (\theta_{M_1}^{q+1} - \psi_{M_2}^q) \Delta t \\
& \leq k_1 \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^n)^2 \Delta x + \frac{2k_1 (a_1 + a_2) \alpha_1^2}{1 + a_1^2} \left(\sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t \right. \\
& \quad \left. + \sum_{q=0}^{n-1} \sum_{j=M_2}^M \left(\frac{\psi_j^{q+1} - \psi_j^q}{\Delta t} \right)^2 \Delta x \Delta t \right). \quad (2.22)
\end{aligned}$$

We obtain similar estimates to (1.14) and combine with (2.22) to get

$$\begin{aligned}
 & k_1(1 + a_1^2) \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t + k_2(d + a_2^2) \sum_{q=0}^{n-1} \sum_{j=M_2}^M \left(\frac{\psi_j^{q+1} - \psi_j^q}{\Delta t} \right)^2 \Delta x \Delta t \\
 & + \frac{k_1}{1 + a_1^2} \sum_{q=1}^n \sum_{j=0}^{M_1} (\delta^2 \theta_j^q)^2 \Delta x \Delta t + \frac{k_2}{d + a_2^2} \sum_{q=1}^n \sum_{j=M_2}^M (\delta^2 \psi_j^q)^2 \Delta x \Delta t \\
 & + k_1 \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^n)^2 \Delta x + k_2 \sum_{j=M_2}^{M-1} (\delta \psi_{j+1/2}^n)^2 \Delta x + 2k \sum_{q=0}^{n-1} \left(\frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \right)^2 \\
 & \times (\theta_{M_1}^{q+1} - \psi_{M_2}^q) \Delta t - 2k \sum_{q=0}^{n-1} \left(\frac{\psi_{M_2}^{q+1} - \psi_{M_2}^q}{\Delta t} \right) (\theta_{M_1}^{q+1} - \psi_{M_2}^{q+1}) \Delta t \\
 & \leq k_1 \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^0)^2 \Delta x + k_2 \sum_{j=M_2}^{M-1} (\delta \psi_{j+1/2}^0)^2 \Delta x + 2(a_1 + a_2) \left(\frac{k_1 \alpha_1^2}{1 + a_1^2} + \frac{k_2 \alpha_2^2}{d + a_2^2} \right) \\
 & \times \left(\sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t + \sum_{q=0}^{n-1} \sum_{j=M_2}^M \left(\frac{\psi_j^{q+1} - \psi_j^q}{\Delta t} \right)^2 \Delta x \Delta t \right). \quad (2.23)
 \end{aligned}$$

The two terms just before the inequality sign in (2.23) can be estimated by

$$\begin{aligned}
 & 2k \sum_{q=0}^{n-1} \left(\frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \right) (\theta_{M_1}^{q+1} - \psi_{M_2}^q) \Delta t - 2k \sum_{q=0}^{n-1} \left(\frac{\psi_{M_2}^{q+1} - \psi_{M_2}^q}{\Delta t} \right) (\theta_{M_1}^{q+1} - \psi_{M_2}^{q+1}) \Delta t \\
 & = 2k \sum_{q=0}^{n-1} \left[\frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} (\theta_{M_1}^{q+1} - \psi_{M_2}^{q+1}) - \frac{\psi_{M_2}^{q+1} - \psi_{M_2}^q}{\Delta t} (\theta_{M_1}^{q+1} - \psi_{M_2}^{q+1}) \right] \Delta t \\
 & + 2k \Delta t \sum_{q=0}^{n-1} \frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \frac{\psi_{M_2}^{q+1} - \psi_{M_2}^q}{\Delta t} \Delta t \\
 & = 2k \sum_{q=0}^{n-1} (\theta_{M_1}^{q+1} - \psi_{M_2}^{q+1}) ((\theta_{M_1}^{q+1} - \psi_{M_2}^{q+1}) - (\theta_{M_1}^q - \psi_{M_2}^q)) \\
 & + 2k \Delta t \sum_{q=0}^{n-1} \frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \frac{\psi_{M_2}^{q+1} - \psi_{M_2}^q}{\Delta t} \Delta t \\
 & \geq 2k \sum_{q=0}^{n-1} (\theta_{M_1}^{q+1} - \psi_{M_2}^{q+1})^2 - k \left[\sum_{q=0}^{n-1} (\theta_{M_1}^{q+1} - \psi_{M_2}^{q+1})^2 + \sum_{q=0}^{n-1} (\theta_{M_1}^q - \psi_{M_2}^q)^2 \right] \\
 & - k \Delta t \sum_{q=0}^{n-1} \left[\left(\frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \right)^2 + \left(\frac{\psi_{M_2}^{q+1} - \psi_{M_2}^q}{\Delta t} \right)^2 \right] \Delta t \\
 & \geq k(\theta_{M_1}^n - \psi_{M_2}^n)^2 - k(\theta_{M_1}^0 - \psi_{M_2}^0)^2 \\
 & - k \Delta t \sum_{q=0}^{n-1} \left[\left(\frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \right)^2 + \left(\frac{\psi_{M_2}^{q+1} - \psi_{M_2}^q}{\Delta t} \right)^2 \right] \Delta t \text{ [by using (2.9)]}
 \end{aligned}$$

$$\begin{aligned}
&= k(\theta_{M_1}^n - \psi_{M_2}^n)^2 - k(\theta_{M_1}^0 - \psi_{M_2}^0)^2 \\
&\quad - k\Delta x^p \sum_{q=0}^{n-1} \left[\left(\frac{\theta_{M_1}^{q+1} - \theta_{M_1}^q}{\Delta t} \right)^2 + \left(\frac{\psi_{M_2}^{q+1} - \psi_{M_2}^q}{\Delta t} \right)^2 \right] \Delta x \Delta t. \tag{2.24}
\end{aligned}$$

If we put (2.24) back to (2.23), we then get

$$\begin{aligned}
&k_1(1 + a_1^2) \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t + k_2(d + a_2^2) \sum_{q=0}^{n-1} \sum_{j=M_2}^M \left(\frac{\psi_j^{q+1} - \psi_j^q}{\Delta t} \right)^2 \Delta x \Delta t \\
&\quad + \frac{k_1}{1 + a_1^2} \sum_{q=1}^n \sum_{j=0}^{M_1} (\delta^2 \theta_j^q)^2 \Delta x \Delta t + \frac{k_2}{d + a_2^2} \sum_{q=1}^n \sum_{j=M_2}^M (\delta^2 \psi_j^q)^2 \Delta x \Delta t \\
&\quad + k_1 \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^n)^2 \Delta x + k_2 \sum_{j=M_2}^{M-1} (\delta \psi_{j+1/2}^n)^2 \Delta x + k(\theta_{M_1}^n - \psi_{M_2}^n)^2 \\
&\leq k_1 \sum_{j=0}^{M_1-1} (\delta \theta_{j+1/2}^0)^2 \Delta x + k_2 \sum_{j=M_2}^{M-1} (\delta \psi_{j+1/2}^0)^2 \Delta x + k(\theta_{M_1}^0 - \psi_{M_2}^0)^2 \\
&\quad + \left(k\Delta x^p + 2(a_1 + a_2) \left(\frac{k_1 \alpha_1^2}{1 + a_1^2} + \frac{k_2 \alpha_2^2}{d + a_2^2} \right) \right) \\
&\quad \times \left(\sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t + \sum_{q=0}^{n-1} \sum_{j=M_2}^M \left(\frac{\psi_j^{q+1} - \psi_j^q}{\Delta t} \right)^2 \Delta x \Delta t \right). \tag{2.25}
\end{aligned}$$

(2.12) follows from (2.25) by using (2.10). \blacksquare

By Using Theorem 2.2, we can get a few more estimates, given in the next theorem.

