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Risk Management and the Credit Risk Premium

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Risk Management and the Credit Risk Premium

Abstract

Firms use various strategies to manage financial risks. This is the case even among firms that face very similar risk exposures, such as gold mining corporations. This paper develops a theoretical model to show that this diversity can be explained by differences in firms' credit risk premia. The model shows that if the credit risk premium is relatively small, firms use convex hedging strategies. If the credit risk premium is relatively large, firms use concave hedging strategies. Firms in between those two extremes use strategies that feature both convex and concave elements, e.g. collar strategies. Finally, firms that are unlevered, invest little and are exposed to few non-hedgeable risks are the most likely to use linear approximations of the optimal hedging strategy. The model replicates essentially all observed hedging strategies in the gold mining industry.

Keywords: Corporate risk management, hedging strategies, financing strategies, financial constraints, credit risk, default premium, security design

JEL Classification: G32

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1 Introduction[1]

Financial markets have witnessed a dramatic increase in the variety of risk management products. Given that firms face many different types of exposures, such diversity is to be expected. However, it is puzzling that even among firms that face practically identical risk exposures, such as gold producers, we observe a similar diversity in risk management strategies. For example, to manage their exposure to gold prices gold mining firms have been using forward and spot-deferred contracts, put and call options, and gold loans.[2] The resulting risk management portfolios have provided firms with payoff schedules that are linear, concave or convex functions of the future gold price, or payoff schedules that are neither convex nor concave (collars). Figure 1 demonstrates this variety by depicting the payoff schedules of risk management portfolios of four gold producers.[3]

The objective of this paper is to explain this diversity and to show how a firm should hedge that faces financial constraints. Froot, Scharfstein and Stein [1993] showed that financial constraints can motivate the use of derivatives. Building on their framework Mello and Parsons [1999] examined what financially constrained firms should hedge (firm value vs.

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[2] A spot-deferred contract is a forward contract which allows the short to defer delivery. A gold loan is essentially a structured debt contract whose face value depends on the future price of gold. It can be replicated by a standard loan and a short position of forward contracts on gold

[3] These four cases are representative examples. Most risk management programs in the gold mining industry fit into one of the cases in Figure 1.

sales vs. net cash flows, etc.). Extending this line of research, this paper derives *how* financially constrained firms should hedge, i.e. what combination of instruments should be chosen, and how other parameters, such as size and direction of positions, strike prices, etc. should be set. Thus, while Mello and Parsons [1999] derive solutions for the optimal hedge ratio, this paper explains why some firms chose convex or concave hedging strategies while others use strategies that are linear or neither concave nor convex.

There are several papers that examine under what conditions firms prefer non-linear to linear hedging strategies. This literature has shown that non-linear strategies may be optimal in the presence of basis risk,[4] borrowing or short-selling constraints,[5] if hedgeable and non-hedgeable risks are correlated,[6] if a firm can make certain production decisions after observing the state of nature,[7] if a firm's capital requirement is a non-linear function of the state of nature,[8] or if a firm minimizes VaR.[9] The one insight common to all of these papers is that hedging a non-linear exposure optimally requires a non-linear strategy. Thus, this literature does not explain why firms with practically identical exposures would implement different hedging strategies.

This paper shows that the diversity in non-linear hedging strategies can be explained by differences in firms' credit risk premia or the cost differential between internal and external

[4] See Wolf [1987], and Moschini and Lapan [1995].

[5] See Detemple and Adler [1988].

[6] See Detemple and Adler [1988], Moschini and Lapan [1995], and Brown and Toft [2001].

[7] See Moschini and Lapan [1992].

[8] See Froot, Scharfstein and Stein [1993].

[9] See Ahn, Boudoukh, Richardson and Whitelaw [1999].

funds. It is well known that financial market imperfections can drive a wedge between the cost of internal and external funds.[10] Consequently, firms prefer to rely on internal rather than external capital to finance their investments. To ensure sufficient internal funding for current and future investments, firms can use derivatives to alter the sequence of their cash inflows from operations to coincide with the sequence of their cash outflows. However, securing funding for current and future investments are competing goals.[11] Hence, firms face a trade-off when providing funding for current and for future investments. The cost differential between internal and external financing, i.e. the credit risk premium, determines a firm's discount factor, and hence which of the two sides dominates. If the cost differential is small, the firm's financial policy focuses primarily on future capital requirements. In this case the optimal financial portfolio generates additional cash flows in only those states in which the firm expects a cash shortfall. This strategy generally requires the purchase of put options (convex strategy). If the cost differential is large, the firm's financial policy focuses primarily on current capital requirements. In this case the optimal financial portfolio generates funds for current investments. Rather than issuing regular debt, the firm raises cash by selling call options on its output (concave strategy) in order to minimizes the probability of default. For cost differentials between these two extremes, the optimal financial policy contains both convex and concave elements (collar strategy).

[10] Myers and Majluf [1984] attribute differences in the cost of external capital to asymmetric information, while Myers [1977], Jensen and Meckling [1976], and Jensen [1986] rely on agency costs.

[11] A simple example can illustrate this point: Issuing debt may solve a current cash shortfall but will worsen a firm's future funding situation, when principal and interest need to be repaid. On the other hand, holding liquidity to pre-finance future investments reduces the amount of capital available for current investments.

The non-linearity in the optimal financial portfolio is caused by a firm's option to default. Hedging future capital requirements implies that a firm gives up this option, and thus considers its value as an additional cost. The greater this value, which depends on a firm's current leverage (financial or operational), its future capital expenditures, and the level of production risk, the more non-linear is the optimal financial portfolio. Thus, a linear approximation of the optimal financial portfolio works best if a firm's current leverage, future capital requirements, and production risk are all relatively low. The theoretical model replicates essentially all observed hedging strategies in the gold mining industry as shown in Figure 1.

The predictions of this model are similar to Franke, Stapleton and Subrahmanyam [1998] who show that risk-averse agents who face both hedgeable and non-hedgeable risk use convex hedging strategies, while agents who face only non-hedgeable risk use concave hedging strategies. Similarly, Leland [1980] shows that agents whose risk tolerance increases with income purchase portfolio insurance from agents whose risk tolerance increases less rapidly. Neither paper, however, explains why firms use linear or collar strategies.

The remaining paper is organized as follows. Sections 2 and 3 develop the one- and two-period models respectively. The paper's main results are derived from the two-period model. Section 4 relates the results to risk management practices in the gold mining industry, and Section 5 concludes.

2 The One-Period Model

Consider a firm that produces a single product using a non-divisible, decreasing returns-to-scale production technology with standard properties: $f'(.) > 0$, $f'(0) = \infty$, $f'(\infty) = 0$, and $f''(.) < 0$.[12] The firm faces two types of risks: production and price risk. Production risk, represented by ξ, is defined by the probability distribution $H(\xi)$. Outcomes of the technology shock are not observable to outsiders. Hence, financial contracts based on ξ are not available. Actual output, given by $\xi f(k)$ and known only to the firm's managers, can be sold at the competitive market price θ. The random variable θ is governed by the probability distribution $G(\theta) : [0, \infty) \to [0, 1]$ and is common knowledge. For simplicity, production and price shocks are assumed to be independent.[13] Production requires an investment of k dollars at the beginning of the period and causes operating expenses of $c(k)$. One can think of $c(k)$ as the firm's operating leverage. At the time of investment neither θ nor ξ are known. The firm's operating net cash flow at the end of the period is given by $\theta\xi f(k) - c(k)$.

To finance its investment the firm has access to a financial market. It consists of a complete set of contingent claims with respect to θ. This allows us to treat the firm's financing strategy as a security design problem. Each claim promises to pay one dollar at the end of the period if a particular output price occurs and nothing otherwise. The price of a contingent claim is denoted by $p(\theta)$ to reflect its dependence on the realization of θ. The

[12] Decreasing returns-to-scale imply the existence of an optimal investment level in this model. Hence, the firm is not indifferent with respect to the future level of investment.

[13] The effects of correlation between price and quantity risk have already been analysed by Wolf [1987], Detemple and Adler [1988], and Moschini and Lapan [1995].

number of claims the firm purchases (or sells) for state θ is given by $z(\theta)$. If $z(\theta)$ is positive, the firm acquires long positions that generate a cash outflow at the beginning of the period but an expected cash inflow at the end of the period. If $z(\theta)$ is negative, the firm acquires short positions, thus increasing its current cash balance but reducing its expected future cash flows. The cash flow of the financial portfolio at the beginning of the period is given by $\int p(\theta)z(\theta)\theta$. The firm is free to choose a mix of long and short positions for different states, i.e. no restrictions are placed on $z(\theta)$.

At the beginning of the period the firm's net cash flow π_1 is determined by the firm's internal equity (e) investment expenditures (k) and the cash flow of the financing portfolio: $\pi_1 = e - k - \int p(\theta)z(\theta)d\theta$.[14] At the end of the period, after θ and ξ are revealed, the firm's total net cash flow is the sum of its operating and financing cash flows, i.e. $\pi_2(\theta,\xi) = \theta\xi f(k) - c(k) + z(\theta)$.

