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MINIMUM ENERGY TRIGGERING SIGNALS

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In switching problems the efficiency of the switching circuits is often of considerable importance. In the past the basic switching mechanism has been actuated by pulses or simple functions of voltage, current, or force, and efforts to obtain increased efficiency have led to the use of transistors, development of better relays, and the improvement of other components. This dissertation considers, as an alternative or complementary approach to the problem, the optimization of the function of time which is used for triggering so as to minimize the energy required of the triggering signal.

The general problem of determining the optimum triggering signal for a lumped-constant, linear circuit is treated. The optimum signal is defined as that which produces a given current through, or a voltage across, a resistive output element at time $t = T$ while at the same time requiring a minimum of energy from the generator driving the circuit. The output resistance is considered as characterizing the input terminals of a bistable element such as a thyratron, multivibrator, or a magnetic relay.

General equations characterizing the optimum signal are derived, and the conditions under which they are valid are noted. There are two pathological types of circuits for which characteristic equations are not obtained. However, both of these types of circuits are unrealistic in the sense that they do not allow for generator internal resistance or stray capacitance across the circuit input terminals. For realizable circuits and realizable generators a characteristic equation is obtained which is always valid.

Methods of solution of the characteristic integral equations are discussed, and it is shown that the Laplace transform can be used to reduce

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the integral equations into algebraic ones which are susceptible of simple solution. Finally, several sample problems are proposed and the solutions obtained. These examples, in addition to demonstrating general solutions, also demonstrate a method for finding the undetermined constants involved in the equations.

CHAPTER I

INTRODUCTION

Switching is of great importance in almost all fields of engineering endeavor. The problems range all the way from the trivial ones of mechanically turning an equipment on or off to the complicated problems associated with automatic telephone exchanges. Regardless of the complexity, however, the basic switching mechanism in the past has been actuated by pulses or predetermined functions in time of voltage, current, or force. In some of the complex problems in switching much attention has been paid to the efficiency of the switching process. However, the improvements in efficiency have resulted from reducing the energy requirements of the basic switching elements by building better relays and other terminal devices, and from increasing the efficiency of the associated circuits such as might result through the use of transistors. Thus, improved switching has resulted from the improvement of the switching circuits.

One might consider an alternative or complementary approach to the switching problem. In many cases it might be desirable to operate a remotely located switch by radio. Here the energy required of the transmitter may be of primary importance and one may or may not have optimized the receiver efficiency. Thus, in this type of problem the receiver may be fixed and one may only be free to adjust the transmitter parameters. Here it might be quite inefficient to select arbitrarily some given type of triggering signal, such as a pulse, for example, and it may be of considerable benefit to optimize the function of time which is chosen for the triggering signal. This paper considers the optimization of the time function of the signal to be used for triggering some given circuit.

1.1 Comparison of the Optimum Triggering Signal and Signal Detection Problems

Before attempting a solution to the optimum triggering signal problem it may be well to review some of the accomplishments in the field of signal detection, because at first glance it appears that the solution to the signal detection problem might yield a solution to the optimum triggering signal problem. However, it will be shown that the two problems are very different in nature, and the solution to one suggests nothing in the way of a solution to the other.

The part of the signal detection problem which bears the closest relation to the optimum triggering signal problem is probably best defined by quoting from Van Vleck and Middleton's treatment of the problem.[1]

"In the visual case, the best filter, in the first approximation is that which gives a maximum value of the ratio of the peak signal to the noise background. This statement can be formulated more precisely as follows: coming into the IF filter there is a fluctuating noise voltage and also a signal $S_1(t)$, which is a definite function of time. After filtering and rectification, the video amplitude is not expressible as a simple sum of pure signal and pure (i.e., normal, random) noise, as there are complicated modulation or interference effects between the two, especially for the linear detector. If the signal is periodically repeated, there will be a particular epoch at which the amplitude on the oscilloscope screen is on the average a maximum. The excess of the average of the amplitude at this epoch over the mean amplitude in the absence of signal we denote as s_V. We say on the average, because sometimes the noise may interfere con-

1. J. H. Van Vleck and David Middleton, "A Theoretical Comparison of the Visual, Aural, and Meter Reception of Pulse Signals in the Presence of Noise," Research Laboratory of Physics, Harvard University, Cambridge, Massachusetts, Journal of Applied Physics, November, 1946, pp 940-971.

structively and sometimes destructively with the signal. We must seek, therefore, to make $s_V(t)$ stand out as much as possible relative to the r.m.s. deflection n_V caused by the noise background."

The first theoretical treatment of this problem was made by D. O. North.[2] He showed that to maximize the ratio of the peak signal, known exactly, to the r.m.s. noise, the filter characteristic should be the complex conjugate of the Fourier transform of the pulse.

In addition to optimizing the filter characteristic for the detection of signals known exactly, there have also been investigations leading to optimization of pulse widths, repetition rates, scanning rates, etc.[1,3] These optimizations lead to the choice of the parameters of some signal of a given type. However, the general problem of determining, if possible, the best function of time for peaking the filter output for a given filter does not appear in the literature.

Without a critical examination, it appears at first glance that North's determination of the best filter for peaking the ratio of the output signal to the r.m.s. noise might be applied conversely to the determination of the best signal for a given filter. However, there is a basic difference between the two problems. Suppose first that we have some signal known exactly and that we want to determine the best filter for maximizing the ratio of peak output signal to r.m.s. noise. In this problem it is assumed that there is some given noise level at the filter input. It is clear that we do not want to simply select the filter characteristic to maximize the output signal, because this in general might also peak the output noise.

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2. D. O. North, in an unpublished report PTR-6C, entitled "Analysis of Factors Which Determine Signal-Noise Discrimination in Pulsed Carrier Systems," RCA, Princeton, June 25, 1943.

3. J. L. Lawson and G. E. Uhlenbeck, Threshold Signals, Vol. 24, Rad. Lab. Series, McGraw-Hill Book Co., New York, 1948.

For example, the best circuit for simply peaking the output signal would be an ideal transformer with infinite turns ratio. However, this transformer would also peak the output noise and thus would not maximize the signal-noise ratio. In signal detection, therefore, the choice of the filter involves inherently a choice of the noise output.

In the optimum triggering signal problem noise plays no role whatsoever. In this case it is assumed that the filter or circuit is given. But if the filter is given, it then follows that the noise output from the filter is also given. The choice of the optimum triggering signal therefore involves no choice of the noise output from the circuit. In this case, optimizing the input signal to maximize the peak output also maximizes, on the average, the ratio of the peak output signal to the noise output. Thus, one would not expect in general that the optimum signal is, using North's theorem in converse, the complex conjugate of the inverse Fourier transform of the filter characteristic.

CHAPTER II

OPTIMUM TRIGGERING SIGNALS

2.1 Statement of the Problem

The problem which is considered is that of activating a critical bistable element, located at the output terminals of a circuit, with a minimum of energy delivered to the input terminals of the circuit. The critical element is envisioned, for example, as a biased thyratron or a bistable multivibrator. These types of critical elements require that a minimum instantaneous voltage be exceeded at least once between a sensitive pair of tube elements in order that triggering occur and conduction take place. Another example would be a relay which requires some minimum current through the pull-in coil in order for the relay to be activated. In general, the circuit with the critical element at the output terminals is considered to be stable and made up of lumped-constant, linear elements, such as resistances, inductances, capacitances, and linear vacuum tubes. The critical element to be triggered, without loss of generality, can always be considered as purely resistive; i.e., thyratrons, multivibrators, and relay coils can all be considered to have a resistive component of input impedance and the reactive components can all be lumped into the remainder of the network. The description of the critical element as a pure resistance always allows the activating output signal to be described in terms of a current through a resistance. Thus this type of subterfuge allows all triggering problems to be resolved into a single type in which one desires to establish some minimum current through an output resistance at some instant in time. It should be noted, however, that a resistive description of the critical element is not necessary, but has been used only for convenience. The resistive description renders the triggering of both tubes and relays subject to a single treatment.

Consider then the circuit as characterized by the block diagram below.

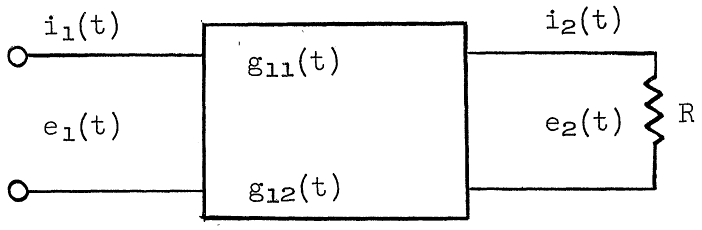

Fig. 1. Block diagram characterizing the circuit to be triggered.

Here $g_{11}(t)$ is the current $i_1(t)$ which flows in response to a voltage $e_1(t)$ when $e_1(t)$ is a unit impulse. $g_{12}(t)$ is the current $i_2(t)$ which flows in response to $e_1(t)$ when $e_1(t)$ is a unit impulse. With these definitions we can write[4]

$$i_1(t) = \int_0^t e_1(\tau)\, g_{11}(t-\tau)d\tau \quad , \tag{1}$$

and

$$i_2(t) = \int_0^t e_1(\tau)\, g_{12}(t-\tau)d\tau \quad , \tag{2}$$

where $e_1(t)$ is now an arbitrary driving function but equal to zero for $t < 0$. Note, that if R is the internal resistance of a relay to be triggered, then we want to describe the triggering function as a current [given by equation (2) above]. However, should R be the input resistance of, say, a thyratron, then we would want to describe the triggering function as a voltage as follows.

$$\begin{aligned} e_2(t) &= i_2(t)R \\ &= R\int_0^t e_1(\tau)g_{12}(t-\tau)d\tau \quad . \end{aligned} \tag{3}$$

4. Gardner and Barnes, Transients in Linear Systems, Vol. 1, John Wiley and Sons, Inc., New York, 1952.

Let us now consider a time interval $0 \leq t \leq T$, where T defines both the instant of triggering and also the allowable elapsed time interval of the driving function. If T represents the instant of triggering, then $i_2(T)$ or $e_2(T)$ must equal some critical minimum current or voltage which will establish the conduction of a tube or the pulling in of a relay. Thus we want

$$i_2(T) = I_c = \int_0^T e_1(\tau)\, g_{12}(T-\tau)d\tau \tag{4}$$

or

$$e_2(T) = E_c = I_c{}'R = R\int_0^T e_1(\tau)\, g_{12}(T-\tau)d\tau \quad . \tag{5}$$

Here the prime on I_c does not indicate differentiation, but is used simply to distinguish the critical current in equation (4) from that in equation (5).

Now, the energy input into the network up to the time t_1 is given by

$$\begin{aligned} E(t_1) &= \int_0^{t_1} P_{in}\, dt \\ &= \int_0^{t_1} e_1(t)\, i_1(t)\, dt \\ &= \int_0^{t_1} e_1(t)\left[\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau\right] dt \quad . \end{aligned} \tag{6}$$

The final total energy input in the interval $0 \leq t \leq T$ is then

$$E(T) = \int_0^T e_1(t)\int_0^t e_1(\tau)\, g_{11}(t-\tau)d\tau\, dt \quad . \tag{7}$$

Expressed mathematically, then, the optimization problem becomes that of minimizing the integral

$$E(T) = \int_0^T e_1(t)\int_0^t e_1(\tau)\, g_{11}(t-\tau)d\tau\, dt$$

subject to the constraint that

$$i_2(T) = I_c = \int_0^T e_1(\tau)\, g_{12}(T-\tau)\, d\tau$$

or

$$e_2(T) = E_c = R\int_0^T e_1(\tau)\, g_{12}(T-\tau)\, d\tau \quad .$$

Note that the constraint equations can also be written with t instead of τ as the variable of integration. Thus the constraint equations can have the same variables of integration as the energy equation. The constraint equations therefore become

$$i_2(T) = I_c = \int_0^T e_1(t)\, g_{12}(T-t)\, dt \qquad (8)$$

or

$$e_2(T) = E_c = R\int_0^T e_1(t)\, g_{12}(T-t)dt \quad . \qquad (9)$$

2.2 Derivation of an Equation Characterizing Solutions to the Linear Problem

General problems involving the minimization of integrals are treated by that field of mathematics known as the Calculus of Variations. The simple variational problem involves a determination of a function appearing in the integrand of an integral so as to minimize the integral. The Variational Calculus is also capable of treating problems involving the minimization of integrals subject to the condition that secondary restraints involving the function to be determined are also satisfied. Those variational calculus problems in which there are secondary restraints are known as isoperimetrical problems. For a treatment of the Variational Calculus see references 5 and 6. In the following treatment of the optimum trigger-

5. A. R. Forsyth, Calculus of Variations, Cambridge University Press, 1927.
6. C. Fox, An Introduction to the Calculus of Variations, Oxford University Press, 1950.

ing signal problem the Variational Calculus will be used. $e_1(t)$ will be selected so as to minimize the value of

$$E = \int_0^T e_1(t) \int_0^t e_1(\tau)\, g_{11}(t-\tau)d\tau\, dt \quad , \tag{10}$$

and also at the same time satisfy

$$i_2(T) = I_c = \int_0^T e_1(t)\, g_{12}(T-t)dt \quad . \tag{11}$$

We first assume that there is some function $e_1(t)$ satisfying the above conditions, and then examine what happens when we use a function perturbed away from the optimum, instead of using the optimum driving function. Thus, if we let $e_1(t)$ be the optimum driving function, we will consider a perturbed driving function

$$e_1(t) + \epsilon\eta(t) \quad , \tag{12}$$

where ϵ is some small arbitrary constant and $\eta(t)$ is some arbitrary continuous function of t with $\eta(o) = \eta(T) = 0$. Also, in order to incorporate both equations (10) and (11) into a single equation, we will make use of the Lagrangian multiplier λ, where λ is an arbitrary constant to be chosen finally to satisfy the boundary conditions of the problem.[7] We therefore consider the minimization of

$$E - \lambda i_2(T) = \int_0^T \left[e_1(t) \int_0^t e_1(\tau)g_{11}(t-\tau)d\tau - \lambda e_1(t)g_{12}(T-t) \right] dt. \tag{13}$$

Now in equation (13) let us substitute for $e_1(t)$ the perturbed driving function given by (12). Equation (13) then becomes

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7. I. S. Sokolnikoff, Advanced Calculus, McGraw-Hill Book Co., New York, 1939.