Theorem 2.3. *Assume that schemes (1.13)–(1.20) have been solved to t^n ($t^n \geq \Delta t$), and C is the constant as described in Theorem 2.2. If $a_1, a_2, \alpha_1, \alpha_2$ are small enough and $\Delta x, \Delta t$ are chosen to satisfy (2.1), (2.10), and (2.11), then*

$$\sum_{j=0}^{M_1} \theta_j^{n2} \Delta x + \sum_{j=M_2}^M \psi_j^{n2} \Delta x \leq CC_0, \tag{2.26}$$

and

$$\begin{aligned}
&\sum_{j=1}^{M_1} (\delta^2 \theta_j^n)^2 \Delta x + \sum_{j=M_2+1}^M (\delta^2 \psi_j^n)^2 \Delta x + \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\delta \theta_{j+1/2}^{q+1} - \delta \theta_{j+1/2}^q}{\Delta t} \right)^2 \Delta x \Delta t \\
&\quad + \sum_{q=0}^{n-1} \sum_{j=M_2}^M \left(\frac{\delta \psi_{j+1/2}^{q+1} - \delta \psi_{j+1/2}^q}{\Delta t} \right)^2 \Delta x \Delta t \leq CC_0. \tag{2.27}
\end{aligned}$$

Proof. By using the first part of (1.17), we have

$$\theta_j^n - \theta_0^n = \theta_j^n = \sum_{l=0}^{j-1} \delta \theta_{j+1/2}^l \Delta x, \tag{2.28}$$

we then square both sides of (2.28), and use the Cauchy–Schwarz inequality to get

$$\begin{aligned}\theta_j^{n2} &= \left(\sum_{l=0}^{j-1} \delta\theta_{l+1/2}^n \Delta x \right)^2 \leq \left(\sum_{l=0}^{j-1} \Delta x \right) \left(\sum_{l=0}^{j-1} \delta\theta_{l+1/2}^{n2} \Delta x \right) \\ &\leq l_1 \left(\sum_{l=0}^{j-1} \delta\theta_{l+1/2}^{n2} \Delta x \right) \leq l_1 \left(\sum_{j=0}^{M_1-1} \delta\theta_j^{n2} \Delta x \right),\end{aligned}\quad (2.29)$$

we then multiply both sides of (2.29) by Δx , sum over j , and use (2.12) to get

$$\sum_{j=0}^{M-1} \theta_j^{n2} \Delta x \leq l_1 \left(\sum_{j=0}^{M_1} \Delta x \right) \left(\sum_{j=0}^{M_1} \delta\theta_j^{n2} \Delta x \right) \leq l_1^2 \left(\sum_{j=0}^{M_1-1} \delta\theta_j^{n2} \Delta x \right) \leq CC_0, \quad (2.30)$$

the other term in (2.26) can be proved similarly, this proves (2.26).

To prove (2.27), we apply the δ operator to both sides of (1.13), notice that the right-hand side of (1.13) does not depend on j , so we get

$$(1 + a_1^2) \frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} - \delta^3\theta_{j+1/2}^{q+1} = 0. \quad (2.31)$$

We then multiply both sides of (2.31) by $\frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} \Delta x \Delta t$, sum over q, j and use (1.17) to get

$$\begin{aligned}(1 + a_1^2) \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} \right)^2 \Delta x \Delta t + \sum_{q=0}^{n-1} \sum_{j=1}^{M_1} \delta^2\theta_j^{q+1} (\delta^2\theta_j^{q+1} - \delta^2\theta_j^q) \Delta x \\ + \sum_{q=0}^{n-1} \delta^2\theta_0^{q+1} \frac{\delta\theta_{1/2}^{q+1} - \delta\theta_{1/2}^q}{\Delta t} \Delta t - \sum_{q=0}^{n-1} \delta^2\theta_{M_1+1}^{q+1} \frac{\delta\theta_{M_1+1/2}^{q+1} - \delta\theta_{M_1+1/2}^q}{\Delta t} \Delta t = 0.\end{aligned}\quad (2.32)$$

By (1.20),

$$\delta^2\theta_0^{q+1} = \frac{\theta_1^{q+1} - 2\theta_0^{q+1} + \theta_{-1}^{q+1}}{\Delta x^2} = 0, \quad (2.33)$$

so the third term of (2.32) vanishes. We assume the schemes are extended, so that

$$\delta^2\theta_{M_1+1}^{q+1} = \frac{\theta_{M_1+2}^{q+1} - 2\theta_{M_1+1}^{q+1} + \theta_{M_1}^{q+1}}{\Delta x^2} = 0, \quad (2.34)$$

so the last term on the right-hand side of (2.32) also vanishes. We now estimate the second term on the right-hand side of (2.32) by

$$\begin{aligned}\sum_{q=0}^{n-1} \sum_{j=1}^{M_1} ((\delta^2\theta_j^{q+1})^2 - \delta^2\theta_j^{q+1} \delta^2\theta_j^q) \Delta x &\geq \frac{1}{2} \sum_{q=0}^{n-1} \sum_{j=1}^{M_1} (\delta^2\theta_j^{q+1})^2 \Delta x - \frac{1}{2} \sum_{q=0}^{n-1} \sum_{j=1}^{M_1} (\delta\theta_j^q)^2 \Delta x \\ &\geq \frac{1}{2} \sum_{j=1}^{M_1} (\delta^2\theta_j^n)^2 \Delta x - \frac{1}{2} \sum_{j=1}^{M_1} (\delta^2\theta_j^0)^2 \Delta x.\end{aligned}\quad (2.35)$$

So (2.32) becomes

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{M_1-1} (\delta^2 \theta_j^n)^2 \Delta x + (1 + a_1^2) \sum_{q=0}^{n-1} \sum_{j=0}^{M_1} \left(\frac{\delta \theta_{j+1/2}^{q+1} - \delta \theta_{j+1/2}^q}{\Delta t} \right)^2 \Delta x \Delta t \\ \leq \frac{1}{2} \sum_{j=1}^{M_1-1} (\delta^2 \theta_j^0)^2 \Delta x \leq \frac{1}{2} C_0. \end{aligned} \quad (2.36)$$

The other terms of (2.27) about ψ_j^q can be estimated similarly. \blacksquare

III. WEAK TRUNCATION ERRORS

In this section, we will obtain the truncation errors to schemes (1.13)–(1.20). Let $h = (\Delta x, \Delta t)$, we first construct the approximate solutions $\{\theta^h, \psi^h, g^h\}$, which are linear interpolations of $\{\theta_j^q\}$, $\{\psi_j^q\}$, and $\{g(\theta_j^q, \psi_j^q)\}$.