The firm's objective is to maximize the present value of expected net cash flows, by choosing an investment level k and a financial policy function $z(\theta)$.

$$\max_{k,z(\theta)} \quad \pi_1 + \frac{1}{1+r} E \max\{0, \pi_2(\theta,\xi)\} \tag{1}$$

$$\begin{aligned} \text{s.t.} \quad \pi_1 &= e - k - \int p(\theta)z(\theta)d\theta \\ \pi_2(\theta,\xi) &= \theta\xi f(k) - c(k) + z(\theta) \\ \pi_1 &\geq 0 \end{aligned}$$

[14] The initial endowment can be positive or negative. It is bounded below by $e = -(E\theta\xi f(k^*) - c(k^*))$ to ensure that the firm's investment project has a positive NPV, i.e. the project's return must cover at least the start-up cost e.

Without loss of generality the risk-free interest rate r is assumed to equal zero.[15] The restriction that $\pi_1 \geq 0$ is innocuous because solutions will be derived for various levels of e.

Default is triggered at the end of the period if $\theta\xi f(k) - c(k) + z(\theta) < 0$. In this event ownership of the firm is transferred to creditors, i.e. the buyers of the firm's contingent claims. The transfer of ownership and subsequent liquidation causes the value of the firm to diminish by D to $\max\{0, \theta\xi f(k) - c(k) - D\}$.[16] The liquidation proceeds are distributed equally among all outstanding claims. The price of a contingent claim in a risk-neutral world is the expected value of its payoff. Hence,

$$p(\theta) = g(\theta) \cdot \begin{cases} 1 & \text{if } z(\theta) \geq 0 \\ 1 - H\left(\frac{c(k)-z(\theta)}{\theta f(k)}\right) + H\left(\frac{c(k)-z(\theta)}{\theta f(k)}\right) L & \text{if } z(\theta) < 0 \end{cases} \tag{2}$$

where

$$L \equiv \frac{E_\xi \left[\max\{0, \theta\xi f(k) - c(k) - D\} | \theta\xi f(k) - c(k) + z(\theta) < 0\right]}{-z(\theta)}$$

represents the expected liquidation proceeds per outstanding claim in the case of default. The function $g(\theta)$ denotes the density of θ and hence expresses the probability that a particular state occurs. If $z(\theta)$ is positive, a financial claim yields one dollar in state θ with certainty. Its value is therefore given by $g(\theta)$. If $z(\theta)$ is negative, the firm commits to deliver $z(\theta)$ dollars

[15] The objective of this paper is to analyze financing decisions if external capital is costly. In order not to cloud the analysis by other issues that give rise to hedging behavior the model is analyzed in a risk-neutral environment.

[16] This deadweight loss can be thought of as the direct and indirect costs of bankruptcy, such as transactions costs, loss of managerial ability, customer relationships, and the loss of growth or investment opportunities. The existence of this deadweight loss is the only friction in this model that gives rise to a meaningful financial policy. Otherwise a Modigliani-Miller world would persist. Since these costs are eventually borne by the firm in form of a credit risk premium, debt is more costly than internal financing. To maintain generality, bankruptcy costs are assumed to consist of both fixed and proportional components. That is, $D = d_1 + d_2 z(\theta)$, where $d_1 > 0$ and $d_2 < 0$. Note that default can only be triggered if $z(\theta)$ is negative.

if state θ occurs. With probability $1 - H\left(\frac{c(k)-z(\theta)}{\theta f(k)}\right)$ it will honor this commitment, but with probability $H\left(\frac{c(k)-z(\theta)}{\theta f(k)}\right)$ the firm defaults and a financial claim yields only L dollars. It is easy to show that $0 \leq L < 1$.

Although not denoted explicitly, the value of a contingent claim depends on the firm's policy functions k and $z(\theta)$, and the default costs D in addition to θ. This is because the price of each claim incorporates a risk premium to compensate creditors for any loss in the event of default. The probability of default naturally depends on both the firm's investment and financial decisions. For example, if the firm has outstanding claims ($z(\theta) < 0$) the following partial derivatives can be signed.[17]

$$\frac{\partial p(\theta)}{\partial k} \geq 0, \; \frac{\partial p(\theta)}{\partial z(\theta)} \geq 0, \; \frac{\partial p(\theta)}{\partial D} \leq 0 \tag{3}$$

The value of a contingent claim increases as investment and hence expected output increases. The value decreases as the number of claims *issued* increases, i.e. $z(\theta)$ decreases, because the probability of default raises as more claims are issued. Finally, the value of a contingent claim decreases as financial distress costs increase. Hence, the financial market in this model resembles the OTC market in which default risk is incorporated into the price of securities rather than exchange markets which do not expose traders to counter-party risk.

The financial claims that the firm can issue resemble debt contracts. In particular, the firm cannot raise equity capital. This assumption is less restrictive than it may appear at first glance. Diamond [1984] showed that debt is the optimal financial contract if a firm's income is not costlessly observable by outsiders, because debt minimizes monitoring costs.

[17] If $z(\theta) \geq 0$, all partial derivatives are zero.

This argument applies here. In fact, the financial objective in this model is similar to the financial objective in the Diamond model. It is to generate funds for current and future investments in a way that minimizes monitoring costs.

2.1 The Optimal Financing Strategy in the One-Period Model

Consider first the benchmark case in which there are no financial distress costs ($D = 0$). In this case Stiglitz's [1969] irrelevance theorem holds. As long as the firm's project has a positive NPV, i.e. $e \geq -[E\theta\xi f(k) - c(k)]$, it can always secure sufficient funding at a rate that adequately reflects the project's risk. The optimal investment level k^* is determined by the Euler equation

$$E\left[\theta\xi f'(k) - c'(k)|\theta\xi f(k) > c(k)\right] \cdot \Pr\left(\theta\xi f(k) > c(k)\right) = 1. \tag{4}$$

This equation says that the expected return from investing an additional unit of capital, the expected marginal return conditional on no default multiplied by the probability of no default, must equal the value of a unit of capital. Note that k^* is independent of the firm's financial policy. The financing policy must only satisfy the budget constraint

$$\int z(\theta)g(\theta)d\theta = e - k^* - \pi_1. \tag{5}$$

The solution is not uniquely defined. To see this, simply substitute $z(\theta) = c_1 + c_2\theta$ into equation (5).

A similar solution prevails if the firm's internal equity e is sufficient to finance the optimal investment level k^*, i.e. if $e > k^*$, even if financial distress costs are positive. In this case $z(\theta) \geqslant 0$, which implies that the firm does not require external financing so that the possibility of default is not an issue.

The financing problem is interesting only if $e < k^*$ and $D > 0$. In this case, the firm must raise external capital to support its investment program. However, since financial distress costs are positive, external capital is more costly than internal capital because outside lenders require compensation for the expected deadweight loss in form of a credit risk premium. For example, the value of a one-dollar liability is less than one dollar, i.e.

$$\int p(\theta)d\theta < 1, \tag{6}$$

which follows directly from equation (2). Raising external finance is therefore justified only if the expected marginal return is greater than one. This implies that if the firm raised external capital, i.e. $z(\theta)$ is negative for at least some θ, it would not pay out any cash flows to its owners ($\pi_1 = 0$) and it would not take any long positions in the financial assets. Either action would result in a marginal benefit to the firm of only one dollar.

Appendix A states the first-order conditions to the maximization problem defined by equations (1) and (2), and derives a solution for $z(\theta)$. If ξ is uniformly distributed, the optimal financial portfolio must satisfy

$$z(\theta) = \min\{0, -c_1\theta f(k) + c_2 c(k) + c_3\}, \quad c_{1,2,3} > 0, \tag{7}$$

an example of which is depicted in Figure 2. Although equation (7) is only a characterization of the optimal solution, it nevertheless provides useful insights. Note that the payoff of the optimal financial portfolio is like the payoff of a short call. Three points are noteworthy about this solution. First, the firm does not sell a regular bond to fund its current investments but a state-contingent contract. This contract obligates the firm to make high repayments in good states and low repayments in bad states, thereby minimizing the probability of default. This contract is similar to an income bond, but the firm has the advantage that income is a function of θ which is costlessly observable.[18]

The second point refers to the non-linearity of the solution. Why does the firm not sell contingent claims in low states? The answer is that creditors are not willing to enter into contracts that pay off only if $\theta < \frac{c(k)+D}{(1+a)f(k)}$. The existence of senior claims, represented by $c(k)$ and D, implies zero recovery for any new claims issued that pay off in this region. Therefore, the value of such claims would be zero.

Third, the firm sells a constant fraction (c_1) that is proportional to its future expected output. This implies that all marginal claims issued for each state θ bear the same credit risk, and hence the same credit risk premium. If this were not the case, then the firm could raise the same amount of cash at $t = 1$, but sell fewer contingent claims of low value and more contingent claims of high value, and thus reduce financing costs.

2.2 An Example

To gain a deeper understanding of the mechanics of the model consider the case in which the financial portfolio is linear in θ, i.e. $z(\theta) = -c\theta f(k)$. Sufficient conditions for this case are that operating costs $c(k)$ and constant bankruptcy costs d_1 are both zero. Appendix

[18] One can think of the optimal financing strategy also in terms of the Diamond [1984] framework. Debt is optimal because it minimizes monitoring costs. Monitoring takes place as a function of both ξ and θ, but observing θ is costless. Therefore, the optimal financial portfolio minimizes the number of states ξ in which creditors monitor. This number should be the same for each contingent claim issued.

