RECURSIVE IDENTIFICATION OF VIBRATING STRUCTURES FROM NOISE-CORRUPTED OBSERVATIONS

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ABSTRACT

In this paper the multiple-input multiple-output (MIMO) *recursive* structural dynamics identification problem is considered. The emphasis is on the development of methods capable of overcoming the inconsistency problem of Recursive Least Squares (RLS) and offering accurate modal parameter estimates from noise-corrupted data. A number of approaches, each one based on a discrete, stochastic estimation scheme and an appropriate modal parameter extraction procedure, are considered. The estimation schemes are multivariable extensions of a recently introduced Recursive Pseudo-Linear Regression (RPLR), the Recursive Extended Least Squares (RELS), and the Recursive Maximum Likelihood (RML) algorithms. The significance of the modal parameter extraction procedure in obtaining accurate mode shape estimates is illustrated, and its close relationship to experiment design and model structure selection issues revealed. Two such procedures are then presented and incorporated in the proposed methods.

The performance of the proposed recursive modal analysis methods is examined through simulated and experimental vibration data, and issues such as estimated model accuracy, convergence, noise rejection, and computational complexity, are critically evaluated with structures characterized by well-separated as well as closely-spaced vibrational modes.

1 INTRODUCTION

The field of Experimental Structural Dynamics Identification has grown rapidly in recent years as new techniques have been introduced and significant advances realized (Ibrahim amd Mikulcik, 1973; Vold and Rocklin, 1982; Eman and Kim, 1983; Juang and Pappa, 1985; Leuridan et al, 1986; Braun and Ram, 1987; Snoyes et al, 1987; Fassois et al, 1989a). The vast majority of the currently available methods are based on batch (nonrecursive) procedures which operate on available data records in an off-line fashion. For a number of applications, however, including the modeling of structures with time-dependent characteristics, adaptive modal control, and the detection of unforeseen structural changes, *recursive* techniques are necessary. These techniques sequentially update a structural dynamics model every time that new information becomes available, and also offer a number of advantages, such as dramatically reduced computer memory requirements and the possibility of data processing until sufficient accuracy (as indicated by the directly available estimator covariance matrix) is achieved, that make them useful alternatives to off-line methods.

Nevertheless, recursive experimental modal analysis methods have been so far discussed in a rather limited number of studies. Davies and Hammond (1984) compared the Recursive Least Squares (RLS) to the Instrumental Variable (IV) and the classical Fourier and Prony methods, and concluded that the RLS can provide accurate modal parameter estimates only in cases where the noise-to-signal (N/S) ratio approaches zero. Indeed, the RLS method is known to give estimates which are highly biased in the presence of noise (Åström and Eykhoff, 1971); a fact that severely limits its applicability

in experimental modal analysis (also see Section 3 of the present paper). The same authors (Davies and Hammond, 1986) utilized the RLS method for the determination of a rough or gross description of multi-modal systems based on estimated lower-order models. Sundararajan and Montgomery (1985) used a recursive least-squares lattice filter for the on-line identification of the order, mode shapes, and modal amplitudes of structural systems. This information was then processed by a recursive, gradient-type equation-error identification algorithm for modal parameter estimation. The method was successfully utilized in an adaptive modal control scheme proposed by Sundararajan et al (1985), and also the identification of the structural dynamics characteristics of a two-dimensional grid structure (Montgomery and Sundararajan, 1985). This approach is, however, also known to perform well only in low N/S ratio environments, and tends to fit the noise dynamics otherwise (Sundararajan et al, 1985).

It is thus evident that the techniques that have been so far proposed for *recursive* structural dynamics identification are inherently *deterministic*, and thus inappropriate for any environment where the N/S ratio is not negligible. Moreover, the issue of *modal parameter extraction* is essentially neglected, and the impulse-invariance discrete-to-continuous transformation typically used without any particular attention paid to the form of the excitation signal and its consequences on the estimated modal parameter accuracy (Davies and Hammond, 1984; Leuridan et al, 1986). In this paper it is shown that accurate mode shape estimation is an issue closely related to experiment design, critically depending upon the excitation signal type (specifically the *intersample* behavior of the excitation), the model structure, and the particular discrete-to-continuous transformation.

The main objective of the present paper is the development of effective *Multiple-Input Multiple-Output (MIMO), recursive, stochastic* structural dynamics identification methods capable of providing accurate modal parameter estimates at realistic N/S ratios and reasonable computational costs. A number of methods are presented based on combinations of MIMO versions of three recursive stochastic estimation algorithms and two modal parameter extraction techniques. The estimation algorithms are multivariable versions of a Recursive Pseudo-Linear Regression (RPLR) algorithm introduced in Fassois (1986) and effectively used in an adaptive control scheme discussed in Fassois et al (1989b), the Recursive Extended Least Squares (RELS), and the Recursive Maximum Likelihood (RML) (Söderström et al, 1978). The modal parameter extraction techniques are based on either impulsive-type excitation combined with model structures with no time-delay and the Impulse Invariance (I.I.) transformation, or, random step excitation combined with model structures with unit time-delay and the Step Invariance (S.I.) transformation. The performance of the proposed modal analysis methods is examined by using both simulated and actual experimental data, and issues such as estimated model accuracy, convergence, noise rejection, and computational complexity are critically evaluated with structures characterized by *well-separated* as well as *closely-spaced* modes.

The paper is organized as follows: The Modal Analysis problem is formulated in Section 2, and the bias error associated with the RLS method briefly discussed in Section 3, where its asymptotic form is derived. The three recursive, stochastic, MIMO estimation algorithms are also presented in Section 3, whereas the issue of modal parameter extraction is addressed in Section 4. Simulation results and a critical evaluation of the

proposed modal analysis methods with structures characterized by both well-separated and closely-spaced modes are presented in subsection 5.1, and their application to the structural dynamics identification of a free-free beam from experimental data in 5.2. The conclusions from this work are finally summarized in Section 6.

2 PROBLEM FORMULATION

The dynamics of a linear, viscously damped structure may be represented by a vector differential equation of the form :

$$\mathbf{M}\ddot{\mathbf{v}}(t) + \mathbf{C}\dot{\mathbf{v}}(t) + \mathbf{K}\mathbf{v}(t) = \mathbf{u}(t) \tag{1}$$

or, equivalently :

$$\mathbf{V}(s) = [\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}]^{-1}\mathbf{U}(s) = \mathbf{G}(s)\mathbf{U}(s) \tag{2}$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K}$ represent the mass, damping and stiffness matrices, $\{\mathbf{u}(t)\}$, $\{\mathbf{v}(t)\}$ the force excitation and resulting displacement vectors, respectively, s the Laplace Transform variable, and $\mathbf{G}(s)$ the corresponding transfer matrix. Each element $G_{ij}(s)$ of $\mathbf{G}(s)$ is a transfer function that may be expressed as :

$$G_{ij}(s) = \sum_{k=1}^{p} \frac{(A_{ij_k} + A^*_{ij_k})s - (A_{ij_k}\mu^*_k + A^*_{ij_k}\mu_k)}{s^2 + 2\xi_k\omega_{n_k}s + \omega^2_{n_k}} \tag{3}$$

where p represents the number of degrees of freedom, $*$ complex conjugation, and A_{ij_k}, μ_k, ω_{n_k}, ξ_k the k-th mode residue, eigenvalue, natural frequency, and damping factor, respectively.

The Experimental Structural Dynamics Identification problem may be then stated as follows : Given excitation $\{\mathbf{u}(t)\}$ and corrupted response $\{\mathbf{x}(t)\}$ data, where :

$$\mathbf{x}(t) = \mathbf{v}(t) + \mathbf{n}(t) \tag{4}$$

with $\{\mathbf{n}(t)\}$ representing a zero-mean, stationary stochastic noise process with autocovariance $\{\mathbf{\Gamma}_{nn}(\tau)\}$, estimate a model of the form (1), or equivalently its corresponding natural frequencies, damping factors, and mode shapes, that characterize the structure under study.

In achieving this objective both stochastic discrete estimation algorithms and appropriate discrete-to-continuous dynamic system transformation techniques for modal parameter extraction are required.

3 MIMO STOCHASTIC RECURSIVE ESTIMATION ALGORITHMS

A sampled-data representation of eqs. (2) and (4) is :

$$\mathbf{x}[k] = \mathbf{G}(B)\mathbf{u}[k] + \mathbf{n}[k] = \mathbf{v}[k] + \mathbf{n}[k] \tag{5}$$

where k represents discrete time, $\mathbf{G}(B)$ the discrete transfer function matrix corresponding to $\mathbf{G}(s)$, B the backshift operator ($B\ \mathbf{x}[k] = \mathbf{x}[k-1]$), and $\{\mathbf{u}[k]\}$, $\{\mathbf{v}[k]\}$, $\{\mathbf{x}[k]\}$, $\{\mathbf{n}[k]\}$, the sampled versions of the excitation, uncorrupted response, corrupted response, and noise signals, respectively. The transfer matrix $\mathbf{G}(B)$ may be parametrized as (El-Sherief and Sinha, 1979) :

$$\mathbf{G}(B) = \begin{bmatrix} \frac{B_{11}(B)}{A_1(B)} & \cdots\cdots & \frac{B_{1p}(B)}{A_1(B)} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{B_{p1}(B)}{A_p(B)} & \cdots\cdots & \frac{B_{pp}(B)}{A_p(B)} \end{bmatrix} \tag{6}$$

with the polynomials $B_{ij}(B)$, $A_i(B)$ being of the form :

$$B_{ij}(B) = b_{o,ij} + b_{1,ij}B + ... + b_{m_{ij},ij}B^{m_{ij}} \tag{7}$$

$$A_i(B) = 1 + a_{1,i}B + ... + a_{n_i,i}B^{n_i} \tag{8}$$

with $A_i(B)$ $(1 \leq i \leq p)$ constrained to be minimum phase. Furthermore, by assuming that the noise autocovariance matrix $\Gamma_{nn}(\tau)$ is diagonal for every τ (a reasonable assumption implying that the scalar noise components that corrupt the response of the structure at each measurement location are uncross-correlated), the multivariable discrete system (5) may be decomposed into p multiple-input single-output (MISO) subsystems of the form :

$$A_i(B)x_i[k] = \sum_{j=1}^{p} B_{ij}(B)u_j[k] + \tilde{n}_i[k] \qquad (1 \leq i \leq p) \tag{9}$$

where $\{x_i[k]\}$,$\{u_i[k]\}$, represent the i-th component of the corrupted response and excitation signals, respectively, and $\{\tilde{n}_i[k]\}$ a noise term defined as :

$$\tilde{n}_i[k] = A_i(B)n_i[k] \tag{10}$$

By further assuming that the process $\{n_i[k]\}$ is characterized by a rational spectral density, the spectral factorization theorem (Åström, 1970) allows for the following Moving Average (MA) innovations representation of $\{\tilde{n}_i[k]\}$:

$$\tilde{n}_i[k] = C_i(B)w_i[k] \tag{11}$$

where $C_i(B)$ is a minimum phase polynomial of the form :

$$C_i(B) = 1 + c_{1,i}B + ... + c_{l_i,i}B^{l_i} \tag{12}$$

and possibly infinite order l_i, and $\{w_i[k]\}$ a zero-mean, white noise process with variance $\sigma^2_{w_i}$.

One of the simplest possible approaches for the recursive estimation of the polynomials $A_i(B)$ and $B_{ij}(B)$ $(1 \leq i \leq p)$ in the subsystem equation (9) then is the one based on the Recursive Least Squares (RLS) algorithm (Åström and Eykhoff, 1971). In the time interval (t, t+1) the RLS attempts to fit an AutoRegressive with eXogeneous inputs (ARX) model of the form :

$$A(B, \mathbf{p}[t]).x[k] = \sum_{j=1}^{p} B_j(B, \mathbf{p}[t]).u_j[k] + e[k, \mathbf{p}[t]] \Longleftrightarrow$$

$$\Longleftrightarrow x[k] = \mathbf{r}^T[k].\mathbf{p}[t] + e[k, \mathbf{p}[t]] \quad (0 \leq k \leq t) \tag{13}$$

to the input/output data. In this representation the subscript i was dropped for the sake of simplicity, the polynomials $B_j(B, \mathbf{p}[t])$, $A(B, \mathbf{p}[t])$ are of the form (7) and (8), respectively, $\{e[k, \mathbf{p}[t]]\}$ represents the prediction error sequence associated with the model, $\mathbf{r}[k]$ the regression vector :

$$\mathbf{r}[k] = [-x[k-1]... - x[k-n] \mid u_1[k]...u_1[k-m_1] \mid .... \mid u_p[k]...u_p[k-m_p]]^T \tag{14}$$

and $\mathbf{p}[t]$ the parameter vector to be estimated :

$$\mathbf{p}[t] = [a_1....a_n \mid b_{o,1}....b_{m_1,1} \mid .... \mid b_{o,p}....b_{m_p,p}]^T \tag{15}$$

Despite its simplicity and low computational complexity, the RLS algorithm is known (Åström and Eykhoff, 1971) to provide estimates that are highly inconsistent (asymptotically biased) in the presence of noise. As an illustration of this phenomenon, consider the RLS-based modal analysis of the two degree-of-freedom system with natural frequencies and damping factors presented in Table 1. The modal parameter estimates are very

accurate in the noise-free case, but, as expected, the situation drastically changes as noise is added, and high estimation errors are observed in the 10% N/S case.