$$E - \lambda i_2(T) = \int_0^T \left\{ [e_1(t) + \epsilon\eta(t)] \int_0^t [e_1(\tau) + \epsilon\eta(\tau)] g_{11}(t-\tau) d\tau \right\} dt$$

$$- \int_0^T \lambda[e_1(t) + \epsilon\eta(t)] \; g_{12}(T-t) \; dt$$

$$= \int_0^T \left[e_1(t) \int_0^t e_1(\tau) g_{11}(t-\tau) d\tau - \lambda e_1(t) g_{12}(T-t) \right] dt$$

$$+ \epsilon \int_0^T \left[\eta(t) \int_0^t e_1(\tau) g_{11}(t-\tau) d\tau + e_1(t) \int_0^t \eta(\tau) g_{11}(t-\tau) d\tau - \lambda\eta(t) g_{12}(T-t) \right] dt$$

$$+ \epsilon^2 \int_0^T \eta(t) \int_0^t \eta(\tau) g_{11}(t-\tau) d\tau \; dt \quad . \qquad (14)$$

It can now be argued that if $e_1(t)$ is such as to minimize equation (13), then any perturbation can only serve to increase the value of equation (14). Note that the first term of equation (14) is the original unperturbed function given by equation (13). The second term of equation (14) has a multiplier ϵ. If we allow ϵ to have either positive or negative values, then to guarantee that the perturbation has increased the original function [this implies that $e_1(t)$ minimizes the function $E - \lambda i_2(T)$] the coefficient of ϵ must be zero. The third term has ϵ^2 as a multiplier, which is always positive, and, in addition, the coefficient of ϵ^2 is the energy input to the network due to the perturbing signal acting alone, which must also be positive. Thus if we set the coefficient of ϵ equal to zero, we will have established a condition on $e_1(t)$ such that any perturbations of $e_1(t)$ always result in an increase in the value of equation (13). This is exactly the condition which minimizes the quantity given by equation (10) and satisfies equation (11). Equating the coefficient of ϵ to zero gives

$$0 = \int_0^T \eta(t) \int_0^t e_1(\tau)g_{11}(t-\tau)d\tau\ dt + \int_0^T e_1(t) \int_0^t \eta(\tau)g_{11}(t-\tau)d\tau\ dt$$

$$- \lambda \int_0^T \eta(t)g_{12}(T-t)dt \quad . \qquad (15)$$

As it stands, it would appear that solution of this equation for $e_1(t)$ would lead to a function of the network parameters, g_{11} and g_{12}, and also of $\eta(t)$. But if $e_1(t)$ is given in terms of the perturbation then we have not truly found an optimizing function, but have optimized only with respect to some particular perturbation. However, if the right-hand side of equation (15) can be resolved into the form

$$\int_0^T \eta(t)f\ [e_1(t)\ ,\ g_{11}(t)\ ,\ g_{12}(t)]\ dt \quad ; \qquad (16)$$

then we could satisfy equation (15) by insisting that $f[e_1(t), g_{11}(t), g_{12}(t)]$ be identically equal to zero. If this is possible, then the description of $e_1(t)$ will be independent of the perturbation and will be given in terms of only the network functions $g_{11}(t)$ and $g_{12}(t)$.

CHAPTER III

RESTRICTIONS ON $g_{11}(t)$ FOR STABLE, LUMPED-CONSTANT, LINEAR NETWORKS, AND NATURE OF THE CORRESPONDING SOLUTIONS

3.1 Restrictions on $g_{11}(t)$ for Stable, Lumped-Constant, Linear Networks

Before looking for solutions to equation (15), we will first consider some of the general characteristics of $g_{11}(t)$ and some of the restrictions on $e_1(t)$ since they will be important in the mathematical treatment of the problem.

Consider now only the input terminals of the network to be triggered.

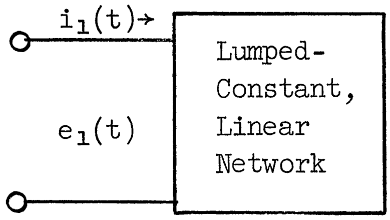

Fig. 2. Block diagram showing input terminals of the network to be triggered.

The Laplace transformed equation relating input current to input voltage is

$$i_1(s)z_{11}(s) = e_1(s) \quad , \tag{17}$$

where $z_{11}(s)$ is a complex impedance function describing the driving point impedance of the network. If $e_1(t)$ is a unit impulse $u_1(t)$, then $i_1(t)$ is, by definition, equal to $g_{11}(t)$. We have, therefore,

$$g_{11}(s) = \frac{1}{z_{11}(s)} \quad . \tag{18}$$

Now, because $z_{11}(s)$ is a driving point impedance function, it can be written as the ratio of two polynomials in s, subject to the restriction that the degree of the numerator does not differ by more than one from the degree of the denominator. This is in consequence of the restriction that driving point impedances can have no more than simple poles on the imaginary axis.[8] Thus $g_{11}(s)$ can in general be written

$$g_{11}(s) = k_1 s + k_2 + \frac{N(s)}{D(s)} \quad , \tag{19}$$

where k_1 or k_2, or both, may be equal to zero, and where $N(s)$ is at least one degree less than $D(s)$. $g_{11}(t)$ is therefore of the form

$$g_{11}(t) = k_1 u_2(t) + k_2 u_1(t) + \text{other terms} \quad , \tag{20}$$

where $u_2(t)$ is a unit doublet and $u_1(t)$ is a unit impulse. In general, the "other terms" in equation (20) can be expressed as powers of t times damped sinusoids, i.e., in the form

$$t^n e^{-at} \sin(\omega t + \theta) \quad . \tag{21}$$

This form results because the zeros of $D(s)$ lie either on the negative real axis or else appear in complex conjugate pairs. Because of this, $N(s)/D(s)$ can be expanded in partial fractions yielding a sum of terms of the form

$$\frac{N'(s)}{[(s + a)^2 + b^2]^n} \quad , \tag{22}$$

where $N'(s)$ is of degree less than that of the denominator. The inverse Laplace transform of a function of the form (22) has the form (21).

Because of equation (20), we can characterize $g_{11}(t)$ as belonging to one of three possible classes. These are:

(1) $g_{11}(t)$ contains neither a unit doublet nor a unit impulse.
(2) $g_{11}(t)$ contains no unit doublet, but does contain a unit impulse.
(3) $g_{11}(t)$ contains a unit doublet.

3.2 Nature of Solutions for the Three Possible Classes of $g_{11}(t)$

It was shown in Section 3.1 that for lumped-constant, linear networks

8. E. A. Guillemin, "Modern Methods of Network Synthesis," appearing in *Advances in Electronics*, Vol. 3, Academic Press, Inc., New York, 1951.

one could write the Laplace transform of $i_1(t)$ as

$$i_1(s) = g_{11}(s)\, e_1(s) \quad . \tag{23}$$

It was also shown that because of the restrictions on driving point impedances

$$g_{11}(s) = k_1 s + k_2 + \frac{N(s)}{D(s)} \quad , \tag{24}$$

where any term or combination of terms on the right side can be zero. Thus, $i_1(s)$ can be written

$$i_1(s) = \left[k_1 s + k_2 + \frac{N(s)}{D(s)}\right] e_1(s) \quad , \tag{25}$$

where the degree of $N(s)$ is at least one less than that of $D(s)$. Now, because we are in general looking for solutions to the equation

$$0 = \eta(t) \int_0^t e_1(\tau) g_{11}(t-\tau) d\tau + e_1(t) \int_0^t \eta(\tau) g_{11}(t-\tau) d\tau - \lambda \eta(t) g_{12}(T-t), \tag{26}$$

which we hope will be independent of $\eta(t)$ and will be in terms of only $g_{11}(t)$ and $g_{12}(t)$, we will consider $e_1(s)$ to be the ratio of two polynomials in consequence of $g_{11}(s)$ and $g_{12}(s)$ being ratios of polynomials and the integrals being convolution integrals. Thus, we will consider some of the implications of equation (25) under the restriction that $e_1(s)$ is the ratio of two polynomials.

<u>Case 1, $k_1 \neq 0$</u> [this corresponds to the third class of $g_{11}(t)$].—Suppose now that a driving signal $e_1(t)$ (not necessarily optimum) contains a unit impulse and that $k_1 \neq 0$ in equation (25). $e_1(s)$ then contains a constant term, and therefore [from equation (25)] $i_1(s)$ contains an s term. Thus $i_1(t)$ contains a unit doublet. Under these conditions let us now compute the energy input contribution due to the impulse of voltage and the doublet

of current. The unit impulse is symbolically defined as[4]

$$u_1(t) = \lim_{a \to 0} \frac{u(t) - u(t-a)}{a} , \tag{27}$$

where u(t) is a unit step function. A sketch of the unit impulse before the limit is taken appears in Fig. 3.

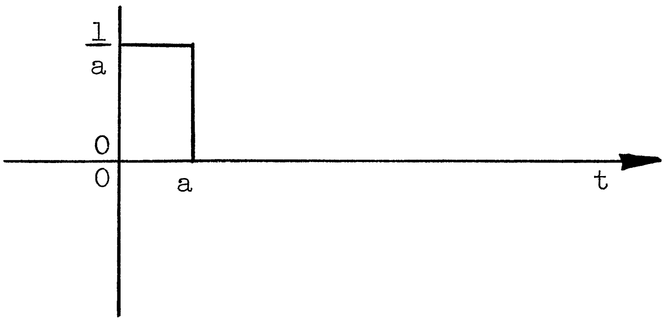

Fig. 3. Unit impulse.

The unit doublet is symbolically defined as[4]

$$u_2(t) = \lim_{a \to 0} \frac{u(t) - 2u(t-a) + u(t-2a)}{a^2} . \tag{28}$$

A sketch of this function before the limit is taken appears in Fig. 4.

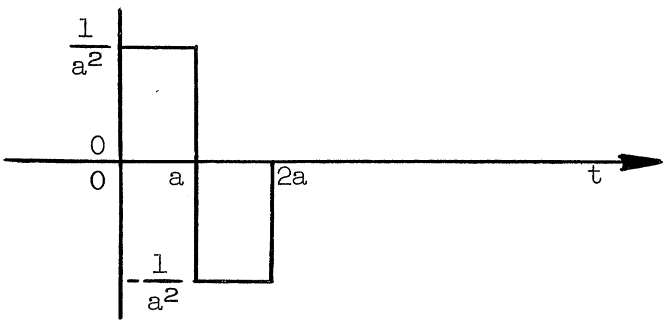

Fig. 4. Unit doublet.

The energy input to the network in the interval $0 \leq t \leq T$ resulting for $e_1(t) = K_1u_1(t)$ and $i_1(t) = K_2u_2(t)$ is

$$E_{in} = \int_0^T e_1(t)\, i_1(t)\, dt$$

$$= \lim_{a \to 0} \int_0^a \frac{K_1}{a} \frac{K_2}{a^2}\, dt$$

$$= \lim_{a \to 0} \frac{K_1 K_2}{a^2} \to \infty \quad . \tag{29}$$

Similarly, for $k_1 \neq 0$ in equation (25), higher order terms (unit doublets, unit triplets, etc.) in $e_1(t)$ also result in infinite energy input to the network. However, for any linear, lumped-constant network, one can always realize a given output voltage at time $t = T$ with a properly chosen step function. Furthermore, any unit step function can only result in a finite input energy to a network in a finite time interval $0 \leq t \leq T$ (except for the trivial case where there is a short circuit across the input terminals). Thus, if $g_{11}(t)$ contains a unit doublet, unbounded driving functions cannot be optimum.

Case 2, $k_1 = 0$ and $k_2 \neq 0$ [this corresponds to the second class of $g_{11}(t)$].—As in the previous case, let us assume a driving function $e_1(t)$ which contains a unit impulse. Then $i_1(s)$ contains a constant term [from equation (25)]. $i_1(t)$ therefore contains a unit impulse. Now, the energy input to the network in the interval $0 \leq t \leq T$ resulting for $e_1(t) = K_1 u_1(t)$ and $i_1(t) = K_2 u_1(t)$ is

$$E_{in} = \int_0^T e_1(t) i_1(t)\, dt$$

$$= \lim_{a \to 0} \int_0^a \frac{K_1 K_2}{a^2}\, dt$$

$$= \lim_{a \to 0} \frac{K_1 K_2}{a} \to \infty \quad . \tag{30}$$

Similarly, for $k_2 \neq 0$ in equation (25), higher order terms (unit doublets, unit triplets, etc.) in $e_1(t)$ also result in infinite energy input to the network. As in the previous case then, if $k_2 \neq 0$, we conclude that the optimum signal is some bounded function in the interval $0 \leq t \leq T$.

Case 3, $k_1 = k_2 = 0$ [this corresponds to the first class of $g_{11}(t)$].— For this case

$$i_1(s) = e_1(s) \frac{N(s)}{D(s)} , \tag{31}$$

where N(s) is at least one degree less than D(s). Thus an input impulse of voltage results in only a bounded current. Let K_2 be the maximum value of $i_1(t)$ on the interval $0 \leq t \leq T$. Then the energy input is

$$\begin{aligned} E_{in} &= \int_0^T e_1(t) i_1(t)\, dt \\ &\leq \lim_{a \to 0} \int_0^a \frac{K_1}{a} K_2\, dt = K_1 K_2 \quad . \end{aligned} \tag{32}$$

Thus, for this case, we see that an impulse is admissible in the optimum driving function because an impulse results in only a finite energy input to the network. From equation (31) we see, however, that if $e_1(t)$ contains a unit doublet or higher order function, then $i_1(t)$ will contain a unit impulse or higher order function, and just as in the previous cases the energy input to the network would be infinite.

Conclusions

Because of the above input energy considerations, we conclude that:

THEOREM 1 An optimum driving function can never be of higher order than a unit impulse;

and

THEOREM 2 A sufficient condition for the optimum driving function to be bounded is for $g_{11}(t)$, the input impulse response of the network, to contain a unit impulse or a unit doublet.

CHAPTER IV

EQUATIONS CHARACTERIZING SOLUTIONS TO THE OPTIMUM TRIGGERING SIGNAL PROBLEM FOR THE THREE CLASSES OF $g_{11}(t)$

4.1 General Considerations

The basic equation characterizing optimum driving signals for linear, lumped-constant networks was derived in Chapter II. This equation was

$$\int_0^T \left[\eta(t)\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau + e_1(t)\int_0^t \eta(\tau)g_{11}(t-\tau)d\tau - \lambda\eta(t)g_{12}(T-t)\right]dt = 0 \; . \tag{33}$$

It was also suggested in Chapter II that if this equation could be written in the form

$$\int_0^T \eta(t)f[e_1(t),g_{11}(t),g_{12}(t)]\; dt = 0 \; , \tag{34}$$

we could set

$$f[e_1(t),g_{11}(t),g_{12}(t)] = 0 \tag{35}$$

and obtain solutions independent of $\eta(t)$. Whether or not we can resolve equation (33) into the form of equation (34) is critically dependent on the nature of $g_{11}(t)$ and $e_1(t)$, i.e., in the treatment of integrals, if the integrands are continuous, then certain mathematical theorems apply. If, however, we have integrands with discontinuous, or perhaps even unbounded, integrands then we will have to be very careful in the treatment of them.