B lists the first-order conditions and discusses the simulation method.[19] Figure 3 depicts the optimal solutions for c, k, the Lagrange multiplier λ which represents the marginal value of internal funds, and the expected \$ payoff of contingent claims as functions of the firm's internal equity e. If $e \geq 25$, then the benchmark case applies. The firm can attain the optimal investment level ($k^* = 25$) without raising external funds. Each contingent claim then pays one dollar in the second period, and $\lambda = 1$ indicates that there is no cost differential between internal and external funds. As the firm's internal equity falls, however, it sells an increasing fraction of its expected output to raise funds. As more contingent claims are issued the value of each claim declines because the probability of default increases. As the value declines, the cost differential between internal and external capital increases. To justify the higher cost of capital the firm underinvests in its core business which raises the marginal return on investment. Hence, for $e < 25$, the firm invests less than the optimal investment level k^*.

Figure 4 depicts three value functions of the one-period model as functions of the internal equity e. The first refers to the benchmark case of a frictionless market (dashed line). In a frictionless market there is no underinvestment. The value function is therefore linear. The second value function (dotted line) applies if financial markets are open but there are market frictions, e.g. financial distress costs etc. In this case the firm underinvests if its internal equity is insufficient to attain k^*. The resulting loss of value causes the value function to

[19] The simulation assumes the following specifications: $f(k) = Ak^{\alpha}$, where $A = 10$ and $\alpha = 0.5$, $\xi \sim U(1-a, 1+a)$, where $a = 0.9$, and $d_2 = 0.4$. This parametrization implies an optimal investment level of $k^* = 25$.

become concave for $e < k^*$. The third value function (solid line) is also locally concave and applies if financial markets are closed ($z(\theta) = 0$). The underinvestment is severer than in the previous case, and the concavity of the value function is more pronounced. The overall shape, however, is similar. In the third case, a closed form solution for the value function exists, even if $c(k) > 0$. For computational convenience the simulation of the two-period model assumes that financial markets are closed in the second period. Given the similar shapes of both value functions, the qualitative results should not be affected by this simplification.

3 The Two-Period Model

In the two-period model the financial objective is twofold. First, as in the one-period model, the firm must secure funding for current investments. Second, the firm must also secure funding for future investments. There are several ways to secure funding for future investments. One way is to pre-finance by holding excess cash. This strategy, however, is costly because it reduces the amount of capital available for current investments. It is also inefficient in the sense that it secures funding in good states in which additional funds are not necessary. A less capital intensive alternative would be to purchase options that generate additional cash inflows only in those states in which the firm expects a cash shortfall. This strategy only requires the payment of the option premium. In both cases, however, securing funding for future investments commits capital, and hence comes at the expense of current investments. In this sense the two financial objectives are incompatible. The firm's credit risk premium, or equivalently the cost differential between internal and external capital, determines which of these two objectives dominates. If the credit risk premium is high, because the firm's current financial condition is poor, then the current funding requirement primarily determines the risk management strategy. If the credit risk premium is low, because the firm's current financial condition is good, then future funding requirements become the dominant factor.

The two-period model is a straight forward extension of the one-period model. In the first period the firm invests and decides upon a financial policy. In the second period the firm undertakes a second investment decision. Price and production shocks are assumed to be iid across periods. The firm now solves

$$\max_{k,z(\theta)} \quad \pi_0 + EV\left(\pi_1(\theta,\xi)\right) \tag{8}$$

$$\begin{aligned} \text{s.t.} \quad \pi_0 &= e_0 - k - \int p(\theta) z(\theta) d\theta \\ \pi_1(\theta,\xi) &= \theta\xi f(k) - c(k) + z(\theta) \\ \pi_0 &\geq 0, \end{aligned}$$

where k and $z(\theta)$ are the firm's choice variables in the first period, and $V(.)$ is the value function of the one-period model in the second period. The first-order conditions are stated in Appendix C. As closed-form solutions are not available in general, the solutions for $z(\theta)$ are derived numerically by evaluating the first-order conditions with respect to $z(\theta)$. Analog to equation (22) the financial policy function must satisfy

$$\begin{aligned} &(1-H(b_3))\int_{b_1}^{b_3} V'(\theta\xi f(k) - c(k) + z(\theta))dH(\xi) \\ =\ & \lambda\left[1 - H(b_1) - \frac{D}{\theta f(k)}h(b_1) + d_2\left[H\left(b_1\right) - H\left(b_2\right)\right]\right] \qquad \forall\theta.^{20} \end{aligned} \tag{9}$$

Equation (9) defines $z(\theta)$ as a function of the investment level k and the Lagrange multiplier λ. Solutions for the financial policy function $z(\theta)$ are therefore calculated for different combinations of k and λ. The level of investment k never affected the shape of the optimal financial portfolios in any noticeable way. Simulation results are therefore only reported for different levels of λ.

Figure 5 graphs the future payoff schedule of five (representative) financial portfolios as a function of the future spot price θ. Each case assumes a different level of λ which increases successively from Case I to V.[24] The firm selects one of these portfolios depending on how much cash must be raised for current investments.

In Case I, the payoff schedule of the financial portfolio is linear. It guarantees that future investments can be financed internally in all states of the world. To attain this portfolio, the firm would need to hold substantial amounts of short-term liquid assets. Note, that the risk-neutral pricing assumption of the model implies a forward price of $E\theta$, which equals 1 in the simulation. Forward contracts alone are therefore not sufficient to attain this case. A firm that would choose this strategy would not be considered financially constrained because $\lambda = 1$ implies that there is no cost differential between internal and external funds. This is an extreme case and it is unlikely that any firm would fall into this category.

In Case II, the payoff schedule of the financial portfolio in convex. The firm's internal equity is not sufficient to simply pre-finance future investments as in Case I. Instead the firm

[24] The following values for λ are used: 1 (Case I), 1.07 (Case II), 1.3 (Case III), 1.5 (Case IV), 1.8 (Case V). For example, $\lambda = 1.3$ implies that there is a 30% premium on the cost of external capital over the cost of internal capital. See Appendix C for further parameter specifications.

sells forward contracts on its output and purchases put options to generate additional cash flows if the future spot price and hence revenue turns out to be low. A Case II portfolio would be close to self-financing because any linear portfolio that contains the point $z(1) = 0$ is a zero-cost portfolio. It thus would be chosen by firms that do not need to raise funds for current investments but would like to reduce the likelihood of a cash shortfall in the next period.

In Case III, the optimal financial portfolio is neither convex nor concave. It resembles a collar strategy. It could be approximated by purchasing out-of-the-money puts and selling out-of-the-money calls. In addition, the firm would issue regular debt and take short positions in forwards. A firm that selects this portfolio needs to secure funding for future *and* current investments. The collar strategy highlights the trade-off between the two financial goals.

In Cases IV and V, the optimal financial portfolio is again strictly speaking neither convex nor concave. However, a reasonable approximation could be obtained by a strategy that has a concave payoff schedule. For example, the firm could sell more calls than it purchases puts of equal strike price. Or it could short forwards, sell call options, and issue regular debt. In either case, the financial portfolio generates significant amounts of cash for current investments. Note that the weight between securing funding for current and future investments has now shifted in favor of current investments. In contrast, in Case III the current and future financing situations seem to be of similar importance.

Note how the shape of $z(\theta)$ changes from a convex to a collar and finally to a concave strategy as a function of a firm's *current* capital requirement. As the firm must raise more and more external capital to fund current investments the firm's credit risk premium increases successively. As a result, the financial objective shifts from securing funding for future investments to securing funding for current investments. The first objective generally requires a convex strategy while the second objective generally requires a concave strategy.

To gain a deeper understanding about the exact shapes of the financial portfolios, consider Figure 6 which depicts six different regions in which the firm faces different probabilities of financial distress and different probabilities of an internal cash shortage in the next period. If the payoff profile of the financial portfolio $z(\theta)$ lies in Regions 1-3, the firm will avoid default with certainty because the cash flow from the financial portfolio exceeds the minimum cash flow from operations, i.e. $z(\theta) > -(1-a)\theta f(k) + c(k)$. If $z(\theta)$ lies in Regions 4-5, the firm faces a positive probability of default because the cash flow from the financial portfolio is in between the minimum and maximum cash flows from operations, i.e. $-(1-a)\theta f(k) + c(k) \geq z(\theta) \geq -(1+a)\theta f(k) + c(k))$. In Region 6 default is certain because the cash flow from the financial portfolio is less than the maximum cash flow from operations, i.e. $-(1+a)\theta f(k) + c(k) > z(\theta)$. The firm could not sell any contingent claims in Region 6 since their expected recovery rates and hence their values would be zero.

If the payoff profile of the financial portfolio lies in Region 1, the firm always has sufficient internal capital available to finance next period's investments because the minimum total cash flow exceeds the capital requirement of the next period, i.e. $(1-a)\theta f(k) - c(k) + z(\theta) > k^*$. In Regions 2 and 4, a cash shortage will occur with positive probability because

$(1-a)\theta f(k) - c(k) + z(\theta) \geq k^* \geq (1+a)\theta f(k) - c(k) + z(\theta)$, and in Regions 3, 5 and 6, the firm will experience a cash shortage with certainty because $k^* > (1+a)\theta f(k) - c(k) + z(\theta)$.