This bias error depends on the second-order properties of the noise and excitation signals as well as the actual system characteristics, and it is very significant even at relatively low N/S ratios. An expression of the asymptotic form of the error for the single-input case may be derived as (see Appendix A):

$$\delta\mathbf{p} = \begin{bmatrix} \delta\mathbf{a} \\ \\ \delta\mathbf{b} \end{bmatrix} = - \begin{bmatrix} (\Gamma_{vv} + \Gamma_{nn} - \Gamma_{vu}\Gamma_{uu}^{-1}\Gamma_{vu}^{T})^{-1}\gamma_{\tilde{n}n} \\ \\ \Gamma_{uu}^{-1}\Gamma_{vu}^{T}(\Gamma_{vv} + \Gamma_{nn} - \Gamma_{vu}\Gamma_{uu}^{-1}\Gamma_{vu}^{T})^{-1}\gamma_{\tilde{n}n} \end{bmatrix} \tag{16.a}$$

where $\delta\mathbf{a}$, $\delta\mathbf{b}$ represent the bias errors associated with the estimates of the coefficients of the A and B polynomials, respectively, Γ_{vv} the covariance of the uncorrupted output vector $\mathbf{v}[k] = [v[k-1]....v[k-n]]^T$, Γ_{uu} the covariance of the input vector $\mathbf{u}[k] = [u[k]....u[k-m]]^T$, Γ_{nn} the covariance of the noise vector $\mathbf{n}[k] = [n[k-1]....n[k-n]]^T$, Γ_{vu} the cross-covariance between $\mathbf{v}[k]$ and $\mathbf{u}[k]$, and :

$$\gamma_{\tilde{n}n} = [\sum_{k=0}^{n} a_k\gamma_{nn}[k-1] \;\; \sum_{k=0}^{n} a_k\gamma_{nn}[k-2] \;\; .... \;\; \sum_{k=0}^{n} a_k\gamma_{nn}[k-n]]^T \tag{16.b}$$

with $\{\gamma_{nn}[k]\}$ representing the autocovariance of $\{n[k]\}$, and $a_o = 1$. From this expression it is evident that the asymptotic bias error will be nonzero even in the uncorrelated noise $\{n[k]\}$ case. Indeed (16.*b*) then becomes :

$$\gamma_{\tilde{n}n} = \sigma_n^2[a_1 \; a_2....a_n]^T = \sigma_n^2\mathbf{a} \tag{16.c}$$

and :

$$\delta\mathbf{p} = -\sigma_n^2 \begin{bmatrix} (\Gamma_{vv} + \sigma_n^2\mathbf{I} - \Gamma_{vu}\Gamma_{uu}^{-1}\Gamma_{vu}^{T})^{-1}\mathbf{a} \\ \\ \Gamma_{uu}^{-1}\Gamma_{vu}^{T}(\Gamma_{vv} + \sigma_n^2\mathbf{I} - \Gamma_{vu}\Gamma_{uu}^{-1}\Gamma_{vu}^{T})^{-1}\mathbf{a} \end{bmatrix} \tag{16.d}$$

where σ_n^2 represents the variance of $\{n[k]\}$ and $\mathbf{I}$ the identity matrix.

As a result of the inconsistency problem the RLS algorithm will not be further discussed, and the emphasis will be on stochastic algorithms capable of offering consistent estimates in the presence of noise. Three such algorithms, based on a Recursive Pseudo-Linear Regression (RPLR) method (Fassois, 1986), the Recursive Extended Least Squares (RELS), and the Recusive Maximum Likelihood (RML) schemes (Södeström et al, 1978), are presented in the sequel.

3.1 The Recursive Pseudo-Linear Regression (RPLR) Algorithm

In this algorithm an output-error model of the form:

$$\begin{aligned} A(B,\mathbf{p}[t]).x[k] &= \sum_{j=1}^{p} B_j(B,\mathbf{p}[t]).u_j[k] + A(B,\mathbf{p}[t]).e_o[k,\mathbf{p}[t]] \Longleftrightarrow \\ \Longleftrightarrow x[k] &= \mathbf{r}_o^T[k,\mathbf{p}[t]].\mathbf{p}[t] + e_o[k,\mathbf{p}[t]] \qquad (0 \leq k \leq t) \end{aligned} \tag{17}$$

is to be estimated in the time interval $(t, t+1)$. In this model structure $\{e_o[k,\mathbf{p}[t]]\}$ represents the output error sequence (defined as the difference between the corrupted system and model responses), $\mathbf{p}[t]$ the parameter vector defined by (15), and $\mathbf{r}_o[k,\mathbf{p}[t]]$ the regression vector :

$$\mathbf{r}_o[k,\mathbf{p}[t]] = \frac{1}{A(B,\mathbf{p}[t])}[0...0 \mid u_1[k]...u_1[k-m_1] \mid ... \mid u_p[k]...u_p[k-m_p]]^T \tag{18}$$

The estimator $\hat{\mathbf{p}}[t]$ is then defined as :

$$\hat{\mathbf{p}}[t] = arg\ min J_o(\mathbf{p}[t]) = arg\ min \sum_{k=1}^{t} \gamma[t,k].e_o^2[k,\mathbf{p}[t]] \tag{19}$$

where *arg min* stands for minimizing argument, and $\gamma[t,k]$ is a forgetting factor introduced in order to enhance the tracking capability of the algorithm. Because of the

dependence of the regression vector $\mathbf{r}_o[k, \mathbf{p}[t]]$ on the parameter vector $\mathbf{p}[t]$, the minimization of $J_o(\mathbf{p}[t])$, however, is a highly nonlinear problem, and a number of difficulties, including high computational complexity and the possible existence of local minima, are encountered.

In order to overcome these problems, the Recursive Pseudo-Linear Regression method is based on the following modified model structure :

$$\begin{aligned} A(B, \mathbf{p}[t]).x[k] &= \sum_{j=1}^{p} B_j(B, \mathbf{p}[t]).u_j[k] + A(B, \hat{\mathbf{p}}[t-1]).e_1[k, \mathbf{p}[t]] \Longleftrightarrow \\ \Longleftrightarrow x_t^F[k] &= \mathbf{r}_1^T[k, \hat{\mathbf{p}}[t-1]].\mathbf{p}[t] + e_1[k, \mathbf{p}[t]] \qquad (0 \leq k \leq t) \end{aligned} \tag{20}$$

with $\{e_1[k, \mathbf{p}[t]]\}$ representing the prediction error sequence associated with the model, and $\mathbf{r}_1[k, \hat{\mathbf{p}}[t-1]]$ the regression vector :

$$\mathbf{r}_1[k, \hat{\mathbf{p}}[t-1]] = \tfrac{1}{A(B, \hat{\mathbf{p}}[t-1])}\mathbf{r}[k] =$$

$$= [-x_t^F[k-1] \ldots - x_t^F[k-n] \mid u_{1t}^F[k] \ldots u_{1t}^F[k-m_1] \mid \ldots. \mid u_{pt}^F[k] \ldots u_{pt}^F[k-m_p]]^T \tag{21}$$

where :

$$u_{jt}^F[k] = \frac{1}{A(B, \hat{\mathbf{p}}[t-1])}.u_j[k] \qquad (1 \leq k \leq t \; ; \; 1 \leq j \leq p) \tag{22.a}$$

$$x_t^F[k] = \frac{1}{A(B, \hat{\mathbf{p}}[t-1])}.x[k] \qquad (1 \leq k \leq t) \tag{22.b}$$

Because of the linear form of the dependence of the model (20) on the parameter vector $\mathbf{p}[t]$, the estimator :

$$\hat{\mathbf{p}}[t] = arg\ min J(\mathbf{p}[t]) = \sum_{k=1}^{t} \gamma[t, k].e_1^2[k, \mathbf{p}[t]] \tag{23}$$

may be now expressed in the simple linear least squares form. Furthermore, by assuming small variations in the coefficients of the estimated autoregressive $A(B)$ polynomial at two consecutive sampling instants such that :

$$A(B, \hat{\mathbf{p}}[t]) \cong A(B, \hat{\mathbf{p}}[t-1]) \tag{24}$$

the model structure (20) approximates the output-error structure (17), and hence the estimator (23) the output-error estimator (19).

By further assuming that $|\gamma[t,k]| < 1 \ \ (t \neq k)$, as is typically the case, and that the estimated autoregressive polynomial $A(B, \hat{\mathbf{p}}[t])$ varies slowly with time, the estimator (23) may be recursively updated. In this work a U-D version (Ljung, 1983) of the algorithm, in which the covariance matrix $\mathbf{P}[t]$ of the estimator $\hat{\mathbf{p}}[t]$ is factored as :

$$\mathbf{P}[t] = \mathbf{U}[t].\mathbf{F}[t].\mathbf{U}^T[t] \tag{25}$$

where $\mathbf{U}[t]$ represents an upper triangular matrix with diagonal elements equal to unity and $\mathbf{F}[t]$ a diagonal matrix, is used. This approach guarantees the positive definiteness of the covariance matrix and improves the numerical properties of the algorithm. The recursive U-D version of the RPLR algorithm may be then shown to be :

1 - Compute :

$$\mathbf{f} = \mathbf{U}^T[t-1].\mathbf{r}_1[t, \hat{\mathbf{p}}[t-1]] \tag{26.a}$$

$$\mathbf{g} = \mathbf{F}[t-1].\mathbf{f} \tag{26.b}$$

$$\beta_o = \lambda[t] \tag{26.c}$$

2 - For $j = 1, ..., d$ (where d is the number of parameters to be estimated) perform steps 3-5 :

3 - Compute :

$$\beta_j = \beta_{j-1} + f_j . g_j \tag{26.d}$$

$$\mathbf{F}_{jj}(t) = \frac{\beta_{j-1}\mathbf{F}_{jj}[t-1]}{\beta_j \lambda[t]} \tag{26.e}$$

$$v_j = g_j \tag{26.f}$$

$$u_j = -\frac{f_j}{\beta_{j-1}} \tag{26.g}$$

4 - For $i = 1, ..., j-1$ perform step 5 (if $j = 1$ skip step 5)

5 - Compute :

$$\mathbf{U}_{ij}[t] = \mathbf{U}_{ij}[t-1] + v_i . u_j \tag{26.h}$$

$$v_i = v_i + \mathbf{U}_{ij}[t-1] . v_j \tag{26.i}$$

6 - Compute :

$$\bar{\mathbf{l}}(t) = (v_1 ..... v_d)^T \tag{26.j}$$

$$\mathbf{l}(t) = \frac{\bar{\mathbf{l}}(t)}{\beta_d} \tag{26.k}$$

$$\bar{e}_1[t] = x_t^F[t] - \mathbf{r}_1^T[t, \hat{\mathbf{p}}[t-1]] . \hat{\mathbf{p}}[t-1] = e_1[t, \hat{\mathbf{p}}[t-1]] \tag{26.l}$$

$$\hat{\mathbf{p}}(t) = \hat{\mathbf{p}}(t-1) + \mathbf{l}(t).\bar{e}_1[t] \tag{26.m}$$

where $\{\lambda[t]\}$ represents a forgetting factor sequence defined as :

$$\gamma[t,k] = \prod_{j=k+1}^{t} \lambda[j] \qquad \gamma[t,t] = 1 \tag{27}$$

and updated through the recursion :

$$\lambda[t] = \begin{cases} \lambda_o.\lambda[t-1] + 1 - \lambda_o & t \leq t_o \\ \lambda & t > t_o \end{cases} \tag{28}$$

where λ_o, $\lambda[0]$, λ are constants selected to be equal or slightly smaller than unity, and t_o represents an appropriately selected time instant. The algorithm is typically initialized by selecting :

$$\mathbf{U}[0] = \mathbf{I} \tag{29.a}$$

$$\mathbf{F}[0] = \alpha\mathbf{I} \qquad \alpha \gg 0 \tag{29.b}$$

$$\mathbf{p}[0] = \mathbf{0} \tag{29.c}$$

The algorithm (26) – (29) updates the parameter vector as soon as the new input and output data samples $u_j[t]$ $(1 \leq j \leq p)$ and $x[t]$, respectively, become available. In forming the regression vector $\mathbf{r}_1[t, \hat{\mathbf{p}}[t-1]]$, however, the sequences $\{u_{jt}^F[k]\}_{k=t-m_j}^{t}$ $(1 \leq j \leq p)$ and $\{x_t^F[k]\}_{k=t-n}^{t}$ need to be evaluated during each recursion. Because of the recursive nature of the filtering required (see eqs. (22)), some approximation is

necessary. In the context of the present work this is done by approximating the Infinite Impulse Response (IIR) filter $A^{-1}(B, \hat{\mathbf{p}}[t-1])$ by a truncated Finite Impulse Response (FIR) filter of the form (Fassois, 1986):

$$H(B, \hat{\mathbf{p}}[t-1]) = 1 + h_1[t-1].B + ... + h_b[t-1].B^b \tag{30}$$

with coefficients calculated through the expression :

$$h_i[t-1] = -\sum_{j=1}^{n} \hat{a}_j[t-1].h_{i-j}[t-1] \qquad (i = 1, 2, ..., b) \tag{31}$$

with $h_o[t-1] = 1$, $h_{i-j}[t-1] = 0$ for $i < j$, and $\{\hat{a}_j[t-1]\}$ $(1 \leq j \leq n)$ representing the AR parameter estimates at time t-1. The required filtered signal samples are then obtained as :

$$u_{jt}^F[k] \cong u_j[k] + \sum_{i=1}^{b} h_i[t-1].u_j[k-i] \qquad (1 \leq j \leq p\ ;\ t - m_j \leq k \leq t) \tag{32.a}$$

$$x_t^F[k] \cong x[k] + \sum_{i=1}^{b} h_i[t-1].x[k-i] \qquad (t - n \leq k \leq t) \tag{32.b}$$

Although somewhat simpler approximate filtering procedures are possible, the aforementioned approach is used since it was found to give the best accuracy. In addition, and in order to assure proper operation of the algorithm, the stability of the filter $A^{-1}(B, \hat{\mathbf{p}}[t-1])$ needs to be examined during every recursion. This may be done by using the Jury criterion (Ackermann, 1985), and in case of instability the filtering may be performed by using the last estimated A(B) polynomial.