$$\begin{aligned} \theta^h(x, t) &= \theta_j^q + \frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} (t - t^q) + \delta \theta_{j+1/2}^q (x - x_j) \\ &\quad + \frac{\delta \theta_{j+1/2}^{q+1} - \delta \theta_{j+1/2}^q}{\Delta t} (x - x_j)(t - t^q), (x, t) \in [x_j, x_{j+1}] \\ &\quad \times (t^q, t^{q+1}], 0 \leq j \leq M_1, q \geq 0, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \psi^h(x, t) &= \psi_j^{q-1} + \frac{\psi_j^q - \psi_j^{q-1}}{\Delta t} (t - t^q) + \delta \psi_{j+1/2}^{q-1} (x - x_j) \\ &\quad + \frac{\delta \psi_{j+1/2}^q - \delta \psi_{j+1/2}^{q-1}}{\Delta t} (x - x_j)(t - t^q), (x, t) \in [x_j, x_{j+1}] \\ &\quad \times (t^q, t^{q+1}], M_2 - 1 \leq j \leq M - 1, q \geq 0, \end{aligned} \quad (3.2)$$

and

$$g^h(t) = g(\theta^q, \psi^q) + \frac{g(\theta^{q+1}, \psi^{q+1}) - g(\theta^q, \psi^q)}{\Delta t} (t - t^q), t \in (t^q, t^{q+1}], q \geq 0. \quad (3.3)$$

Notice that, with the above constructions, we can obtain the following boundary conditions to θ^h, ψ^h :

$$\theta^h(0, t) = 0, -k_1 \theta_x^h(l_1, t) = k(\theta^h(l_1, t) - \psi^h(l_2, t)), \quad (3.4)$$

$$\psi^h(1, t) = 0, -k_2 \psi_x^h(l_2, t) = k(\theta^h(l_1, t) - \psi^h(l_2, t)). \quad (3.5)$$

Both functions θ^h, ψ^h are constructed slightly beyond their ranges $[0, l_1]$ and $[l_2, 1]$ in order for us to take care of the boundary conditions. A direct result of Theorems 2.2, 2.3 will give the following theorem.

Theorem 3.1. *Let $\{\theta^h, \psi^h\}$ be the approximate solutions constructed in (3.1), (3.2), $g^h(t)$ be the function given in (3.3). Then for any $T > 0$,*

$$\begin{aligned} & \|\theta^h(x, t)\|_{H^1(0, l_1)} + \|\theta^h(x, t)\|_{H^1((0, l_1) \times (0, T))} + \|\theta_{xt}^h(x, t)\|_{L^2((0, l_1) \times (0, T))} \\ & + \|\theta^h(l_1, t)\|_{H^1(0, T)} + \|\psi^h(x, t)\|_{H^1(l_2, 1)} + \|\psi^h(x, t)\|_{H^1((l_2, 1) \times (0, T))} \\ & + \|\psi_{xt}^h(x, t)\|_{L^2((l_2, 1) \times (0, T))} + \|\psi^h(l_2, t)\|_{H^1(0, T)} + \|g^h(t)\|_{L^2(0, T)} \leq CC_0^{1/2}, \end{aligned} \quad (3.6)$$

where C is the constant as described in Theorem 2.2.

Let T be a fixed time, we define the truncation errors τ_1, τ_2 as follows:

$$\int_0^T \int_0^{l_1} ((1 + a_1^2)\theta_t^h - \alpha_1 g_t^h)\phi_1 dx dt + \int_0^T \int_0^{l_1} \theta_x^h \phi_{1x} dx dt = \int_0^T (\theta_x^h \phi_1)(l_1, t) dt + \tau_1 \quad (3.7)$$

$$\int_0^T \int_{l_2}^1 ((d + a_2^2)\psi_t^h - \alpha_2 g_t^h)\phi_2 dx dt + \int_0^T \int_{l_2}^1 \psi_x^h \phi_{2x} dx dt = - \int_0^T (\psi_x^h \phi_2)(l_2, t) dt + \tau_2, \quad (3.8)$$

where ϕ_1, ϕ_2 are test functions to be described later. The next two theorems will give the exact measurements of τ_1, τ_2 .

Theorem 3.2. *Let T be a fixed time and ϕ_1 be a test function that satisfies*

$$\phi_1(0, t) = 0, C(\phi_1) = \|\phi_1\|_{H^1((0, l_1) \times (0, T))} + \|\phi_1(x, t)\|_{L^2(0, l_1)} + \|\phi_1(l_1, t)\|_{L^2(0, T)} < \infty. \quad (3.9)$$

If $a_1, a_2, \alpha_1, \alpha_2$ are small and $\Delta x, \Delta t$ are chosen so that they satisfy (2.1), (2.10), and (2.11), then

$$|\tau_1| \leq CC_0^{1/2} C(\phi_1) \Delta x^p, 0 < p \leq 1, \quad (3.10)$$

where C is the constant as described in Theorem 2.2.

Proof. For simplicity, we assume $T = N\Delta t$ for a positive integer N . By using schemes (1.13)–(1.20) and definitions of θ^h, g^h , we get

$$\begin{aligned} & (1 + a_1^2)\theta_t^h - \alpha_2 g_t^h \\ & = (1 + a_1^2) \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} + \frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} (x - x_j) \right) - \alpha_1 \frac{g(\theta_j^{q+1}, \psi_j^{q+1}) - g(\theta_j^q, \psi_j^q)}{\Delta t} \\ & = \delta^2 \theta_j^{q+1} + (1 + a_1^2) \frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} (x - x_j) = \delta^2 \theta_j^{q+1} + (1 + a_1^2) \theta_{xt}^h (x - x_j). \\ & \theta_x^h = \delta\theta_{j+1/2}^q + \frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} (t - t^q). \end{aligned} \quad (3.11)$$

So τ_1 can be simplified to

$$\tau_1 = \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} (1 + a_1^2) \theta_{xt}^h (x - x_j) \phi_1 dx dt$$

$$\begin{aligned}
& + \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} \frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} (t - t^q) \phi_{1_x} dx dt \\
& + \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} (\delta^2\theta_j^{q+1} \phi_1 + \delta\theta_{j+1/2}^{q+1} \phi_{1_x}) dx dt - \int_0^T \theta_x^h(l_1, t) \phi_1(l_1, t) dt.
\end{aligned} \tag{3.12}$$

The first two terms can be estimated by using (3.6) and (2.27) to get

$$\begin{aligned}
& \left| \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} \theta_{xt}^h(x - x_j) \phi_1 dx dt \right. \\
& \quad \left. + \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} \frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} (t - t^q) \phi_{1_x} dx dt \right| \\
& \leq \Delta x \|\theta_{xt}^h\|_{L^2((0, l_1) \times (0, T))} \|\phi_1\|_{L^2((0, l_1) \times (0, T))} \\
& \quad + \Delta t \left(\sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \left(\frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} \right)^2 \Delta x \Delta t \right)^{1/2} \|\phi_{1_x}\|_{L^2((0, l_1) \times (0, T))} \\
& \leq CC_0^{1/2} C(\phi_1) (\Delta x + \Delta t).
\end{aligned} \tag{3.13}$$

In order to estimate the last two terms of (3.12), we introduce a new function θ^* :

$$\begin{aligned}
\theta^* & = \delta\theta_{j+1/2}^q + \frac{\delta\theta_{j+1/2}^{q+1} - \delta\theta_{j+1/2}^q}{\Delta t} (t - t^q) + \delta^2\theta_j^q (x - x_j) \\
& \quad + \frac{\delta^2\theta_j^{q+1} - \delta^2\theta_j^q}{\Delta t} (x - x_j) (t - t^q), (x, t) \in (x_j, x_{j+1}] \\
& \quad \times (t^q, t^{q+1}], 0 \leq j \leq M_1, 0 \leq q \leq N - 1.
\end{aligned} \tag{3.14}$$

Then, the results of Theorems 2.2, 2.3 show that

$$\|\theta^*\|_{H^1((0, l_1) \times (0, T))} \leq CC_0^{1/2}. \tag{3.15}$$

We now simplify the last two terms of (3.12).