Default and future cash shortages both cause costs. In the case of default the firm looses the return from its future investments, and financial contracts contain a credit risk premium to compensate creditors for the expected deadweight loss D. In the case of a future cash shortfall the firm must raise costly external finance in the next period. If the firm's internal equity is insufficient to support the optimal investment level in the first period, the firm must trade off the costs of underinvestment against the costs of financial distress. The firm designs the payoff profile of the financial portfolio in order to minimize the sum of these costs. This implies that the marginal claims in each state θ bear the same sum of costs with respect to default, future capital shortage, and current underinvestment.

Finally, let's consider what causes the financial strategies to be non-linear. The model is one of a levered firm. Leverage, represented by $c(k)$, can be interpreted as either operational of financial. Due to this leverage default is a valuable option because the firm can walk away from some of its existing obligations. In hedging its future capital requirements, however, the firm gives up the option to default. The value of this option becomes a cost of hedging. It causes the optimal financial strategy to be convex, because the firm needs to generate cash not only for future investments but also to cover those obligations that the firm would forgo if it declared bankruptcy. Thus, the convexity in the optimal solution increases with both leverage and the firm's capital requirement.

A third cause for non-linearity is the level of non-hedgeable risks (ξ). Non-hedgeable risk, such as production risk, increases the value of the option to default, and hence the convexity of the optimal financial portfolio. Thus, the model predicts that firms that are unlevered, make small net investments relative to the size of their operations, and face moderate amounts of production risk are most likely to use linear approximations to the optimal financial portfolio.

4 Hedging Strategies in the Gold Mining Industry

The model replicates the four primary hedging strategies in the gold mining industry. In 1998, the fractions of gold producers that used linear, convex, concave, or collar strategies were 32%, 18%, 25% and 25% respectively.[25] The issue of state-contingent debt, so called gold loans, is also common in the industry. More specifically, the results in Adam [2001] indicate that the industry's largest gold producers tend to adopt convex hedging strategies. These firms maintain the highest debt levels in the industry and are least likely to face financial constraints (measured by diversification, dividend policy, and the existence of a credit rating). In contrast, firms that use concave hedging strategies appear to be the most financially constrained firms. They are among the smallest firms in the industry, are almost exclusively financed by equity, keep the highest liquidity levels, and rarely pay dividends. Firms that use collar strategies appear to be somewhat in between convex and concave hedgers in terms of financial constraints. Finally, firms whose net investments are small relative to the scale of their operations tend to use linear strategies. These results are

[25] See Gold & Silver Hedge Outlook, April 1999, Scotia Capital Markets Equity Research.

consistent with the predictions of the theoretical model.

5 Conclusion

This paper shows how financially constrained firms should hedge. It thus extends the work of Froot, Scharfstein and Stein [1993], who showed why firms hedge, and Mello and Parsons [1999], who examined what firms should hedge that face financial constraints. Intuitively, financial constraints affect the marginal value of financial capital within a corporation. Many hedging strategies, e.g. those using options, shift capital not only across states of nature but also across time. The magnitude of financial constraints affects hedging strategies through its effect on intertemporal decisions. Thus, while the mere *existence* of financial constraints gives rise to a corporate demand for derivatives, the *magnitude* of these financial constraints determines the exact structure of the optimal risk management portfolio.

The model generates distinct financial strategies that can be approximated by convex, concave, collar, or linear strategies. If a firm's credit risk (premium) is relatively low, which implies that its current financial condition is sound, the firm focuses mostly on avoiding a cash shortfall in the future. The firm hedges its future capital requirements which requires a convex strategy. Intuitively, the hedging cash flows must cover not only the future capital requirements but also those obligations that the firm would forgo if it declared bankruptcy. Thus, the firm's option to default causes the optimal hedging strategy to be convex. If a firm's credit risk (premium) is relatively high, which implies that its current financial condition is tense, the firm is mostly concerned with raising sufficient cash for current investments.

To minimize the probability of default it issues state contingent contracts under which repayment obligations are high in good states and low in bad states. Due to the possibility of default, financial claims that pay off in good states are more valuable than financial claims that pay off in bad states. Since the firm sells the most valuable claims first, in order to minimize financing costs, the optimal financial strategy is concave. If a firm's credit risk (premium) is in between those two cases, both current and future capital requirements determine the firm's financial strategy. It consists of both convex and concave segments (collar strategy).

The non-linearity in the optimal financial portfolio is caused by a firm's option to default. Hedging future capital requirements implies that a firm gives up this option, and thus, considers its value as an additional cost. The greater this value, which depends on a firm's current leverage (financial or operational), its future capital expenditures, and the level of production risk, the more non-linear is the optimal financial portfolio. Thus, a linear approximation of the optimal financial portfolio works best if a firm's current leverage, future capital requirements, and production risk are all relatively low.

The predictions of the model are consistent with the diversity of risk management strategies observed in the gold mining industry. The four primary strategies are indeed linear, convex, concave and collar. The model thus explains why even a relatively homogeneous group of firms that face practically identical risk exposures use different hedging strategies.

A Appendix: The One-Period Model

In a frictionless economy, i.e. $D = 0$, the optimal investment level k^* is uniquely defined by the Euler equation

$$\int\int_{\frac{c(k)}{\theta f(k)}} \theta\xi f'(k) - c'(k) dH(\xi) dG(\theta) = 1, \tag{A.10}$$

assuming that the production function $f(k)$ is concave and the cost function $c(k)$ is weakly convex. Equation (10) is an exact analog of equation (4). If bankruptcy costs are positive ($D > 0$) and if the firm's internal equity is insufficient to finance its investment expenditures, i.e. $e < k^*$, then the firm solves

$$\max_{k, z(\theta)} \quad \int\int \max\{0, \theta\xi f(k) - c(k) + z(\theta)\} dH(\xi) dG(\theta) \tag{A.11}$$

$$\text{s.t. } k = e - \int p(\theta) z(\theta) d\theta$$

where

$$p(\theta) = \begin{cases} g(\theta) & \text{if } z(\theta) \geq 0 \\ g(\theta)\left[1 - H\left(\frac{c(k)-z(\theta)}{\theta f(k)}\right) + H\left(\frac{c(k)-z(\theta)}{\theta f(k)}\right) L\right] & \text{if } z(\theta) < 0 \end{cases} \tag{A.12}$$

and the expected liquidation proceeds per outstanding claim in the case of default are defined by

$$L \equiv \frac{E_\xi \left[\max\{0, \theta\xi f(k) - c(k) - D\} | \theta\xi f(k) - c(k) + z(\theta) < 0\right]}{-z(\theta)}. \tag{A.13}$$

The conditional expectation in (13) is defined by

$$E_\xi \max\{...\} \equiv \frac{\int^{\frac{c(k)-z(\theta)}{\theta f(k)}} \max\{0, \theta\xi f(k) - c(k) - D\} dH(\xi)}{H\left(\frac{c(k)-z(\theta)}{\theta f(k)}\right)}. \tag{A.14}$$

Using (14) and assuming $z(\theta) < 0$, the pricing function simplifies to

$$p(\theta) = g(\theta)\left[1 - H\left(b_2\right) - \frac{\theta f(k)}{z(\theta)} \int_{b_1}^{b_2} (\xi - b_1) dH(\xi)\right], \tag{A.15}$$

where $b_1 \equiv \frac{c(k)+D}{\theta f(k)}$ and $b_2 \equiv \frac{c(k)-z(\theta)}{\theta f(k)}$. These two boundaries have an economic interpretation. If $\xi < b_1$, the liquidation value of the firm is insufficient to cover even operating or bankruptcy costs. Creditors receive nothing for their claims. If $\xi > b_2$, then there will be no default. The firm can affect both boundaries via its investment and financial policies.

The first-order conditions of the one-period model are derived by differentiating the Lagrangian

$$\int\int_{b_2} \theta\xi f(k) - c(k) + z(\theta) dH(\xi) dG(\theta) - \lambda\left(k + \int p(\theta) z(\theta) d\theta - e\right) \tag{A.16}$$

with respect to k, $z(\theta)$, and the Lagrange multiplier λ. The first-order conditions are given by

$$\begin{aligned}
& \int\int_{b_2} \theta\xi f'(k) - c'(k) dH(\xi) dG(\theta) \\
&= \lambda\left(1 + \int \frac{\partial p(\theta)}{\partial k} z(\theta) d\theta\right) && \text{(A.17)} \\
g(\theta)\left[1 - H\left(b_2\right)\right] &= \lambda\left(p(\theta) + \frac{\partial p(\theta)}{\partial z(\theta)} z(\theta)\right) \quad \forall\theta && \text{(A.18)} \\
k - e &= -\int p(\theta) z(\theta) d\theta. && \text{(A.19)}
\end{aligned}$$

Using the Leibnitz Rule the partial derivatives in equations (17) and (18) are given by

$$\frac{\partial p(\theta)}{\partial k} z(\theta) = g(\theta)\left[D h(b_2)\left[\frac{c'(k)}{\theta f(k)} - b_2 \frac{f'(k)}{f(k)}\right] - \int_{b_1}^{b_2} \theta\xi f'(k) - c'(k) dH(\xi)\right] \tag{A.20}$$

$$p(\theta) + \frac{\partial p(\theta)}{\partial z(\theta)} z(\theta) = g(\theta)\left[1 - H(b_2) - \frac{D}{\theta f(k)} h(b_2) + \frac{\partial D}{\partial z(\theta)}\left[H\left(b_2\right) - H\left(b_1\right)\right]\right] \tag{A.21}$$

Interpretation of Equation (17): The LHS of (17) denotes the expected marginal return on investment conditional on no default. The RHS of (17) denotes the cost of investing an additional dollar. The Lagrange multiplier λ represents the internal value of an additional unit of capital. One can interpret λ as the wedge between the cost of internal and external capital. If $\lambda = 1$, the cost of internal and external capital are equal. If $\lambda > 1$, external capital is more costly than internal capital. In this case, the firm still has investment opportunities that would yield a positive net present value at the internal cost of capital but not at the external costs of capital. Investing an additional dollar earns the expected marginal return on investment. The opportunity cost of this additional dollar is 1, but investing it also yields a reduction in credit risk for all outstanding claims, because the probability of default decreases as k increases. Note that the partial derivative in (17) is positive if $z(\theta) < 0$.