The following remarks may be finally made with respect to the RPLR algorithm :

(a) Because of the structure of the model used in this algorithm (compare with the system structure of eqs. (9), (11)), consistency can be achieved in case that the corrupting noise $\{\eta[k]\}$ is uncorrelated.

(b) The computational complexity of the algorithm is : $\frac{3}{2}d^2+\frac{13}{2}d$ multiplication/division operations for the sequence (26), $(2n + m + p + 1)b - \frac{1}{2}n(n + 1)$ $(m = \sum_{j=1}^{p} m_j)$ multiplications for approximate recursive filtering, and $\frac{3}{2}n^2 + \frac{9}{2}n - 4$ multiplications for the stability test.. This gives a total of $\frac{3}{2}d^2 + \frac{13}{2}d + n^2 + 4n + (2n + m + p + 1)b - 4$ multiplication/division operations per recursion.

3.2 The Recursive Extended Least Squares (RELS) Algorithm

The RELS algorithm, which is also known as Recursive Maximum Likelihood version 1 (RML1) (Söderström et al, 1978; Friedlander, 1982), uses an AutoRegressive Moving Average with eXogeneous inputs (ARMAX) model structure, which, in the time interval (t,t+1), may be expressed as :

$$A(B,\mathbf{p}[t]).x[k] = \sum_{j=1}^{p} B_j(B,\mathbf{p}[t]).u_j[k] + C(B,\mathbf{p}[t]).e_2[k,\mathbf{p}[t]] \qquad (0 \le k \le t) \qquad (33)$$

where $\{e_2[k,\mathbf{p}[t]]\}$ represents the prediction error sequence, $B_j(B,\mathbf{p}[t])$, $A(B,\mathbf{p}[t])$ are of the form (7), (8), respectively, $C(B,\mathbf{p}[t])$ is defined as :

$$C(B,\mathbf{p}[t]) = 1 + c_1B + ... + c_lB^l \qquad (34)$$

and $\mathbf{p}[t]$ is the parameter vector to be estimated defined as:

$$\mathbf{p}(t) = [a_1....a_n \mid b_{o,1}....b_{m_1,1} \mid .... \mid b_{o,p}....b_{m_p,p} \mid c_1....c_l]^T \qquad (35)$$

By defining the regression vector :

$$\mathbf{r}_2[k,\mathbf{p}[t]] = [-x[k-1]... - x[k-n] \mid u_1[k]...u_1[k-m_1] \mid .... \mid$$

$$u_p[k]...u_p[k-m_p] \mid e_2[k-1,\mathbf{p}[t]]...e_2[k-l,\mathbf{p}[t]]]^T \tag{36}$$

the ARMAX model may be rewritten as :

$$x[k] = \mathbf{r}_2^T[k,\mathbf{p}[t]].\mathbf{p}[t] + e_2[k,\mathbf{p}[t]] \qquad (0 \leq k \leq t) \tag{37}$$

For the recursive evaluation of the estimate $\hat{\mathbf{p}}[t]$, however, the quantity $\mathbf{r}_2[t,\mathbf{p}[t]]$ needs to be known. This is apparently impossible, since $\hat{\mathbf{p}}[t]$ is yet unavailable, and, in addition, the prediction errors have to be recursively estimated from (33), starting with $k = 0$ and iterating through $k = t$. In the context of the RELS algorithm this difficulty is circumvented by calculating only one approximate prediction error during every recursion. In the recursion implemented in the time interval (t,t+1), the quantity $\bar{e}_2[t]$, defined as :

$$\bar{e}_2[t] = x[t] - \mathbf{r}_2^T[t].\hat{\mathbf{p}}[t-1] \tag{38}$$

is thus calculated, and $\mathbf{r}_2[t,\mathbf{p}[t]]$ is then approximated as :

$$\mathbf{r}_2[t] = [-x[t-1]...-x[t-n] \mid u_1[t]...u_1[t-m_1] \mid .... \mid u_p[t]...u_p[t-m_p] \mid \bar{e}_2[t-1]...\bar{e}_2[t-l]]^T \tag{39}$$

with the approximate prediction errors $\{\bar{e}_2[t-i]\}_{i=1}^{l}$ determined in preceding recursions.

The recursive U-D version of the RELS is then identical to that of the RPLR algorithm after substituting $\mathbf{r}_1[t,\hat{\mathbf{p}}[t-1]]$ by $\mathbf{r}_2[t]$ in (26.a), (26.l) by (38), and $\bar{e}_1[t]$ by $\bar{e}_2[t]$ in (26.m). As a final remark it is noted that the RELS algorithm requires $\frac{3}{2}d^2 + \frac{13}{2}d$ multiplications/divisions per recursion, and, because of the ARMAX model structure used (which corresponds to the actual system structure of eqs.(9), (11)), it has the capability of providing consistent estimates in the presence of either correlated or uncorrelated noise.

3.3 The Recursive Maximum Likelihood (RML) Algorithm

The RML algorithm, which is also known as Recursive Maximum Likelihood version 2 (RML2) (Söderström et al, 1978; Friedlander, 1982), is a recursive, approximate version of the off-line Maximum Likelihood method of Åstrom and Bohlin (1965) based on the ARMAX model structure of eqs. (33) – (37). In this case, however, the regression vector $\mathbf{r}_2[t, \hat{\mathbf{p}}[t]]$ is approximated as :

$$\mathbf{r}_3[t] = [-x[t-1]...-x[t-n] \mid u_1[t]...u_1[t-m_1] \mid .... \mid u_p[t]...u_p[t-m_p] \mid e_3[t-1]...e_3[t-l]]^T \tag{40}$$

with $e_3[k]$ calculated as :

$$e_3[k] = x[k] - \mathbf{r}_3^T[k].\hat{\mathbf{p}}[k] \qquad (t-l \leq k \leq t-1) \tag{41}$$

and used in approximating the prediction error $e_2[k, \mathbf{p}[t]]$.

The recursive U-D version of the RML is then identical to that of the RPLR algorithm after substituting $\mathbf{r}_1[t, \hat{\mathbf{p}}[t-1]]$ by $\mathbf{r}_3^F[t]$, defined as :

$$\mathbf{r}_3^F[t] = \frac{1}{C(B, \hat{\mathbf{p}}[t-1])}.\mathbf{r}_3[t] \tag{42}$$

in $(26.a)$, changing $(26.l)$ to :

$$\bar{e}_3[t] = x[t] - \mathbf{r}_3^T[t].\hat{\mathbf{p}}[t-1] \tag{43}$$

and substituting $\bar{e}_1[t]$ by $\bar{e}_3[t]$ in $(26.m)$. Once again, and because of the recursive filtering operation (42), some approximation is necessary, and for the reasons discussed earlier, the approximate filtering method proposed in the context of the RPLR algorithm is used here as well. The stability of the filter $C^{-1}(B, \hat{\mathbf{p}}[t-1])$ has to be similarly confirmed before filtering is attempted, and this is done by using the Jury criterion.

It is finally remarked that, like the RELS, the RML algorithm has the capability of providing consistent estimates in the presence of correlated or uncorrelated noise, whereas its computational complexity is the highest among all three algorithms, and equals $\frac{3}{2}d^2+\frac{13}{2}d+l^2+4l-4+(n+1)b+(2l+1)b+(m+p)b+(l-1)d$ multiplication/division operations per recursion.

4 MODAL PARAMETER EXTRACTION TECHNIQUES

After the discrete-time models have been obtained, the modal parameters may be evaluated by transforming the estimated models back into the continuous-time domain. As already mentioned, this procedure is crucial for appropriate mode shape estimation, and closely related to experiment design. This becomes apparent by considering the general form of the transformation relationship between the discrete transfer function $G(z^{-1}) = \frac{B_j(z^{-1})}{A(z^{-1})}$ and the corresponding continuous one $G(s) = \frac{B_j(s)}{A(s)}$ (see Appendix B):

$$\begin{aligned} G(s) &= \sum_{k=1}^{p} \frac{(A_k + A_k^*)s - (A_k \ln \lambda_k^* + A_k^* \ln \lambda_k)T^{-1}}{s^2 + 2\xi_k\omega_{n_k}s + \omega_{n_k}^2} \Leftrightarrow \\ \Leftrightarrow G(z^{-1}) &= \sum_{k=1}^{p} \frac{A_k.B(\lambda_k) + A_k^*.B(\lambda_k^*) - (A_k.B(\lambda_k).\lambda_k^* + A_k^*.B(\lambda_k^*).\lambda_k)z^{-1}}{1 - (\lambda_k + \lambda_k^*)z^{-1} + \lambda_k\lambda_k^* z^{-2}} \end{aligned} \quad (44)$$

where (A_k, A_k^*) represent a pair of complex conjugate residues corresponding to a pair of complex conjugate eigenvalues of $G(s)$, (λ_k, λ_k^*) a pair of complex conjugate eigenvalues of $G(z^{-1})$, T the sampling period, z the z-transform variable, and $B_k(\lambda_k)$ the quantity :

$$B(\lambda_k) = Z\{\lambda_k^i \int_{(i-1)T}^{iT} \lambda_k^{-\frac{\tau}{T}} u_j(\tau)d\tau\}.U_j^{-1}(z^{-1}) \quad (45)$$

with i denoting discrete time (t=iT), $Z\{.\}$ z-transform, and $U_j(z^{-1}) = Z\{u_j[k]\}$. Indeed, from this mapping relationship it is evident that for an appropriately selected sampling

period the k-th mode natural frequency and damping factor may be uniquely determined as (Pandit and Wu, 1983) :

$$\omega_{n_k} = \frac{1}{T}\sqrt{[\frac{\ln \lambda_k \lambda_k^*}{2}]^2 + [\arccos \frac{\lambda_k + \lambda_k^*}{2\sqrt{\lambda_k \lambda_k^*}}]^2} \tag{46.a}$$

$$\xi_k = \sqrt{\frac{[\ln \lambda_k \lambda_k^*]^2}{[\ln \lambda_k \lambda_k^*]^2 + 4[\arccos \frac{\lambda_k + \lambda_k^*}{2\sqrt{\lambda_k \lambda_k^*}}]^2}} \tag{46.b}$$

whereas for the determination of the corresponding mode shape vector :

$$\Phi_k = [1 \quad \frac{A_{i2k}}{A_{i1k}} \quad ..... \quad \frac{A_{ipk}}{A_{i1k}}]^T \tag{47}$$

the values of $B(\lambda_k)$, $B(\lambda_k^*)$, and hence the *intersample* behavior of the excitation signals $\{u_j(t)\}_{i=1}^p$ needs to be precisely known. This information also determines the structure of the required discrete-time model, since it indicates its time-delay and the order of the exogeneous polynomial (the degree of the numerator polynomial of the transfer function). As a consequence appropriate mode shape estimation requires careful excitation signal selection and the utilization of the an appropriate model structure and discrete-to-continuous transformation. In this work two such approaches, the first based on the impulse, and the second on the step invariance transformations, will be considered.

(a) The Impulse Invariance (I.I.) Transformation Approach

In this approach the excitation signals consist of trains of "impulsive" functions occuring at the sampling instants. The quantity $B(\lambda_k)$ may be then shown to be equal to unity, and the required discrete-time models of the form ARX(2p,2p-1) or ARMAX(2p,2p-1,l) ($l > 0$) with no time-delay. The continuous-time models thus obtained have impulse response functions coinciding with those of the corresponding discrete-time models at

the sampling instants. The mode shapes may be calculated from (47) after the residues $\{A_{ijk}\}$ have been determined from (44). This method is simple, and despite the fact that it is exact only for impulsive excitation signals (ideally the excitation should consist of a train of mathematical impulses) it is often used in Modal Analysis (Eman and Kim, 1983; Davies and Hammond, 1984; Fassois et al, 1989a).