$$\begin{aligned}
& \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} (\delta^2\theta_j^{q+1} \phi_1 + \delta\theta_{j+1/2}^{q+1} \phi_{1_x}) dx dt - \int_0^T (\theta_x^h \phi_1)(l_1, t) dt \\
& = \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} \left(\theta_x^* \phi_1 - \frac{\delta^2\theta_j^{q+1} - \delta^2\theta_j^q}{\Delta t} (t - t^q) \phi_1 + \delta\theta_{j+1/2}^{q+1} \phi_{1_x} \right) dx dt \\
& = \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} (-\theta^* \phi_{1_x} + \delta\theta_{j+1/2}^{q+1} \phi_{1_x}) dx dt
\end{aligned}$$

$$- \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} \frac{\delta^2 \theta_j^{q+1} - \delta^2 \theta_j^q}{\Delta t} (t - t^q) \phi_1 dx dt + \int_0^T (\theta^* \phi_1 - \theta_x^h \phi_1)(l_1, t) dt. \quad (3.16)$$

We estimate the middle term of (3.16) by using (2.11), (2.27), (3.6), and (3.14) to get

$$\begin{aligned} & \left| \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} \frac{\delta^2 \theta_j^{q+1} - \delta^2 \theta_j^q}{\Delta t} (t - t^q) \phi_1 dx dt \right| \\ & \leq C \frac{\Delta t}{\Delta x} \|\phi_1\|_{L^2((0, l_1) \times (0, T))} \left(\sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \left(\frac{\delta \theta_j^{q+1} - \delta \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t \right)^{1/2} \\ & \leq C C_0^{1/2} C(\phi_1) \Delta x^p. \end{aligned} \quad (3.17)$$

By the definitions of θ^* , θ^h , the last term of (3.16) vanishes. We finally estimate the first term of (3.16) by using the Cauchy-Schwarz inequality, (2.27) and (3.14).

$$\begin{aligned} & \left| \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} (-\theta^* \phi_{1_x} + \delta \theta_{j+1/2}^{q+1} \phi_{1_x}) dx dt \right| \\ & = \left| \sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \int_{t^q}^{t^{q+1}} \int_{x_j}^{x_{j+1}} \left(\frac{\delta \theta_{j+1/2}^{q+1} - \delta \theta_{j+1/2}^q}{\Delta t} (t - t^q) + \delta^2 \theta_j^q (x - x_j) \right. \right. \\ & \quad \left. \left. + \frac{\delta^2 \theta_j^{q+1} - \delta^2 \theta_j^q}{\Delta t} (t - t^q) (x - x_j) \right) \phi_{1_x} dx dt \right| \\ & \leq C(\phi_1) \left(\Delta t \left(\sum_{q=0}^{N-1} \sum_{j=0}^{M_1-1} \left(\frac{\delta \theta_{j+1/2}^{q+1} - \delta \theta_{j+1/2}^q}{\Delta t} \right)^2 \Delta x \Delta t \right)^{1/2} \right. \\ & \quad \left. + \Delta x \left(\sum_{q=0}^N \sum_{j=0}^{M_1-1} (\delta^2 \theta_j^q)^2 \Delta x \Delta t \right)^{1/2} \right) \leq C C_0^{1/2} C(\phi_1) \Delta x. \end{aligned} \quad (3.18)$$

(3.10) follows from (3.12), (3.13), (3.16), (3.17), and (3.18). ■

Theorem 3.3. *Let ϕ_2 be a test function that satisfies*

$$\phi_2(1, t) = 0, C(\phi_2) = \|\phi_2\|_{H^1((l_2, 1) \times (0, T))} + \|\phi_2(x, t)\|_{L^2(l_2, 1)} + \|\phi_2(l_2, t)\|_{L^2(0, T)} < \infty. \quad (3.19)$$

If $a_1, a_2, \alpha_1, \alpha_2$ are small and $\Delta x, \Delta t$ are chosen to satisfy (2.1), (2.10), and (2.11), then

$$|\tau_2| \leq C C_0^{1/2} C(\phi_2) \Delta x^p, 0 < p \leq 1, \quad (3.20)$$

C is the constant as described in Theorem 2.2.

Proof. The proof is similar to the one for Theorem 3.2, and so is omitted. ■

IV. CONVERGENCE AND ERROR BOUND

Theorem 4.1. *Let $\{\theta^h\}, \{\psi^h\}$ be the approximate solutions constructed in (3.1), (3.2) and T be a positive time. If $\Delta x, \Delta t, a_1, a_2, \alpha_1, \alpha_2$ satisfy (2.1), (2.10), and (2.11), then there exists a subsequence $\{h_l\}$ of $\{h\}$, functions θ defined on $[0, l_1] \times [0, T]$, and ψ defined $[l_2, 1] \times [0, T]$, such that*

$$\lim_{h_l \rightarrow 0} \theta^{h_l} = \theta \text{ pointwise uniformly on } [0, l_1] \times [0, T], \quad (4.1)$$

$$\lim_{h_l \rightarrow 0} \psi^{h_l} = \psi \text{ pointwise uniformly on } [l_2, 1] \times [0, T], \quad (4.2)$$

$$\lim_{h_l \rightarrow 0} g^{h_l} = g(\theta, \psi) \text{ pointwise uniformly on } [0, T]. \quad (4.3)$$

Remark 4.1. θ, ψ are unique, the uniqueness will be proved in Theorem 4.3.

Proof. We can use (3.6) to show that the functions $\{\theta^h\}, \{\psi^h\}$ are uniformly bounded and uniformly equicontinuous on the indicated sets. Appropriate subsequences, therefore, converge pointwise uniformly, which proves (4.1), (4.2). We now prove (4.3). Let $t \in (t^q, t^{q+1}]$ for some q , then

$$\begin{aligned} |g^{h_l}(t) - g(\theta^{h_l}, \psi^{h_l})(t)| &\leq |g^{h_l}(t) - g^{h_l}(t^q)| \\ &\quad + |g^{h_l}(t^q) - g(\theta^{h_l}, \psi^{h_l})(t^q)| + |g(\theta^{h_l}, \psi^{h_l})(t^q) - g(\theta^{h_l}, \psi^{h_l})(t)|. \end{aligned} \quad (4.4)$$

To estimate the first term of (4.4), we use (3.3) and an estimate similar to (2.19) to get

$$\begin{aligned} &|g(\theta^{h_l}, \psi^{h_l})(t^q) - g(\theta^{h_l}, \psi^{h_l})(t)| \\ &= \left| \frac{g(\theta^{q+1}, \psi^{q+1}) - g(\theta^q, \psi^q)}{\Delta t} (t - t^q) \right| \\ &\leq C\Delta t \left(\left(\sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \right)^{1/2} + \left(\sum_{j=M_2}^M \left(\frac{\psi_j^{q+1} - \psi_j^q}{\Delta t} \right)^2 \Delta x \right)^{1/2} \right) \\ &\leq C\Delta t^{1/2} \left(\left(\sum_{q=0}^{N-1} \sum_{j=0}^{M_1} \left(\frac{\theta_j^{q+1} - \theta_j^q}{\Delta t} \right)^2 \Delta x \Delta t \right)^{1/2} \right. \\ &\quad \left. + \left(\sum_{q=0}^{N-1} \sum_{j=M_2}^M \left(\frac{\psi_j^{q+1} - \psi_j^q}{\Delta t} \right)^2 \Delta x \Delta t \right)^{1/2} \right) \\ &\leq CC_0^{1/2} \Delta t^{1/2} \leq CC_0^{1/2} \Delta x^{\frac{1+p}{2}}. \end{aligned} \quad (4.5)$$

The other two terms can be estimated similarly, we get

$$|g^{h_l}(t) - g(\theta^{h_l}, \psi^{h_l})(t)| \leq CC_0^{1/2} \Delta x^{\frac{1+p}{2}}. \quad (4.6)$$

Since $g(\theta^h, \psi^h)$ is continuous for θ^h, ψ^h , (4.3) follows from (4.1), (4.2), and (4.6). ■

We can further prove by using Theorem 3.1 and Theorem 4.1 that there exist new subsequences $\{\theta^{h_l}\}, \{\psi^{h_l}\}$, which converge to θ, ψ in somewhat stronger sense. In order to simplify our notations, we will still use $\{h_l\}$ to represent the new indexes, and here is the theorem.

