NON-LINEAR MODELING OF MAXWELL'S EQUATIONS

by

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Preface

This is the thirty-eighth in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross section of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers; and (d) low and high density ionization phenomena.

K. M. Siegel

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Non-Linear Modeling of Maxwell's Equations

Abstract

A basic stumbling block to carrying out many experiments in the laboratory is the reliance which is placed on the theory of linear modeling. In aerodynamics and electromagnetics one is seldom able to construct a precise linear model; one must appeal to physical reasoning to show that the linear model one is able to construct gives results in close agreement with those which would have been obtained with a precise one. For example, as long as good conductors are used, conductivity is not modeled.

Until recent years the limitation on indoor radar cross section ranges involved the size of the model that could be handled, and the magnitude of the frequency of the coherent source. Even now no precise linear modeling experiments exist for bomber cross sections at very high microwave frequencies.

With the growth of interest in I. C. B. M. programs it has become desirable (as measured in dollars) to use modeling to determine the radar

cross sections of ablating warheads, and to determine attenuation effects and scattering effects in or by plasmas. Linear modeling is of very limited utility for such experiments. One must model a lossy dielectric at high frequencies or not do the experiment in the laboratory. One must model some plasma parameters, or risk missing crucial first order effects.

Previous analysis predicted the feasibility of non-linear modeling. The first such modeling results are given in Reference 1. In this paper several non-linear models of equations of mathematical physics are presented. Maxwell's equations are non-linearly modeled. Experiments making use of non-linear models are discussed.

Introduction

Just exactly what is meant by modeling is not at all simple to define. By modeling we mean more than what the theorist implies when he reasons by analogy with some known and believed "similar" physical problem. If we are faced with finding a solution to a physical problem which (1) cannot be solved analytically by known techniques, (2) could be solved analytically but at too great a cost, or (3) would involve measurements, in an experimental approach, which are either too costly or too difficult to do on a full scale basis (at least initially), we must turn to a model-experiment approach. Thus we see the economic factor usually comes into the question of modeling. An aerodynamicist would not consider building a new aircraft until he had made wind tunnel tests, or for that matter would he obtain financing without such tests. We should also keep in mind that in this country it might be, and probably would be, easier to "sell" results based on "modeling" experiments than it would be to "sell" results based on reasonable approximations to exact analytic results. In the latter case most people would require an experimental "check".

When we discuss modeling we state one measurement system on a small model will allow us to obtain results which we can relate to the difficult or expensive full scale experimental results desired. The way this relationship is usually made is via linear transformations. To clarify the meaning of linear modeling we will now go through a standard textbook example (Ref. 2).

A fundamental example in fluid dynamics would be based on the Navier-Stokes equation. Consider the equation of motion

$$\frac{\partial \underline{q}}{\partial t} + (\underline{q} \cdot \nabla)\, \underline{q} = - \nabla \left(\frac{p}{\rho} \right) + \nu \nabla^2 \underline{q} \tag{1}$$

where $\underline{q}$ is the velocity, p the pressure, ρ the density, t the time and ν the kinematic viscosity.

Now consider a scaled motion which differs only in the scales of length and time. Let us denote the quantities describing the second motion with dashes.

$$\frac{\partial \underline{q}'}{\partial t'} + (\underline{q}' \cdot \nabla')\, \underline{q}' = - \nabla' \left(\frac{p'}{\rho'} \right) + \nu' \nabla'^2 \underline{q}' . \tag{2}$$

The motion is said to be "similar" if we can proceed from equation (1) to equation (2) by multiplying every term in equation (1) by the same constant factor α. By hypothesis we consider only scaling length and time

$$\underline{r} = \lambda \underline{r}', \quad t = K t' ; \tag{3}$$

for similarity we must have

$$\frac{q'}{t'} = \alpha \frac{q}{t}, \quad \frac{q'^2}{r'} = \alpha \frac{q^2}{r}, \quad \frac{p'}{\rho' r'} = \alpha \frac{p}{\rho r}$$

$$\frac{\nu' q'}{r'^2} = \frac{\alpha \nu q}{r^2} .$$

We note by division

$$\frac{q'}{q} = \frac{r'/t'}{r/t}, \quad \frac{p'}{p} = \frac{\rho' q'^2}{\rho q^2}, \quad \frac{q' r'}{\nu'} = \frac{q r}{\nu} = R$$

where R is the Reynolds number. This must be the same for the two motions. Since the equation of continuity as well as the first two of the above conditions is satisfied by (3), it follows that the equality of R is both necessary and sufficient for the similarity of motions. In experiments made with models in the wind tunnel the quantities q' and r' for the tunnel and model are less than for the full scale case, while ν is the same for both. This has led to the use of compressed air wind tunnels where $\nu = \frac{\mu}{\rho}$ $\left[\mu = \text{coefficient of viscosity} \right]$ is decreased by taking an increased value of ρ.

Now that we understand what similarity implies, and the methods which are used in linear modeling, let us go on to non-linear modeling. Let us first consider an elementary example, the simple harmonic oscillator, to clarify the method of approach.

Consider the initial dynamic system as

$$m \ddot{x} = - k^2 x \tag{4}$$

The modeling transformations are

$$\begin{aligned} x_1 &= \phi(x) = \sum_i a_i x^i \\ m_1 &= \alpha m \\ k_1 &= \gamma k \\ t_1 &= \beta t \end{aligned} \tag{5}$$

An unnecessary but nevertheless perhaps desirable requirement would be that the experiment in the laboratory be governed by the same equation of the motion. Thus we require

$$m_1 \frac{d^2 x_1}{dt_1^2} = - k_1^2 x_1 . \tag{6}$$

Now it is clear that if $\phi(x)$ were equal to a constant times x we would have linear modeling. Since there is nothing which a priori precludes non-linear modeling we shall attempt it here, but first we will present some ground rules. We pretend we cannot solve (4), and we perform transformations (5) and attempt to determine the values of a_i such that we arrive at (6). We again pretend we cannot solve (6), and that the purpose of all this is that the experiment given by (6) is more economical than the experiment given by (4). Since we know the solutions to (4) and (6) we are in a position to check the concepts of non-linear modeling but we must not use this knowledge of the solutions to (4) and (6) in deriving the a_i. Substituting relationships (5) into (6) we obtain

$$\frac{\alpha m}{\gamma^2 \beta^2} \ddot{\phi} = - k^2 \phi .$$

If we let

$$\frac{\gamma^2 \beta^2}{\alpha} = \lambda^2 ,$$

then

$$m \ddot{\phi} = - \lambda^2 k^2 \phi .$$

By use of the chain rule, and of (4), this becomes

$$m\left[\frac{d^2\phi}{dx^2}\,\dot{x}^2 + \frac{(-k^2x)}{m}\,\frac{d\phi}{dx}\right] = -\lambda^2 k^2 \phi \,. \tag{7}$$