(b) The Step Invariance (S.I.) Transformation Approach

In this approach the excitation signals are allowed to vary only at the sampling instants; an effect that may be realized through a zero-order holding device. The inter-sample behavior of $\{u_j(t)\}$ is thus modeled as :

$$u_j(t) = u_j[(k-1)T] \qquad (k-1)T < t \leq kT \tag{48}$$

and $B(\lambda_k)$ may be evaluated as :

$$B(\lambda_k) = T\frac{\lambda_k - 1}{\ln \lambda_k} z^{-1} \tag{49}$$

The required discrete-time models are of the form ARX(2p,2p) or ARMAX(2p,2p,l) ($l > 0$) with unit time-delay. The continuous-time model obtained through this procedure has a step response that coincides with the step response of the corresponding discrete-time model at the sampling instants, and the mode shapes are calculated similarly to the previous case, by using eqs. (44), (47), and (49).

Examples of mode shape vectors evaluated through each one of the aforementioned approaches are presented in Section 5.

5 MODAL ANALYSIS RESULTS BASED ON SIMULATED AND EXPERIMENTAL DATA

In this section the performance characteristics of the recursive modal analysis methods based on the RPLR, RELS, RML estimation schemes and the Impulse-Invariance (I.I.) and Step-Invariance (S.I.) transformation approaches for modal parameter extraction, are examined with simulated as well as experimental vibration data.

5.1 Simulation Results

In the simulations two two degree-of-freedom systems, one with *well-separated* and the other with *closely-spaced* modes are considered, and issues such as estimation accuracy, speed of convergence, noise rejection, and computational complexity, discussed.

The exact design of each simulation experiment depends upon the particular modal parameter extraction method used. Impulsive-type force excitation, with values forming a zero-mean, uncorrelated stochastic process, and ARX(4,3) or ARMAX(4,3,2) models with no time-delay are thus used in conjunction with the impulse-invariance transformation approach, whereas excitation signals consisting of uncorrelated random steps occuring at the sampling instants and ARX(4,4) or ARMAX(4,4,2) models with unit time-delay are used in conjunction with the step-invariance transformation approach. In all cases data sets consisting of 2000 points each are employed for identification, and the estimation algorithms are initialized by using $\hat{\mathbf{p}}[0] = \mathbf{0}$, $\alpha = 10^{12}$, and the following forgetting factors :

RPLR : $\lambda[0] = 0.97 \;\; \lambda_o = 0.80 \;\; ; \;\; \lambda[t] = 1 \qquad (t \geq 600)$

RELS, RML : $\lambda[0] = 0.97 \;\; \lambda_o = 0.80 \;\; ; \;\; \lambda[t] = 0.999 \qquad (t \geq 600)$

Identification results obtained by using both noise-free and noise-corrupted data are presented. In the latter case the noise signals used are zero-mean uncorrelated stochastic processes that are uncross-correlated with the force signals, and the N/S ratio is defined as the ratio of the standard deviation of the noise over that of the corresponding uncorrupted response :

$$N/S = \sqrt{\frac{Var[n_i[k]]}{Var[v_i[k]]}} \times 100\% \tag{50}$$

where Var[.] denotes variance.

5.1.1 Vibratory System with Well-Separated Modes

In this case identification of the vibratory system of Figure 1 with modal parameters presented in Table 2 is considered based on force and displacement data sampled with a sampling period $T = 0.5325$ secs.

Modal analysis results obtained through the RPLR scheme and both the impulse-invariance and step-invariance transformations approaches are presented in Table 3 for three (0, 2, and 10%) N/S ratios. As is clearly observed, the natural frequency and damping factor estimates are very accurate in all cases, whereas the accuracy of the mode shape estimates starts to deteriorate at the 10% N/S ratio case. It is nevertheless apparent that no consistency problem (such as that associated with the RLS scheme) is encountered, and the results are considered to be quite accurate. Regarding the impulse and step invariance transformation approaches, they both appear to offer comparable accuracy, with the first one being probably slightly better at the higher N/S cases. The results obtained through the RELS and RML schemes and the impulse invariance transformation are presented in Table 4 for the same three N/S ratios. Both schemes

seem to perform similarly, with the RML results being slightly more accurate. When compared to the RPLR, both the RELS and RML offer comparable or slightly inferior accuracy. This is especially true for the damping factor estimates which deteriorate rather significantly at the 10% N/S case. The results of both schemes coupled with the step invariance approach are analogous to those of the RPLR and are therefore not presented here (Ben Mrad, 1988).

The frequency response curves, as estimated by the RPLR, RELS, and RML schemes in conjunction with the impulse invariance transformation approach, are presented in Figures 2, 3, and 4, respectively, for both the 0 and 10% N/S ratio cases, and very good overall agreement with the theoretical curves is observed. Figures 5 and 6 show the convergence patterns of the estimated transfer function parameters for the RPLR and the RELS schemes coupled with the impulse invariance transformation for the 10% N/S ratio case (the RML convergence patterns are not presented since they are almost identical to those of RELS (Ben Mrad, 1988)). From these plots it is observed that the autoregressive (A) parameters tend to converge faster than the exogeneous (B) parameters, and also that the parameters estimated by the RPLR converge faster than those estimated by the RELS (and RML) schemes. In certain cases convergence problems were encountered with all methods, and covariance resetting had to be used. This technique was found to be more often required, but also more effective, with the RPLR-based methods.

It is finally mentioned that among the methods requiring data prefiltering (RPLR and RML), the RML-based methods are the ones that have most stability problems since the estimated moving-average (C) polynomial appears to be often unstable. Such problems are rather rare with the RPLR methods since the estimated autoregressive (A)

polynomial remains stable for most of the time.

5.1.2 Vibratory System with Closely-Spaced Modes

In this case identification of the vibratory system of Figure 7 with modal parameters presented in Table 5 is considered based on force and displacement data sampled with a sampling period T = 0.0244 secs. The modal analysis results obtained through the RPLR scheme and both the impulse-invariance and step-invariance transformation approaches are presented in Table 6 for three (0, 2, and 10%) N/S ratios. The estimates obtained through the impulse-invariance transformation are accurate in the 0 and 2% N/S ratio cases, but problems are encountered in the 10% case, where no complete mode shape information is obtained (a pair of eigenvalues are estimated as real), and the error associated with the damping factor estimates increases. The estimates obtained through the step-invariance transformation are comparable to those of the impulse-invariance transformation in the 0% and 2% N/S ratio cases, and better in the 10% N/S ratio case. The results obtained through the RELS and RML schemes and the impulse-invariance transformation are presented in Table 7, and are comparable to the corresponding results of the RPLR scheme, with the RML being, once again, somewhat more accurate than the RELS. Similar remarks may be made about the versions of the methods using the step-invariance transformation approach. Regarding the estimated frequency response curves and the parameter convergence patterns, the remarks of the previous subsection are applicable here as well (Ben Mrad, 1988).

In summary, the recursive modal analysis methods are found to be effective in distinguishing the two closely-spaced vibrational modes and providing accurate modal pa-

rameter estimates. When their performance is compared to that associated with systems characterized by well-separated modes, somewhat higher damping factor estimation errors in the noisy cases, and failure to determine complete mode shape information in the 10% N/S ratio (when the impulse-invariance transformation approach was used), were observed. Attempts to overcome these difficulties through model order overdetermination did not prove fruitful (Ben Mrad, 1988).

5.1.3 Discussion

Based on the simulation results presented, the recursive modal analysis methods based on the RPLR, RELS, and RML estimation schemes and the impulse and step-invariance transformation approaches are considered to be quite effective for the identification of systems with well-separated and closely-spaced modes. The inconsistency problem of the deterministic RLS algorithm is overcome, and accurate estimates are obtained in the noise-corrupted data case as well. The accuracy of the methods is comparable, with the RML-based methods being consistently slightly better than the RELS-based ones. On the other hand, the RPLR-based methods appear to give the best convergence characteristics, and, in contrast to the RML methods, instabilities in the required filtering operations are rarely encountered. Regarding computational complexity, the RELS methods offer the lowest load, with the RPLR and RML methods following.

5.2 Experimental Results: Modal Analysis of A Free-Free Beam

The recursive modal analysis methods are now employed for the structural dynamics identification of a free-free beam from laboratory data. The beam used for this purpose is

made of steel, has a rectangular cross section, and only the parameters of its first three modes, having natural frequencies that may be approximately (by neglecting added mass effects of measuring devices, etc, and using linear theory) calculated as ω_n = 482.06, 1328.84, and 2605.07 (r/s), are to be estimated. The damping factors cannot be analytically determined but are expected to be small. The complete experimental configuration is depicted in Figure 8. The beam was excited by a zero-mean band-limited (0 – 2827 r/s) white Gaussian force signal applied through an electromagnetic exciter and measured through a load cell, and the resulting vibration signals were measured through accelerometers. Both excitation and response signals were low-pass filtered with a cut-off frequency of 2827 (r/s) and subsequently sampled with a sampling frequency of 10053 (r/s). In this analysis two sets of data, refered to as Sets 1 and 2 (Figure 9), are utilized.

The determination of the necessary model orders was achieved through the Akaike Information Criterion (AIC) (Akaike, 1981), according to which the model that yields the minimum value of the index :

$$AIC = L.\ln(\frac{RSS}{L}) + 2d \tag{51}$$

with L representing the number of data used, d the number of estimated parameters, and RSS the sum of squares of the prediction errors, is selected as adequate. Furthermore, the relative importance of each vibrational mode is quantified in terms of *dispersion percentages*, defined as (Fassois et al, 1989a):

$$D_k = \frac{d_k}{\sum_{i=1}^{p} d_i} \times 100\% \tag{52}$$

with d_k representing the part of the energy due to the k-th mode and p the total number

of degrees of freedom. The estimation algorithms were initialized exactly as in the simulations.

Identification results obtained through the RPLR, RELS, and RML techniques combined with the impulse-invariance transformation are presented in Tables 8 and 9 for the Data Sets 1 and 2, respectively. In these two tables the adequate model orders, along with natural frequency, damping factor, residue magnitude and phase, as well as dispersion percentage estimates, are shown. As is immediately observed order overdetermination is often necessary with all methods, and four or five degree-of-freedom models are typically required. Such models are indicated as adequate by the AIC criterion, as exemplified in Table 10 for the case of the method based on the RPLR and the impulse invariance transformation approach applied to Data Set 2. Although, model order overdetermination produces extra (false) eigenvalues, those are either real or characterized by frequencies lying outside the frequency range of interest and small dispersion percentages, and are therefore easy to detect. The results of Tables 8 and 9 are in good overall agreement, and the damping factor estimates are considered to be reasonable. The only exception relates to the RML-based estimation for Data Set 2, and this is due to the rather slow convergence of the recursive estimator. The convergence patterns of the transfer function parameters estimated by the RPLR, RELS, and RML-based methods for Data Set 1 are depicted in Figure 10. The convergence of the RPLR method is quite good, but that of both the RELS and RML techniques appears to be very sluggish; a result which is consistent with those of Section 3. Figure 11 finally depicts the magnitude curves of the frequency response characteristics as estimated by all three methods from Data Set 1.

6 CONCLUDING REMARKS

In this paper recursive, stochastic, modal analysis methods that overcome the inconsistency problem of the deterministic RLS approach, were presented. These methods are based on multiple-input multiple-output (MIMO) versions of a recently introduced Recursive Pseudo-Linear Regression (RPLR), the Recursive Extended Least Squares (RELS), and the Recursive Maximum Likelihood (RML) estimation algorithms combined with the Impulse and Step-Invariance discrete-to-continuous transformation approaches for modal parameter extraction. The performance of the methods was evaluated with both simulated and experimental vibration data.

The RPLR-based methods were shown to offer very good accuracy coupled with fast convergence and excellent noise rejection properties. Covariance resetting was often necessary, but also very effective in accelerating convergence. Instability problems in the required data prefiltering were rarely encountered, and model order overdetermination was necessary in certain cases, but certainly not as often as with the other approaches. The computational load of these methods is modest, higher than that of RELS but lower than that of RML-based approaches.

The RELS-based methods were shown to be the least accurate, especially at the higher N/S ratios and in conjunction with systems characterized by closely-spaced modes. Their convergence is significantly slower than that of the RPLR methods, but very comparable to that of RML. Covariance resetting did not seem to accelerate convergence, and model order overdetermination was often necessary for accurate modal parameter identification. The main advantage of these methods is that no data prefiltering is

required and the computational load is therefore lowest.

The RML-based methods were shown to offer good accuracy, which is in general comparable to that of the RPLR methods, and also good noise rejection properties. Their convergence was, however, significantly slower than that of the RPLR methods, and, similarly to RELS, sometimes quite slow, with covariance resetting not providing any essential benefit. Instability problems in the data prefiltering operation was relatively often encountered, so that the stability checking procedure appears to be absolutely necessary. Model order overdetermination was (similarly to RELS) often needed, and the computational load of the methods is highest.