In Chapter III the nature of the functions $g_{11}(t)$ and $e_1(t)$ [factors in the integrands of equation (33)] were discussed and resolved into different categories. It was shown that there are definite restrictions on $g_{11}(t)$ resulting from the fact that this function has the nature of a driving point admittance. It was also shown that there are definite restrictions on $e_1(t)$ imposed by the nature of the problem, i.e., $e_1(t)$ is to be a minimum energy type signal.

Except for the class of networks which do not contain a unit impulse or a unit doublet in the input impulse response function, the optimum signal was shown to have a boundedness requirement. Incidentally, this boundedness requirement exists for any problems of practical significance, because one in general cannot realize practically a voltage generator without an internal resistance, and also, any real circuit will have some stray capacitance across the input terminals. This input configuration guarantees the appearance of an impulse of $g_{11}(t)$. Thus, in general we will consider $e_1(t)$ to be bounded except for pathological circuits. In addition, if $e_1(t)$ is finally to satisfy equation (33), then it must be capable of being written in terms of the bounded factors appearing in $g_{11}(t)$ and $g_{12}(t)$. But these bounded factors all have the form of equation (21) and are thus continuous, well-behaved functions in the interval $0 \leq t \leq T$. In equation (33) we will therefore consider $e_1(t)$ to be bounded and continuous in $0 \leq t \leq T$. Note, that for $g_{11}(t)$ continuous, an impulse is admissible in $e_1(t)$. Therefore, if we mechanically treat equation (33) under the assumption that $e_1(t)$ is continuous and later find a discontinuous solution, we must be suspicious of the validity of that solution and subject the particular problem to further investigation.

Finally, then, we will consider discontinuities in equation (33) as resulting only from discontinuities in $g_{11}(t)$ and $g_{12}(t)$. We now consider the three classes of $g_{11}(t)$ as separate cases.

4.2 Derivation of Characteristic Equation for $g_{11}(t)$ Continuous

Note that the first and third terms of equation (33) are already in the form of equation (34), but the second term is not. In the second term of equation (33), η appears inside the inner integral. The order of integration can be reversed if $e_1(t)$, $\eta(t)$, and $g_{11}(t)$ are continuous functions, allowing the second term on the left-hand side of equation (33) to be written in the form of equation (34). We first write

$$\int_0^T e_1(t)\int_0^t \eta(\tau)g_{11}(t-\tau)d\tau\ dt = \int_0^T\int_0^t e_1(t)\eta(\tau)g_{11}(t-\tau)d\tau\ dt \ . \quad (36)$$

This simply states that because the inner integration does not involve t, $e_1(t)$ can be taken inside. In this integration we are summing first in τ from 0 to t and then summing in t from 0 to T. We are thus integrating over a region defined by the shaded area in Fig. 5.

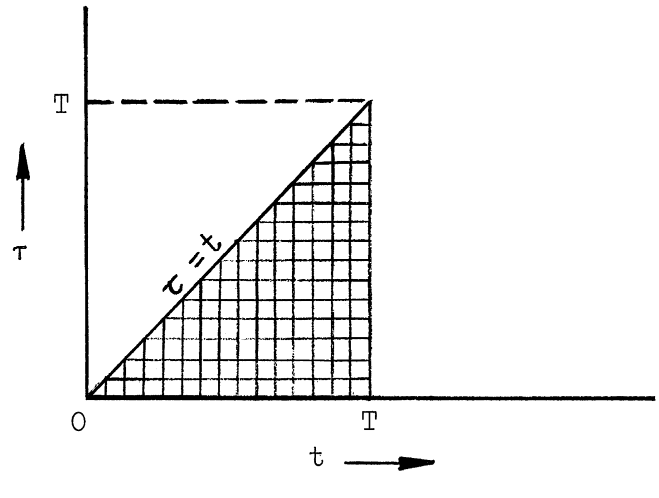

Fig. 5. Region of integration in the τ,t plane.

We could equally well reverse this order of integration by integrating first in t and then in τ. Note, however, that in order to do this we must change the limits of integration, because if we integrate first in t, t will vary from τ to T, and τ will then range from 0 to T (see reference 7, pp 131-136). Thus,

$$\int_0^T e_1(t)\int_0^t \eta(\tau)g_{11}(t-\tau)d\tau\, dt = \int_0^T\int_0^t e_1(t)\eta(\tau)g_{11}(t-\tau)d\tau\, dt$$

$$= \int_0^T\int_\tau^T e_1(t)\eta(\tau)g_{11}(t-\tau)dt\, d\tau$$

$$= \int_0^T \eta(\tau)\int_\tau^T e_1(t)g_{11}(t-\tau)dt\, d\tau \quad . \quad (37)$$

Finally, we can interchange the names of the variables of integration, so that the final form of equation (37) will be of the same form as the other two terms of equation (33). Thus equation (37) can be written

$$\int_0^T e_1(t)\int_0^t \eta(\tau)g_{11}(t-\tau)d\tau\, dt = \int_0^T \eta(t)\int_t^T e_1(\tau)g_{11}(\tau-t)d\tau\, dt \quad .(38)$$

Equation (33) finally becomes

$$\int_0^T \eta(t)\left[\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau + \int_t^T e_1(\tau)g_{11}(\tau-t)d\tau - \lambda g_{12}(T-t)\right]dt = 0. \quad (39)$$

This can be satisfied independent of η by setting

$$\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau + \int_t^T e_1(\tau)g_{11}(\tau-t)d\tau = \lambda g_{12}(T-t) \quad . \quad (40)$$

It is sometimes convenient, when attempting numerical solutions, to have the second integral of this equation expanded as follows:

$$\int_t^T e_1(\tau)g_{11}(\tau-t)d\tau = \int_0^T e_1(\tau)g_{11}(\tau-t)d\tau - \int_0^t e_1(\tau)g_{11}(\tau-t)d\tau \quad . \quad (41)$$

Equation (40) then becomes

$$\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau - \int_0^t e_1(\tau)g_{11}(\tau-t)d\tau + \int_0^T e_1(\tau)g_{11}(\tau-t)d\tau = \lambda g_{12}(T-t) \quad . \quad (42)$$

In general, if solutions of this equation for $e_1(t)$ exist, the solutions will be in terms of the network functions $g_{11}(t)$ and $g_{12}(t)$ and these solutions will be optimum in the sense as characterized by the statement of the problem. Note that if $e_1(t)$ is discontinuous, which is possible in this case, then equation (37) is not valid, and the problem requires special treatment (see Chapter VII, Example 2).

4.3 Derivation of the Characteristic Equation for $g_{11}(t)$ Containing a Unit Impulse Plus Continuous Terms

For this case we write

$$g_{11}(t) = k_1 u_1(t) + k_2 g_{11_c}(t) \quad , \tag{43}$$

where $g_{11_c}(t)$ contains only continuous terms. Using this expression in equation (33), we get

$$\int_0^T \left\{ \eta(t) \int_0^t e_1(\tau)\, [k_1 u_1(t-\tau) + k_2 g_{11_c}(t-\tau)]\, d\tau \right.$$

$$\left. + \, e_1(t) \int_0^t \eta(\tau)[k_1 u_1(t-\tau) + k_2 g_{11_c}(t-\tau)]d\tau - \lambda\eta(t) g_{12}(T-t) \right\} dt = 0 \quad . \tag{44}$$

We now write the above integrals of sums as the sum of integrals.

$$\int_0^T \left[\eta(t) \int_0^t e_1(\tau) k_1 u_1(t-\tau) d\tau + \eta(t) \int_0^t e_1(\tau) k_2 g_{11_c}(t-\tau) d\tau \right.$$

$$+ \, e_1(t) \int_0^t \eta(\tau) k_1 u_1(t-\tau) d\tau + e_1(t) \int_0^t \eta(\tau) k_2 g_{11_c}(t-\tau) d\tau$$

$$\left. - \, \lambda\eta(t) g_{12}(T-t) \right] dt = 0 \quad . \tag{45}$$

As in the previous case, we would like to write equation (45) in the form of equation (34), if possible. Except for the third and fourth terms of the integrand, the equation is already in the desired form. Now, the integrand of the fourth term is continuous, and we can therefore resolve this term into the desired form just as was done in Section 4.2. The fourth term then becomes

$$\int_0^T e_1(t) \int_0^t \eta(\tau)k_2 g_{11_c}(t-\tau)d\tau \, dt = \int_0^T \eta(t) \int_t^T e_1(\tau)g_{11_c}(\tau-t)k_2 d\tau dt . \quad (46)$$

Now, the third term contains a discontinuous function in the integrand; therefore, we cannot simply interchange the order of integration as in Section 4.2. Instead consider simply the integral

$$\int_0^t \eta(\tau)u_1(t-\tau)d\tau$$

using the definition for $u_1(t)$ as given by equation (27).

$$\int_0^t \eta(\tau)u_1(t-\tau)d\tau = \int_0^t \eta(\tau) \lim_{a\to 0} \frac{u(t-\tau) - u(t-\tau-a)}{a} d\tau$$

$$= \lim_{a\to 0} \frac{1}{a} \int_0^t \eta(\tau) \, [u(t-\tau) - u(t-\tau-a)]d\tau \quad . \quad (47)$$

The justification for taking the limit outside the integral is based upon an interpretation of the meaning of the limit. In a strict sense,

$$\lim_{a\to 0} \frac{u(t) - u(t-a)}{a}$$

does not exist. Equation (47), however, can be interpreted variously as the response to the limit of a function or the limiting response to a function. In the physical sense these two interpretations have the same meaning. We proceed then to the evaluation of equation (47). Notice that the integrand is zero except between $t - a \leq \tau \leq t$, and there it is just $\eta(\tau)$. Thus,

$$\int_0^t \eta(\tau)u_1(t-\tau)d\tau = \lim_{a\to 0} \frac{1}{a} \int_{t-a}^t \eta(\tau) \, d\tau \quad . \quad (48)$$

Now we use the mean value theorem of integral calculus, which states that if the integrand of an integral is continuous, then the integral is equal to the product of the length of the interval times the value of the function at some point within the interval. We have, therefore,

$$\int_0^t \eta(\tau)u_1(t-\tau)d\tau = \lim_{a\to 0}\frac{1}{a}\, a\eta(t - a + \Theta a) \quad , \tag{49}$$

where $0 \le \Theta \le 1$. Now if we take the limit we get

$$\int_0^t \eta(\tau)u_1(t-\tau)\, d\tau = \eta(t) \quad . \tag{50}$$

Finally, then, equation (45) becomes:

$$\int_0^T \eta(t)\Bigg[\int_0^t e_1(\tau)k_1u_1(t-\tau)d\tau + \int_0^t e_1(\tau)k_2g_{11_c}(t-\tau)d\tau$$

$$+ k_1e_1(t) + \int_t^T e_1(\tau)k_2g_{11_c}(\tau-t)d\tau - \lambda g_{12}(T-t)\Bigg]dt = 0 \ . \tag{51}$$

We can now satisfy this equation independent of η by setting

$$\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau + k_1e_1(t) + \int_t^T e_1(\tau)k_2g_{11_c}(\tau-t)d\tau = \lambda g_{12}(T-t). \tag{52}$$

Solutions of this equation for $e_1(t)$ will be in terms of only $g_{11}(t)$ and $g_{12}(t)$ and optimum in the sense as characterized by the statement of the optimum triggering signal problem.

4.4 Derivation of the Characteristic Equation for $g_{11}(t)$ Containing a Unit Doublet Plus Possibly a Unit Impulse and Continuous Terms

For this case, we write

$$g_{11}(t) = k_1u_2(t) + k_2u_1(t) + k_3g_{11_c}(t) \quad , \tag{53}$$

where $g_{11_c}(t)$ contains only terms continuous in t. Using this expression in equation (33) gives

$$\int_0^T \left[\eta(t) \int_0^t e_1(\tau) g_{11}(t-\tau) d\tau + e_1(t) \int_0^t \eta(\tau) k_1 u_2(t-\tau) d\tau \right.$$

$$+ e_1(t) \int_0^t \eta(\tau) k_2 u_1(t-\tau) d\tau + e_1(t) \int_0^t \eta(\tau) g_{11c}(t-\tau) d\tau$$

$$\left. - \lambda\eta(t) g_{12}(T-t) \right] dt = 0 \quad . \tag{54}$$

We now subject the third and fourth integrals to the same treatment as in Sections 4.2 and 4.3. This allows these terms to be written with $\eta(t)$ as a factor. However, we will now consider the second integral separately, using the definition for $u_2(t)$ as given by equation (28). Thus

$$\int_0^t \eta(\tau) u_2(t-\tau) d\tau = \int_0^t \eta(\tau) \lim_{a \to 0} \frac{1}{a^2} [u(t-\tau) - 2u(t-\tau-a) + u(t-\tau-2a)] d\tau$$

$$= \lim_{a \to 0} \frac{1}{a^2} \int_0^t \eta(\tau) [u(t-\tau) - 2u(t-\tau-a) + u(t-\tau-2a)] d\tau$$

$$= \lim_{a \to 0} \frac{1}{a^2} \left\{ \int_0^t \eta(\tau) \, [u(t-\tau) - u(t-\tau-a)] d\tau \right.$$

$$\left. - \int_0^t \eta(\tau) \, [u(t-\tau-a) - u(t-\tau-2a)] d\tau \right\} \quad . \tag{55}$$

Now, the integrand of the first integral on the right-hand side of equation (55) is zero except between $t - a \leq \tau \leq t$ and there it is $\eta(\tau)$. The second integrand is zero except in the interval $t - 2a \leq \tau \leq t - a$ and there it is likewise $\eta(\tau)$. We have, therefore,

$$\int_0^t \eta(\tau) u_2(t-\tau) d\tau = \lim_{a \to 0} \frac{1}{a^2} \left[\int_{t-a}^t \eta(\tau) d\tau - \int_{t-2a}^{t-a} \eta(\tau) d\tau \right] . \tag{56}$$

Just as in Section 4.3, we again use the mean value theorem and write

$$\int_0^t \eta(\tau) u_2(t-\tau) d\tau = \lim_{a \to 0} \frac{a\eta(t - a + \theta a) - a\eta(t - 2a - \phi a)}{a^2} \quad , \tag{57}$$

where $0 \leq \theta \leq 1$ and $0 \leq \phi \leq 1$. Therefore

$$\int_0^t \eta(\tau)u_2(t-\tau)d\tau = \lim_{a\to 0} \frac{\eta(t - a + \theta a) - \eta(t - 2a + \phi a)}{a} \quad . \qquad (58)$$

But this is the definition of the derivative of a function. Thus

$$\int_0^t \eta(\tau)u_2(t-\tau)d\tau = \frac{d\eta(t)}{dt} \quad . \qquad (59)$$

Therefore, we can finally write equation (54) as follows:

$$\int_0^T \left\{\eta(t)\left[\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau + k_2e_1(t) + k_3\int_t^T e_1(\tau)g_{11c}(\tau-t)d\tau - \lambda g_{12}(T-t)\right] + k_1e_1(t)\frac{d\eta(t)}{dt}\right\}dt = 0 \quad . \qquad (60)$$

In this case we see that the integrand of the characteristic equation cannot be written as the product of $\eta(t)$ times a function of only $e_1(t)$, $g_{11}(t)$, and $g_{12}(t)$ because of the appearance of $[d\eta(t)/dt]$. However, let us now set

$$f(t) = \int_0^t e_1(\tau)g_{11}(t-\tau)d\tau + k_2e_1(t) + k_3\int_t^T e_1(\tau)g_{11c}(\tau-t)d\tau - \lambda g_{12}(T-t) \quad . \qquad (61)$$

Equation (60) then takes the form:

$$\int_0^T \eta(t)f(t)dt + \int_0^T \frac{d\eta(t)}{dt} k_1e_1(t)dt = 0 \quad . \qquad (62)$$

We can now integrate the first integral of equation (62) by parts if f(t) is continuous. This gives:

$$\int_0^T \eta(t)f(t)dt = \left[\eta(t)\int f(t)dt\right]_0^T - \int_0^T \frac{d\eta(t)}{dt}\left[\int f(t)dt\right] dt \quad . \qquad (63)$$

Now, f(t) is the sum of a number of terms, each of which is a current flowing in response to a continuous function $e_1(t)$ plus terms in $g_{12}(T-t)$.