Differentiating (7) with respect to time we obtain

$$m\left[\frac{d^3\phi}{dx^3}\,\dot{x}^3 + 2\dot{x}\,\frac{d^2\phi}{dx^2}\left(\frac{-k^2x}{m}\right) + \left(\frac{-k^2\dot{x}}{m}\right)\frac{d\phi}{dx} + \left(\frac{-k^2x}{m}\right)\dot{x}\,\frac{d^2\phi}{dx^2}\right]$$

$$= -\lambda^2 k^2\,\frac{d\phi}{dx}\,\dot{x}\,. \tag{8}$$

Dividing through by $-k^2\,\dot{x}$ and collecting terms we obtain

$$\left[-\frac{m}{k^2}\,\frac{d^3\phi}{dx^3}\,\dot{x}^2 + 3\,\frac{d^2\phi}{dx^2}\,x + \frac{d\phi}{dx}\right] = +\lambda^2\,\frac{d\phi}{dx}\,. \tag{9}$$

We can eliminate $\dot{x}^2$ between (7) and (9) to obtain

$$\frac{\dfrac{d^3\phi}{dx^3}}{\dfrac{d^2\phi}{dx^2}} + \frac{3\,\dfrac{d^2\phi}{dx^2}\,x + (1-\lambda^2)\,\dfrac{d\phi}{dx}}{\lambda^2\,\phi - \dfrac{d\phi}{dx}\,x} = 0\,. \tag{10a}$$

Now (10a) may be written

$$\frac{d\left[\ln\left(\dfrac{\dfrac{d^2\phi}{dx^2}}{\lambda^2\phi - \dfrac{d\phi}{dx}\,x}\right)\right]}{dx} = \frac{2\,\dfrac{d^2\phi}{dx^2}\,x}{\dfrac{d\phi}{dx}\,x - \lambda^2\,\phi} \,. \qquad (10b)$$

so that if we let

$$\frac{\dfrac{d^2\phi}{dx^2}}{\lambda^2\phi - \dfrac{d\phi}{dx}\,x} = v;$$

(10b) may be written

$$\frac{1}{v}\,\frac{dv}{dx} = -2\,vx \,.$$

A first integral of this is

$$\frac{1}{v} = x^2 - c_1^{\,2}.$$

Substituting for v:

$$(x^2 - c_1^{\,2})\,\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx}\,x - \lambda^2\,\phi = 0.$$

This equation possesses a general solution

$$\boxed{\phi = c_2 \cos(\lambda \cos^{-1} \frac{x}{c_1}) + c_3 \sin(\lambda \sin^{-1} \frac{x}{c_1})} \quad . \quad (11)$$

We note if $\lambda = 1$ we have linear modeling, since

$$\phi = \frac{(c_2 + c_3)}{(c_1)} x.$$

The condition $\lambda = 1$ forces $\gamma^2 \beta^2 = \alpha$. This is the same type similarity condition as we discussed for the Navier-Stokes equation.

Note that (11) defines an infinite set of non-linear modeling functions. In the event that λ is an integer and $c_3 = 0$ we see that ϕ is a Tschebyscheff polynomial of the first kind of order λ. The simplest example of this, that for $\lambda = 2$, is

$$x_1 = \phi(x) = c_2 \left[2 (\frac{x}{c_1})^2 - 1 \right] . \qquad (12a)$$

Let us use this to test the validity of our model. A solution of (4) is

$$x = A \sin \frac{kt}{\sqrt{m}} + B \cos \frac{kt}{\sqrt{m}} .$$

From (12a) we have [with $c_2 = 1$]

$$x_1 = \left[\frac{2 (A \sin \frac{kt}{m} + B \cos \frac{kt}{m})^2}{c_1^2} - 1 \right] . \qquad (12b)$$

Now we use (5) to see if this is really a solution of (6). Since we let $\lambda = 2$ to obtain (12a), we must use this restriction on the modeling constants:

$$\frac{\gamma^2 \beta^2}{\alpha} = 4. \tag{13}$$

Equation (6) has the form

$$m_1 \frac{d^2 x_1}{dt_1{}^2} = - k_1{}^2 x_1, \tag{6}$$

which becomes upon the use of (12b)

$$\frac{2}{c_1{}^2} \frac{\alpha m}{\beta^2} \frac{d^2 (A \sin \frac{kt}{\sqrt{m}} + B \cos \frac{kt}{\sqrt{m}})^2}{dt^2} =$$

$$- \gamma^2 k^2 \left[\frac{2(A \sin \frac{kt}{\sqrt{m}} + B \cos \frac{kt}{\sqrt{m}})^2}{c_1{}^2} - 1 \right] \tag{14}$$

and by use of (13)

$$\frac{m}{c_1{}^2} \frac{d^2}{dt^2} (A \sin \frac{kt}{\sqrt{m}} + B \cos \frac{kt}{\sqrt{m}})^2 = - 2 k^2 \left[\frac{2(A \sin \frac{kt}{\sqrt{m}} + B \cos \frac{kt}{\sqrt{m}})^2}{c_1{}^2} - 1 \right]. \tag{15}$$

Performing the indicated differentiation, and dividing through by $2 k^2$ we obtain

$$\frac{A^2}{c_1^{\,2}} + \frac{B^2}{c_1^{\,2}} = 1. \tag{16}$$

Thus if $c_1^{\,2} = A^2 + B^2$ we see that (12b) is a solution to (6).

We have presented what we have called the simplest type of non-linear modeling. It will be recalled that the equation of motion was the same in both systems in this example. In the next section we shall consider a case of non-linear modeling in which the two systems are governed by different equations of motion.