Theorem 4.2. *Let $\{\theta^h\}, \{\psi^h\}$ be the approximate solutions constructed in (3.1), (3.2) and T be a fixed time. If $\Delta x, \Delta t, a_1, a_2, \alpha_1, \alpha_2$ satisfy (2.1), (2.10), and (2.11), then there exists a subsequence $\{h_l\}$ of $\{h\}$, such that*

$$\lim_{h_l \rightarrow 0} \theta^{h_l}(x, t) = \theta(x, t) \text{ weakly in } L^2((0, l_1) \times (0, T)), \quad (4.7)$$

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l}(x, t) = \theta_x(x, t) \text{ weakly in } L^2((0, l_1) \times (0, T)), \quad (4.8)$$

$$\lim_{h_l \rightarrow 0} \theta_t^{h_l}(x, t) = \theta_t(x, t) \text{ weakly in } L^2((0, l_1) \times (0, T)), \quad (4.9)$$

$$\lim_{h_l \rightarrow 0} \theta^{h_l}(l_1, t) = \theta(l_1, t) \text{ weakly in } L^2(0, T), \quad (4.10)$$

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l}(l_1, t) = \theta_x(l_1, t) \text{ weakly in } L^2(0, T), \quad (4.11)$$

$$\lim_{h_l \rightarrow 0} \psi^{h_l}(x, t) = \psi(x, t) \text{ weakly in } L^2((l_2, 1) \times (0, T)), \quad (4.12)$$

$$\lim_{h_l \rightarrow 0} \psi_x^{h_l}(x, t) = \psi_x(x, t) \text{ weakly in } L^2((l_2, 1) \times (0, T)), \quad (4.13)$$

$$\lim_{h_l \rightarrow 0} \psi_t^{h_l}(x, t) = \psi_t(x, t) \text{ weakly in } L^2((l_2, 1) \times (0, T)), \quad (4.14)$$

$$\lim_{h_l \rightarrow 0} \psi^{h_l}(l_2, t) = \psi(l_2, t) \text{ weakly in } L^2(0, T), \quad (4.15)$$

$$\lim_{h_l \rightarrow 0} \psi_x^{h_l}(l_2, t) = \psi_x(l_2, t) \text{ weakly in } L^2(0, T), \quad (4.16)$$

$$\lim_{h_l \rightarrow 0} g_t^{h_l}(t) = g_t(\theta, \psi)(t) \text{ weakly in } L^2(0, T). \quad (4.17)$$

In addition, the functions θ, φ inherit the following properties:

$$\begin{aligned} & \|\theta(x, t)\|_{H^1((0, l_1) \times (0, T))} + \|\theta(l_1, t)\|_{H^1(0, T)} + \|g_t(t)\|_{L^2(0, T)} \\ & \quad + \|\psi(x, t)\|_{H^1((l_2, 1) \times (0, T))} + \|\psi(l_2, t)\|_{H^1(0, T)} \leq CC_0^{1/2}. \end{aligned} \quad (4.18)$$

Proof. We will just prove (4.8) in detail; (4.7), (4.9)–(4.17) can be proved similarly. (4.1) implies that

$$\lim_{h_l \rightarrow 0} \theta^{h_l}(x, t) = \theta(x, t) \text{ in distribution sense on } (0, l_1) \times (0, T), \quad (4.19)$$

and so

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l}(x, t) = \theta_x(x, t) \text{ in distribution sense on } (0, l_1) \times (0, T). \quad (4.20)$$

On the other hand, (3.6) proves that there is a new subsequence of $\{\theta_x^{h_l}\}$, which converges weakly on $L^2((0, l_1) \times (0, T))$. To simplify our notations, we still use $\{\theta^{h_l}\}$ to represent the new subsequence, so we have

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l}(x, t) = \bar{\theta}(x, t) \text{ weakly on } L^2((0, l_1) \times (0, T)) \quad (4.21)$$

for some function $\bar{\theta}(x, t) \in L^2((0, l_1) \times (0, T))$. In particular,

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l}(x, t) = \bar{\theta}(x, t) \text{ in distribution sense on } (0, l_1) \times (0, T). \quad (4.22)$$

By uniqueness, $\theta_x = \bar{\theta}$, which proves (4.8). (4.18) is a direct result of Theorem 3.1 and (4.7)–(4.17). \blacksquare

In the next theorem, we will prove that θ, ψ are solutions of (1.1)–(1.8). In fact, they are solutions in a sense that is stronger than the sense of distribution, because the test functions ϕ_1, ϕ_2 are in much weaker spaces than C_0^∞ .

Theorem 4.3. *If ϕ_1 satisfies (3.9) and ϕ_2 satisfies (3.19), then the limit functions θ, ψ obtained in Theorem 4.1 are the solutions of (1.1)–(1.8). In other words, they satisfy*

$$\int_0^T \int_0^{l_1} ((1 + a_1^2)\theta_t - \alpha_1 g_t(\theta, \psi))\phi_1 dx dt + \int_0^T \int_0^{l_1} \theta_x \phi_{1_x} dx dt = \int_0^T (\theta_x \phi_1)(l_1, t) dt, \quad (4.23)$$

$$\int_0^T \int_{l_2}^1 ((d + a_2^2)\psi_t - \alpha_2 g_t(\theta, \psi))\phi_2 dx dt + \int_0^T \int_{l_2}^1 \psi_x \phi_{2_x} dx dt = - \int_0^T (\psi_x \phi_2)(l_2, t) dt, \quad (4.24)$$

and

$$\theta(0, t) = 0, -k_1 \theta_x(l_1, t) = k(\theta(l_1, t) - \psi(l_2, t)), \quad (4.25)$$

$$\psi(1, t) = 0, -k_2 \psi_x(l_2, t) = k(\theta(l_1, t) - \psi(l_2, t)). \quad (4.26)$$

Proof. In (3.7), we let $\{h\}$ be the subsequence $\{h_l\}$ in Theorem 4.2, and let $h_l \rightarrow 0$. Then (4.23) is proved by (3.9), (3.10), (4.8), (4.9), (4.10), and (4.17). (4.24) can be proved similarly. By (3.4), (4.1), (4.2), we have

$$\lim_{h_l \rightarrow 0} -k(\theta^{h_l}(l_1, t) - \psi^{h_l}(l_2, t)) = -k(\theta(l_1, t) - \psi(l_2, t)), \text{ pointwise uniformly on } [0, T]. \quad (4.27)$$

By a similar proof as (4.1), we can prove that

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l} = \theta_1(x, t) \text{ pointwise uniformly on } [0, l_1] \times [0, T] \quad (4.28)$$

for some function $\theta_1(x, t)$ defined on $[0, l_1] \times [0, T]$. Further, we can prove by using (4.18) that

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l} = \theta_1(x, t) \text{ weakly on } L^2((0, l_1) \times (0, T)). \quad (4.29)$$

By comparing (4.8) and (4.29), we conclude that $\theta_1 = \theta_x$, therefore,

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l}(x, t) = \theta_x(x, t) \text{ pointwise uniformly on } [0, l_1] \times [0, T], \quad (4.30)$$

in particular

$$\lim_{h_l \rightarrow 0} \theta_x^{h_l}(l_1, t) = \theta_x(l_1, t) \text{ pointwise uniformly on } [0, T]. \quad (4.31)$$

So the second part of (4.25) is obtained from (3.4), (4.27), and (4.31). (4.26) can be proved similarly. \blacksquare

In the next theorem, we will give the error bound estimate.