Therefore, $f(t)$ is continuous if $g_{12}(T-t)$ is continuous. Under these conditions, because $\eta(0) = \eta(T) = 0$, the first term on the right-hand side of equation (63) is zero. Thus, equation (62) becomes

$$\int_0^T \frac{d\eta(t)}{dt}\left[k_1 e_1(t) - \int f(t)dt\right] dt = 0 . \tag{64}$$

This equation can be satisfied independent of η by setting

$$k_1 e_1(t) - \int f(t)dt = 0 . \tag{65}$$

Finally, equation (65) can be differentiated with respect to t, giving:

$$k_1 \frac{de_1(t)}{dt} - \int_0^t e_1(\tau)g_{11}(t-\tau)d\tau - k_2 e_1(t) - k_3\int_t^T e_1(\tau)g_{11c}(\tau-t)d\tau + \lambda g_{12}(T-t) = 0 . \tag{66}$$

Thus if $g_{12}(T-t)$ is continuous in this last and final case, we can solve for $e_1(t)$ in terms of only $g_{11}(t)$ and $g_{12}(t)$. Moreover, the solution will be optimum in the sense as characterized by the statement of the optimum triggering signal problem.

CHAPTER V

METHODS OF SOLUTION OF THE CHARACTERISTIC EQUATIONS FOR THE THREE CLASSES OF $g_{11}(t)$

5.1 Introduction

Characteristic equations for the three classes of $g_{11}(t)$ were derived in Chapter IV. The nature of the characteristic equation was shown to depend intrinsically on the nature of $g_{11}(t)$. Thus there are three separate equations for the three classes of $g_{11}(t)$:

(1) $g_{11}(t)$ continuous;

(2) $g_{11}(t)$ contains an impulse plus continuous terms;

(3) $g_{11}(t)$ contains a unit doublet plus other terms of (2) above.

The characteristic equations for the above three cases are given respectively by equations (42), (52), and (66).

Several methods of solving the characteristic equations have been investigated. However, only one of these appears powerful enough to allow solutions for other than very simple cases. One rather crude method of solution, for some given $g_{11}(t)$ and $g_{12}(t)$, is that of examining the terms of the appropriate characteristic equation with the hope of being able to guess at the general form of the solution. If one could guess the general form of the solution in terms of undetermined coefficients, one could then substitute this form into the characteristic equation and determine the coefficients so as to satisfy the equation. This method of solution might work in some very simple cases. However, if $g_{11}(t)$ and $g_{12}(t)$ take on any complexity whatsoever this approach would break down because of the complexity of the characteristic equation itself.

A second approach to the solution of a characteristic equation would be that of attempting to reduce the integral equation to a differential equation. If this could be done, then the characteristic differential equation might be susceptible of solution by previously established methods. This approach is also fruitless as can be simply shown by differentiating some of the characteristic terms of the integral equations. To differentiate these terms one must use Leibnitz' rule, which is stated below.[7] Let

$$\Theta(\alpha) = \int_{u_o(\alpha)}^{u_1(\alpha)} f(x,\alpha)dx \quad ,$$

where u_o and u_1 are differentiable functions in a closed interval (α_o,α_1); $f(x,\alpha)$ and $f_\alpha(x,\alpha)$ are continuous in the region $\alpha_o \leq \alpha \leq \alpha_1$, $u_o(\alpha) \leq x \leq u_2(\alpha)$. Then,

$$\frac{d\Theta}{d\alpha} = \int_{u_o(\alpha)}^{u_1(\alpha)} \frac{\partial f(x,\alpha)}{\partial\alpha} dx - f(u_o,\alpha)\frac{du_o}{d\alpha} + f(u_1,\alpha)\frac{du_1}{d\alpha} \quad . \tag{67}$$

Consider now,

$$\int_o^t e_1(\tau)g_{11}(t-\tau)d\tau \quad ,$$

a typical term appearing in all three characteristic equations. Differentiating this term gives:

$$\frac{d}{dt}\int_o^t e_1(\tau)g_{11}(t-\tau)d\tau = \int_o^t e_1(\tau)\frac{\partial g_{11}(t-\tau)}{\partial t} d\tau + e_1(t)g_{11}(0) \quad . \tag{68}$$

Similarly, differentiation of the other integrals leads to new integrals. Thus, differentiation of the integral characteristic equations lead to new integral equations which are not inherently any simpler than the original equations. In fact, differentiation of the equations leads to differentiation of g_{11} and g_{12}, thus increasing the complexity. It can also be seen by inspection that repeated differentiation would only lead to equations involving higher order derivatives of g_{11} and g_{12}.

5.2 The Laplace Transform Method of Solution

One method of solution, however, which is sufficiently powerful to allow solutions, even in cases where g_{11} and g_{12} are complicated functions, is that of taking the Laplace transform of the characteristic equation. This reduces the original integral equation into an algebraic equation which is always susceptible of relatively simple solution. Note that there are two possible exceptions to the above statements. These are cases where $g_{11}(t)$ contains a unit doublet and, in addition, $g_{12}(t)$ is a discontinuous function, and the case where $e_1(t)$ is discontinuous. For these cases characteristic equations independent of η were not obtained. Methods of solution for these special cases will be discussed at the end of this chapter.

First, it will be noted that all characteristic equations contain only three different types of terms. These are:

(1) Direct functions of t;

(2) Convolution integrals of the form $\int_0^t f(\tau)g(t-\tau)d\tau$;

(3) Integrals of the form $\int_t^T e_1(\tau)g_{11_c}(\tau-t)d\tau$.

The Laplace transform of terms of the first type are taken directly. For terms of the second type, we use the theorem which states that "The Laplace transform of a convolution integral is the product of the Laplace transforms of the convoluted functions." Thus

$$\mathcal{L}\left[\int_0^t f(t)g(t-\tau)d\tau\right] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$$

$$= F(s)G(s) \quad . \qquad (69)$$

For terms of the third type we first expand the integral as the difference of two integrals as follows:

$$\int_t^T e_1(\tau)g_{11_c}(\tau-t)d\tau = \int_0^T e_1(\tau)g_{11_c}(\tau-t)d\tau - \int_0^t e_1(\tau)g_{11_c}(\tau-t)d\tau \quad , \quad (70)$$

and treat each of the resulting integrals separately. The second integral on the right-hand side of equation (70) is in the standard form of a convolution integral. In this integral, however, $e_1(t)$ is convoluted with $g_{11}(-t)$ and not with $g_{11}(t)$. Thus

$$\mathcal{L}\left[\int_0^t e_1(\tau)g_{11_c}(\tau-t)d\tau\right] = F(s) \cdot \mathcal{L}[g_{11}(-t)] \quad . \qquad (71)$$

The first integral on the right-hand side of equation (70) is not a convolution integral and must therefore be treated in a different manner. We recall now that $g_{11_c}(t)$ is a sum of terms of the form $t^n e^{at} \sin(\omega t + \Theta)$. Now, the integral of a sum can be written as the sum of integrals with each term treated independently. Moreover, each term will be of the same general form so that we need only treat one such term. We therefore consider:

$$\int_0^T e_1(\tau)(\tau-t)^n e^{a(\tau-t)} \sin(\omega\tau - \omega t + \Theta)\, d\tau \quad .$$

Now,

$$(\tau-t)^n e^{a(\tau-t)} \sin(\omega\tau - \omega t + \Theta) = \left[\tau^n - n\tau^{n-1}t + \frac{n(n-1)\tau^{n-2}t^2}{2!} + \ldots + t^n\right] x$$

$$e^{a\tau}e^{-at}\,[\sin(\omega\tau + \Theta)\cos\omega t - \cos(\omega\tau + \Theta)\sin\omega t] \quad . \qquad (72)$$

Notice that this expression contains only terms of the form:

$$K[t^k e^{-at} \sin(\omega t + \Theta)] \cdot [\tau^\ell e^{a\tau} \sin(\omega\tau + \beta)] \quad .$$

Hence, τ and t can be separated, resulting in the product of two functions, one containing only τ and the other containing only t. Thus,

$$\int_0^T e_1(\tau)(\tau-t)^n e^{a(\tau-t)} \sin(\omega\tau - \omega t + \theta)\, d\tau$$

can be written as the sum of terms each of which is a function of t times a definite integral integrated between constant limits (because the integration does not involve t). Each term of this sum is therefore some function of t times some (as yet undetermined) constant. Thus, using this expansion and factoring scheme, the Laplace transform of

$$\int_0^T e_1(\tau) g_{11_C}(\tau-t)\, d\tau$$

is readily obtained.

The methods outlined above can be readily applied to any of the characteristic equations, thus reducing the integral equations to algebraic equations. However, there are two special types of network for which characteristic equations are not obtainable. These are: (1) the type where $g_{11}(t)$ contains a unit doublet and $g_{12}(T-t)$ is not continuous, and (2) $e_1(t)$ is discontinuous. We will now take up the first special case.

We first examine the way in which a unit doublet gets into $g_{11}(t)$. It was shown earlier that the Laplace transform of this function has the nature of a driving point admittance, and can therefore be written as

$$g_{11}(s) = \frac{N(s)}{D(s)} , \qquad (73)$$

where N(s) differs in degree by no more than one from the degree of D(s). The appearance of a unit doublet in $g_{11}(t)$ results when N(s) is one degree larger than D(s), thus resulting in a pole at infinity. But a pole in admittance implies a zero of impedance. Now, except for short circuits, the only way of obtaining a short circuit at infinite frequency is for there to be a purely capacitive path between the input terminals, such as in Figs. 6 and 7.

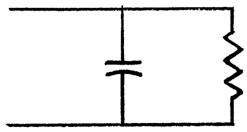

Fig. 6. Circuit containing a doublet in the input impulse response.

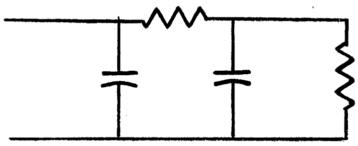

Fig. 7. Circuit containing a doublet in the input impulse response.

Note now that for the circuit of Fig. 6 the transfer impulse response, $g_{12}(t)$, contains an impulse. For the circuit of Fig. 7, however, g_{12} is continuous in $0 \leq t \leq T$. Thus, the characteristic equation (66) can be used to solve for the optimum signal for Fig. 7. Because $g_{12}(t)$ is discontinuous for Fig. 6, equation (66) cannot be applied.

Now in any practical problem one is never concerned with circuits having the nature of those of Fig. 6 or Fig. 7, because of the necessity of working only with voltage generators having an internal resistance. With practical generators, therefore, the circuit to be triggered is always characterized as having an input series resistance. But if the network contains a series input resistance then $g_{11}(t)$ cannot contain a unit doublet, because the resistance prevents the appearance of any zeros in the input impedance. Thus, in practical problems the question of the applicability of equation (66) does not arise.

The problem of obtaining the optimum driving function for a circuit similar to that of Fig. 6 is therefore a purely academic one. However, it still seems intuitively reasonable that an optimum signal exists. The discussion in the above paragraph suggests a method of solution for this case; i.e., we can assume first that there is a series resistance in the input impedance of the circuit. This allows us to use equation (52) to find the optimum signal for the modified circuit. This solution will, in general, involve the added input resistance. Having the solution, we can then let the input resistance tend toward zero and obtain the solution for the original unmodified circuit. An example illustrating this method is presented in Section 7.1.

The characteristic equation for the case where $g_{11}(t)$ is continuous was given by equation (42). This equation was derived assuming that $e_1(t)$ is continuous. However, it was shown in Chapter III that, for this case, discontinuous terms are admissible in $e_1(t)$. However, it was also shown that these terms could be of no higher order than a unit impulse because they would result in infinite energy input to the circuit. Thus, to begin with, an attempt can be made, using equation (42), to obtain a solution. If the solution should turn out to be nonsensical then it is obvious that the equation is not valid, and that in all probability the true solution contains a unit impulse. We can then go back and examine the behavior of the circuit when an impulse is applied and look for a solution by other means. An example illustrating this type of problem is presented in Section 7.2.

CHAPTER VI

AN ALTERNATIVE FORMULATION OF THE PROBLEM; THE OPTIMUM DRIVING CURRENT

6.1 General Discussion

In the previous discussion the optimum driving signal has been characterized as being generated by an ideal voltage generator. The question now naturally arises concerning what happens if, instead of an ideal voltage generator, we have an ideal current generator. That is, the optimum triggering signal problem can equally well be stated as that of finding the current which if applied to a given linear, lumped-constant circuit results in some specified output voltage at time T, such that the energy delivered to the network is minimum.