Non-linear Modeling of the Non-linear Spring Equation.

The non-linear spring equation is

$$\ddot{x} + h(x^2 - 1)\, x = 0. \tag{17}$$

In an attempt to model equation (17) non-linearly with respect to displacement, we use the modeling transformations

$$x_1 = \phi\,(x)$$

$$\mu = bh$$

$$t_1 = ct.$$

Let us assume that the model system is governed by

$$\frac{dx_1^2}{dt_1^{\,2}} - \mu\, x_1 = 0. \tag{18}$$

Substitution of the transformations into (18) yields

$$\frac{1}{c^2}\frac{d}{dt}\left(\frac{d\phi}{dx}\cdot\dot{x}\right) - bh\,\phi = \frac{1}{c^2}\left[\frac{d^2\phi}{dx^2}\dot{x}^2 + \frac{d\phi}{dx}\cdot\ddot{x}\right] - bh\,\phi = 0. \tag{19}$$

Let $\gamma = bc^2$; then (19) becomes

$$\frac{d^2\phi}{dx^2}\dot{x}^2 + \frac{d\phi}{dx}\ddot{x} - \gamma h\phi = 0. \tag{20}$$

Now by (17) we obtain

$$\frac{d^2\phi}{dx^2}\dot{x}^2 - \frac{d\phi}{dx}h(x^2-1)\,x - \gamma h\phi = 0. \tag{21}$$

We differentiate (21) with respect to time obtaining

$$\frac{d^3\phi}{dx^3}\dot{x}^3 + \frac{d^2\phi}{dx^2}2\dot{x}\,\ddot{x} - \frac{d^2\phi}{dx^2}\dot{x}\,h(x^2-1)\,x$$

$$- \frac{d\phi}{dx}h(x^2-1)\,\dot{x} - \frac{d\phi}{dx}h(2x\,\dot{x})x - \gamma h\frac{d\phi}{dx}\dot{x} = 0. \tag{22}$$

Dividing through by $\dot{x}$, we obtain

$$\frac{d^3\phi}{dx^3}\dot{x}^2 = -\frac{d^2\phi}{dx^2}2\ddot{x} + \frac{d^2\phi}{dx^2}h(x^2-1)\,x + \frac{d\phi}{dx}h(x^2-1) + \frac{d\phi}{dx}h\,2x^2 + \gamma h\frac{d\phi}{dx}. \tag{23}$$

Eliminating $\dot{x}^2$ between (21) and (23), we obtain

$$\frac{\dfrac{d^3\phi}{dx^3}}{\dfrac{d^2\phi}{dx^2}} = \frac{\dfrac{d^2\phi}{dx^2}\left[-2\ddot{x} + h(x^2 - 1)x\right] + \dfrac{d\phi}{dx}\left[h(x^2 - 1) + h\,2x^2 + \gamma h\right]}{\dfrac{d\phi}{dx}\,h(x^2 - 1)\,x + \gamma h\phi} . \tag{24}$$

Since

$$\frac{d}{dx}\left[\frac{d\phi}{dx}\,h(x^2 - 1)x + \gamma h\phi\right] \equiv \frac{d^2\phi}{dx^2}\,h(x^2 - 1)x + \frac{d\phi}{dx}\,\gamma h + \frac{d\phi}{dx}\left[h(x^2 - 1) + 2hx^2\right], \tag{25}$$

(24) becomes,

$$\frac{d}{dx}\left[\ln\frac{d^2\phi}{dx^2}\right] = \frac{d}{dx}\ln\left[\frac{d\phi}{dx}\,h(x^2 - 1)x + \gamma h\phi\right] - \frac{\dfrac{d^2\phi}{dx^2}\,2\ddot{x}}{\dfrac{d\phi}{dx}\,h(x^2 - 1)x + \gamma h\phi} . \tag{26}$$

Using the temporary change of variables

$$v = \frac{\dfrac{d^2\phi}{dx^2}}{\dfrac{d\phi}{dx}\,h(x^2 - 1)x + \gamma h\phi} , \tag{27}$$

equation (26) takes on the form

$$\frac{d}{dx}\left[\ln v\right] = -\,v \cdot 2\ddot{x}. \tag{28a}$$

We can rewrite (28a) in the form

$$\frac{dv}{v^2} = 2\left[h\,x^3 - hx\right]dx \tag{28b}$$

which upon integration becomes

$$-\frac{1}{v} = \frac{2\,hx^4}{4} - h\,x^2 + c. \tag{28c}$$

Replacing v by the expression (27), (28c) becomes

$$\left(\frac{hx^4}{2} - h\,x^2 + c\right)\frac{d^2\phi}{dx^2} + (h\,x^3 - hx)\frac{d\phi}{dx} + \gamma\, h\,\phi = 0. \tag{29}$$

Dividing through by h and letting $c_1 = \frac{c}{h}$ we obtain

$$\left(\frac{x^4}{2} - x^2 + c_1\right)\frac{d^2\phi}{dx^2} + (x^3 - x)\frac{d\phi}{dx} + \gamma\phi = 0 \tag{30}$$

an ordinary linear equation with non-constant coefficients.

In order to solve (30), we can take advantage of the fact that it may easily be thrown into self-adjoint form: dividing it by $\sqrt{\frac{x^4}{2} - x^2 + c_1}$ we obtain

$$\sqrt{\frac{x^4}{2} - x^2 + c_1}\,\frac{d^2\phi}{dx^2} + \frac{x^3 - x}{\sqrt{\frac{x^4}{2} - x^2 - c_1}}\,\frac{d\phi}{dx} + \frac{\gamma\,\phi}{\sqrt{\frac{x^4}{2} - x^2 + c_1}} = 0, \tag{31}$$

which can be put in the form

$$\sqrt{\frac{x^4}{2} - x^2 + c_1}\ \frac{d}{dx}\left[\sqrt{\frac{x^4}{2} - x^2 + c_1}\ \frac{d\phi}{dx}\right] + \gamma\phi = 0. \qquad (32)$$