Interpretation of equation (18): Selling an additional contingent claim in state θ raises $p(\theta)$ dollars minus the marginal price reduction for all outstanding claims of the same state due to the increase in credit risk. Note that the partial derivative in (18) is positive if $z(\theta) < 0$. The internal value of these additional funds is given by λ. The cost of selling the additional claim is that with probability $g(\theta)[1 - H(b_2)]$ one dollar must be repaid at the end of the period.

To derive the optimal financial portfolio $z(\theta)$ note that equation (18) must hold $\forall\theta$. Thus, λ is independent of θ. Using equation (21), the first-order condition (18) reduces to

$$1 - H(b_2) = \lambda \left[1 - H(b_2) - \frac{D}{\theta f(k)} h(b_2) + \frac{\partial D}{\partial z(\theta)} \left[H(b_2) - H(b_1)\right]\right] \qquad \forall\theta. \tag{A.22}$$

Equation (22) shows that $\lambda > 1$, because $\frac{\partial D}{\partial z(\theta)} < 0$ and $b_1 < b_2$. Hence, the internal value of capital is greater than the external value of capital. If bankruptcy costs consist of fixed and variable costs, e.g.

$$D = \begin{cases} 0 & \text{if } z(\theta) \geq 0 \\ d_1 + d_2 z(\theta) & \text{if } z(\theta) < 0 \end{cases} \tag{A.23}$$

where $d_1 > 0$ and $-1 < d_2 < 0$, and if $\xi \sim U(1-a, 1+a)$, then the optimal financial portfolio in the one-period model must satisfy

$$z(\theta) = \min\left\{0, \frac{-(1+a)\theta f(k) + c(k) + \frac{\lambda}{\lambda-1} d_1 (1+d_2)}{1 - \frac{\lambda}{\lambda-1} d_2 (2+d_2)}\right\}. \tag{A.24}$$

This follows from equation (22). Equation (24) shows that $z(\theta)$ is a concave function of θ, consisting of two linear segments. A more compact form of equation (24) is given by

$$z(\theta) = \min\{0, -c_1\theta f(k) + c_2 c(k) + c_3\}, \quad c_{1,2,3} > 0. \tag{A.25}$$

It is easy to show that $c_1 < 1 + a$. Hence, the firm does not overhedge.

B An Example

Consider the case in which the optimal financial policy is linear, i.e. $z(\theta) = -c\theta f(k)$. Sufficient conditions for this case are $c(k) = 0$ and $d_1 = 0$, which follows from equation (24). Define $d \equiv d_2 < 0$. Then the first-order conditions of the one-period model that define the solutions for k, c, and λ become

$$E\theta f'(k) \int_c \xi dH(\xi) = \lambda \left[1 + E\theta f'(k) \left(dh(c)c^2 - \int_{-dc}^{c} \xi dH(\xi)\right)\right] \tag{B.26}$$

$$1 - H(c) = \lambda \left[1 - H(c) + dh(c)c + d[H(c) - H(-dc)]\right] \tag{B.27}$$

$$k - e = E\theta f(k) \left[[1 - H(c)]c + \int_{-dc}^{c} (\xi + dc) dH(\xi)\right]. \tag{B.28}$$

Equation (27) must hold $\forall\theta$, which verifies that c is independent of θ. Closed form solutions for c, k, and λ are not available. The model must be solved numerically. To do so, let $\xi \sim U(1-a, 1+a)$. Three cases must now be distinguished. Each case states whether default is possible and whether the firm's liquidation value is sufficient to cover D.

Case I ($-dc < c \leq 1-a < 1+a$): In this case the financial policy is chosen such that default is impossible. The first-order conditions (26), (27), and (28) simplify to

$$E\theta E\xi f'(k) = \lambda \tag{B.29}$$

$$1 = \lambda \tag{B.30}$$

$$k - e = E\theta f(k)c. \tag{B.31}$$

For this, case to apply the firm's internal equity must satisfy

$$k^* - (1-a)E\theta f(k^*) \leq e. \tag{B.32}$$

That is the firm's internal equity must exceed k^* minus next periods minium output, which is the maximum amount that can be borrowed risk-free.

Case II ($-dc \leq 1-a < c < 1+a$): In this case the financial policy is chosen such that default is possible but the firm's liquidation value is always sufficient to cover D.

$$E\theta f'(k)\left[(1+a)^2 - c^2\right] = \lambda\left[4a - E\theta f'(k)\left[(1-2d)c^2 - (1-a)^2\right]\right] \tag{B.33}$$

$$1 + a - c = \lambda\left[1 + a - d(1-a) - (1-2d)c\right] \tag{B.34}$$

$$k - e = E\theta f(k)\left[\frac{(1+a-d(1-a))c}{2a} - \frac{(1-2d)c^2 + (1-a)^2}{4a}\right].$$

For this case to apply the firm's internal equity must satisfy

$$k^{\#} - \alpha(a,d)E\theta f(k^{\#}) < e < k^* - (1-a)E\theta f(k^*), \tag{B.35}$$

where

$$\alpha(a,d) \equiv \frac{(1+a)^2(1+2d)-(1-a)[2d(1+a)+(1-a)]}{4a}, \tag{B.36}$$

and k^* is the optimal capital stock defined by the FOCs.

Case III ($1-a < -dc < c < 1+a$): In this case the financial policy is chosen such that default is possible and the firm's liquidation value may not be sufficient to cover D.

$$E\theta f'(k)\left[(1+a)^2 - c^2\right] = \lambda\left[4a - E\theta f'(k)(1-2d-d^2)c^2\right] \tag{B.37}$$

$$1+a-c = \lambda\left[1+a-(1-2d-d^2)c\right] \tag{B.38}$$

$$k-e = E\theta f(k)\left[\frac{(1+a)c}{2a} - \frac{(1-2d-d^2)c^2}{4a}\right]. \tag{B.39}$$

These three sets of equations can be simplified further.

Case I ($-dc < c \leq 1-a < 1+a$):

$$E\theta f'(k) = 1 \tag{B.40}$$

$$E\theta f(k)c = k-e. \tag{B.41}$$

Case II ($-dc \leq 1-a < c < 1+a$):

$$E\theta f'(k)\left(4a - d(1-a^2) + cd(1+3a)\right) = 4a \tag{B.42}$$

$$E\theta f(k)\left[2c(1+a-d(1-a)) - (1-2d)c^2 - (1-a)^2\right] = 4a(k-e) \tag{B.43}$$

Case III ($1-a < -dc < c < 1+a$):

$$E\theta f'(k)(1+a)(1+a+cd(2+d)) = 4a \tag{B.44}$$

$$E\theta f(k)\left[2c(1+a) - (1-2d-d^2)c^2\right] = 4a(k-e). \tag{B.45}$$

These equations are solved numerically for k and c, assuming a production function of Cobb-Douglas form, i.e. $f(k) = Ak^{\alpha}$. The pricing function of a contingent claim in each of the three cases is given by

$$p(\theta) = g(\theta) \begin{cases} 1 & \text{Case I} \\ \frac{(1+a)(1+d)-c(1-d)}{2a} + \frac{c-(1-a)^2/c}{4a} & \text{Case II} \\ \frac{(1-a)-c(1-d(1+d))}{2a} + \frac{c-(1-d)^2}{4a} & \text{Case III} \end{cases} \tag{B.46}$$

The value function of the one-period model is given by

$$V(e) = \begin{cases} E\theta\xi f(k^*) + e - k^* & \text{if } e \geq k^* \\ E\theta f(k(e)) \int_{c(e)} (\xi - c(e))\, dH(\xi) & \text{if } \underline{e} < e < k^* \\ 0 & \text{if } e \leq \underline{e} \end{cases} \tag{B.47}$$

and graphed in Figure 5. If $e \leq \underline{e}$, the firm cannot secure external financing because the project has a negative NPV. In this case the value function of the firm is zero. If $e \geq k^*$, the firm does not require any external financing to operate at the optimal level k^*.[26] The value function is linear. If $\underline{e} < e < k^*$ the value function is concave. Interestingly, the general form of the value function is preserved even if financial markets are closed, i.e. $z(\theta) = 0$. In this case a closed-form solution of the value function exists even if $c(k) > 0$. It is given by

$$V(e) = \begin{cases} E\theta\xi f(k^*) - c(k^*) + e - k^* & \text{if } e \geq k^* \\ E\theta\xi f(e) - c(e) & \text{if } 0 < e < k^* \\ 0 & \text{if } e \leq 0 \end{cases} \tag{B.48}$$

[26] The optimal investment level k^* in this example is defined by $E\theta\xi f'(k^*) = 1$.

where k^* is defined by equation (4). This value function is used in the two-period model because a closed-form solution simplifies the numerical simulation, and displays essentially identical characteristics as the value function for the case when financial markets are open.