For the optimum current case we proceed in a manner analogous to that for the optimum voltage case. However, we now describe the circuit in terms of the responses to an input impulse of current rather than an input impulse of voltage. Thus, let $z_{11}(t)$ be the voltage across the input terminals of the circuit which results from a unit impulse of current flowing into the input terminals, and let $z_{12}(t)$ be the voltage across the output terminals of the network which results from a unit impulse of current flowing into the input terminals. With these definitions we can write:

$$e_1(t) = \int_0^t i_1(\tau)\, z_{11}(t-\tau)d\tau \quad , \tag{74}$$

and

$$e_2(t) = \int_0^t i_1(\tau)\, z_{12}(t-\tau)d\tau \quad , \tag{75}$$

where $i_1(t)$ is any input current, $e_1(t)$ is the input voltage resulting from $i_1(t)$, and $e_2(t)$ is the output voltage resulting from $i_1(t)$. We now want to specify that the output voltage at time $t = T$ is some given constant. Thus,

$$e_2(T) = E_c = \int_0^T i_1(\tau)z_{12}(T-\tau)d\tau \quad . \tag{76}$$

In equation (76) we now change the variable of integration from τ to t and obtain:

$$e_2(T) = E_c = \int_0^T i_1(t)z_{12}(T-t)dt \quad . \tag{77}$$

Now the energy input to the network is given by:

$$\begin{aligned} E(t_1) &= \int_0^{t_1} P_{in}\, dt \\ &= \int_0^{t_1} e_1(t)i_1(t)dt \\ &= \int_0^{t_1} i_1(t) \int_0^t i_1(\tau)z_{11}(t-\tau)d\tau\, dt \quad . \end{aligned} \tag{78}$$

The final total energy input in the interval $0 \leq t \leq T$ is then

$$E(T) = \int_0^T i_1(t) \int_0^t i_1(\tau)z_{11}(t-\tau)d\tau\, dt \quad . \tag{79}$$

Expressed mathematically, then, the optimization problem becomes that of minimizing the integral

$$E(T) = \int_0^T i_1(t) \int_0^t i_1(\tau)z_{11}(t-\tau)d\tau\, dt \quad ,$$

subject to the constraint that

$$e_2(T) = E_c = \int_0^T i_1(t)z_{11}(T-t)dt \quad .$$

Therefore, it can be seen that there is exact symmetry between the optimum triggering voltage and the optimum triggering current problems, and that the mathematical statements of the two problems are identical except for the change in symbols. This implies that the two mathematical problems are one and the same, and that the statements regarding the optimum $e_1(t)$ in relation to $g_{11}(t)$, $g_{12}(t)$, and $i_2(T)$ would be exactly those regarding the optimum $i_1(t)$ if the same conditions existed in $z_{11}(t)$ as in $g_{11}(t)$, in $z_{12}(t)$ as in $g_{12}(t)$, and in $e_2(T)$ as in $i_2(T)$. Thus we can simply restate the results of the preceding chapters in terms of the optimum driving current and the network functions $z_{11}(t)$ and $z_{12}(t)$.

6.2 General Equation Characterizing Optimum Driving Currents

This is equation (15) written in new symbols. Thus,

$$0 = \int_0^T \eta(t) \int_0^t i_1(\tau) z_{11}(t-\tau) d\tau \, dt + \int_0^T i_1(t) \int_0^t \eta(\tau) z_{11}(t-\tau) d\tau \, dt - \lambda \int_0^T \eta(t) z_{12}(T-t) dt \quad . \tag{80}$$

6.3 Restrictions on the Nature of $z_{11}(t)$ for Lumped-Constant, Linear Networks

Because driving point impedances have the same general characteristics as driving point admittances we can use the conclusions of Chapter III but stated now relative to $z_{11}(t)$ instead of $g_{11}(t)$. Thus, we can characterize $z_{11}(t)$ as belonging to one of three possible classes. These are:

(1) $z_{11}(t)$ contains neither a unit doublet nor a unit impulse;

(2) $z_{11}(t)$ contains no unit doublet, but does contain a unit impulse;

(3) $z_{11}(t)$ contains a unit doublet.

6.4 Nature of the Optimum Triggering Current for the Three Classes of $z_{11}(t)$

The conclusions of Chapter III, stated in terms of optimum driving currents, are:

> THEOREM 1' An optimum driving current can never be of higher order than a unit impulse;

and

> THEOREM 2' A sufficient condition for the optimum driving current to be bounded is for $z_{11}(t)$, the input impulse response of the network, to contain a unit impulse or a unit doublet.

6.5 Equations Characterizing Solutions to the Optimum Triggering Current Problem for the Three Classes of $z_{11}(t)$

Case 1: $z_{11}(t)$ Continuous and $i_1(t)$ Continuous.—Equation (42) of Chapter IV becomes

$$\int_0^t i_1(\tau)z_{11}(t-\tau)d\tau - \int_0^t i_1(\tau)z_{11}(\tau-t)d\tau + \int_0^T i_1(\tau)z_{11}(\tau-t)d\tau = \lambda z_{12}(T-t) \ . \tag{81}$$

Case 2: $z_{11}(t)$ Contains a Unit Impulse Plus Continuous Terms.—Equation (52) of Chapter IV becomes

$$\int_0^t i_1(\tau)z_{11}(t-\tau)d\tau + k_1 i_1(t) + \int_t^T i_1(\tau)z_{11_c}(\tau-t)d\tau = \lambda z_{12}(T-t) \ . \tag{82}$$

Case 3: $z_{11}(t)$ Contains a Unit Doublet Plus Possibly a Unit Impulse and Continuous Terms.—If $z_{12}(T-t)$ is bounded, then equation (66) of Chapter IV becomes

$$k_1 \frac{di_1(t)}{dt} - \int_0^t i_1(t)z_{11}(t-\tau)d\tau - k_2 i_1(t) - k_3\int_t^T i_1(\tau)z_{11_c}(\tau-t)d\tau + \lambda z_{12}(T-t) = 0 \ . \tag{83}$$

Finally, in addition to being able to translate the optimum driving voltage equations into optimum driving current equations, we can also use the same Laplace transform method of obtaining solutions to particular equations. This method was discussed in Chapter V.

CHAPTER VII

EXAMPLES DEMONSTRATING SOLUTIONS TO PARTICULAR PROBLEMS

The purpose of this chapter is to demonstrate the applicability to particular problems of the theory contained in the preceding chapters. Many notions of practical significance are difficult to discuss in general terms. One such notion, for example, is that of the minimum energy requirement necessary to cause triggering. One can only examine a minimum energy requirement after having found an optimum driving function, and, of course, one cannot find such a function unless g_{11} and g_{12} are specified. One particularly interesting facet of the optimum triggering signal problem is the study of the relationship between energy required and the time allowed for triggering. In the following examples solutions to particular problems will be found, and the energy input to the network for the optimum signal will be discussed. In some cases solutions will be found in terms of some particular parameter of interest such as time allowed for triggering, or, perhaps, some particular circuit element of interest.

7.1 Example 1

Consider the circuit shown in Fig. 8.

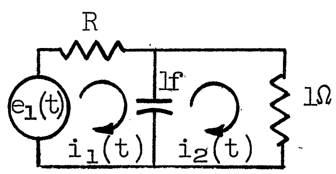

Fig. 8. Circuit of Example 1.

Let us now determine $e_1(t)$ so that the energy delivered to the circuit is minimum and so that $i_2(T)$ has some given value. In this problem we will leave T unspecified and attempt to determine $e_1(t)$ in terms of T. If this is possible, we will finally be able to study the dependency of the minimum triggering energy on T. In addition, we will also leave R unspecified so as to demonstrate a solution to the type of problem in which $g_{11}(t)$ contains a unit doublet and $g_{12}(t)$ is unbounded. For this particular type of problem no characteristic equation was obtained, but a method of solution was suggested in Chapter V. If R = 0 in Fig. 8 then the circuit becomes that of the above indeterminate type. Thus, we will find a solution in terms of R and then let R go to zero as was suggested in Chapter V.

We proceed first to a determination of $g_{11}(t)$ and $g_{12}(t)$ for the circuit of Fig. 8. To do this, we first let $e_1(t)$ be a unit impulse. Now, if $e_1(t)$ is a unit impulse, then $i_1(t)$ and $i_2(t)$ are, by definition, $g_{11}(t)$ and $g_{12}(t)$, respectively. Thus, the Laplace transformed equations for the circuit are:

$$g_{11}(s)\left(R + \frac{1}{s}\right) - g_{12}(s)\,\frac{1}{s} = 1 \tag{84}$$

$$-\,g_{11}(s)\,\frac{1}{s} + g_{12}(s)\left(1 + \frac{1}{s}\right) = 0 \ . \tag{85}$$

These equations have the solutions:

$$g_{11}(s) = \frac{1}{R}\,\frac{s + 1}{s + \frac{R+1}{R}} = \frac{1}{R} - \frac{1}{R^2}\,\frac{1}{s + \frac{R+1}{R}} \ , \tag{86}$$

and

$$g_{12}(s) = \frac{1}{R}\,\frac{1}{s + \frac{R+1}{R}} \ . \tag{87}$$

Taking the inverse Laplace transforms of equations (86) and (87) gives

$$g_{11}(t) = \frac{1}{R} u_1(t) - \frac{1}{R^2} e^{-\frac{R+1}{R}t} , \qquad (88)$$

and

$$g_{12}(t) = \frac{1}{R} e^{-\frac{R+1}{R}t} . \qquad (89)$$

Thus, we see that this problem is of the type discussed in Section 4.3; i.e., $g_{11}(t)$ contains a unit impulse plus continuous terms. We therefore want to use characteristic equation (52) for this problem. Thus, $e_1(t)$ must satisfy

$$\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau + k_1e_1(t) + \int_t^T e_1(\tau)k_2g_{11_c}(\tau-t)d\tau = \lambda g_{12}(T-t) \quad . \qquad (90)$$

In this case it is convenient to have the second integral of equation (90) expanded into the sum of two integrals. Thus, we rewrite equation (90) as

$$\int_0^t e_1(\tau)g_{11}(t-\tau)d\tau + k_1e_1(t) - \int_0^t e_1(\tau)g_{11_c}(\tau-t)k_2d\tau + \int_0^T e_1(\tau)k_2g_{11_c}(\tau-t)d\tau$$

$$= \lambda g_{12}(T-t) \quad . \qquad (91)$$

We now take the Laplace transform of equation (91) and obtain:

$$e_1(s)\left(\frac{1}{R} - \frac{1}{R^2}\frac{1}{s+\frac{R+1}{R}} + \frac{1}{R} + \frac{1}{R^2}\frac{1}{s-\frac{R+1}{R}}\right) - \mathcal{L}\left(e^{\frac{R+1}{R}t}\int_0^T \frac{1}{R^2} e_1(\tau)e^{-\frac{R+1}{R}\tau}d\tau\right)$$

$$= \mathcal{L}\left(e^{\frac{R+1}{R}t}\frac{\lambda}{R}e^{-\frac{R+1}{R}T}\right) . \qquad (92)$$

This reduces to:

$$e_1(s)\left[\frac{s^2-\left(\frac{R+1}{R}\right)^2+\left(\frac{R+1}{R^2}\right)}{s^2-\left(\frac{R+1}{R}\right)^2}\right] = \frac{\lambda' R}{2}\,\frac{1}{s-\frac{R+1}{R}}\;, \qquad (93)$$

where

$$\lambda' = \frac{\lambda}{R}\,e^{-\frac{R+1}{R}T} + \frac{1}{R^2}\int_0^T e_1(\tau)e^{-\frac{R+1}{R}\tau}\,d\tau\;. \qquad (94)$$

Whence,

$$e_1(s) = \frac{\lambda' R}{2}\,\frac{s+\frac{R+1}{R}}{s^2-\left(\frac{R+1}{R}\right)^2+\left(\frac{R+1}{R^2}\right)}\;. \qquad (95)$$

Now, the right-hand side of equation (95) can be expanded into partial fractions, giving

$$e_1(s) = \frac{\lambda' R}{2}\left[\frac{\frac{1}{2}+\frac{\frac{R+1}{2R}}{\sqrt{\left(\frac{R+1}{R}\right)^2-\left(\frac{R+1}{R^2}\right)}}}{s-\sqrt{\left(\frac{R+1}{R}\right)^2-\left(\frac{R+1}{R^2}\right)}}+\frac{\frac{1}{2}-\frac{\frac{R+1}{2R}}{\sqrt{\left(\frac{R+1}{R}\right)^2-\left(\frac{R+1}{R^2}\right)}}}{s+\sqrt{\left(\frac{R+1}{R}\right)^2-\left(\frac{R+1}{R^2}\right)}}\right]. \qquad (96)$$

Finally, taking the inverse Laplace transformation of equation (96) gives

$$e_1(t) = \frac{\lambda' R}{2}\left[\left(\frac{1}{2}+\frac{\frac{R+1}{2R}}{\sqrt{\left(\frac{R+1}{R}\right)^2-\left(\frac{R+1}{R^2}\right)}}\right)e^{\sqrt{\left(\frac{R+1}{R}\right)^2-\left(\frac{R+1}{R^2}\right)}\,t}\right.$$

$$\left.+\left(\frac{1}{2}-\frac{\frac{R+1}{2R}}{\sqrt{\left(\frac{R+1}{R}\right)^2-\left(\frac{R+1}{R^2}\right)}}\right)e^{-\sqrt{\left(\frac{R+1}{R}\right)^2-\left(\frac{R+1}{R^2}\right)}\,t}\right]. \qquad (97)$$

This can also be written

$$e_1(t) = \frac{\lambda' R}{2}\left[\cosh\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}\; t + \right.$$

$$\left. + \frac{\frac{R+1}{R}}{\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}} \sinh\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}\, t\right] . \quad (98)$$

Now, $e_1(t)$, as given by either equation (97) or equation (98), still has the undetermined multiplier λ'. We now solve for $i_2(t)$ and then determine λ' in terms of $i_2(T)$. The Laplace transform of $i_2(t)$ is given by

$$i_2(s) = e_1(s)g_{12}(s) = \frac{\lambda'}{2}\,\frac{1}{s^2 - \left(\frac{R+1}{R}\right)^2 + \left(\frac{R+1}{R^2}\right)} . \quad (99)$$

Now, this can be expanded into partial fractions, giving

$$i_2(s) = \frac{\lambda'}{2}\left[\frac{\dfrac{1}{2\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}}}{s - \sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}} - \frac{\dfrac{1}{2\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}}}{s + \sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}}\right] . \quad (100)$$

The inverse Laplace transformation of equation (100) evaluated at $t = T$ is

$$i_2(T) = \frac{\lambda'}{2}\left[\frac{e^{\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}\,T}}{2\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}} - \frac{e^{-\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}\,T}}{2\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}}\right]$$

$$= \frac{\lambda'}{2\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}} \sinh\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}\; T . \quad (101)$$

Thus,

$$\lambda' = \frac{2\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}\, i_2(T)}{\sinh\sqrt{\left(\frac{R+1}{R}\right)^2 - \left(\frac{R+1}{R^2}\right)}\, T} . \quad (102)$$