If we define:

$$\sqrt{\frac{x^4}{2} - x^2 + c_1}\ \frac{d}{dx} = \frac{d}{dy}\ ,$$

then (32) becomes

$$\frac{d^2\phi}{dy^2} + \gamma\phi = 0 :\ \phi = c_2 e^{\pm i\sqrt{\gamma}\, y}\ . \qquad (33a)$$

The above definition of y implies that

$$dy = \frac{dx}{\sqrt{\frac{x^4}{2} - x^2 + c_1}}\ ,$$

an expression which may be integrated by quadrature to yield

$$y = \sqrt{2}\int_0^x \frac{dt}{\sqrt{t^4 - 2t^2 + c'}} = \sqrt{\frac{2}{b}}\int_0^{x/\sqrt{a}} \frac{ds}{\sqrt{(1 - s^2)(1 - \frac{a}{b}s^2)}}\ ,$$

where $c' = 2c_1 = ab$, $b + a = 2$, and $s = \frac{t}{\sqrt{a}}$. Since the integral appearing above is in the standard form of the elliptic integral of the first kind with modulus $\sqrt{\frac{a}{b}}$, expressed $F\left(\sqrt{\frac{a}{b}}, \frac{x}{\sqrt{a}}\right)$, the modeling function we seek is

$$\boxed{\phi(x) = c_2 \exp\left[\pm i \sqrt{\frac{2\gamma}{b}}\, F\left(\sqrt{\frac{a}{b}}, \frac{x}{\sqrt{a}}\right)\right]} \qquad (33b)$$

If $c' = 1$ we have the following simple special case.

$$y = \sqrt{2} \int_0^x \frac{dt}{1 - t^2} = \frac{+1}{\sqrt{2}} \left[\ln \frac{1+t}{1-t} \right]_0^x = \frac{1}{\sqrt{2}} \ln\left(\frac{1+x}{1-x}\right)$$

$$\phi(x) = c_2 \left(\frac{1+x}{1-x}\right)^{\pm i \sqrt{2\gamma}} .$$

Now we are ready to discuss more general systems.

General Non-linear Oscillator

Consider as a prototype a system governed by the equation

$$\frac{d^2 x}{dt^2} + f(x) = 0, \qquad (34)$$

which describes the free, undamped, non-linear vibrations of quite general systems having one degree of freedom. If the model is required to satisfy the equation

$$\frac{d^2y}{ds^2} + f^*(y) = 0, \tag{35}$$

in which f* differs from f only by a linear change in parameters, then it is desired to obtain the functional relation y = F(x) when the independent variables are related by s = a t. This last relation permits one to write (35) in the form

$$\frac{d^2y}{dt^2} + a^2 f^*(y) = 0, \tag{36}$$

and by use of the chain rule this can be written as

$$\frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} + a^2 f^*(y) = 0. \tag{37}$$

Substitution for $\frac{d^2x}{dt^2}$ from (34) and use of the identity

$$\frac{d^2x}{dt^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{dx}{dt}\right)^2,$$

permits one to write (37) in the form

$$\frac{d^2y}{dx^2} + \frac{f(x)}{2\int f(x)\,dx}\frac{dy}{dx} - \frac{a^2 f^*(y)}{2\int f(x)\,dx} = 0. \tag{38}$$

Equation (38) is the differential equation whose solution gives the desired functional relationship between x and y. It does not seem possible to obtain its solution for arbitrary f; however, the equation can be written in a form more suitable for solution once f is prescribed. Setting

$$g(x) = \sqrt{\int f(x)\, dx}$$

(38) can be put in the form

$$g(x) \frac{d}{dx}\left(g(x) \frac{dy}{dx}\right) - \frac{a^2}{2} f^* (y) = 0, \tag{39}$$

and by the change of variable

$$dm = \frac{dx}{g(x)} ,$$

(39) is reduced to

$$\frac{d^2y}{dm^2} - \frac{a^2}{2} f^* (y) = 0. \tag{40}$$

Equation (40) can be written in the form

$$\frac{d}{dy} \left(\frac{dy}{dm}\right)^2 - a^2 f^* (y) = 0,$$

which is solvable for m. Hence

$$m = \frac{1}{a} \int \frac{dy}{g^*(y)} , \tag{41}$$

where $g^*(y) = \sqrt{\int f^*(y)\, dy}$.

The definition of m together with (41) imply that

$$\int \frac{dx}{g(x)} = \frac{1}{a} \int \frac{dy}{g^*(y)}, \tag{42}$$

which is the desired functional relationship between x and y.

The non-linear spring equation (17) is of course a special case of the results just presented. For if in (34) we let $f(x) = \mu (x^3 - x)$, then $f^*(y) = \mu' (y^3 - y)$, where $\mu' = b\mu$, and (42) becomes

$$\int \frac{dx}{\sqrt{\int (x^3 - x)\, dx}} = \frac{1}{a\sqrt{b}} \int \frac{dy}{\sqrt{\int (y^3 - y)\, dy}}. \tag{43}$$

If the initial conditions are

$$\frac{dx}{dt} = 0 \qquad \text{when } x = x_o,$$

$$\frac{dy}{dt} = 0 \qquad \text{when } y = y_o,$$

and if $y = y_1$ when $x = 0$, then (43) becomes

$$\int_0^x \frac{du}{\sqrt{\int_{x_o}^{u} (v^3 - v)\, dv}} = \frac{1}{a\sqrt{b}} \int_{y_1}^{y} \frac{du}{\sqrt{\int_{y_o}^{u} (v^3 - v)\, dv}} \tag{44}$$

By performing two of the integrals in (44), and defining

$$A,\ B = 1 \mp \sqrt{1 + {x_o}^4 - 2{x_o}^2}$$

$$C,\ D = 1 \mp \sqrt{1 + {y_o}^4 - 2{y_o}^2} \ ,$$

we obtain the expression

$$F\left(\sqrt{\frac{C}{D}},\ \frac{y}{\sqrt{C}}\right) = F\left(\sqrt{\frac{C}{D}},\ \frac{y_1}{\sqrt{C}}\right) + a\sqrt{\frac{bD}{B}}\ F\left(\sqrt{\frac{A}{B}},\ \frac{x}{\sqrt{A}}\right)$$

where F (,) is the elliptic integral of the first kind as defined above in equation (33b). It should be noted that the above treatment of the non-linear spring differs from that in the preceding section in that both model and prototype satisfy equations of motion of the same form. One reason for its inclusion was to show how easily boundary conditions may be handled.