Theorem 4.4. *Let T be fixed, $d, a_1, a_2, \alpha_1, \alpha_2, l_1, l_2, k_1, k_2, k$ be the physical constants described in Section I and C is the constant described in Theorem 2.2. Let $a_1, a_2, \alpha_1, \alpha_2, \Delta x, \Delta t$ be chosen to satisfy (2.1), (2.10), and (2.11). Let $\{\theta^h, \psi^h\}$ be the approximate solutions constructed in (3.1), (3.2). Then there exist unique functions θ, ψ obtained in Theorem 4.1, such that they are the exact solutions of (1.1)–(1.8), and the whole sequences $\{\theta^h\}, \{\psi^h\}$ converge to θ, ψ , respectively, as $h \rightarrow 0$. Moreover, if for any $0 \leq t \leq T$, we define*

$$\eta(t) = \|[(\theta - \theta^h)(x, t)]\|_{L^2(0, l_1)} + \|[(\psi - \psi^h)(x, t)]\|_{L^2(l_2, 1)} + \left(\int_0^t (\theta(l_1, s) - \psi(l_2, s)) ds \right)^2, \quad (4.32)$$

then

$$\eta(t) \leq CC_0^{1/2}(\eta(0) + \Delta x^{\frac{p}{2}}), 0 < p \leq 1. \quad (4.33)$$

Remark 4.2. Notice that we actually prove that the whole sequences $\{\theta^h\}, \{\psi^h\}$ converge to the unique solutions θ, ψ , not just converging subsequences.

Proof. We subtract (4.23) from (3.7), multiply both sides by k_1 , introduce a new notation $\Delta\theta = \theta - \theta^h$, take $\phi_1 = \int_t^T \Delta\theta(x, s) ds$, which satisfies (3.9), and we get

$$\begin{aligned} & k_1(1 + a_1^2) \int_0^T \Delta\theta_t \left(\int_t^T \Delta\theta ds \right) dx dt + k_1 \int_0^T \int_0^{l_1} \Delta\theta_x \left(\int_t^T \Delta\theta_x ds \right) dx dt \\ & = \alpha_1 k_1 \int_0^T \int_0^{l_1} (g_t(\theta, \psi) - g_t^h) \left(\int_t^T \Delta\theta ds \right) dx dt \\ & \quad + k_1 \int_0^T \Delta\theta_x(l_1, t) \left(\int_t^T \Delta\theta(l_1, s) ds \right) dx + k_1 \tau_1. \end{aligned} \quad (4.34)$$

Similarly, we subtract (4.24) from (3.8), multiply both sides by k_2 , introduce $\Delta\psi = \psi - \psi^h$, take $\phi_2 = \int_t^T \Delta\psi(x, s) ds$, which satisfies (3.19), and combine with (4.34) to get

$$I_1 + I_2 = I_3 + I_4 + k_1 \tau_1 + k_2 \tau_2, \quad (4.35)$$

where

$$\begin{aligned} I_1 = & k_1(1 + a_1^2) \int_0^T \int_0^{l_1} \Delta\theta_t \left(\int_t^T \Delta\theta ds \right) dx dt \\ & + k_2(d + a_2^2) \int_0^T \int_{l_2}^1 \Delta\psi_t \left(\int_t^T \Delta\psi ds \right) dx dt, \end{aligned} \quad (4.36)$$

$$I_2 = k_1 \int_0^T \int_0^{l_1} \Delta\theta_x \left(\int_t^T \Delta\theta_x ds \right) dx dt + k_2 \int_0^T \int_{l_2}^1 \Delta\psi_x \left(\int_t^T \Delta\psi_x ds \right) dx dt, \quad (4.37)$$

$$\begin{aligned} I_3 = & \alpha_1 k_1 \int_0^T \int_0^{l_1} (g_t(\theta, \psi) - g_t^h) \left(\int_t^T \Delta\theta ds \right) dx dt \\ & + \alpha_2 k_2 \int_0^T \int_{l_2}^1 (g_t(\theta, \psi) - g_t^h) \left(\int_t^T \Delta\psi ds \right) dx dt, \end{aligned} \quad (4.38)$$

$$I_4 = k_1 \int_0^T \Delta\theta_x(l_1, t) \left(\int_t^T \Delta\theta(l_1, s) ds \right) dt - k_2 \int_0^T \Delta\psi_x(l_2, t) \left(\int_t^T \Delta\psi(l_2, s) ds \right) dt. \quad (4.39)$$

We now estimate I_1, I_2, I_3 , and I_4 . The first term of I_1 can be estimated by

$$\begin{aligned} & k_1(1+a_1^2) \int_0^T \int_0^{l_1} \Delta\theta_t \left(\int_t^T \Delta\theta ds \right) dx dt \\ &= k_1(1+a_1^2) \int_0^{l_1} \Delta\theta \left(\int_0^T \Delta\theta ds \right) dx \Big|_0^T - k_1(1+a_1^2) \int_0^T \int_0^{l_1} (-\Delta\theta^2) dx dt \\ &= -k_1(1+a_1^2) \int_0^{l_1} \Delta\theta(x, 0) \left(\int_0^T \Delta\theta dt \right) dx + k_1(1+a_1^2) \int_0^T \int_0^{l_1} \Delta\theta^2 dx dt \\ &\geq -\frac{k_1(1+a_1^2)}{2} \int_0^{l_1} \Delta\theta(x, 0)^2 dx - \frac{k_1(1+a_1^2)}{2} \int_0^{l_1} \left(\int_0^T \Delta\theta(x, t) dt \right)^2 dx \\ &\quad + k_1(1+a_1^2) \int_0^T \int_0^{l_1} \Delta\theta^2 dx dt \\ &\geq -\frac{k_1(1+a_1^2)}{2} \int_0^{l_1} \Delta\theta(x, 0)^2 dx + k_1(1+a_1^2) \left(1 - \frac{l_1}{2}\right) \int_0^T \int_0^{l_1} \Delta\theta^2 dx dt. \end{aligned} \quad (4.40)$$

We can estimate the other term of I_1 similarly to get

$$\begin{aligned} I_1 \geq & k_1(1+a_1^2) \left(1 - \frac{l_1}{2}\right) \int_0^T \int_0^{l_1} \Delta\theta^2 dx dt + k_2(d+a_2^2) \left(1 - \frac{l_2}{2}\right) \int_0^T \int_{l_2}^1 \Delta\psi^2 dx dt \\ & - \frac{k_1(1+a_1^2)}{2} \int_0^{l_1} \Delta\theta(x, 0)^2 dx - \frac{k_2(d+a_2^2)}{2} \int_{l_2}^1 \Delta\psi(x, 0)^2 dx. \end{aligned} \quad (4.41)$$