The modal parameter extraction approaches based on the Impulse and Step-Invariance transformations were shown to perform equally well. This was clearly expected since the experimental conditions (the model structure and excitation form) were selected to match the exact requirements of each transformation; a fact that was shown to be important for accurate mode shape estimation.

In summary, the performance of the recursive modal analysis methods of this paper is considered to be quite good at all N/S ratios considered, the discrimination of closely-spaced frequencies excellent, and the improvements over deterministic techniques tremendous.

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Appendix A : The Asymptotic Bias Error Associated with the RLS Method

For simplicity the single-input case will be considered; generalization to the multiple-input case is straightforward. The linear least squares (LLS) estimator of the parameter vector $\mathbf{p}[t]$ of the model (13) may be expressed as (Söderström et al, 1978) :

$$\hat{\mathbf{p}}[t] = (\frac{1}{t+1}\sum_{k=0}^{t}\mathbf{r}[k].\mathbf{r}^T[k])^{-1}.(\frac{1}{t+1}\sum_{k=0}^{t}\mathbf{r}[k].x[k]) \tag{A.1}$$

By rewritting the system equation (9) as :

$$x[k] = \mathbf{r}^T[k].\mathbf{p} + \tilde{n}[k] \tag{A.2}$$

and substituting into $(A.1)$ the expression :

$$\hat{\mathbf{p}}[t] = \mathbf{p} + (\frac{1}{t+1}\sum_{k=0}^{t}\mathbf{r}[k].\mathbf{r}^T[k])^{-1}.(\frac{1}{t+1}\sum_{k=0}^{t}\mathbf{r}[k].\tilde{n}[k]) \tag{A.3}$$

is obtained. The asymptotic bias error is then equal to :

$$\begin{aligned}\delta\mathbf{p} &= p\lim_{t\to\infty}(\frac{1}{t+1}\sum_{k=0}^{t}\mathbf{r}[k].\mathbf{r}^T[k])^{-1}.(\frac{1}{t+1}\sum_{k=0}^{t}\mathbf{r}[k].\tilde{n}[k]) \\ &= (p\lim_{t\to\infty}\frac{1}{t+1}\sum_{k=0}^{t}\mathbf{r}[k].\mathbf{r}^T[k])^{-1}.(p\lim_{t\to\infty}\frac{1}{t+1}\sum_{k=0}^{t}\mathbf{r}[k].\tilde{n}[k])\end{aligned} \tag{A.4}$$

where $p\lim$ denotes limit in probability and Frechet's theorem (Cramer, 1946) was used. By invoking standard ergodicity results (Söderström , 1975), $(A.4)$ may be written as :

$$\delta\mathbf{p} = (E\{\mathbf{r}[k].\mathbf{r}^T[k]\})^{-1}.E\{\mathbf{r}[k].\tilde{n}[k]\} \tag{A.5}$$

where $E\{.\}$ denotes expectation. Furthermore, by decomposing $\mathbf{r}[k]$ as (see (5) and (14)):

$$\mathbf{r}[k] = \begin{bmatrix} -\mathbf{v}[k] \\ \mathbf{u}[k] \end{bmatrix} + \begin{bmatrix} -\mathbf{n}[k] \\ \mathbf{0} \end{bmatrix} \tag{A.6}$$

where :

$$\mathbf{v}[k] = [v[k-1]....v[k-n]]^T \tag{A.7}$$

$$\mathbf{u}[k] = [u[k]....u[k-m]]^T \tag{A.8}$$

$$\mathbf{n}[k] = [n[k-1]....n[k-n]]^T \tag{A.9}$$

substituting into $(A.5)$, and using the fact that the sequence $\{n[k]\}$, as well as its filtered version $\{\tilde{n}[k]\}$, are orthogonal (uncorrelated and zero-mean) to $\{u[k]\}$, and therefore $\{v[k]\}$, the following expression is obtained :

$$\delta\mathbf{p} = -\begin{bmatrix} \Gamma_{vv} + \Gamma_{nn} & -\Gamma_{vu} \\ -\Gamma_{vu}^T & \Gamma_{uu} \end{bmatrix}^{-1} . \begin{bmatrix} \gamma_{\tilde{n}n} \\ \mathbf{0} \end{bmatrix} \tag{A.10}$$

where $\Gamma_{vv} = E\{\mathbf{v}[k].\mathbf{v}^T[k]\}$, $\Gamma_{uu} = E\{\mathbf{u}[k].\mathbf{u}^T[k]\}$, $\Gamma_{vu} = E\{\mathbf{v}[k].\mathbf{u}^T[k]\}$ and :

$$\gamma_{\tilde{n}n} = E\{[n[k-1].\tilde{n}[k].....n[k-n].\tilde{n}[k]]^T\} \tag{A.11}$$

By using (10), the vector $\gamma_{\tilde{n}n}$ may be expressed as :

$$\gamma_{\tilde{n}n} = [\sum_{k=0}^{n} a_k\gamma_{nn}[k-1] \quad \sum_{k=0}^{n} a_k\gamma_{nn}[k-2]..... \sum_{k=0}^{n} a_k\gamma_{nn}[k-n]]^T \tag{A.12}$$

where $\{\gamma_{nn}[k]\}$ represents the autocovariance of $\{n[k]\}$, and $a_o = 1$. Since the matrix to be inverted in $(A.10)$ is positive definite hermitian (because it is a covariance matrix and the input signal $\{x[k]\}$ is persistently exciting of sufficient order), the "inside-out rule" of matrix algebra (Lewis and Odell, 1971) may be applied to give :

$$\delta\mathbf{p} = \begin{bmatrix} \delta\mathbf{a} \\ \delta\mathbf{b} \end{bmatrix} = -\begin{bmatrix} (\Gamma_{vv} + \Gamma_{nn} - \Gamma_{vu}\Gamma_{uu}^{-1}\Gamma_{vu}^T)^{-1}.\gamma_{\tilde{n}n} \\ \Gamma_{uu}^{-1}\Gamma_{vu}^T(\Gamma_{vv} + \Gamma_{nn} - \Gamma_{vu}\Gamma_{uu}^{-1}\Gamma_{vu}^T)^{-1}.\gamma_{\tilde{n}n} \end{bmatrix} \tag{A.13}$$

where $\delta\mathbf{a}$, $\delta\mathbf{b}$ represent the asymptotic bias errors associated with the estimates of the coefficients of the A and B polynomials, respectively.

Appendix B : The General Form of the Discrete-To-Continuous Transformation

Consider the continuous-time transfer function $G(s) = \frac{V(s)}{U_j(s)}$ in the modal decomposition form :

$$G(s) = \sum_{k=1}^{2p} \frac{A_k}{s - \mu_k} \tag{B.1}$$

where μ_k, A_k represent the k-th eigenvalue and residue, respectively. The part of the model response dùe to the k-th eigenvalue will then be :

$$v_k(t) = A_k \,.\exp(\mu_k t) \int_0^t \exp(-\mu_k \tau).u_j(\tau)d\tau \tag{B.2}$$

Discretization (with $t = lT$, T representing the sampling period) and further manipulating of $(B.2)$ yields :

$$[1 - \exp(\mu_k T)z^{-1}].V_k(z^{-1}) = A_k.\ Z\{\exp(\mu_k lT) \int_{(l-1)T}^{lT} \exp(-\mu_k \tau).u_j(\tau)d\tau\} \tag{B.3}$$

where $Z\{.\}$ denotes z-transform, $V_k(z^{-1}) = Z\{v_k[lT]\}$, and z the z-transform variable. From this expression it is evident that the discrete eigenvalue corresponding to μ_k is :

$$\lambda_k = \exp(\mu_k T) \tag{B.4}$$

and $(B.3)$ may be equivalently rewritten as :

$$\frac{V_k(z^{-1})}{U_j(z^{-1})} = \frac{A_k.B(\lambda_k)}{1 - \lambda_k z^{-1}} \tag{B.5}$$

with :

$$B(\lambda_k) = Z\{\lambda_k^l \int_{(l-1)T}^{lT} \lambda_k^{-\frac{\tau}{T}} u_j(\tau)d\tau\}.U_j^{-1}(z^{-1}) \tag{B.6}$$

and $U_j(z^{-1}) = Z\{u_j[lT]\}$. The discrete-time transfer function model $G(z^{-1}) = \frac{V(z^{-1})}{U_j(z^{-1})}$ corresponding to $(B.1)$ then is :

$$G(z^{-1}) = \sum_{k=1}^{2p} \frac{A_k.B(\lambda_k)}{1 - \lambda_k z^{-1}} \tag{B.7}$$

and for the structural systems with complex conjugate eigenvalues this correspondence may be expressed as :

$$\begin{aligned} G(s) &= \sum_{k=1}^{p} \frac{(A_k + A_k^*)s - (A_k.\ln\lambda_k^* + A_k^*.\ln\lambda_k).T^{-1}}{s^2 + 2\xi_k\omega_{n_k}s + \omega_{n_k}^2} \Leftrightarrow \\ \Leftrightarrow G(z^{-1}) &= \sum_{k=1}^{p} \frac{A_k.B(\lambda_k) + A_k^*.B(\lambda_k^*) - [A_k.B(\lambda_k).\lambda_k^* + A_k^*.B(\lambda_k^*).\lambda_k]z^{-1}}{1 - (\lambda_k + \lambda_k^*)z^{-1} + \lambda_k\lambda_k^*z^{-2}} \end{aligned} \tag{B.8}$$

where ω_{n_k}, ξ_k denote the k-th mode natural frequency and damping factor, respectively, the star complex conjugate, and $(B.4)$ was used.

Table 1 : RLS-based modal analysis for a two degree-of-freedom system.

	Theoretical Parameters	RLS Estimates	
		0 % N/S	10 % N/S
ω_n (rad/sec)	62.444	62.439	62.601
	64.377	64.385	80.008
ξ	0.0482	0.0482	0.0275
	0.1776	0.1776	0.5515

Table 2 : Theoretical modal parameters for the system with well-separated modes.

ω_n (rad/sec)	ξ	Mode Shapes
1.516	0.0758	(1 , 2.351 + j 0.0)
2.950	0.1475	(1 , -0.851 + j 0.0)

Table 3 : Modal parameters of the system with well-separated modes estimated via the RPLR scheme and the I.I. and S.I. transformation approaches.

	Impulse Invariance			*Step Invariance*		
N/S Ratio (%)	0	2	10	0	2	10
ω_n	1.516	1.516	1.516	1.516	1.516	1.516
(rad/sec)	2.948	2.949	2.961	2.950	2.953	2.970
% Errors in	0.0092	0.0125	0.0079	0	0.0026	0.0031
ω_n	0.0576	0.0244	0.3850	0	0.0979	0.6936
ξ	0.0758	0.0755	0.0743	0.076	0.0761	0.0777
	0.1472	0.1475	0.1475	0.147	0.1478	0.1518
% Errors in	0.0699	0.4261	1.9431	0	0.4643	2.5900
ξ	0.1911	0.0258	0.0325	0	0.2359	2.9341
Mode Shapes	(1,2.351	(1,2.358	(1,2.513	(1,2.351	(1,2.347	(1,2.271
	+j0.0)	+ j0.0018)	+j0.1549)	+j0.0)	-j0.002)	-j0.0078)
	(1,-0.851	(1,-0.852	(1,-0.443	(1,-0.851	(1,-0.872	(1,-1.928
	+j0.0)	- j0.0005)	+ j0.161)	+j0.0)	-j0.0065)	+j0.6872)
% Errors in	0.0122	0.3152	9.5312	0.0029	0.0872	3.4213
Mode Shapes	0.0178	0.1623	51.556	0.0108	2.6310	150.1534

Table 4 : Modal parameters of the system with well-separated modes estimated via the RELS and RML schemes with the I.I. transformation approach.

	RELS			*RML*		
N/S Ratio (%)	0	2	10	0	2	10
ω_n	1.516	1.519	1.531	1.516	1.518	1.522
(rad/sec)	2.948	2.968	2.976	2.948	2.964	2.990
% Errors in	0.0092	0.0211	0.9709	0.0092	0.1451	0.3706
ω_n	0.0573	0.6308	0.8959	0.0573	0.4840	1.3661
ξ	0.0758	0.0758	0.0794	0.0757	0.0759	0.0795
	0.1472	0.1490	0.1749	0.1472	0.1483	0.1674
% Errors in	0.0699	0.0435	4.7188	0.0699	0.1411	4.8903
ξ	0.1911	1.0224	19.5623	0.1911	0.5885	13.5519
Mode Shapes	(1,2.351	(1,2.332	(1,2.508	(1,2.351	(1,2.341	(1,2.261
	+j0.0)	- j0.0607)	+j0.4372)	+j0.0)	-j0.0447)	+j0.1878)
	(1,-0.851	(1,-0.879	(1,-0.499	(1,-0.851	(1,-0.852	(1,-0.916
	+j0.0)	- j0.0111)	+ j0.562)	+j0.0)	+j0.0059)	+j0.209)
% Errors in	0.0003	2.6944	19.7611	0.0003	1.9504	8.8474
Mode Shapes	0.0131	3.5750	77.9592	0.0131	0.7132	25.7419

Table 5 : Theoretical modal parameters for the system with closely-spaced modes.