Euler's Equation

Consider a hydrodynamic system governed by Euler's equation

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho} \nabla p + \vec{G}. \qquad (45)$$

In the event that this system is either an adiabatic or an isothermal gas, ρ may be expressed as a function of p by an equation of state, $\rho = f(p)$, and (45) may be written

$$\frac{d\vec{v}}{dt} = -\frac{1}{f(p)} \nabla p + \vec{G} \qquad (45')$$

If for this prototype we specify a model system obeying

$$\frac{d\vec{w}}{ds} = -\frac{1}{f^*(q)} \nabla' q + \vec{G}', \qquad (46)$$

it is desired to obtain the functional relation $p = F(q)$ which arises when the other variables of the two systems are related by

$$\vec{w} = a\vec{v}, \; s = bt, \; \nabla' = \frac{1}{c} \nabla, \; \vec{G}' = e\vec{G},$$

and when f^* differs from f only by some linear change of parameters.

These relations permit us to rewrite (46) as

$$\frac{a}{b} \frac{d\vec{v}}{dt} = -\frac{1}{c\, f^*(q)} \frac{dq}{dp} \nabla p + e\vec{G}. \qquad (47)$$

Solution of (47) for $\vec{G}$ yields

$$\vec{G} = \frac{a}{be} \frac{d\vec{v}}{dt} + \frac{1}{ce\, f^*(q)} \frac{dq}{dp} \nabla p,$$

and substitution of this result in (45) gives us

$$\frac{d\vec{v}}{dt}\left(1 - \frac{a}{be}\right) + \nabla p \left(\frac{1}{f(p)} - \frac{dq}{dp} \frac{1}{ce\, f^*(q)}\right) = 0.$$

If the stipulation is made that a = be, the above reduces to

$$\frac{dq}{dp} - \frac{ce\, f^*(q)}{f(p)} = 0, \tag{48}$$

since we may assume that ∇p is not identically zero. The variables in this equation are separable, so that if f(p) is specified then p = F(q) may be found.

For example, in the event that the system is an ideal adiabatic gas, then

$$\rho = K p^{1/\gamma},$$

and (48) becomes

$$\frac{dq}{dp} - \frac{ac}{b} \frac{K' q^{1/\gamma'}}{K p^{1/\gamma}} = 0.$$

This is immediately integrable to yield

$$p = \left[\frac{\gamma'(\gamma - 1)}{\gamma(\gamma' - 1)} \; \frac{bK}{acK'} \; q^{\frac{\gamma' - 1}{\gamma'}} \right]^{\frac{\gamma}{\gamma - 1}} + C.$$

Note that the modeling is linear if both model and prototype have the same specific heat ratios ($\gamma = \gamma'$).

Boltzmann Equation

Consider a statistical distribution of particles whose distribution function, F, satisfies the Boltzmann equation

$$\frac{\partial F}{\partial t} + V \cdot \nabla F + A \cdot \nabla_V F + I(F) = 0, \tag{49}$$

where I(F) is the collision integral of the system. Now the first three terms in (49) are simply the expanded form of the material derivative $\frac{DF}{Dt}$. Hence (49) can be written in the form

$$\frac{DF}{Dt} + I(F) = 0. \tag{50}$$

This form lends itself to modeling more rapidly than does (49). Hence if the model is required to satisfy an equation of the form

$$\frac{DG}{Ds} + I^* (G) = 0, \tag{51}$$

where I^* differs from I only by a linear change in parameters, and if $S = at$, then a simple manipulation permits (50) to be written in the form

$$a\, I^*(G) \frac{dF}{dG} - I(F) = 0. \tag{52}$$

The variables are separable in (52); hence it may presumably be solved once I is prescribed.

For example, in the event that both systems are neutral or weakly ionized gases, then I is given approximately by

$$I(F) = \frac{F - F_o}{\tau} ,$$

where τ is the characteristic time of the system, and F_o is its Maxwellian distribution. Hence

$$I^*(G) = \frac{G - G_o}{\tau'} ,$$

and the solution of (52) is

$$F = F_o + C(G - G_o)^{\tau'/a\tau} , \tag{53}$$

where C is a constant of integration. Inspection of (53) shows that the modeling is non-linear except when $\tau' = a\tau$.

Non-linear Modeling in Acoustics

Let us next consider the scalar wave equation

$$\nabla^2 \phi + k^2 \phi = 0, \quad k = 2\pi/\lambda \tag{54}$$

where λ = wavelength. This equation we plan to non-linearly model by means of an equation of the same form

$$\nabla^2 \psi + \ell^2 \psi = 0 \tag{55}$$

wherein we take $x_i' = a\, x_i$ and $\lambda' = b\,\lambda$. If we find we cannot do an economical experiment when b = a , then linear modeling is out of the question, since two distances λ and the x_i are modeled differently. In all cases known to the authors the experimentalists have used the laws of linear modeling and have forced the model size to be in the same ratio to the true size as the modeled wavelength is to the true wavelength. However, since in our case we cannot do so, let us find the non-linear modeling function

$$\phi = \phi(\psi) \tag{56}$$

which allows a and χ to be arbitrary. From (56) we obtain

$$\begin{aligned} \nabla\phi &= \phi' \nabla\psi , \\ \nabla^2 \phi &= \phi'' (\nabla\psi)^2 + \phi' \nabla^2 \psi \\ &= \phi'' (\nabla \psi)^2 - \ell^2 \psi \phi', \end{aligned} \tag{57}$$

where $(\nabla\psi)^2$ means $\nabla\psi \cdot \nabla\psi$ and use has been made of the fact that $\nabla^2\psi = -\ell^2\psi$. Substitution of (57) into (54) gives

$$(\nabla\psi)^2 \phi'' - \ell^2 \psi\phi' + k^2 \phi = 0. \tag{58}$$

To solve (58) one may proceed as follows: First, take the gradient of (58) to obtain

$$\phi'''(\nabla\psi)^2 \nabla\psi + 2\phi''(\nabla \nabla\psi)\cdot\nabla\psi - \ell^2 \psi\phi''\nabla\psi + (k^2 - \ell^2)\phi'\nabla\psi = 0. \tag{59}$$