Finally, using this value for λ' in equation (98) and factoring (R+1)/R out of all the radicals, we obtain:

$$e_1(t) = \frac{(R+1)\sqrt{1-\frac{1}{R+1}}\; i_2(T)}{\sinh \frac{R+1}{R}\sqrt{1-\frac{1}{R+1}}\; T}\left[\cosh \frac{R+1}{R}\sqrt{1-\frac{1}{R+1}}\; t + \frac{1}{\sqrt{1-\frac{1}{R+1}}}\sinh \frac{R+1}{R}\sqrt{1-\frac{1}{R+1}}\; t\right]. \tag{103}$$

Let us now consider the behavior of equation (103) as R tends toward zero. If R becomes very small compared to unity, then:

$$e_1(t) \to \frac{i_2(T)}{e^{T/R}}\left[e^{t/R}\right] = i_2(T)e^{(t-T)/R} . \tag{104}$$

Now, as R tends toward zero the above function will be very small for $t < T$ because the exponent will tend towards minus infinity. However, as t approaches T the numerator of the exponent also tends toward zero, and, in the limit, if t tends toward T at the same time as R tends toward zero, the function becomes exactly zero for $t < T$, but indeterminant at $t = T$. Equation (104) therefore suggests that the optimum driving function is of such a nature that it is zero except for the time $t = T$. The functions which have this nature are unit steps, impulses, doublets, etc. In addition we know that for finite energy input, impulses and higher order functions are not admissible. Let us now go back and examine the physical circuit of Fig. 8 when $R = 0$. It becomes quite clear that the optimum signal in this case is indeed a unit step. The voltage across the load exactly equals the driving voltage, and thus a unit step starting at $t = T$ produces an instantaneous voltage across the load. Moreover, this voltage across the resistive load exists for zero time in the interval $0 \leq t \leq T$; therefore, there is no energy dissipated in the load resistance. Also, the nature of the circuit and the problem dictates that at time $t = T$ a voltage will appear across the capacitor. Thus, if $i_2(T) = 1$, then the energy delivered to the circuit is given by equation (105).

$$\text{Energy} = \frac{1}{2} C E^2 = \frac{1}{2} \text{ joule} . \quad (105)$$

Let us now return to equation (103), where $R \neq 0$, and determine the energy input to the network when driven by this optimum function. Also, in order to simplify the mathematics let us consider the network when $R = 1$. For this case, $e_1(t)$ becomes:

$$e_1(t) = \frac{\sqrt{2}\, i_2(T)}{\sinh \sqrt{2}\, T} (\cosh \sqrt{2}\, t + \sqrt{2} \sinh \sqrt{2}\, t)$$

$$= \frac{\sqrt{2}/2\, i_2(T)}{\sinh \sqrt{2}\, T} \left[(1 + \sqrt{2})\, e^{\sqrt{2}\, t} + (1 - \sqrt{2})\, e^{-\sqrt{2}\, t} \right] . \quad (106)$$

We now determine $i_1(t)$. The Laplace transform of $i_1(t)$ is given by

$$i_1(s) = e_1(s) g_{11}(s)$$

$$= \frac{\sqrt{2}/2\, i_2(T)}{\sinh \sqrt{2}\, T} \left(\frac{1 + \sqrt{2}}{s - \sqrt{2}} + \frac{1 - \sqrt{2}}{s + \sqrt{2}} \right) \left(\frac{s + 1}{s + 2} \right) . \quad (107)$$

After combining terms and performing the indicated multiplications, this equation can be expanded into partial fractions, giving

$$i_1(s) = \frac{i_2(T)}{2 \sinh \sqrt{2}\, T} \left[\frac{\sqrt{2} + 1}{s - \sqrt{2}} + \frac{\sqrt{2} - 1}{s + \sqrt{2}} \right] . \quad (108)$$

Taking the inverse Laplace transform of equation (108) gives

$$i_1(t) = \frac{i_2(T)}{2 \sinh \sqrt{2}\, T} \left[(\sqrt{2} + 1)\, e^{\sqrt{2}\, t} + (\sqrt{2} - 1)\, e^{-\sqrt{2}\, t} \right] . \quad (109)$$

We can now find the energy input to the network as follows:

$$E = \int_0^T e_1(t) i_1(t) dt$$

$$= \frac{\sqrt{2}\, i_2^2(T)}{4 \sinh^2 \sqrt{2}\, T} \int_0^T \left[(\sqrt{2}+1)^2 e^{2\sqrt{2}\, t} - (\sqrt{2}-1)^2 e^{-2\sqrt{2}\, t} \right] dt$$

$$= \frac{i_2^2(T)}{8 \sinh^2 \sqrt{2}\, T} \left[(\sqrt{2}+1)^2 e^{2\sqrt{2}\, t} + (\sqrt{2}-1)^2 e^{-2\sqrt{2}\, t} \right]_0^T$$

$$= \frac{i_2^2(T)[(3 + 2\sqrt{2})\, e^{2\sqrt{2}\, T} + (3 - 2\sqrt{2})\, e^{-2\sqrt{2}\, T} - 6]}{2(e^{2\sqrt{2}\, T} + e^{-2\sqrt{2}\, T} - 2)} \quad . \qquad (110)$$

We can now examine this energy as a function of T, the time allowed for triggering. In particular, we are interested in determining T so as to minimize the energy requirement. To do this we take the derivative of E with respect to T. Since $i_2(T)$ is constant, we have

$$\frac{dE}{dT} = \frac{-8 i_2^2(T)}{\sqrt{2}\,(e^{2\sqrt{2}\, T} - 2 + e^{-2\sqrt{2}\, T})} \quad . \qquad (111)$$

Now this function is zero only at infinity. Moreover, we see [from equation (110)] that as T goes to infinity E approaches a finite limit. Also it can be shown that this limit is not a maximum, and thus minimum energy results when T is infinite. Thus,

$$E_{min} = i_2^2(T)\ (3/2 + \sqrt{2}) \quad . \qquad (112)$$

A plot of equation (110) appears in Fig. 9. From Fig. 9 we see that the function has essentially reached its minimum value by T = 2 and, therefore, there is no advantage in allowing more than 2 seconds for the triggering to take place. If we were able to choose T = 2, then the optimum driving function would become

$$e_1(t) = 0.0835\ i_2(T)\ (\cosh \sqrt{2}\, t + \sqrt{2} \sinh \sqrt{2}\, t) \quad . \qquad (113)$$

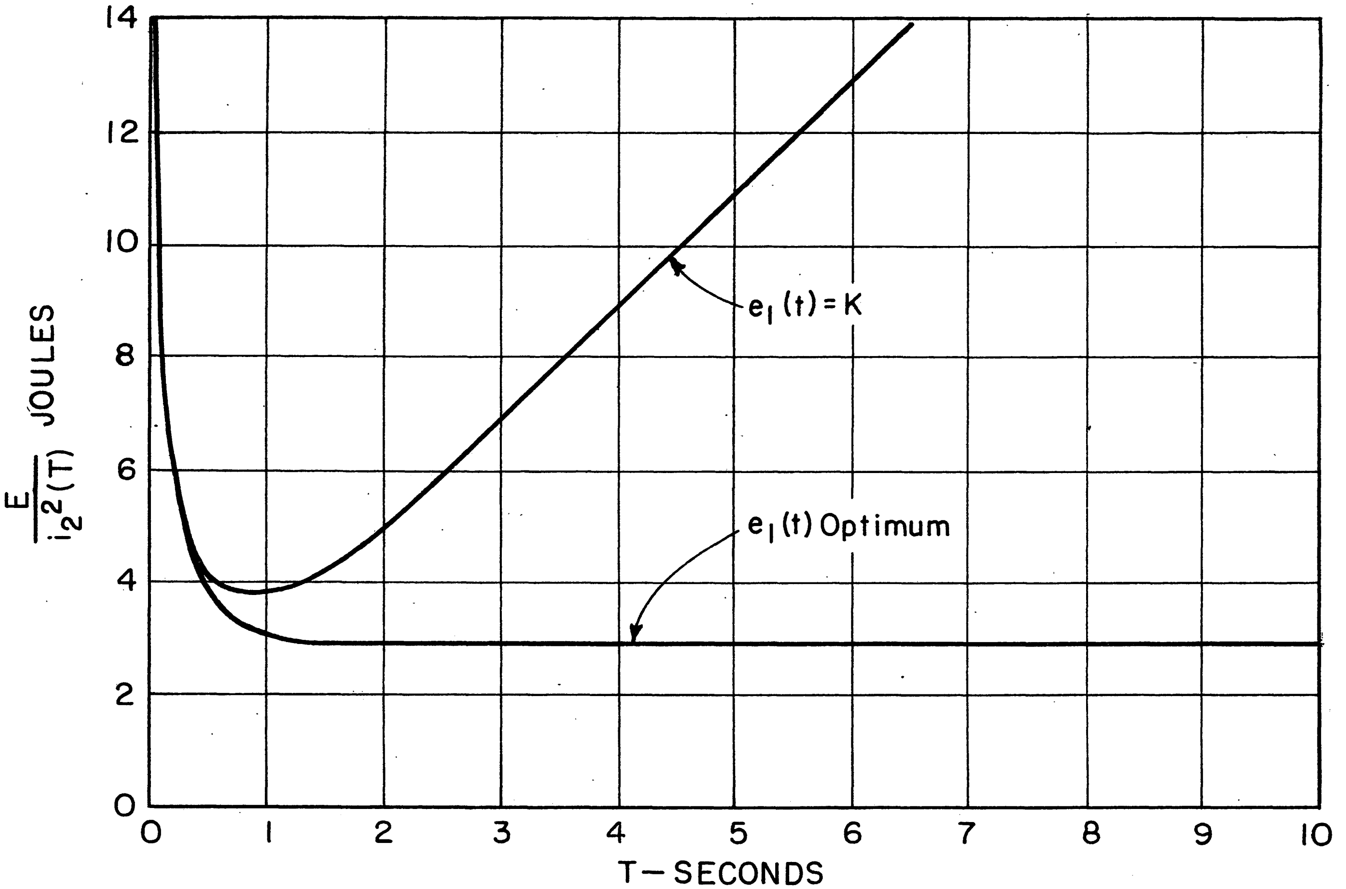

Fig. 9. $E/i_2^2(T)$ vs T.

One other interesting result of this problem is that the energy required varies directly with the square of $i_2(T)$. Thus, if anything can be done to reduce the $i_2(T)$ requirement, a considerable saving in the input energy requirement might result.

It is interesting now to compare the energy required of the optimum triggering signal with that required of a more familiar type of signal. For the sake of comparison, let us examine the energy input of a step of voltage of sufficient amplitude to cause triggering at time $t = T$. For this case, we let

$$e_1(t) = k \quad . \tag{114}$$

Now,

$$\begin{aligned} i_2(s) &= g_{12}(s)\, e_1(s) \\ &= \frac{k}{s(s+2)} \quad . \end{aligned} \tag{115}$$

This can be expanded into partial fractions, giving

$$i_2(s) = \frac{k}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right] . \tag{116}$$

The inverse Laplace transform of equation (116) is

$$i_2(t) = \frac{k}{2}\left[1 - e^{-2t}\right] \quad . \tag{117}$$

Thus,

$$i_2(T) = \frac{k}{2}\left[1 - e^{-2T}\right] \quad , \tag{118}$$

from which we get:

$$e_1(t) = k = \frac{2i_2(T)}{1 - e^{-2T}} \quad . \tag{119}$$

Now,

$$i_1(s) = g_{11}(s)e_1(s)$$

$$= \frac{k(s + 1)}{s(s + 2)} \quad . \tag{120}$$

Equation (120) can be expanded into partial fractions, giving

$$i_1(s) = \frac{k}{2}\left[\frac{1}{s} + \frac{1}{s + 2}\right] \quad . \tag{121}$$

The inverse Laplace transform of equation (121) is

$$i_1(t) = \frac{k}{2}\left[1 + e^{-2t}\right] = \frac{i_2(T)(1 + e^{-2t})}{1 - e^{-2T}} \quad . \tag{122}$$

For this signal the energy input to the network is the integral between 0 and T of the product of $e_1(t)$ and $i_1(t)$, or

$$E = \frac{2i_2^2(T)}{(1 - e^{-2T})^2}\int_0^T (1 + e^{-2t})dt = \frac{i_2^2(T)(2T + 1 - e^{-2T})}{(1 - e^{-2T})^2} \quad . \tag{123}$$

This equation is also plotted in Fig. 9. From this figure it can be seen that the optimum triggering signal always requires less energy than a step of voltage.

It is also seen, however, that for T small the energy required of the step function approaches that required of the optimum function. This behavior results because $e_1(t)$ is a sum of exponentials which is not equal to zero at $t = 0$. Thus, for t or T small the optimum function is very similar to a step function. For this reason, if separate considerations should dictate that the allowable triggering time is small, then there is little advantage to the optimum signal and one would expend very little more energy by using a step function approximation. It should be remembered, however, that the above statements apply to a particular example and are not necessarily true for some other circuit to be triggered.

7.2 Example 2

The previous example demonstrated the solution to a problem for which there was no characteristic equation. There is also another type of circuit, with a different behavior, for which there is no characteristic equation. This second type of problem results when $g_{11}(t)$ is continuous and where $e_1(t)$ may be discontinuous. It was suggested at the end of Chapter V that if $g_{11}(t)$ is continuous we can try using equation (42). If we get a discontinuous answer then the answer may or may not be valid because equation (42) is not valid for $e_1(t)$ discontinuous. However, equation (42) is only invalid for $e_1(t)$ discontinuous; therefore, we can suspect the appearance of an impulse in $e_1(t)$. The following example illustrates this type of difficulty. Consider the circuit shown in Fig. 10.

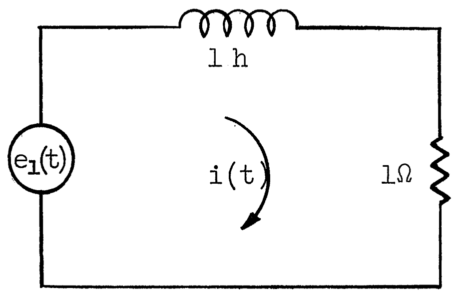

Fig. 10. Circuit to be triggered.