ω_n (rad/sec)	ξ	Mode Shapes
62.444	0.0482	(1 , 1.476 + j 0.1541)
64.377	0.1776	(1 , -0.661 + j 0.0704)

Table 6 : Modal parameters of the system with closely-spaced modes estimated via the RPLR scheme and the I.I. and S.I. transformation approaches.

	Impulse Invariance			*Step Invariance*		
N/S Ratio (%)	0	2	10	0	2	10
ω_n	62.401	62.411	62.427	62.442	62.442	62.441
(rad/sec)	64.349	64.341	64.553	64.385	64.495	65.271
% Errors in	0.0696	0.0530	0.0275	0.0091	0.0033	0.0057
ω_n	0.0437	0.0569	0.2732	0.0122	0.1832	1.3883
ξ	0.0481	0.0476	0.0454	0.0482	0.0481	0.0474
	0.1773	0.1811	0.2008	0.1776	0.1784	0.1851
% Errors in	0.3358	1.2896	5.9545	0.1490	0.2357	1.5775
ξ	0.1418	1.9640	13.0568	0.0567	0.4645	4.2501
Mode Shapes	(1,1.478	(1,1.495	(1,2.170	(1,1.487	(1,1.500	(1,1.676
	+j0.1525)	+ j0.1906)	+j0.2219)	+j0.1476)	+j0.1348)	+j0.077)
	(1,-0.663	(1,-0.677	2 real	(1,-0.650	(1,-0.660	(1,-0.499
	+j0.0717)	+ j0.0914)	roots	+j0.0644)	+j0.0603)	-j0.1236)
% Errors in	0.1449	2.7771	46.9571	0.8219	2.0615	20.5834
Mode Shapes	0.3676	4.0574	——	1.8116	1.5266	25.6657

Table 7 : Modal parameters of the system with closely-spaced modes estimated via the RELS and RML schemes with the I.I. transformation approach.

	RELS			*RML*		
N/S Ratio (%)	0	2	10	0	2	10
ω_n	62.401	62.423	62.792	62.401	62.406	62.446
(rad/sec)	64.349	64.217	63.417	64.349	64.312	64.233
% Errors in	0.0696	0.0339	0.5566	0.0696	0.0615	0.0017
ω_n	0.0437	0.2480	1.4910	0.0437	0.1009	0.2235
ξ	0.0481	0.0484	0.0427	0.0480	0.0481	0.0485
	0.1773	0.1758	0.1934	0.1773	0.1769	0.1822
% Errors in	0.3358	0.4519	11.3866	0.3358	0.1285	0.8127
ξ	0.1418	1.0146	8.9013	0.1418	0.3727	2.6397
Mode Shapes	(1,1.4779	(1,1.902	(1,2.164	(1,1.478	(1,1.902	(1,2.032
	+j0.1526)	+ j0.1344)	+j0.4650)	+j0.1526)	+j0.1298)	+j0.1596)
	(1,-0.663	(1,-0.403	2 real	(1,-0.663	(1,-0.348	2 real
	+j0.0717)	+ j0.0203)	roots	+j0.0717)	+j0.0346)	roots
% Errors in	0.1431	28.6835	50.8245	0.1435	28.7439	37.4210
Mode Shapes	0.3582	39.4940	——	0.3582	47.2946	——

Table 8 : Modal parameters of the beam estimated from Data Set 1 (I.I. transformation approach).

Method	RPLR	RELS	RML
Model	ARX(6,5)	ARMAX(8,7,3)	ARMAX(10,9,3)
Natural Frequency (rad/sec)	455.626 1300.648 2519.342	443.083 1299.910 2505.942 3624.761	448.0478 1289.3639 2535.4385 2534.1314 2 real roots
Damping Factor	0.00533 0.00093 0.00074	0.06546 0.00296 0.00208 0.13969	0.11691 0.00586 0.00597 0.08405
Residue Magnitude	0.0681 0.1003 0.3003	0.0525 0.0988 0.3241 0.2364	0.0368 0.0887 0.3636 0.1647
Residue Phase (rad)	-1.3741 1.9773 2.3874	-1.6427 2.0521 2.4046 -0.0378	-0.8898 1.8001 2.6259 0.9093
Dispersion Percentage (%)	3.2602 14.1583 82.5815	0.4395 11.1156 87.7805 0.6644	-3.1087 -12.4967 105.9337 2.2918

Table 9 : Modal parameters of the beam estimated from Data Set 2 (I.I. transformation approach).

Method	RPLR	RELS	RML
Model	ARX(8,7)	ARMAX(10,9,3)	ARMAX(8,7,3)
Natural Frequency (rad/sec)	462.052 1300.158 2521.719 2 real roots	461.608 1299.174 2522.311 3752.775 2 real roots	369.228 1287.184 2534.449 2 real roots
Damping Factor	0.00411 0.00072 0.00072	0.05324 0.00123 0.00047 0.17983	0.09941 0.02102 0.00285
Residue Magnitude	0.0608 0.0272 0.0547	0.0593 0.0275 0.0556 0.0176	0.1310 0.0216 0.0601
Residue Phase (rad)	-1.3463 2.1387 -1.0675	-1.4503 2.0653 -1.0644 0.1993	-0.9886 3.1365 -1.0235
Dispersion Percentage (%)	42.0922 17.2770 35.8130	-8.4499 -27.5732 -151.8973 0.6120	358.9265 14.7624 -352.1869

Table 10 : AIC values for different model orders (RPLR and I.I. transformation approach, Data Set 2).

Model Order	AIC
ARX(6,5)	583.02
ARX (8,7)	478.93
ARX(10,9)	569.92

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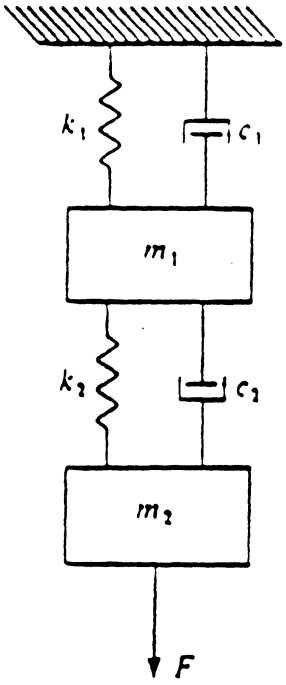


$m_1 = 2\,kg \quad m_2 = 1\,kg$
$c_1 = 1\,N.s/m \quad c_2 = 0.4\,N.s/m$
$k_1 = 10\,N/m \quad k_2 = 4\,N/m$

Figure 1 : Two degree-of-freedom system with well-separated modes.

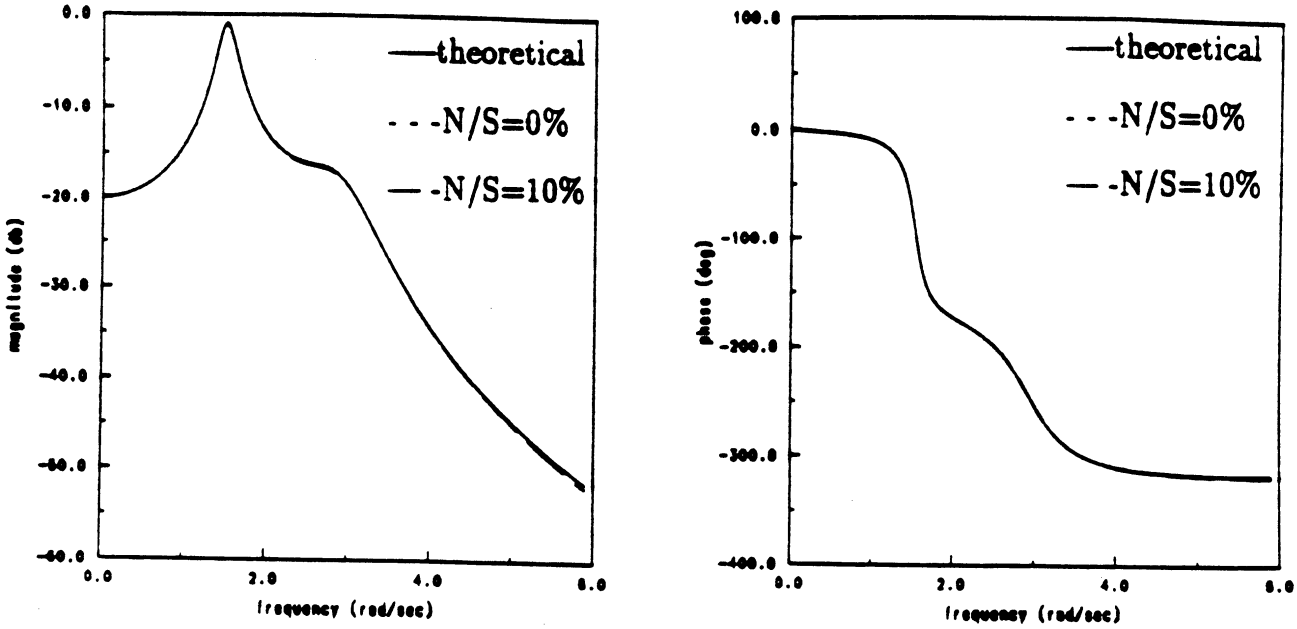


Figure 2 : Frequency response functions estimated by the RPLR scheme and the I.I. transformation for the system with well-separated modes.

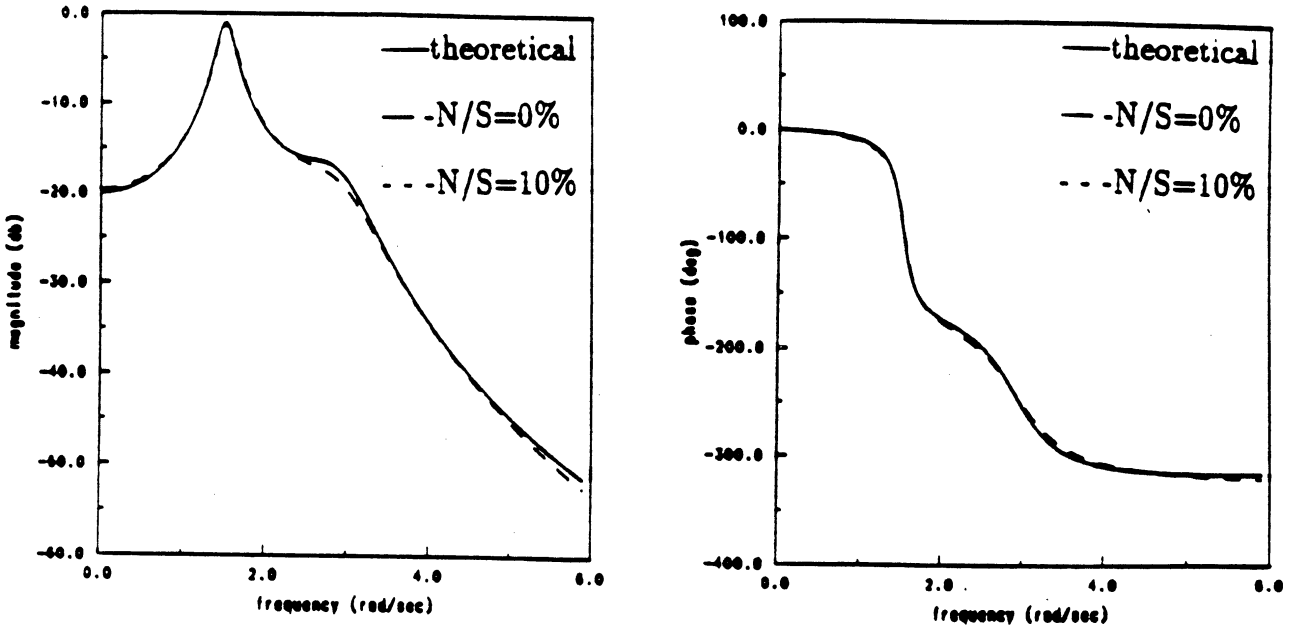


Figure 3 : Frequency response functions estimated by the RELS scheme and the I.I. transformation for the system with well-separated modes.