Next, divide (59) by $|\nabla\psi|$, on the assumption that $\nabla\psi \neq 0$, to get

$$\phi'''(\nabla\psi)^2 \hat{n} + 2\phi''(\nabla\nabla\psi)\cdot\hat{n} - \ell^2\psi\phi''\hat{n} + (k^2 - \ell^2)\phi'\hat{n} = 0,$$

where $\hat{n} = \dfrac{\nabla\psi}{|\nabla\psi|}$ is the unit normal to any solution surface ψ = constant. Introducing the unit dyadic II, one can write this last equation in the form

$$\left[\phi'''(\nabla\psi)^2\, \mathrm{II} + 2\phi''\nabla\nabla\psi - \ell^2\psi\phi''\,\mathrm{II} + (k^2 - \ell^2)\,\phi'\,\mathrm{II}\right]\cdot\hat{n} = 0. \tag{60}$$

Now since the surface ψ = const., to which $\hat{n}$ is the normal, is quite arbitrary it may be concluded from (60) that

$$\phi'''(\nabla\psi)^2\, \mathrm{II} + 2\phi''\nabla\nabla\psi - \ell^2\psi\phi''\,\mathrm{II} + (k^2 - \ell^2)\phi'\,\mathrm{II} = 0. \tag{61}$$

Next, take the divergence of (61) to get

$$\phi^{iv}(\nabla\psi)^2\nabla\psi + 4\phi'''(\nabla\nabla\psi)\cdot\nabla\psi + (k^2 - 4\ell^2)\phi''\nabla\psi - \ell^2\psi\phi'''\nabla\psi = 0. \tag{62}$$

In obtaining (62) the following results have been used:

(i) $\nabla \cdot \mathrm{II} = 0,$

(ii) $\vec{A} \cdot \mathrm{II} = \vec{A},$ for all $\vec{A}$,

(iii) $\nabla \cdot (\nabla \nabla \psi) = \nabla(\nabla^2 \psi) = -\ell^2 \nabla \psi,$

(iv) $\nabla \nabla \psi$ is self-conjugate.

Of these results, (i), (ii), and (iv) follow directly from the properties and definitions of the dyadics involved. The first part of (iii) follows directly upon expansion of $\nabla \times (\nabla \times \nabla \psi) = 0$, and the second part is a consequence of (55). Next, from (58) one finds that

$$(\nabla \psi)^2 = \ell^2 \left(\frac{\psi \phi' - \chi^2 \phi}{\phi''} \right), \tag{63}$$

where $\chi^2 = k^2/\ell^2$, and from (59) and (63) it follows that

$$2(\nabla \nabla \psi) \cdot \nabla \psi = \ell^2 \left(\frac{\phi'' \left[\psi \phi'' + (1 - \chi^2)\phi' \right] - \phi'''(\psi \phi' - \chi^2 \phi)}{\phi''^2} \right) \nabla \psi. \tag{64}$$

Substitution of (63) and (64) into (62) gives (after division by ℓ^2)

$$\left\{ \frac{\phi^{iv}}{\phi''}(\psi\phi' - \chi^2\phi) + \frac{2\phi'''}{\phi''^2}\left[\phi''(\psi\phi'' + (1-\chi^2)\phi') - \phi'''(\psi\phi' - \chi^2\phi) \right] \right\} \nabla \psi + \left\{ (\chi^2 - 4)\phi'' - \psi\phi''' \right\} \nabla \psi = 0$$

Now it has been assumed that $\nabla \psi \neq 0$, hence this last result implies, after some rearrangement, that

$$\frac{\phi^{iv}}{\phi''}(\psi\phi' - \chi^2\phi) + 2(1-\chi^2)\frac{\phi'\phi'''}{\phi''} - 2\frac{\phi'''^2}{\phi''^2}(\psi\phi' - \chi^2\phi) + (\chi^2-4)\phi'' + \psi\phi''' = 0. \tag{65}$$

A first integral of (65) is[+]

$$\frac{\phi'''}{\phi''^2}(\psi\phi' - \chi^2\phi) = (1-\chi^2)\frac{\phi'}{\phi''} + 3\psi + 2C_1, \tag{66}$$

where $2C_1$ is a constant of integration. Multiplication of (66) by ϕ'' and a rearrangement of the result gives

$$\frac{\phi'''}{\phi''} = \frac{\psi\phi'' + (1-\chi^2)\phi'}{\psi\phi' - \chi^2\phi} + (2C_1 + 2\psi)\frac{\phi''}{\psi\phi' - \chi^2\phi},$$

and if the substitution

$$u = \frac{\phi''}{\psi\phi' - \chi^2\phi} \tag{67}$$

is made, the previous equation becomes

$$\frac{d}{d\psi}(\ln u) = (2C_1 + 2\psi)\,u.$$

Integration of this equation gives

$$\frac{1}{u} = C_2^2 - \psi^2 - 2C_1\psi, \tag{68}$$

[+] See Appendix

where the constant of integration has been taken as $-C_2^2$. Equations (67) and (68) now give

$$(C_2^2 - \psi^2 - 2C_1\psi)\phi'' - \psi\phi' + \chi^2 \phi = 0. \qquad (69)$$

Setting $S = \psi + C_1$ reduces (69) to

$$(m^2 - S^2)\frac{d^2\phi}{dS^2} + (C_1 - S)\frac{d\phi}{dS} + \chi^2 \phi = 0, \qquad (70)$$

where $m^2 = C_1^2 + C_2^2$. Equation (70) is similar in form to the standard hypergeometric equation and under the change of independent variable $S = m(1-2x)$, it becomes

$$x(1-x)\frac{d^2\phi}{dx^2} + \left(\frac{1}{2} - \frac{C_1}{2m} - x\right)\frac{d\phi}{dx} + \chi^2 \phi = 0. \qquad (71)$$

The hypergeometric equation referred to is of the form

$$x(1-x)\frac{d^2 y}{dx^2} + \left[\gamma - (\alpha+\beta+1)x\right]\frac{dy}{dx} - \alpha\beta y = 0, \qquad (72)$$

and comparison of (72) and (71) shows that $\alpha = \pm\chi$, $\beta = \mp\chi$, and $\gamma = 1/2 - C_1/2m$. The analysis from this point depends quite heavily on the value of γ which in turn depends on the constants C_1 and C_2. It is known that the hypergeometric equation has two linearly independent solutions regardless of the value of γ, but when $\gamma = 0, \pm 1, \pm 2, \ldots$, one or both of these solutions possess singularities of various types. Since the experimental parameters which make up γ are under our control, we may avoid these troublesome integral values, however.