We proceed first to the determination of $g_{11}(t)$ and $g_{12}(t)$. In this case

$$g_{11}(t) = g_{12}(t) \quad . \tag{124}$$

The Laplace transformed equation for $g_{11}(t)$ is

$$g_{11}(s)(s + 1) = 1 \quad . \tag{125}$$

Therefore,

$$g_{11}(s) = \frac{1}{s + 1} \quad . \tag{126}$$

Taking the inverse Laplace transform of equation (126) gives

$$g_{11}(t) = e^{-t} . \tag{127}$$

Using this expression in equation (42) finally results in

$$e_1(s)\left(\frac{1}{s+1} - \frac{1}{s-1}\right) = \frac{\lambda e^{-T} - K}{s-1} , \tag{128}$$

where

$$K = \int_0^T e_1(\tau)\, e^{-\tau} d\tau . \tag{129}$$

Solving finally for $e_1(s)$ we have

$$e_1(s) = -(s+1)\,\frac{\lambda e^{-T} - K}{2} . \tag{130}$$

Now, the inverse Laplace transform of equation (130) is

$$e_1(t) = -\frac{\lambda e^{-T} - K}{2}\,[u_2(t) + u_1(t)] . \tag{131}$$

We know that an optimum triggering function cannot contain terms of higher order than a unit impulse and thus the solution contained here is nonsensical. This means that $e_1(t)$ must contain a unit impulse. In fact, we can see that an impulse is the optimum function. An impulse of properly chosen amplitude will result in the desired triggering current in zero time. Moreover, it is obvious that a step of current beginning at $t = T$ will result in zero energy dissipation in the interval $0 \leq t \leq T$, and the only energy delivered to the circuit is stored in the inductance. This energy is

$$E_{in} = \frac{1}{2} L\, i_2^2(T) . \tag{132}$$

For this simple case then, we see that a solution is obtainable. However, the solution was obtained through other than the theory contained in this paper. In some more complicated examples the theory also breaks down, and in some cases one is not able to guess at the answer as in the case above. It might be emphasized again, however, that $g_{11}(t)$ is continuous only in circuits without practical significance. It is seen by inspection that $g_{11}(t)$ is continuous only when all of the input current to the circuit is forced to flow through inductive paths. In practical problems one can use only voltage generators with internal resistance, and, in general, the circuit to be driven can always be considered as containing a stray capacitance across the input terminals. For practical circuits, therefore, the input impulse response contains a unit impulse and the characteristic equation (52) should be used. Note also that equation (52) is always valid.

Thus, for $g_{11}(t)$ continuous, one cannot always obtain solutions. However, one can obtain as good an approximation to the solution as is desired by simply adding an arbitrarily large but finite shunt resistance to ground across the input terminals, thus creating an impulse in the input impulse response. Because the true solution is unbounded, however, one cannot, after obtaining a solution with the added resistance, let the resistance go to infinity and expect the limit of the solution to approach anything which is recognizable. Therefore, a solution only has meaning for the input resistance finite. Obviously, a resistance of 1000^{1000} is finite and for all practical purposes one would not be able to determine its existance. A resistance of this magnitude, however, placed across the input terminals of a circuit, with $g_{11}(t)$ continuous, allows solution for the optimum triggering signal.

7.3 Example 3

Having discussed two examples with pathological circuits, we will now spend the remainder of this chapter with a meaningful circuit of practical significance, i.e., a circuit containing an impulse but not a doublet in the input impulse response.

A type of problem which might be of practical interest is one involving vacuum tubes. Let us now consider the optimum driving voltage to be applied to a single stage RC coupled amplifier whose output is used to trigger a biased thyratron. For purposes of simplicity we will use simple values for the circuit constants. Consider the circuit of Fig. 11.

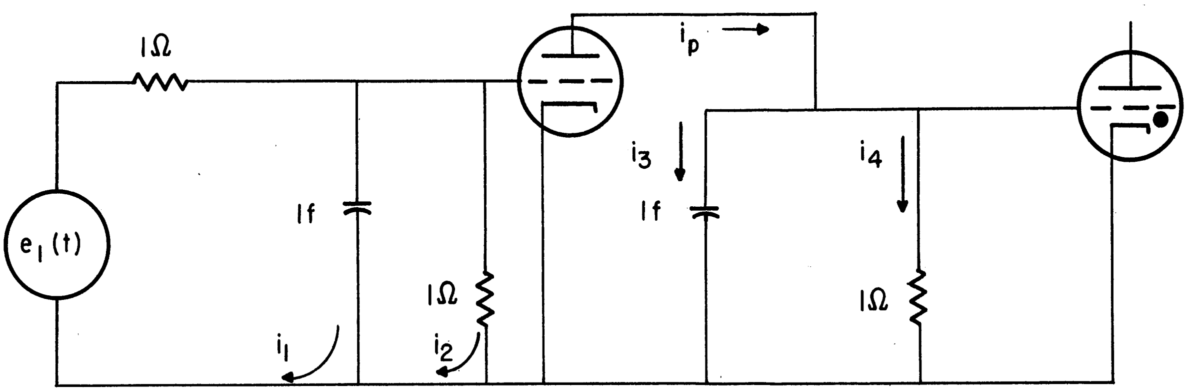

Fig. 11. Circuit to be triggered, containing an RC coupled amplifier and a biased thyratron.

In this case we wish to establish some critical $i_4(t)$ at some given time t = T with a minimum of energy delivered to the input circuit by the generator $e_1(t)$. We proceed first to the determination of the input impulse response, $g_{11}(t)$, and the transfer impulse response, $g_{12}(t)$. The transfer current in this case is $i_4(t)$. For $e_1(t)$ equal to a unit impulse, we have

$$i_1(s)(1 + 1/s) - i_2(s)\,1/s = 1 \tag{133}$$

$$-i_1(s)\,1/s + i_2(s)(1 + 1/s) = 0 \; . \tag{134}$$

These equations have the solutions

$$i_1(s) = \frac{s+1}{s+2} \; , \tag{135}$$

and

$$i_2(s) = \frac{1}{s+2} \; . \tag{136}$$

Now, because the grid leak resistance is one ohm we have

$$e_g(s) = i_2(s) \; , \tag{137}$$

and

$$i_p(s) = -g_m e_g(s) = \frac{-g_m}{s+2} \; . \tag{138}$$

For the plate circuit we have the following equations:

$$i_3(s) + i_4(s) = i_p(s) = \frac{-g_m}{s+2} \; ; \tag{139}$$

and

$$-i_3(s)\,1/s + i_4(s) = 0 \; . \tag{140}$$

These equations have the solutions

$$i_3(s) = \frac{-g_m s}{(s+1)(s+2)} \; , \tag{141}$$

and

$$i_4(s) = \frac{-g_m}{(s+1)(s+2)} \; . \tag{142}$$

Therefore,

$$g_{11}(s) = i_1(s) = \frac{s+1}{s+2} = 1 - \frac{1}{s+2} \; , \tag{143}$$

and

$$g_{12}(s) = i_4(s) = \frac{-g_m}{(s+1)(s+2)} . \qquad (144)$$

Taking the inverse Laplace transforms of equation (143) and equation (144) gives

$$g_{11}(t) = u_1(t) - e^{-2t} , \qquad (145)$$

and

$$g_{12}(t) = -g_m e^{-t} + g_m e^{-2t} . \qquad (146)$$

For this problem, then, $g_{11}(t)$ contains a unit impulse plus continuous terms. We therefore want to use characteristic equation (52). In using this equation it is useful to have the following:

$$g_{11_c}(t) = -e^{-2t} , \qquad (147)$$

$$g_{11_c}(-t) = -e^{2t} , \qquad (148)$$

$$g_{11_c}(\tau-t) = -e^{-2\tau}e^{2t} , \qquad (149)$$

$$g_{12}(T-t) = -g_m e^{-T}e^{t} + g_m e^{-2T}e^{2t} . \qquad (150)$$

Substituting the above values in equation (52) results in

$$e_1(s)\left(2 - \frac{1}{s+2} + \frac{1}{s-2}\right) - \frac{1}{s-2}\int_0^T e_1(\tau)e^{-2\tau}d\tau = -\lambda g_m\left[\frac{e^{-T}}{s-1} - \frac{e^{-2T}}{s-2}\right] .$$

(151)

Now we let

$$k = \int_0^T e_1(\tau)e^{-2\tau}d\tau . \qquad (152)$$

Equation (151) then reduces to

$$e_1(s) = \frac{s^2(k+\lambda g_m e^{-2T} - \lambda g_m e^{-T}) + s(k+\lambda g_m e^{-2T}) + (4\lambda g_m e^{-T} - 2\lambda g_m e^{-2T} - 2k)}{2(s-1)(s-\sqrt{2})(s+\sqrt{2})} \quad . \tag{153}$$

This can be expanded into partial fractions, giving

$$e_1(s) = \frac{1}{2}\left[\frac{-3\lambda g_m e^{-T}}{s-1} + \frac{(1/2+\sqrt{2}/2)k + (1+\sqrt{2}/2)\lambda g_m e^{-T} + (1/2+\sqrt{2}/2)\lambda g_m e^{-2T}}{s-\sqrt{2}}\right.$$
$$\left. + \frac{(1/2-\sqrt{2}/2)k + (1-\sqrt{2}/2)\lambda g_m e^{-T} + (1/2-\sqrt{2}/2)\lambda g_m e^{-2T}}{s+\sqrt{2}}\right] . \tag{154}$$

The inverse Laplace transform of equation (154) is

$$\begin{aligned} e_1(t) = {} & [-(3/2)\lambda g_m e^{-T}]e^{t} \\ & + [(1/4+\sqrt{2}/4)k + (1/2+\sqrt{2}/4)\lambda g_m e^{-T} + (1/4+\sqrt{2}/4)\lambda g_m e^{-2T}]e^{\sqrt{2}\,t} \\ & + [(1/4-\sqrt{2}/4)k + (1/2-\sqrt{2}/4)\lambda g_m e^{-T} + (1/4-\sqrt{2}/4)\lambda g_m e^{-2T}]e^{-\sqrt{2}\,t} \;. \end{aligned} \tag{155}$$

We have now two undermined constants, k and λ. We proceed first to the determination of k using the definition given by equation (152).

$$\begin{aligned} k = {} & \int_0^T e_1(\tau)e^{-2\tau}d\tau \\ = {} & [-(3/2)\lambda g_m e^{-T}]\,[-e^{-T}+1] \\ & + [(1/4+\sqrt{2}/4)k + (1/2+\sqrt{2}/4)\lambda g_m e^{-T} + (1/4+\sqrt{2}/4)\lambda g_m e^{-2T}]\left[\frac{-e^{-(2-\sqrt{2})T}+1}{2-\sqrt{2}}\right] \\ & + [(1/4-\sqrt{2}/4)k + (1/2-\sqrt{2}/4)\lambda g_m e^{-T} + (1/4-\sqrt{2}/4)\lambda g_m e^{-2T}]\left[\frac{-e^{-(2+\sqrt{2})T}+1}{2+\sqrt{2}}\right] . \end{aligned} \tag{156}$$

From this equation one can solve for k. k turns out to be

$$k = \lambda g_m \left[\frac{5e^{-2T} + (-3/2 - \sqrt{2})e^{-(3-\sqrt{2})T} + (-3/2+\sqrt{2})e^{-(3+\sqrt{2})T} + (-1-3\sqrt{2}/4)e^{-(4-\sqrt{2})T}}{(1 + 3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1 - 3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} \right.$$

$$\left. + \frac{(-1 + 3\sqrt{2}/4)e^{-(4+\sqrt{2})T}}{(1 + 3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1 - 3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} \right]$$

$$= \lambda g_m \left[\frac{5e^{-2T} + (-3/2 - \sqrt{2})e^{-(3-\sqrt{2})T} + (-3/2 + \sqrt{2})e^{-(3+\sqrt{2})T}}{(1 + 3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1 - 3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} \right] - \lambda g_m e^{-2T} . \tag{157}$$

Substituting this value into equation (155) gives for $e_1(t)$

$$e_1(t) = \lambda g_m \left\{ \left[\frac{(-3/2 - 9\sqrt{2}/8)e^{-(3-\sqrt{2})T} + (-3/2 + 9\sqrt{2}/8)e^{-(3+\sqrt{2})T}}{(1 + 3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1 - 3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} \right] e^{t} \right.$$

$$+ \left[\frac{(5/4 + 5\sqrt{2}/4)e^{-2T} + (1/4 - \sqrt{2}/4)e^{-(3+\sqrt{2})T}}{(1 + 3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1 - 3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} \right] e^{\sqrt{2}\, t}$$

$$\left. + \left[\frac{(5/4 - 5\sqrt{2}/4)e^{-2T} + (1/4 + \sqrt{2}/4)e^{-(3-\sqrt{2})T}}{(1 + 3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1 - 3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} \right] e^{-\sqrt{2}\, t} \right\} . \tag{158}$$

Now, we still have one undetermined constant left, namely, λ. We select λ so as to satisfy the boundary condition that the output current, $i_4(t)$, is some specified value at time $t = T$. To find $i_4(t)$ we use the equation

$$i_4(s) = g_{12}(s)e_1(s) \tag{159}$$

Now $e_1(s)$ is given by the Laplace transform of equation (158), or

$$e_1(s) = \frac{\lambda g_m}{(1 + 3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1 - 3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} \times$$

$$\left[\frac{(-3/2 - 9\sqrt{2}/8)e^{-(3-\sqrt{2})T} + (-3/2 + 9\sqrt{2}/8)e^{-(3+\sqrt{2})T}}{s - 1}\right.$$

$$+ \frac{(5/4 + 5\sqrt{2}/4)e^{-2T} + (1/4 - \sqrt{2}/4)e^{-(3+\sqrt{2})T}}{s - \sqrt{2}}$$

$$\left. + \frac{(5/4 - 5\sqrt{2}/4)e^{-2T} + (1/4 + \sqrt{2}/4)e^{-(3-\sqrt{2})T}}{s + \sqrt{2}}\right] . \qquad (160)$$

This reduces to

$$e_1(s) = \lambda g_m (s+2) \left\{ \frac{s[(5/2)e^{-2T} + (-5/4 - 7\sqrt{2}/8)e^{-(3-\sqrt{2})T} + (-5/4 + 7\sqrt{2}/8)e^{-(3+\sqrt{2})T}]}{[(1+3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1-3\sqrt{2}/4)e^{-(2+\sqrt{2})T}](s-1)(s-\sqrt{2})(s+\sqrt{2})}\right.$$

$$\left. + \frac{(-5/2)e^{-2T} + (7/4+5\sqrt{2}/4)e^{-(3-\sqrt{2})T} + (7/4-5\sqrt{2}/4)e^{-(3+\sqrt{2})T}}{[(1+3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1-3\sqrt{2}/4)e^{-(2+\sqrt{2})T}](s-1)(s-\sqrt{2})(s+\sqrt{2})} \right\} . \qquad (161)$$