We now estimate the first term of I_2 to get

$$\begin{aligned} & k_1 \int_0^T \int_0^{l_1} \Delta\theta_x \left(\int_t^T \Delta\theta_x ds \right) dx dt \\ &= k_1 \int_0^T \int_0^{l_1} \left(-\frac{d}{dt} \int_t^T \Delta\theta_x(x, s) ds \right) \left(\int_t^T \Delta\theta_x ds \right) dx dt \\ &= -\frac{k_1}{2} \int_0^{l_1} \left(\int_0^T \Delta\theta_x(x, s) ds \right)^2 \Big|_0^T dx = \frac{k_1}{2} \int_0^{l_1} \left(\int_0^T \Delta\theta_x(x, t) dt \right)^2 dx. \end{aligned} \quad (4.42)$$

We can estimate the other term of I_2 similarly to get

$$I_2 = \frac{k_1}{2} \int_0^{l_1} \left(\int_0^T \Delta\theta(x, t) dt \right)^2 dx + \frac{k_2}{2} \int_{l_2}^1 \left(\int_0^T \Delta\psi(x, t) dt \right)^2 dx. \quad (4.43)$$

We integrate by parts to the first term of I_3 to get

$$\begin{aligned}
 & \alpha_1 k_1 \int_0^T \int_0^{l_1} (g_t(\theta, \psi) - g_t^h) \left(\int_t^T \Delta\theta(x, s) ds \right) dx dt \\
 &= \alpha_1 k_1 \int_0^{l_1} \left((g(\theta, \psi) - g^h) \left(\int_0^T \Delta\theta(x, s) ds \right) \right) \Big|_0^T dx \\
 &+ \alpha_1 k_1 \int_0^T \int_0^{l_1} (g(\theta, \psi) - g^h) \Delta\theta dx dt \\
 &= -\alpha_1 k_1 (g(\theta, \psi) - g^h)(0) \int_0^T \int_0^{l_1} \Delta\theta dx dt + \alpha_1 k_1 \int_0^T \int_0^{l_1} (g(\theta, \psi) - g^h) \Delta\theta dx dt \\
 &\leq \frac{\alpha_1 k_1}{2} ((g(\theta, \psi) - g^h)(0))^2 + \frac{\alpha_1 k_1}{2} \int_0^T \int_0^{l_1} \Delta\theta^2 dx dt \\
 &+ \frac{\alpha_1 k_1}{2} \int_0^T \int_0^{l_1} (g(\theta, \psi) - g^h)^2 dx dt + \frac{\alpha_1 k_1}{2} \int_0^T \int_0^{l_1} \Delta\theta^2 dx dt \\
 &\leq \frac{\alpha_1 k_1}{2} ((g(\theta, \psi) - g(\theta^h, \psi^h))(0))^2 + \frac{\alpha_1 k_1}{2} ((g(\theta^h, \psi^h) - g^h)(0))^2 \\
 &+ \alpha_1 k_1 \int_0^T \int_0^{l_1} \Delta\theta^2 dx dt + \frac{\alpha_1 k_1}{2} \int_0^T \int_0^{l_1} (g(\theta, \psi) - g(\theta^h, \psi^h))^2 dx dt \\
 &+ \frac{\alpha_1 k_1}{2} \int_0^T \int_0^{l_1} (g(\theta^h, \psi^h) - g^h)^2 dx dt. \tag{4.44}
 \end{aligned}$$

By using (4.5), we can estimate the second and last term of (4.44) as

$$\frac{\alpha_1 k_1}{2} ((g(\theta^h, \psi^h) - g^h)(0))^2 + \frac{\alpha_1 k_1}{2} \int_0^T \int_0^{l_1} (g(\theta^h, \psi^h) - g^h)^2 dx dt \leq C \Delta x^{1+p}. \tag{4.45}$$

We now estimate the first and fourth term of (4.44). By the definition of g and a known formula $\max\{r, 0\} = \frac{1}{2}r + \frac{1}{2}|r|$, we get

$$\begin{aligned}
 & g(\theta, \psi) - g(\theta^h, \psi^h) \\
 &= \max \left\{ a_1 \int_0^{l_1} \theta dx + a_2 \int_{l_2}^1 \psi dx - \tilde{g}; 0 \right\} - \max \left\{ a_1 \int_0^{l_1} \theta^h dx + a_2 \int_{l_2}^1 \psi^h dx - \tilde{g}; 0 \right\} \\
 &= \frac{a_1}{2} \int_0^{l_1} \Delta\theta dx + \frac{a_2}{2} \int_{l_2}^1 \Delta\psi dx \\
 &+ \frac{1}{2} \left| a_1 \int_0^{l_1} \theta dx + a_2 \int_{l_2}^1 \psi dx - \tilde{g} \right| - \frac{1}{2} \left| a_1 \int_0^{l_1} \theta^h dx + a_2 \int_{l_2}^1 \psi^h dx - \tilde{g} \right|. \tag{4.46}
 \end{aligned}$$

So,

$$\begin{aligned}
& |g(\theta, \psi) - g(\theta^h, \psi^h)| \\
& \leq \frac{a_1}{2} \left| \int_0^{l_1} \Delta\theta dx \right| + \frac{a_2}{2} \left| \int_{l_2}^1 \Delta\psi dx \right| + \frac{1}{2} \left| a_1 \int_0^{l_1} \Delta\theta dx + a_2 \int_{l_2}^1 \Delta\psi dx \right| \\
& \leq a_1 \left| \int_0^{l_1} \Delta\theta dx \right| + a_2 \left| \int_{l_2}^1 \Delta\psi dx \right|. \tag{4.47}
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{\alpha_1 k_1}{2} ((g(\theta, \psi) - g(\theta^h, \psi^h))(0))^2 + \frac{\alpha_1 k_1}{2} \int_0^T \int_0^{l_1} (g(\theta, \psi) - g(\theta^h, \psi^h))^2 dx dt \\
& \leq C \left(\int_0^{l_1} \Delta\theta(x, 0)^2 dx + \int_{l_2}^1 \Delta\psi(x, 0)^2 dx \right) \\
& \quad + \alpha_1 k_1 (a_1^2 + a_2^2) \left(\int_0^T \int_0^{l_1} \Delta\theta^2 dx dt + \int_0^T \int_{l_2}^1 \Delta\psi^2 dx dt \right). \tag{4.48}
\end{aligned}$$

So,

$$\begin{aligned}
& \left| \alpha_1 k_1 \int_0^T \int_0^{l_1} (g_t(\theta, \psi) - g_t^h) \left(\int_t^T \Delta\theta(x, s) ds \right) dx dt \right| \\
& \leq C \left(\int_0^{l_1} \Delta\theta(x, 0)^2 dx + \int_{l_2}^1 \Delta\psi(x, 0)^2 dx \right) \\
& \quad + \alpha_1 k_1 (a_1^2 + a_2^2) \left(\int_0^T \int_0^{l_1} \Delta\theta^2 dx dt + \int_0^T \int_{l_2}^1 \Delta\psi^2 dx dt \right). \tag{4.49}
\end{aligned}$$