C The Two-Period Model

The first-order conditions for the two-period model (8) are similar to the ones of the one-period model.

$$\begin{aligned} & \int\int_{b_1} V'(\theta\xi f(k) - c(k) + z(\theta)) \left[\theta\xi f'(k) - c'(k)\right] dH(\xi) dG(\theta) \\ = & \lambda\left(1 + \int \frac{\partial p(\theta)}{\partial k} z(\theta) d\theta\right) \end{aligned} \tag{C.49}$$

$$\begin{aligned} & g(\theta) \int_{b_1} V'(\theta\xi f(k) - c(k) + z(\theta)) dH(\xi) \\ = & \lambda\left(\frac{\partial p(\theta)}{\partial z(\theta)} z(\theta) + p(\theta)\right) \qquad \forall\theta \end{aligned} \tag{C.50}$$

$$k = e_0 + \int p(\theta) z(\theta) d\theta \tag{C.51}$$

The value function $V(...)$ is given by equation (48). Using equation (21), the first-order condition (50) that defines the optimal financial policy can be rewritten as

$$\begin{aligned} & (1 - H(b_3)) \int_{b_2}^{b_3} V'(\theta\xi f(k) - c(k) + z(\theta)) dH(\xi) \\ = & \lambda\left[1 - H(b_2) - \frac{D}{\theta f(k)} h(b_2) + d_2\left[H(b_2) - H(b_1)\right]\right], \end{aligned} \tag{C.52}$$

where $b_1 \equiv \frac{c(k)+D}{\theta f(k)}$, $b_2 \equiv \frac{c(k)-z(\theta)}{\theta f(k)}$, and $b_3 \equiv \frac{k^*+c(k)-z(\theta)}{\theta f(k)}$. This equation has to hold $\forall\theta$. The economic interpretations for b_1 and b_2 are as before. If $\xi < b_3$, then the firm faces a liquidity shortage in the next period, i.e. the firm's internal cash flows will be insufficient to attain the optimal investment level k^*.

Closed-form solutions are not available in general. The model must again be solved numerically. The solutions for the optimal financial portfolio $z(\theta)$ are derived from equation (52) assuming particular levels for λ and k as before. In simulating the solutions to equation (52), there are again several cases to consider.

- Loss of future investment opportunities impossible/possible/certain
- Default impossible/possible/certain
- Expected liquidation proceeds positive/zero

Each case places certain restrictions on $z(\theta)$ and θ that lead to this case. In all, there are six different cases to consider, depicted as six different regions in Figure 6. Assuming again that $\xi \sim U(1-a, 1+a)$.

Case I: $z(\theta) > -(1-a)\theta f(k) + c(k)$ implies that default is impossible. In this case the LHS of equation (52) is given by

$$LHS = g(\theta)\frac{V((1+a)\theta f(k) - c(k) + z(\theta)) - V((1-a)\theta f(k) - c(k) + z(\theta))}{2a\theta f(k)}. \quad \text{(C.53)}$$

Case II: $-(1-a)\theta f(k) + c(k) > z(\theta) > -(1+a)\theta f(k) + c(k)$ implies that default is possible. Then the LHS of equation (52) is given by

$$LHS = g(\theta)\frac{V((1+a)\theta f(k) - c(k) + z(\theta))}{2a\theta f(k)}. \quad \text{(C.54)}$$

Case IIa: Default is possible. If in addition, $D \leq -z(\theta)$ and $D \leq (1-a)\theta f(k) - c(k)$, then the expected liquidation proceeds are always positive. The RHS of equation (52) is given by.

$$RHS = \lambda g(\theta)\frac{[(1+a) - d_2(1-a)]\,\theta f(k) - c(k)(1-d_2) + z(\theta)(1-2d_2) - d_1}{2a\theta f(k)}. \qquad \text{(C.55)}$$

Case IIb: Default is possible. If in addition, $D \leq -z(\theta)$ and $D > (1-a)\theta f(k) - c(k)$, then the expected liquidation proceeds are zero or positive depending on the realization of ξ.

$$RHS = \lambda g(\theta)\frac{(1+a)\theta f(k) - c(k) + z(\theta)(1-2d_2-d_2^2) - d_1(1+d_2)}{2a\theta f(k)}. \qquad \text{(C.56)}$$

Case IIc: If default is possible and in addition, $D > -z(\theta)$, then the expected liquidation proceeds are always zero.

$$RHS = \lambda g(\theta)\frac{(1+a)\theta f(k) - c(k) + 2z(\theta)}{2a\theta f(k)}. \qquad \text{(C.57)}$$

Since λ must be the same for every state θ, one can solve for those pairs $\{z(\theta), \theta\}$ that imply a constant λ. The results are graphed in Figure 5. Of course, the simulated solutions still depend on k. However, further numerical analyses showed show that the magnitude of k does not alter the general form of the optimal financial portfolios. These results are therefore not reported.

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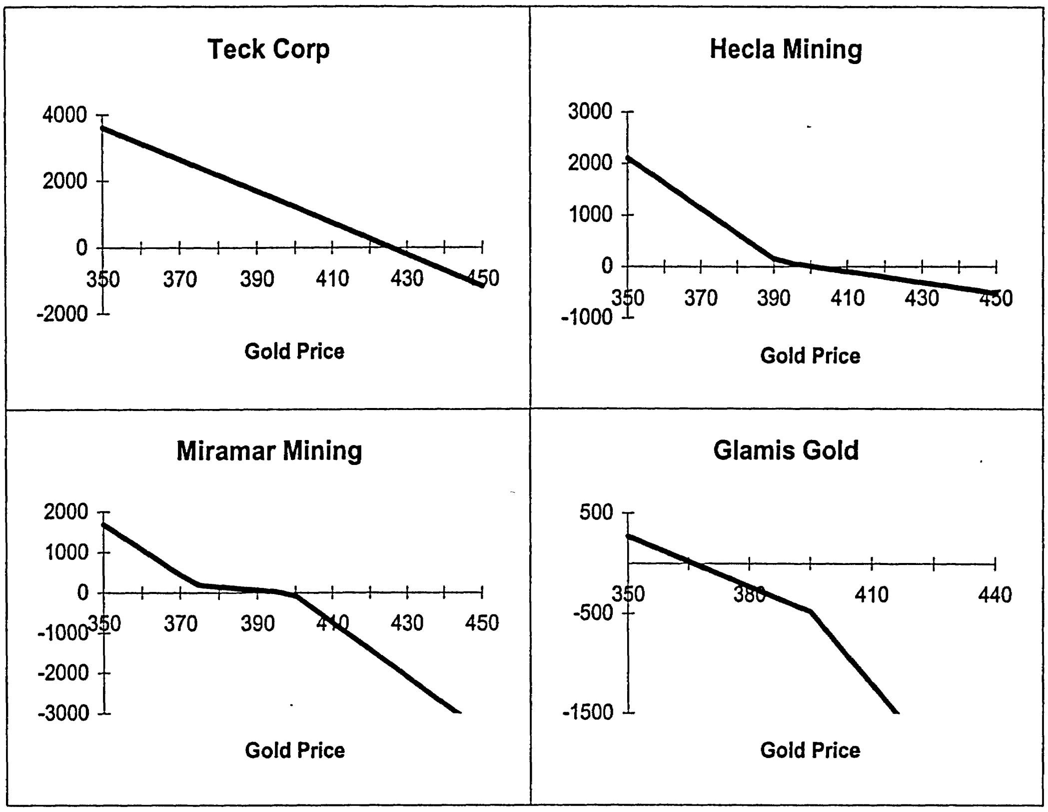

Figure 1: Payoff schedules of risk management portfolios

This figure depicts the future payoff profiles of risk management portfolios of four North American gold producers, for gold prices between US$ 350 and US$ 450 per ounce. The portfolio payoffs are given in thousands of US$. The portfolios consist of the following instruments. Teck: forward contracts; Hecla Mining: forwards and puts; Miramar Mining: forwards, puts and calls; Glamis Gold: forwards and calls.

Source: Gold Hedge Outlook, First Quarter 1995, by Ted Reeve, ScotiaMcLeod Equity Research.