Therefore, equation (159) becomes

$$i_4(s) = \frac{-g_m e_1(s)}{(s+1)(s+2)}$$

$$= -\lambda g_m^{\,2} \left\{ \frac{s[(5/2)e^{-2T} + (-5/4 - 7\sqrt{2}/8)e^{-(3-\sqrt{2})T} + (-5/4 + 7\sqrt{2}/8)e^{-(3+\sqrt{2})T}]}{[(1+3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1-3\sqrt{2}/4)e^{-(2+\sqrt{2})T}](s-1)(s+1)(s-\sqrt{2})(s+\sqrt{2})}\right.$$

$$\left. + \frac{(-5/2)e^{-2T} + (7/4+5\sqrt{2}/4)e^{-(3-\sqrt{2})T} + (7/4-5\sqrt{2}/4)e^{-(3+\sqrt{2})T}}{[(1+3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1-3\sqrt{2}/4)e^{-(2+\sqrt{2})T}](s-1)(s+1)(s-\sqrt{2})(s+\sqrt{2})} \right\} .$$

$$(162)$$

This expression can be expanded into partial fractions and the inverse Laplace transform taken. We have then for $i_4(T)$

$$i_4(T) = -\lambda g_m^2 \left[\frac{(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T} + (1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T} + (3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T}}{(1+3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1-3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} + \frac{(3/2 - 17\sqrt{2}/16)e^{-(4+\sqrt{2})T} - 5e^{-3T}}{(1 + 3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1 - 3\sqrt{2}/4)e^{-(2+\sqrt{2})T}} \right] . \tag{163}$$

This equation can now be solved for λ, giving

$$\lambda = \frac{-i_4(T)\,[(1+3\sqrt{2}/4)e^{-(2-\sqrt{2})T} + (1-3\sqrt{2}/4)e^{-(2+\sqrt{2})T}]}{g_m^2[(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T} + (1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T} + (3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T} + (3/2-17\sqrt{2}/16)e^{-(4+\sqrt{2})T} - 5e^{-3T}]} \tag{164}$$

Finally, for $e_1(t)$, we have

$$e_1(t) = \frac{-i_4(T)}{g_m} \left\{ \frac{[(-3/2-9\sqrt{2}/8)e^{-(3-\sqrt{2})T} + (-3/2+9\sqrt{2}/8)e^{-(3+\sqrt{2})T}]e^{t} + [(5/4+5\sqrt{2}/4)e^{-2T} + (1/4-\sqrt{2}/4)e^{-(3+\sqrt{2})T}]e^{\sqrt{2}t}}{(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T} + (1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T} + (3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T} + (3/2-17\sqrt{2}/16)e^{-(4+\sqrt{2})T} - 5e^{-3T}} + \frac{[(5/4-5\sqrt{2}/4)e^{-2T} + (1/4+\sqrt{2}/4)e^{-(3-\sqrt{2})T}]e^{-\sqrt{2}t}}{(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T} + (1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T} + (3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T} + (3/2-17\sqrt{2}/16)e^{-(4+\sqrt{2})T} - 5e^{-3T}} \right\} . \tag{165}$$

We can now go back and examine the input energy for the optimum driving voltage. To do this we must first find $i_1(t)$. Now the Laplace transform of $i_1(t)$ is given by

$$i_1(s) = g_{11}(s)e_1(s) \quad . \tag{166}$$

$g_{11}(s)$ is given by equation (143) and $e_1(s)$ is given by equation (161). Using these expressions in equation (166) gives:

$$i_1(s) = \frac{i_4(T)}{g_m}\left\{\frac{[5/2e^{-2T}+(-5/4-7\sqrt{2}/8)e^{-(3-\sqrt{2})T}+(-5/4+7\sqrt{2}/8)e^{-(3+\sqrt{2})T}]\frac{s^2}{(s-1)(s^2-2)}}{(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T}+(1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T}+(3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T}+(3/2-17\sqrt{2}/16)e^{-(4+\sqrt{2})T}-5e^{-3T}}\right.$$

$$+\frac{[(1/2+3\sqrt{2}/8e^{-(3-\sqrt{2})T}+(1/2-3\sqrt{2}/8)e^{-(3+\sqrt{2})T}]\frac{s}{(s-1)(s^2-2)}}{(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T}+(1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T}+(3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T}+(3/2-17\sqrt{2}/16)e^{-(4+\sqrt{2})T}-5e^{-3T}}$$

$$\left.+\frac{[(-5/2)e^{-2T}+(7/4+5\sqrt{2}/4)e^{-(3-\sqrt{2})T}+(7/4-5\sqrt{2}/4)e^{-(3+\sqrt{2})T}]\frac{1}{(s-1)(s^2-2)}}{(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T}+(1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T}+(3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T}+(3/2-17\sqrt{2}/16)e^{-(4+\sqrt{2})T}-5e^{-3T}}\right\} . \qquad (167)$$

This expression can be expanded into partial fractions and the inverse Laplace transform taken. This gives:

$$i_1(t) = \frac{-i_4(T)}{g_m}\left\{\frac{[(-1-3\sqrt{2}/4)e^{-(3-\sqrt{2})T}+(-1+3\sqrt{2}/4)e^{-(3+\sqrt{2})T}]e^{t}+[(5/4+5\sqrt{2}/8)e^{-2T}+(-1/4+\sqrt{2}/8)e^{-(3+\sqrt{2})T}]e^{\sqrt{2}t}}{(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T}+(1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T}+(3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T}+(3/2-17\sqrt{2}/16)e^{-(4+\sqrt{2})T}-5e^{-3T}}\right.$$

$$\left.+\frac{[(5/4-5\sqrt{2}/8)e^{-2T}+(-1/4-\sqrt{2}/8)e^{-(3-\sqrt{2})T}]e^{-\sqrt{2}t}}{(1-13\sqrt{2}/16)e^{-(2-\sqrt{2})T}+(1+13\sqrt{2}/16)e^{-(2+\sqrt{2})T}+(3/2+17\sqrt{2}/16)e^{-(4-\sqrt{2})T}+(3/2-17\sqrt{2}/16)e^{-(4+\sqrt{2})T}-5e^{-3T}}\right\} . \qquad (168)$$

Now the input energy can be found from

$$E = \int_0^T e_1(t)i_1(t)dt \quad . \qquad (169)$$

Upon performing the above integration one obtains

$$E = \frac{i_2^2(T)[(7/64+\sqrt{2}/32)-(71/32)e^{-2\sqrt{2}T}+(33/16+39\sqrt{2}/32)e^{-4\sqrt{2}T}+(-99/64-35\sqrt{2}/32)e^{-2T}+(3/32)e^{-(2+2\sqrt{2})T}]}{g_m^2\,[(1-13\sqrt{2}/16)+(1+13\sqrt{2}/16)e^{-2\sqrt{2}T}+(3/2+17\sqrt{2}/16)e^{-2T}+(3/2-17\sqrt{2}/16)e^{-(2+2\sqrt{2})T}-5e^{-(1+\sqrt{2})T}]^2}$$

$$+\frac{i_2^2(T)[(-99/64+37\sqrt{2}/32)e^{-(2+4\sqrt{2})T}+(5/2+15\sqrt{2}/8)e^{-(1+\sqrt{2})T}+(5/4-51\sqrt{2}/16)e^{-(1+3\sqrt{2})T}]}{g_m^2[(1-13\sqrt{2}/16)+(1+13\sqrt{2}/16)e^{-2\sqrt{2}T}+(3/2+17\sqrt{2}/16)e^{-2T}+(3/2-17\sqrt{2}/16)e^{-(2+2\sqrt{2})T}-5e^{-(1+\sqrt{2})T}]^2} . \qquad (170)$$

This equation is shown plotted in Fig. 12.

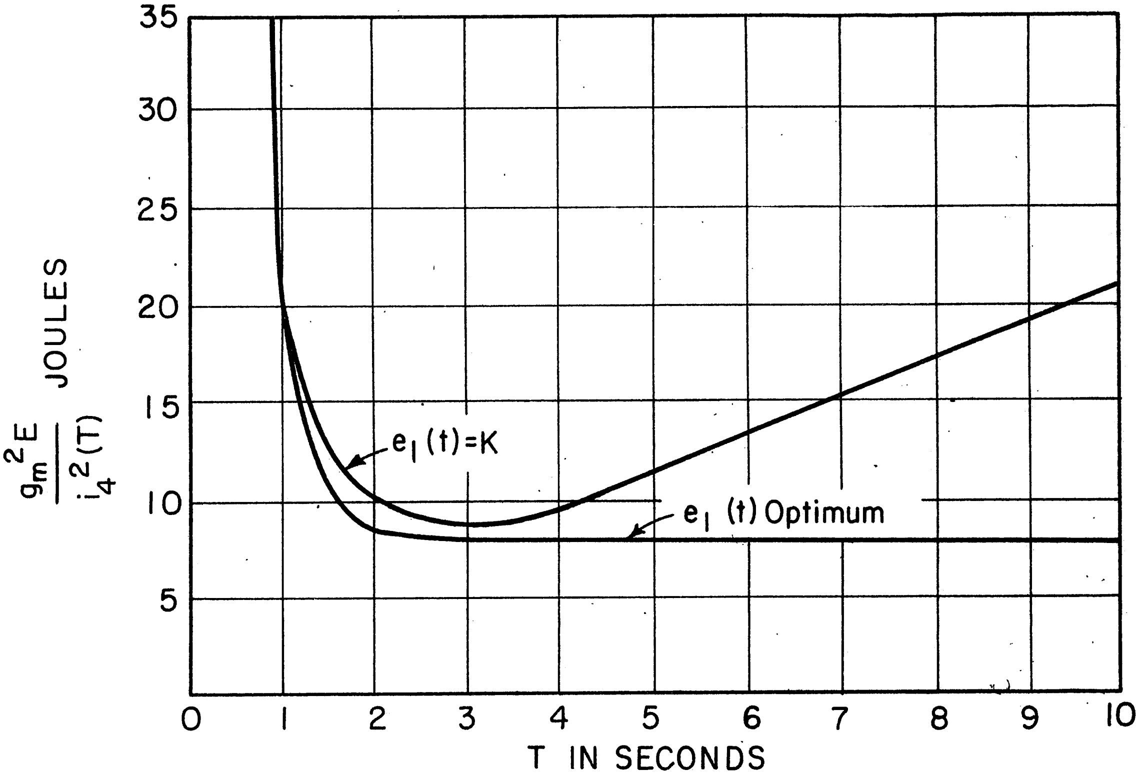

Fig. 12. $\frac{g_m^2 E}{i_4^2(T)}$ vs T.

Let us now compare the energy delivered to the network by the optimum signal with that delivered to the network by a step of voltage of sufficient amplitude for triggering at time $t = T$.

For the step of voltage we have

$$e_1(t) = k \quad . \tag{171}$$

Now,

$$i_4(s) = g_{12}(s)e_1(s) = \frac{-g_m k}{s(s+1)(s+2)} \quad . \tag{172}$$

This can be expanded into partial fractions and the inverse Laplace transform taken. We get

$$i_4(t) \quad = \quad kg_m \; (-1/2 + e^{-t} - 1/2\; e^{-2t}) \quad . \tag{173}$$

From equation (173) we get

$$k \quad = \quad \frac{2i_4(T)}{g_m \; (-1 + 2e^{-T} - e^{-2T})} \quad = \quad e_1(t) \quad . \tag{174}$$

Now,

$$i_1(s) \quad = \quad g_{11}(s)e_1(s) \quad = \quad \frac{k(s + 1)}{s(s + 2)} \quad . \tag{175}$$

This equation can be expanded into partial fractions and the inverse Laplace transform taken. This gives

$$i_1(t) \quad = \quad \frac{i_4(T)(1 + e^{-2t})}{g_m \; (-1 + 2e^{-T} + e^{-2T})} \quad . \tag{176}$$

The input energy for the step of voltage is the integral between 0 and T of the product of $e_1(t)$ and $i_1(t)$. Performing this operation gives, for the input energy,

$$E \quad = \quad \frac{i_4{}^2(T)(2T - e^{-2T} + 1)}{g_m{}^2(1 - 2e^{-T} + e^{-2T})^2} \quad . \tag{177}$$

This equation is also plotted in Fig. 12. It can be seen from this figure that the energy for the optimum signal is always less than that for a unit step. As in the first example we see that for T small the energy required of a step function approaches that required of the optimum signal. This again results because for t or T small the optimum function approaches a step function. Thus, if separate considerations dictate a small allowable triggering time, there is little advantage to the optimum function, and a step function becomes a very good approximation.

Again, however, these statements cannot be considered to be applicable to any problem, but have only resulted for a specific example.

CHAPTER VIII

CONCLUSIONS

8.1 Summary

The general problem of determining the optimum triggering signal for a lumped-constant, linear circuit has been treated. The optimum signal is defined as that which produces a given current through, or a given voltage across, a resistive output element at time $t = T$, while at the same time requiring a minimum of energy from the generator driving the circuit. The output resistance is considered as characterizing the input terminals of a bistable element such as a thyratron, multivibrator, or a magnetic relay.

General equations characterizing the optimum signal were derived, and the conditions under which they are valid were noted. There are two pathological types of circuits for which characteristic equations were not obtained. However, both of these types of circuits are unrealistic in the sense that they do not allow for generator internal resistance or stray capacitance across the circuit input terminals. For realizable circuits and realizable generators a characteristic equation was obtained which is always valid.

Methods of solution for the characteristic equations were discussed, and it was shown that the Laplace transform can be used to reduce the integral equations into algebraic ones which are susceptible of simple solution. Finally, several sample problems were proposed and the solutions obtained. These examples, in addition to demonstrating general solutions, also demonstrated a method for finding the undetermined constants involved in the equations.

8.2 Suggestions for Further Work

In this dissertation the circuits treated have been restricted to those composed of only linear, lumped-constant elements. It is probably true that many areas of endeavor, involving the notion of triggering, are concerned with circuits having either nonlinear or distributed-constant circuit elements. For this type of circuit the linear theory may provide a first approximation to the optimum signal for specific circuits. However, for exact solutions a separate treatment must be made for circuits containing either nonlinear or distributed-constant elements.

The linear theory in this dissertation has been worked out in terms of electrical circuits. However, it is well known that mechanical systems have electrical analogues, and vice versa. Thus, a mechanical system in which the problem is that of obtaining some minimum displacement in a given time interval by means of a forcing function could be analyzed in terms of its electrical analogue, and the optimum forcing function determined. In fact, the whole linear theory could be rewritten in terms of the mechanical elements of mechanical systems.

Finally, the optimum triggering signal theory might have application to the field of measurements. It is well known that the accuracy of measurement of a physical phenomenon is limited because the measurement itself introduces a disturbance into the phenomenon. It is usually of great importance to introduce as little energy change as possible into the system being measured. In some physical systems it might be satisfactory to know only whether some maximum or minimum condition exists or not. Now, this yes-no type of information can be determined in terms of whether or not a measuring circuit is triggered. Thus, if the phenomenon being measured can be made to generate a minimum energy triggering signal for a carefully chosen measurement circuit, then perhaps energy changes introduced by the measurement can be minimized.

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