We do a similar estimate to the other term of I_3 to get

$$\begin{aligned}
I_3 & \leq C \left(\int_0^{l_1} \Delta\theta(x, 0)^2 dx + \int_{l_2}^1 \Delta\psi(x, 0)^2 dx \right) \\
& \quad + (\alpha_1 k_1 + \alpha_2 k_2) (a_1^2 + a_2^2) \left(\int_0^T \int_0^{l_1} \Delta\theta^2 dx dt + \alpha_2 k_2 (1 + a_2) \int_0^T \int_{l_2}^1 \Delta\psi^2 dx dt \right). \tag{4.50}
\end{aligned}$$

Finally, we estimate I_4 . By using the boundary conditions (3.2), (3.3), (4.25), and (4.26), we get

$$\begin{aligned}
I_4 & = -k \int_0^T (\Delta\theta(l_1, t) - \Delta\psi(l_2, t)) \left(\int_t^T \Delta\theta(l_1, s) ds \right) dt \\
& \quad + k \int_0^T (\Delta\theta(l_1, t) - \Delta\psi(l_2, t)) \left(\int_0^T \Delta\psi(l_2, s) ds \right) dt
\end{aligned}$$

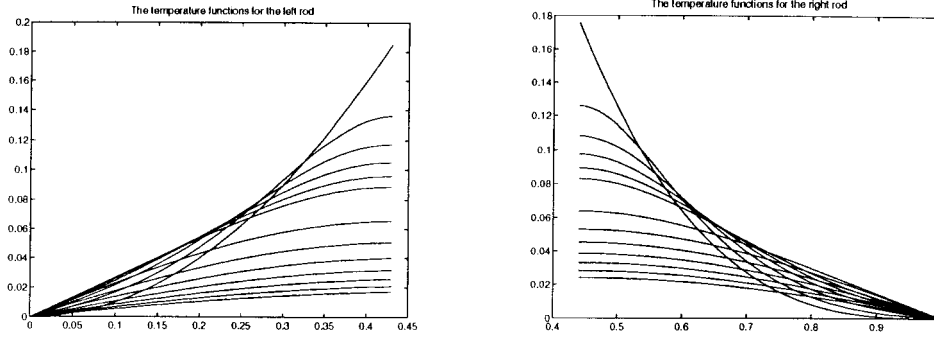


FIG. 2. Temperature functions corresponding to (5.1).

$$\begin{aligned}
 &= -k \int_0^T (\theta(l_1, t) - \psi(l_2, t)) \left(\int_t^T (\Delta\theta(l_1, s) - \Delta\psi(l_2, s)) ds \right) dt \\
 &= k \int_0^T \left(\frac{d}{dt} \int_t^T (\Delta\theta(l_1, s) - \Delta\psi(l_2, s)) ds \right) \left(\int_t^T (\Delta\theta(l_1, s) - \Delta\psi(l_2, s)) ds \right) dt \\
 &= \frac{k}{2} \left(\int_t^T (\Delta\theta(l_1, s) - \Delta\psi(l_2, t)) ds \right)^2 \Big|_0^T \\
 &= -\frac{k}{2} \left(\int_0^T (\Delta\theta(l_1, t) - \Delta\psi(l_2, t)) dt \right)^2. \tag{4.51}
 \end{aligned}$$

We put the estimates of $I_1, I_2, I_3,$ and I_4 back to (4.35) and simplify to get

$$\begin{aligned}
 &\left(k_1(1 + a_1^2) \left(1 - \frac{l_1}{2} \right) - (\alpha_1 k_1 + \alpha_2 k_2)(a_1^2 + a_2^2) \right) \int_0^T \int_0^{l_1} \Delta\theta^2 dx dt \\
 &+ \left(k_2(d + a_2^2) \left(1 - \frac{l_2}{2} \right) - (\alpha_1 k_1 + \alpha_2 k_2)(a_1^2 + a_2^2) \right) \int_0^T \int_{l_2}^1 \Delta\psi^2 dx dt
 \end{aligned}$$

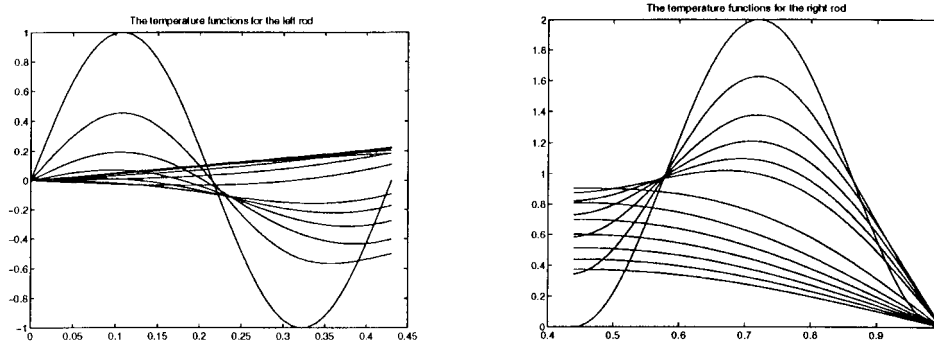


FIG. 3. Temperature functions corresponding to (5.2).

$$\begin{aligned}
& + \frac{k}{2} \left(\int_0^T (\Delta\theta(l_1, t) - \Delta\psi(l_2, t)) dt \right)^2 \\
& \leq C(\eta^2(0) + C_0\tau_1 + C_0\tau_2 + \Delta x^2) \leq CC_0(\eta^2(0) + \Delta x^p).
\end{aligned} \tag{4.52}$$

Since $a_1, a_2, \alpha_1, \alpha_2$ are small, we can assume that

$$\begin{aligned}
k_1(1 + a_1^2) \left(1 - \frac{l_1}{2} \right) - (\alpha_1 k_1 + \alpha_2 k_2)(a_1^2 + a_2^2) &\geq \frac{1}{4}, \\
k_2(d + a_2^2) \left(1 - \frac{l_2}{2} \right) - (\alpha_1 k_1 + \alpha_2 k_2)(a_1^2 + a_2^2) &\geq \frac{1}{4},
\end{aligned} \tag{4.53}$$

we then have

$$\eta^2(t) \leq CC_0(\eta^2(0) + \Delta x^p), \tag{4.54}$$

and (4.33) follows. Since we just proved that the whole sequences $\{\theta^h\}, \{\psi^h\}$ converge, therefore θ, ψ are unique. ■

Remark 4.3. If in (2.9), we take $p = 1$, then (2.9) will become the CFL condition, and the corresponding error bound for the case will be $\Delta x^{1/2}$.

V. COMPUTATIONAL RESULTS

In this section, we will provide some computational results to schemes (1.13)–(1.20). We take $\lambda = 0.5, a_1 = a_2 = 0.001, d = k_1 = 1, k_2 = 0.06, l_1 = 0.43, l_2 = 0.56$, and $\Delta x = 0.01, \Delta t = 2\Delta x^2$. We first take the following two initial functions:

$$\theta(x, 0) = x^2, 0 \leq x \leq 0.43, \psi(x, 0) = (1 - x)^2, 0.44 \leq x \leq 1. \tag{5.1}$$

Then Fig. 1 shows the two temperature functions for the left and right rod at time-steps 0, 20, 40, 60, 80, 100, 200, 300, 400, 500, 600, 700, and 800. We then take the following two initial functions:

$$\theta(x, 0) = \sin\left(\frac{2\pi}{0.43}x\right), 0 \leq x \leq 0.43, \psi(x, 0) = 1 - \cos\left(\frac{\pi}{0.56}(x - 0.44)\right), 0.44 \leq x \leq 1. \tag{5.2}$$

Figure 2 shows the two temperature functions at the same time-steps. Both figures show that the temperatures tend to stabilize.

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