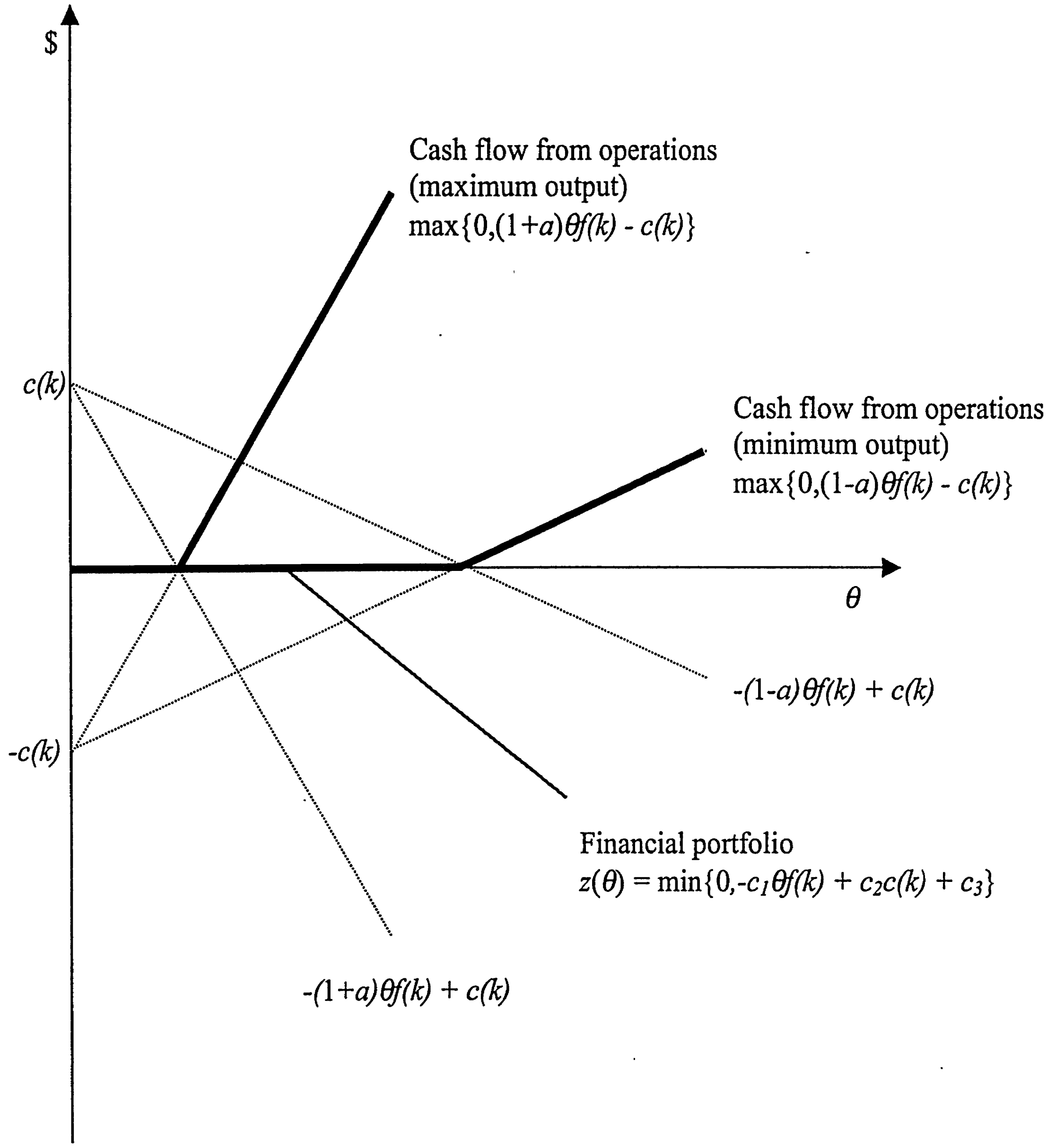

Figure 2: Example of an optimal financing portfolio in the one-period model
The technology shock ξ is assumed to be bounded between 1-a and 1+a. The firm's cash flows from operations are thus bounded from above by $\max\{0,(1+a)\theta f(k) + c(k)\}$ and from below by $\max\{0,(1-a)\theta f(k) + c(k)\}$. The firm's (operational or financial) leverage is denoted by $c(k)$. To raise funds for current investments the firm sells a fraction of its future cash flows, such that all marginal claims issued for each state θ bear the same credit risk.

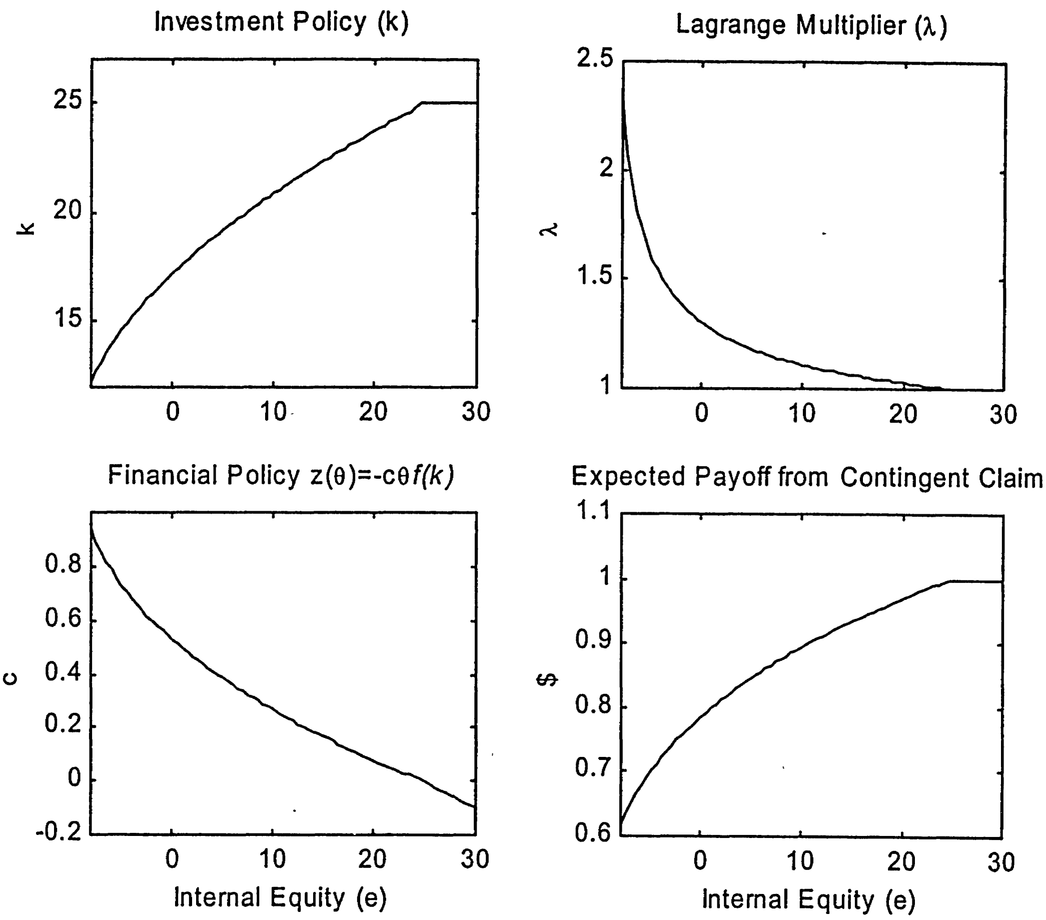

Figure 3: Policy functions of the one-period model

This graph plots the policy functions of the one-period model as functions of the firm's internal equity (e), assuming the financial portfolio is a linear function of θ. The efficient investment level is $k^* = 25$. If $e > k^*$, the firm does not need to raise funds. Instead it saves, i.e. purchases contingent claims, which is implied by the negative value for c. The two subplots to the left show that if the firm's internal equity is insufficient to attain k^*, the firm invests less than the efficient level. This underinvestment is mitigated but not completely eliminated by selling a fraction of future output to raise funds for current investments. As the firm's internal equity declines and sells a higher fraction of its future output, the expected payoff per contingent claim decreases. This is due to the higher probability of financial distress, which raises as the firm sells more claims. Since financial distress causes a real loss to the economy, the cost of external financing raises above the cost of internal financing. This cost differential is captured by the Lagrange multiplier λ. If $\lambda = 1$, then the cost differential is zero. If $\lambda > 1$, then the cost differential is positive.