The general solution of (72), valid for $|x| < 1$, is

$$y = A\,{}_2F_1(\alpha, \beta; \gamma; x) + Bx^{1-\gamma}\,{}_2F_1(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; x),$$

hence the general solution of (71), valid for $\left|\frac{m-S}{2m}\right| < 1$, is

$$\phi = C_3\,{}_2F_1\left(\chi, -\chi; \frac{1}{2} - \frac{C_1}{2m}; \frac{m-S}{2m}\right)$$

$$+ C_4\left(\frac{m-S}{2m}\right)^{\frac{1}{2}+\frac{C_1}{2m}}\,{}_2F_1\left(\chi+\frac{1}{2}+\frac{C_1}{2m}, -\chi+\frac{1}{2}+\frac{C_1}{2m}; \frac{3}{2}+\frac{C_1}{2m}; \frac{m-S}{2m}\right).$$

Thus, if $\left|\frac{m-C_1-\psi}{2m}\right| < 1$ then the desired functional relationship between ϕ and ψ becomes

$$\phi = C_3\,{}_2F_1\left(\chi, -\chi; \frac{1}{2} - \frac{C_1}{2m}; \frac{m-C_1-\psi}{2m}\right)$$

$$+ C_4\left(\frac{m-C_1-\psi}{2m}\right)^{\frac{1}{2}+\frac{C_1}{2m}}\,{}_2F_1\left(\chi+\frac{1}{2}+\frac{C_1}{2m}, -\chi+\frac{1}{2}+\frac{C_1}{2m}; \frac{3}{2}+\frac{C_1}{2m}; \frac{m-C_1-\psi}{2m}\right). \quad (73)$$

The restriction $\left| \frac{m - C_1 - \psi}{2m} \right| < 1$ may seem rather severe, but since C_1 is subject to our choice, we can produce a value such that the inequality is obeyed if we so desire. As a further remark, attention is called to the fact that if $C_1 = 0$, equation (69) becomes

$$(C_2{}^2 - \psi^2)\, \phi'' - \psi\phi' + \chi^2 \phi = 0,$$

the general solution of which is

$$\phi = C_3 \cos\left(\chi \cos^{-1} \frac{\psi}{C_2}\right) + C_4 \sin\left(\chi \sin^{-1} \frac{\psi}{C_2}\right). \qquad (74)$$

This is the same result obtained by us in the Introduction, equation (11) which is the one-dimensional analogue of equation (54). This suggests that for the usual three-dimensional cases $C_1 \neq 0$.

Maxwell's Equations

Following Stratton (Ref. 3), we will first go through the linear modeling procedure for electromagnetic phenomena. Maxwell's equations, which describe these phenomena, may be written

$$\nabla \times E + \mu \frac{\partial H}{\partial t} = 0 \qquad (75)$$

$$\nabla \times H - \epsilon \frac{\partial E}{\partial t} - \sigma E = 0. \qquad (76)$$

Suppose we construct a model system governed by

$$\nabla' \times E + \mu' \frac{\partial H'}{\partial t'} = 0 \tag{77}$$

and

$$\nabla' \times H' - \epsilon' \frac{\partial H'}{\partial t'} - \sigma' E' = 0. \tag{78}$$

whose variables and parameters are related to those of the system of (75) - (76) by

$$\left.\begin{aligned} E &= \alpha E' \\ H &= \beta H' \\ \epsilon &= \gamma \epsilon' \\ \mu &= \delta \mu' \\ \sigma &= a_1 \sigma' \\ x_i &= a_2 x_i \\ t &= a_3 t' \end{aligned}\right\} \tag{79}$$

Substitution of (79) into (75) gives

$$\nabla' \times E' + \mu' \frac{\partial H'}{\partial t'} \underbrace{\left(\frac{\beta \delta a_2}{a_3 \alpha}\right)}_{I} = 0; \tag{80}$$

Also, substitution into (76) gives

$$\frac{\beta}{a_2} \nabla' \times H' - \frac{\gamma \alpha \epsilon'}{a_3} \frac{\partial E'}{\partial t'} - a_1 \alpha \sigma' E' = 0 \qquad (81)$$

Multiplying by a_2/β:

$$\nabla' \times H' - \underbrace{\frac{\gamma a_2 \alpha}{a_3 \beta}}_{\text{II}} \epsilon' \frac{\partial E'}{\partial t'} - \underbrace{\frac{a_1 \alpha a_2}{\beta}}_{\text{III}} \sigma' E' = 0 \qquad (82)$$

I, II, III are the invariants of the linear modeling. We can eliminate the common ratio α/β among them to obtain

$$\gamma \delta \left(\frac{a_2}{a_3} \right)^2 = \text{constant}$$

$$\delta a_1 \frac{a_2^2}{a_3} = \text{constant} \qquad (83)$$

If the two quantities of (83) cannot be made invariant for a particular modeling application, exact linear modeling is impossible. One such case arises when one attempts to obtain the X-band radar cross section of a B-70 aircraft, where object dimensions and wavelength cannot be scaled in the same manner for a reasonable sized experiment. The method of "approximately linear" models, which has been used in the past, is undesirable.

However, non-linear modeling of Maxwell's equations is now possible because of our results on the scalar wave equation.