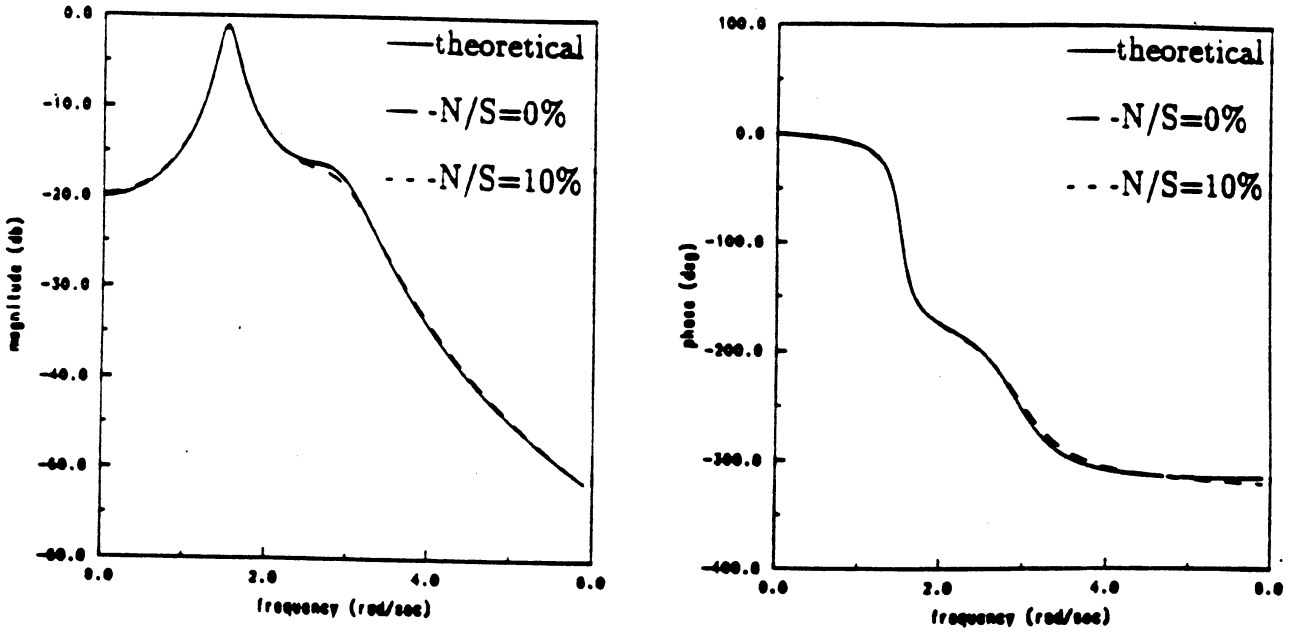


Figure 4 : Frequency response functions estimated by the RML scheme and the I.I. transformation for the system with well-separated modes.

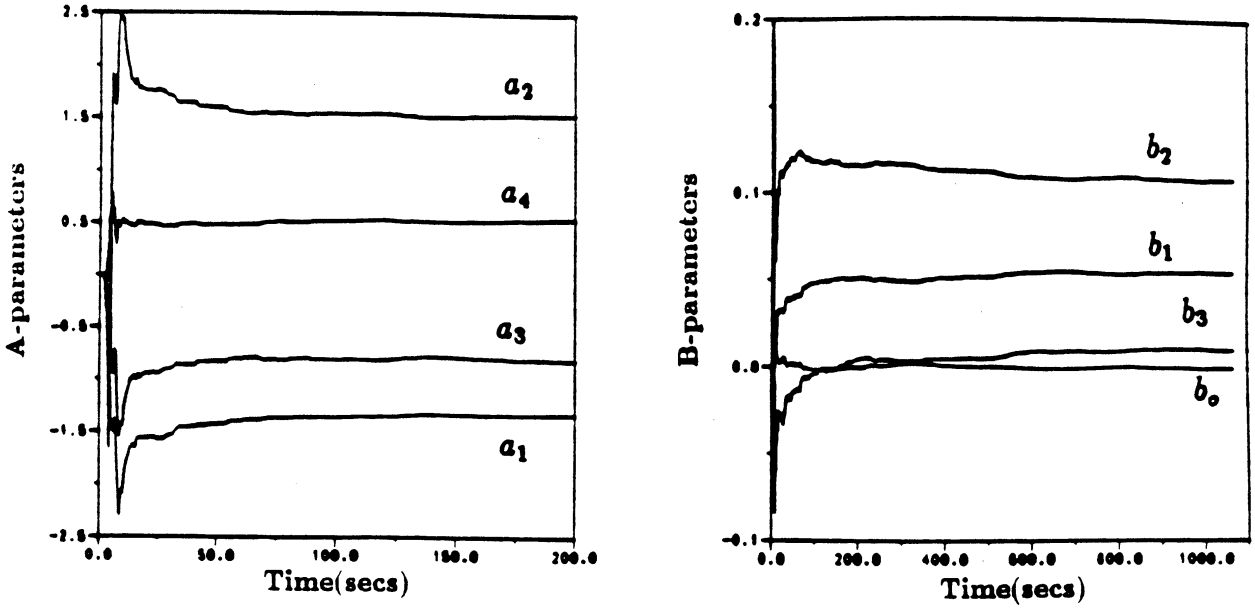


Figure 5 : Model parameters estimated by the RPLR scheme for the system with well-separated modes (N/S = 10 % ; L.L. transformation).

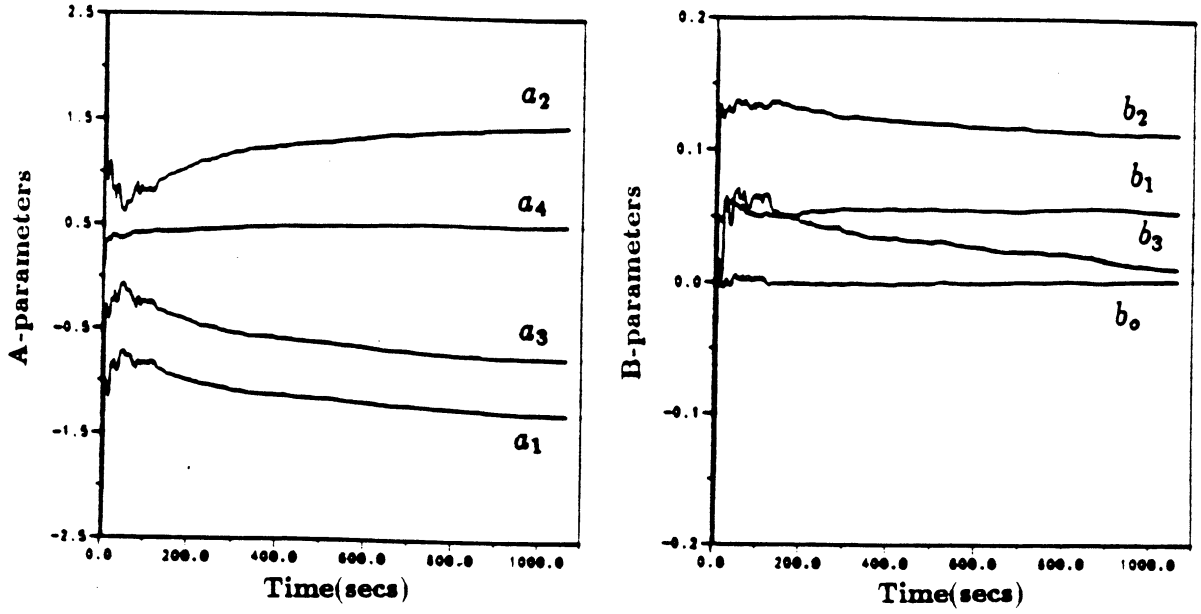


Figure 6 : Model parameters estimated by the RELS scheme for the system with well-separated modes (N/S = 10 % ; I.I. transformation).

$m_1 = 4.5\ kg \quad m_2 = 4.5\ kg$
$c_1 = 45\ N.s/m \quad c_2 = 35\ N.s/m$
$k_1 = 17500\ N/m \quad k_2 = 600\ N/m$
$k_3 = 17500\ N/m \quad c_3 = 15\ N.s/m$

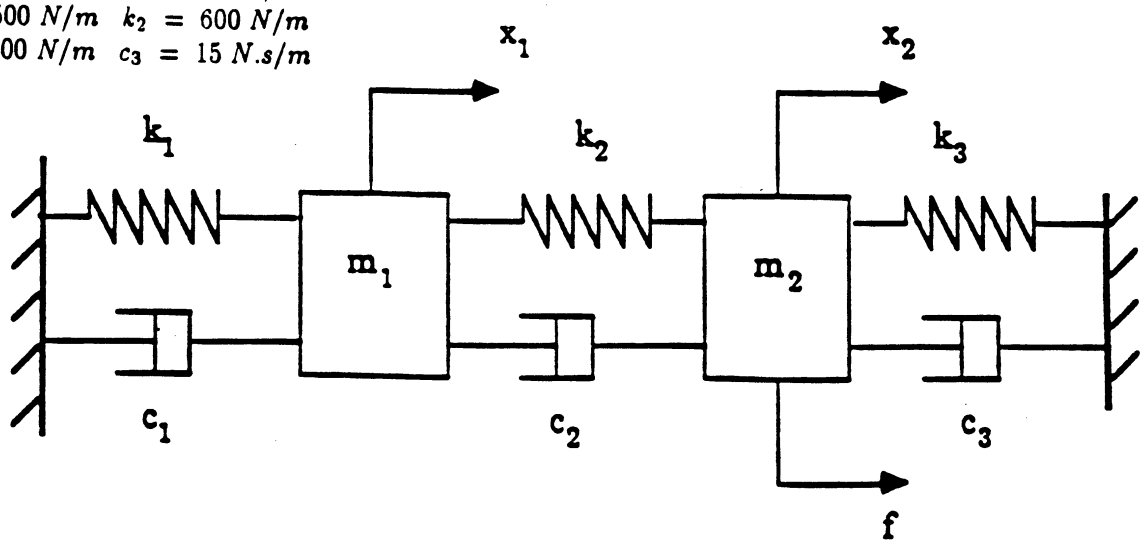


Figure 7 : Two degree-of-freedom system with closely-spaced modes.

1,2 - Accelerometers
3 - Force Sensor
4,5,6 - Amplifier
7 - Vibration Exciter
8 - Power Amplifier
9 - Filter
10 - FFT Analyzer
11 - Data acquisition