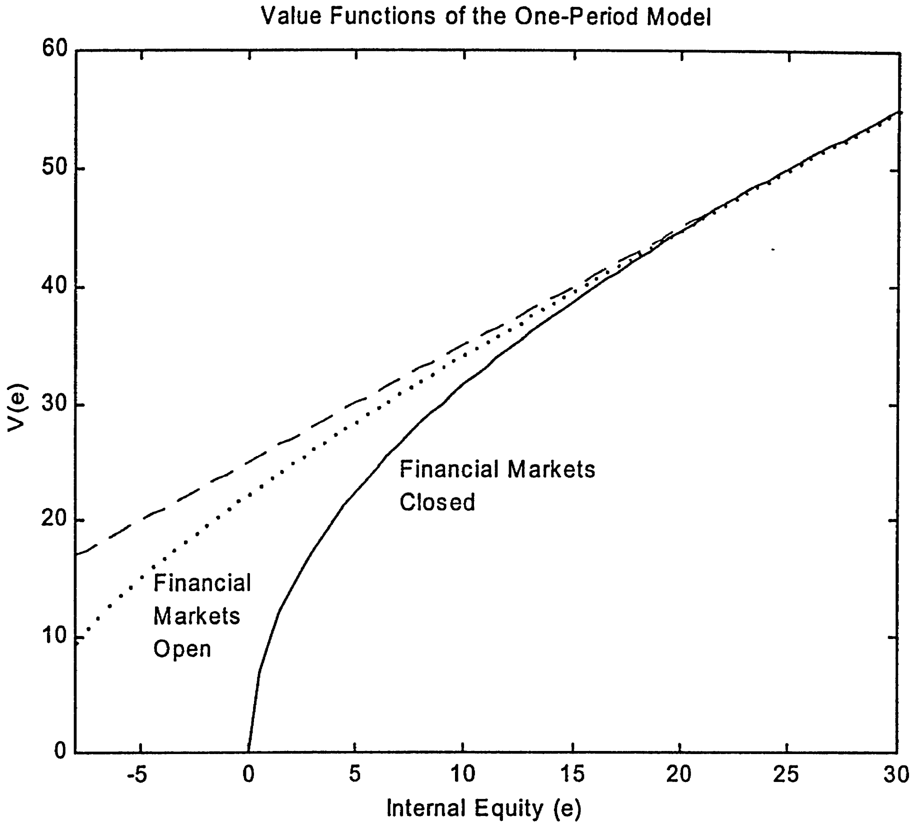

Figure 4: Value functions of the one-period model
This figure graphs the value function of the benchmark case assuming no financial market frictions (dashed line), and the value functions if financial markets are open but there are market frictions, i.e. positive default costs (dotted line), and if financial markets were closed (solid line). The value function of the benchmark case is linear. In the presence of financial market imperfections, however, the value functions are only linear if the firm's internal equity (*e*) is greater than the optimal investment level (*k**) which is 25 in this graph. Otherwise the value functions are concave. The difference between these two value functions and the value function of the benchmark case represents the value loss due to underinvestment and expected default costs.

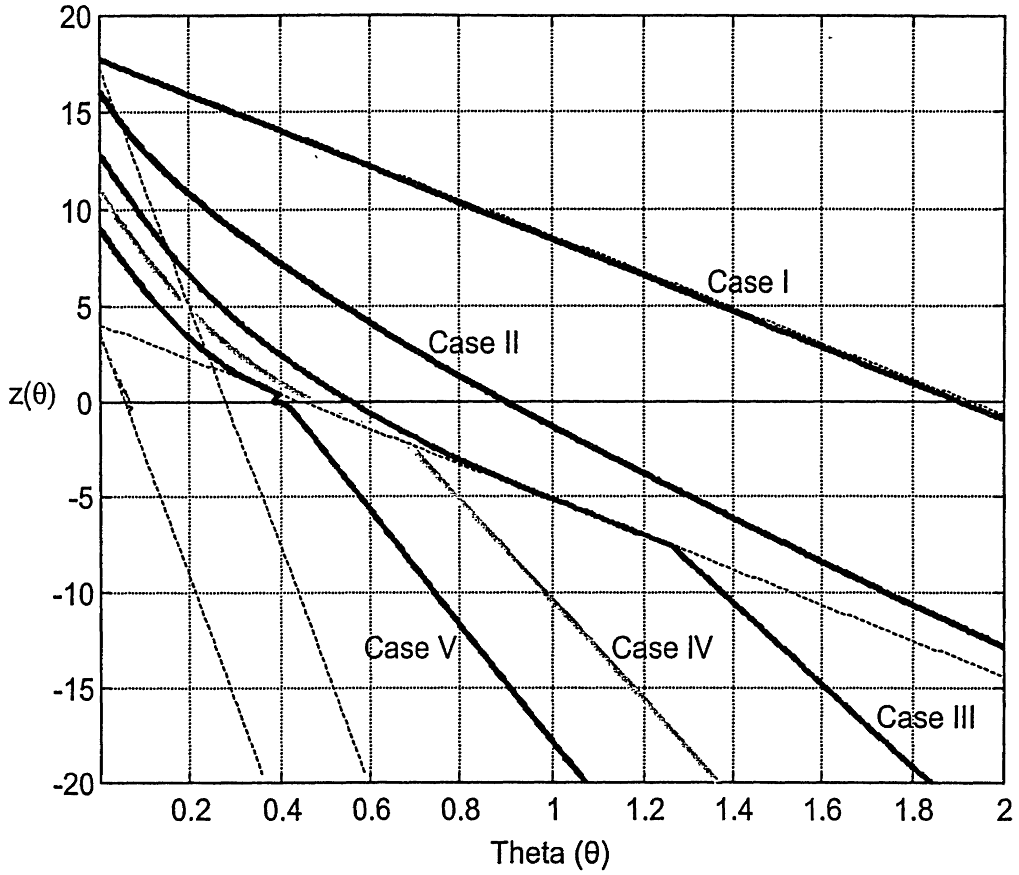

Figure 5: Optimal financial portfolios in the two-period model
This graph depicts the payoff profiles of optimal financial portfolios $z(\theta)$ as functions of the firm's output price θ. If $z(\theta)$ is positive it generates negative cash flows in the current period but a positive expected cash flow in the following period. If $z(\theta)$ is negative it generates positive cash flows in the current period but a negative expected cash flow in the following period. Each case assumes a different cost differential between internal and external financing, represented by λ. The cost differential increases successively from Case I to Case V. The following values for λ are used: 1 (Case I), 1.07 (Case II), 1.3 (Case III), 1.5 (Case IV), 1.8 (Case V). For example, λ =1.3 implies that there is a 30% premium on external capital over the cost of internal capital. The forward price is given by Eθ, which equals 1 in the simulation.

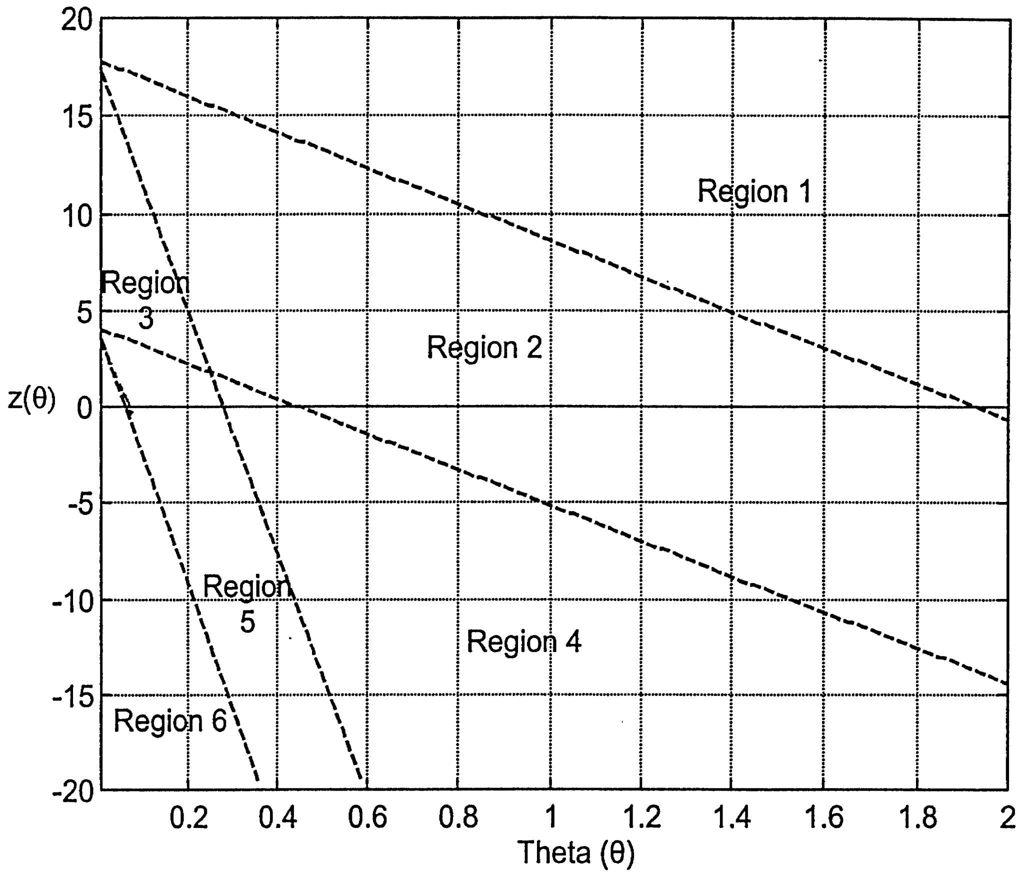

Figure 6: Cost regions in the two-period model

This figure depicts six regions that define whether or not the firm faces the possibility of default and/or the possibility of an internal cash shortfall in the next period. If the financial portfolio lies in Region 1, then there will be no default and no cash shortfall. If the financial portfolio lies in Region 2, there will be no default but there is a positive probability of a cash shortfall in the next period. In Region 3 there will be no default but the firm will experience a cash shortfall with certainty. In Region 4 the firm faces positive probabilities of both default and cash shortage. In Region 5 the firm faces a positive probability of default and a cash shortfall with certainty. In Region 6 default and cash shortfall will occur with certainty. Both default and a cash shortfall cause costs. The firm designs the payoff profile of the financial portfolio $z(\theta)$ such that it minimizes these costs.