It has been found by Schelkunoff (Ref. 4) among others, that the most general electromagnetic field in a source-free region can be described in terms of two <u>scalar wave functions.</u> By this method, if ϕ_1 and ϕ_2 are two scalar functions satisfying

$$\nabla^2 \phi_i + k^2 \phi_i = 0 \quad i = 1, 2 \tag{84}$$

then the electric and magnetic vectors are given by

$$E_x = \frac{1}{\sigma + i\omega\epsilon} \frac{\partial^2 \phi_1}{\partial x \partial z}, \; E_y = \frac{1}{\sigma + i\omega\epsilon} \frac{\partial^2 \phi_1}{\partial y \partial z}, \; E_z = \frac{1}{\sigma + i\omega\epsilon} \left(\frac{\partial^2 \phi_1}{\partial z^2} + k^2 \phi_1 \right) \tag{85}$$

$$H_x = \frac{1}{i\omega\mu} \frac{\partial^2 \phi_2}{\partial x \partial z}, \; H_y = \frac{1}{i\omega\mu} \frac{\partial^2 \phi_2}{\partial y \partial z}, \; H_z = \frac{1}{i\omega\mu} \left(\frac{\partial^2 \phi_2}{\partial z^2} + k^2 \phi_2 \right). \tag{86}$$

Since both ϕ_1 and ϕ_2 are scalar wave functions they can be modeled as was shown in the previous section. This leads us to believe that we may non-linearly model electromagnetic systems in source-free regions, since ϕ_1 and ϕ_2 uniquely represent such systems. The actual techniques which would have to be used to perform this modeling may be summarized as follows: The electric and magnetic fields for the model system are measured. The associated potentials ϕ_1 and ϕ_2 are then found using equations of the form (85), (86). The potentials of the full-scale system may be determined from these by the scalar wave function modeling transformations $\Psi_1 = \Psi_1(\phi_1)$ and $\Psi_2 = \Psi_2(\phi_2)$ given on page 30. The electric and magnetic fields of the full-scale system are found from these by using another set of equations of the form (85), (86).

It should be noted that it is possible to represent an electromagnetic field in terms of two scalar wave functions by other methods which are presumably different from the one described above. Readers are referred to References 5 and 6. The relation of these different scalar wave functions to each other, and to the Debye potentials will be discussed in a forthcoming report.

This paper represents the sum total of our results in non-linear modeling to date. During the next few months we plan to verify the results of this section as was done with the simple harmonic oscillator result in the introduction. If this is successful we plan to perform some experiments using non-linear modeling. This experimental program will be under the direction of Ralph E. Hiatt.

Conclusions

In this paper we have non-linearly modeled the simple harmonic oscillator, the non-linear spring equation, Euler's equation, the Boltzmann equation, a general non-linear oscillator equation, and the scalar wave equation. We have exhibited a method which shows how to non-linearly model Maxwell's equations by utilizing the non-linear modeling of the scalar wave equation.

Despite the a priori expectation that non-linear modeling was at best mathematically very complex, if not impossible, it now appears that there is a possibility of non-linearly modeling more difficult equations, such as the coupled set consisting of Maxwell's equations and the Boltzmann transport equation. Of course, if these non-linear models can be made useful and if results remain general, or at least if the physically interesting cases can be solved, then it appears that one should be able to eliminate a great many expensive facilities which are presently required. It also appears that certain experiments previously considered beyond our present capabilities will now become feasible.

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Appendix

The analysis leading to the first integral, (66) of equation (65), although lengthy, adds little to the main body of the section and is thus included here for the interested reader.

Recall that equation (65) is

$$\frac{\phi^{iv}}{\phi''}(\psi\phi' - \chi^2\phi) + 2(1-\chi^2)\frac{\phi'\phi'''}{\phi''} - 2\frac{\phi'''^2}{\phi''^2}(\psi\phi' - \chi^2\phi) + (\chi^2-4)\phi'' + \psi\phi''' = 0.$$

If this equation is multiplied by $\frac{\phi''^2}{\psi\phi' - \chi^2\phi}$ and the result slightly rearranged, it becomes

$$\phi''\phi^{iv} + (1-\chi^2)\frac{\phi'\phi''' - \phi''^2}{\psi\phi' - \chi^2\phi}\phi'' - \frac{3\phi''^3}{\psi\phi' - \chi^2\phi} + \phi''\phi'''\frac{\psi\phi'' + (1-\chi^2)\phi'}{\psi\phi' - \chi^2\phi} - 2\phi'''^2 = 0.$$

Noting that the coefficient of $\phi''\phi'''$ in the fourth term of this equation is the derivative, with respect to ψ, of $\ln(\psi\phi' - \chi^2\phi)$, one obtains, after solving the equation for this derivative, the result

$$\frac{d}{d\psi}\ln(\psi\phi' - \chi^2\phi) = \frac{2\phi'''}{\phi''} - \frac{\phi^{iv}}{\phi'''} + \frac{3\phi''^2}{\phi'''(\psi\phi' - \chi^2\phi)} - (1-\chi^2)\frac{\phi'\phi''' - \phi''^2}{\phi'''(\psi\phi' - \chi^2\phi)}.$$

Now the first two terms on the right side of this last equation are the derivatives of $\ln(\phi''^2)$ and $-\ln\phi'''$ respectively; hence combining them with the logarithm term on the left, one obtains

$$\frac{d}{d\psi}\ln\left(\frac{\phi'''(\psi\phi'-\chi^2\phi)}{\phi''^2}\right)=\frac{3\phi''^2}{\phi'''(\psi\phi'-\chi^2\phi)}-(1-\chi^2)\frac{\phi'\phi'''-\phi''^2}{(\psi\phi'-\chi^2\phi)\phi'''}\quad. \qquad \text{(A-1)}$$

The last term on the right can be written as

$$-(1-\chi^2)\frac{\phi'^2}{\phi'''(\psi\phi'-\chi^2\phi)}\frac{d}{d\psi}\left(\frac{\phi''}{\phi'}\right),$$

hence, introduction of this result into (A-1) followed by multiplication by

$$\frac{\phi'''(\psi\phi'-\chi^2\phi)}{\phi''^2}\quad\text{gives}$$

$$\frac{d}{d\psi}\left(\frac{\phi'''(\psi\phi'-\chi^2\phi)}{\phi''^2}\right)=3-(1-\chi^2)\frac{1}{(\phi''/\phi')^2}\frac{d}{d\psi}\left(\frac{\phi''}{\phi'}\right)$$

$$=\frac{d}{d\psi}(3\psi+2C_1)+(1-\chi^2)\frac{d}{d\psi}\left(\frac{\phi'}{\phi''}\right),$$

where $2C_1$ is a constant of integration. Equation (66) readily follows.

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