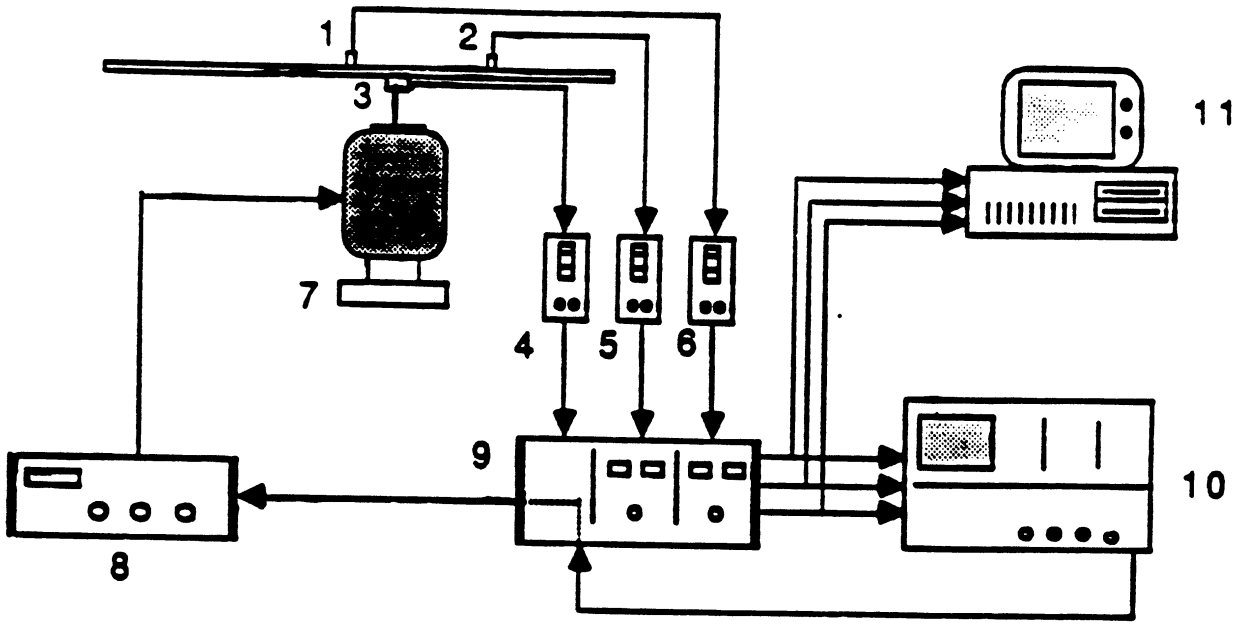
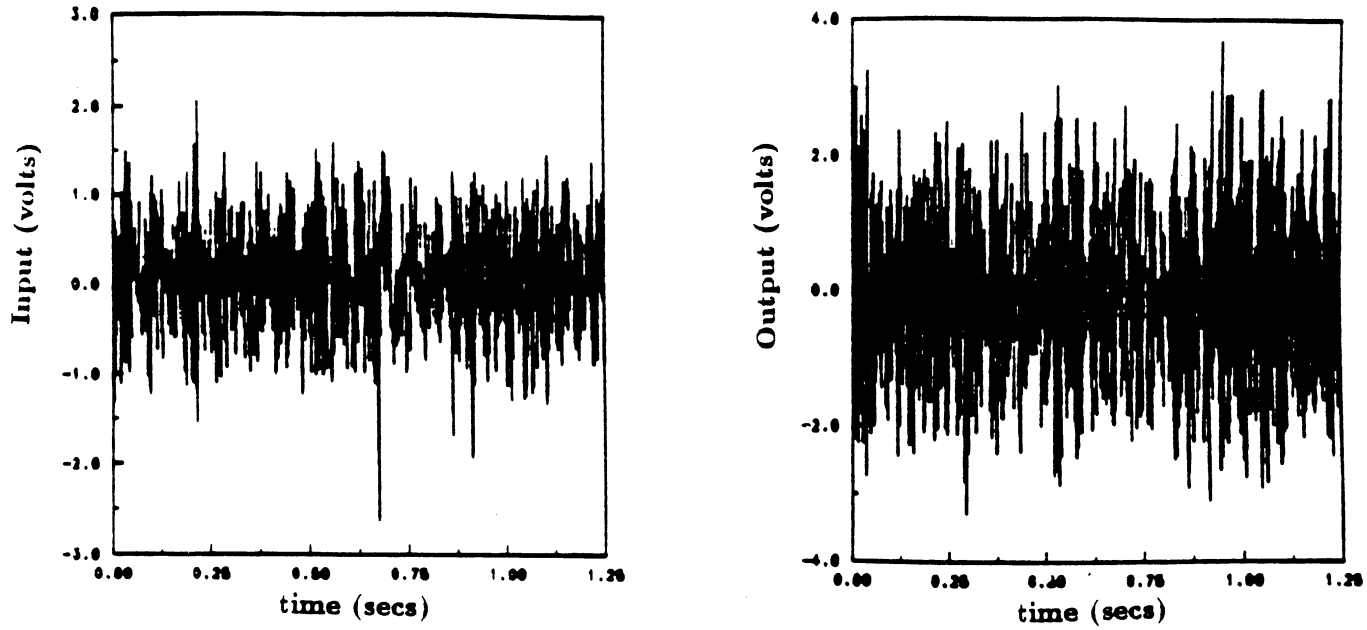
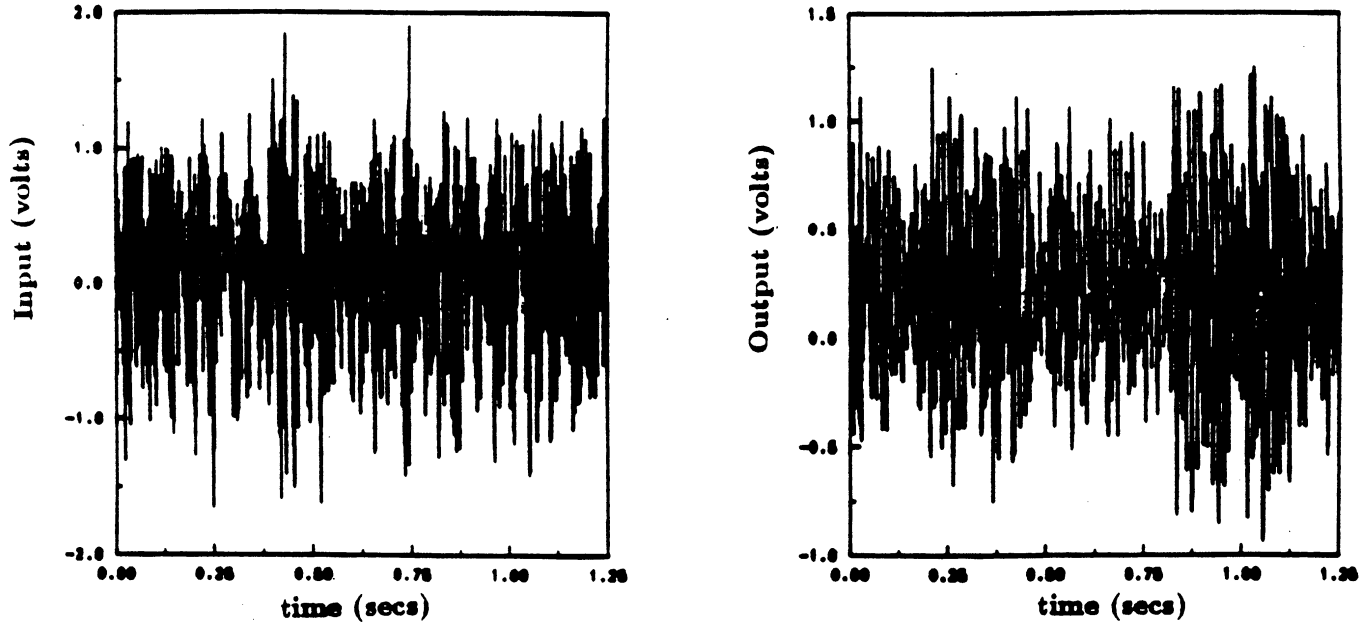


Figure 8 : Experimental set-up for the structural dynamics identification of the free-free beam.

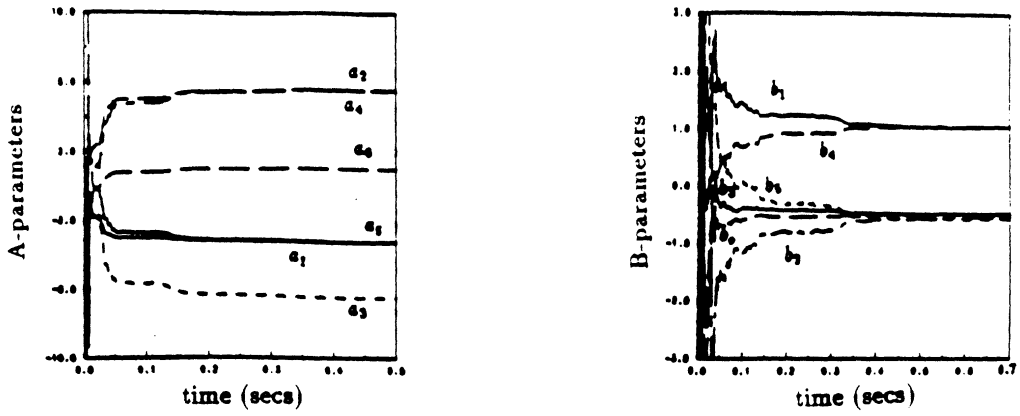


(a)

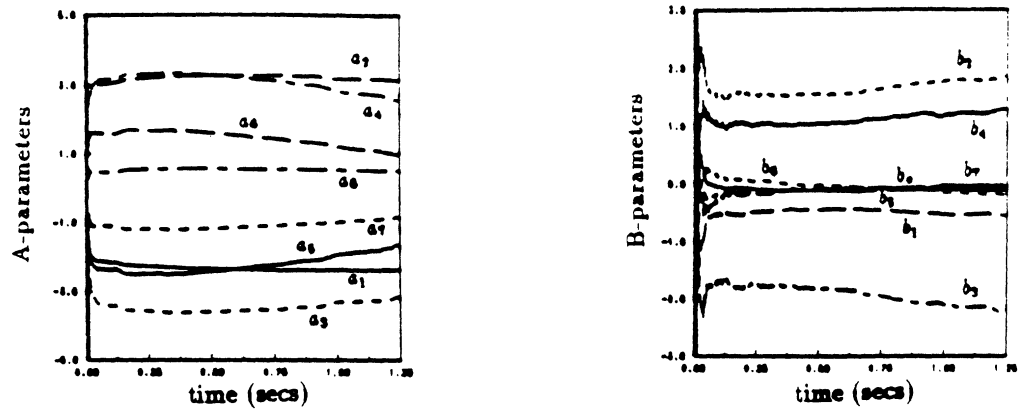


(b)

Figure 9 : The experimental force and acceleration data : (a) Set 1 (b) Set 2.



(a)



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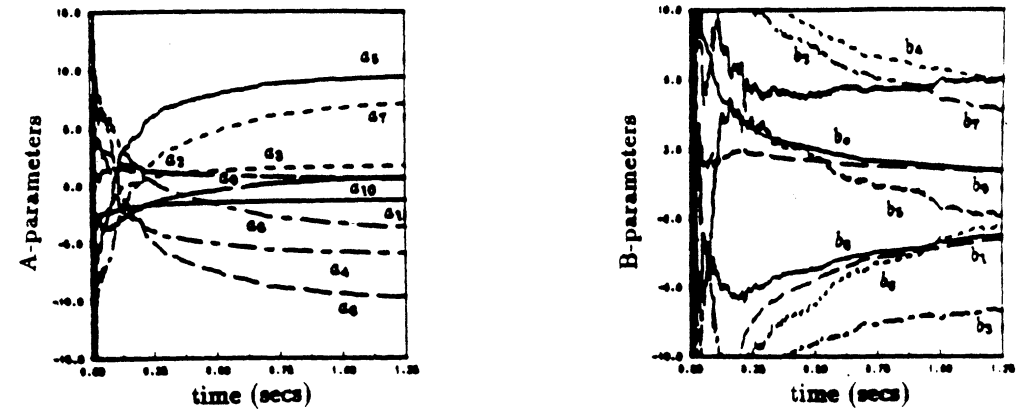
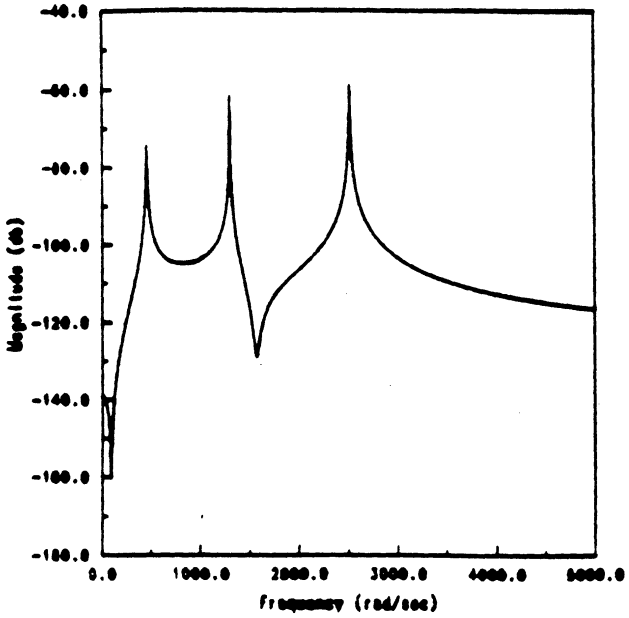
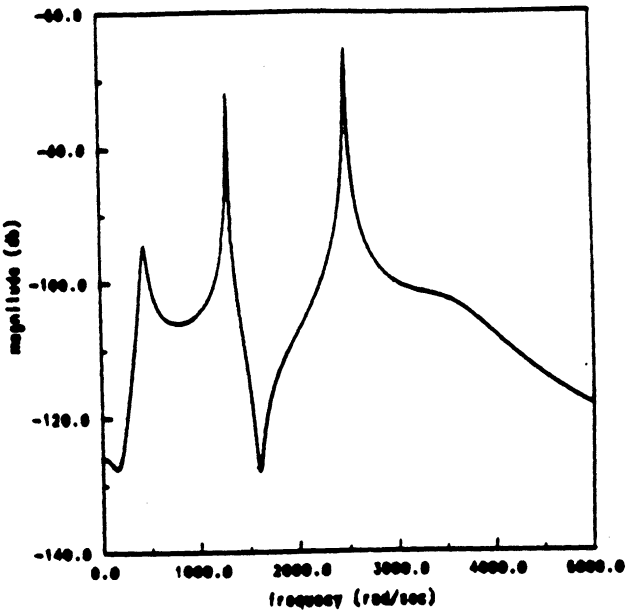


(c)

Figure 10 : Model parameters estimated by the: (a) RPLR (b) RELS and (c) RML schemes (I.I. transformation; Data Set 1).



(a)



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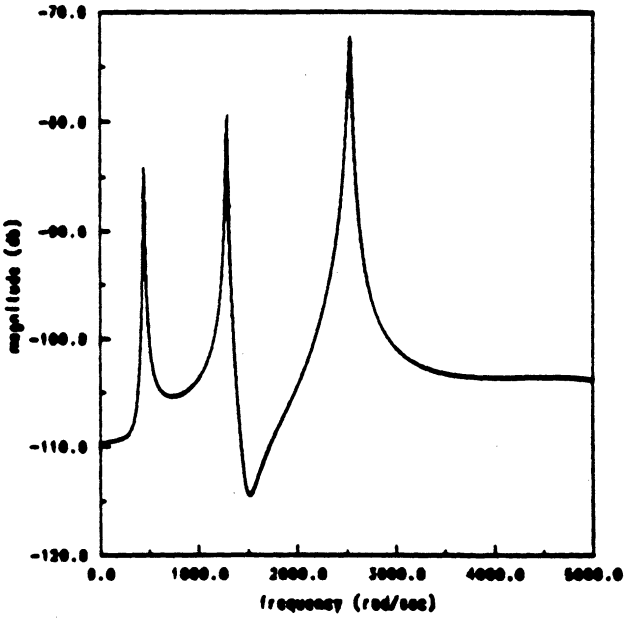

(c)

Figure 11 : Magnitude of the frequency response functions estimated by the : (a) RPLR (b) RELS and (c) RML schemes and the L.L transformation (Data Set 1).

ACKNOWLEDGEMENT

The authors wish to thank Mr. Yoha Hwang of the University of Michigan for providing the experimental data used in one of the application examples presented in this paper.