A Recourse Certainty Equivalent for Decisions Under Uncertainty

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Abstract

We propose a new criterion for **decision-making under uncertainty**. The criterion is based on a **certainty equivalent** (CE) of a random variable Z,

$$S_v(Z) = \sup_z \{z + E_Z v(Z - z)\}$$

where $v(\cdot)$ is the decision maker's **value-risk function**. This CE is derived from considerations of **stochastic optimization with recourse**, and is called **recourse certainty equivalent** (RCE). We study (i) the properties of the RCE, (ii) the recoverability of $v(\cdot)$ from $S_v(\cdot)$ (in terms of the rate of change in risk), (iii) comparison with the "classical CE" $u^{-1}Eu(\cdot)$ in **expected utility** (EU) theory, and Yaari's CE in his **dual theory of choice under risk**, (iv) relation to risk-aversion and (v) applications to models of **production under price uncertainty**, **investment in risky and safe assets** and **insurance**. In these models the RCE gives intuitively appealing answers for all risk-averse decision makers, without the pathologies inherent in the EU model, where the **Arrow-Pratt indices** are used to exclude certain risk averse utilities leading to implausible predictions.

Key words: Stochastic optimization with recourse. Decision-making under uncertainty. Certainty equivalents. Risk aversion. Production under price uncertainty. Investment in risky and safe assets. Insurance.

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Abbreviations and special notation

CE = certainty equivalent	page 2
C_u = classical CE	page 3
DM = decision maker, decision making	
$D(T)$ = distribution functions with compact support	page 25
$D\{x_1, x_2, x_3\}$	page 22
EU = expected utility	page 2
MPIR = mean preserving increase in risk	page 22
MPSIR = mean preserving simple increase in risk	page 22
M_u = u-mean CE	page 3
RCE = recourse CE	page 5
RV = random variable	
SP = stochastic program, stochastic programming	
SPwR = stochastic programming with recourse	page 7
S_v = the RCE	page 5
$\mathcal{U}$ = class of normalized utility functions	page 10
$[\mathbf{x}, \mathbf{p}]$	page 17
(x, p)	page 18
Y_f = Yaari's CE	page 3

1 Introduction

Decision making under uncertainty presupposes the ability to rank random variables, i.e. a **complete order** $\succeq$ on the space of RV's, with $X \succeq Y$ denotes X **preferred** to Y. If the **preference order** $\succeq$ is given in terms of a real valued function CE$(\cdot)$ on the space of RV's,

$$X \succeq Y \iff \mathrm{CE}(X) \geq \mathrm{CE}(Y) \quad \text{for all RV's } X,\ Y$$

we call CE(Z) a **certainty equivalent** (CE) of Z, corresponding to the preference $\succeq$. In particular, a DM is indifferent between a RV Z and a constant[1] z iff $z = \mathrm{CE}(Z)$.

In the **expected utility** (EU) model, the DM is assumed to have a **utility function** $u(\cdot)$ which typically is **strictly increasing** (more is better) and **concave**. The DM's preference is then given by

$$X \succeq Y \iff E\,u(X) \geq E\,u(Y) \tag{1.1}$$

[1]Regarded as a degenerate RV.

$$\iff u^{-1}E\,u(X) \geq u^{-1}E\,u(Y),$$

Accordingly we define the **classical certainty equivalent** (CCE) by

$$C_u(Z) = u^{-1}Eu(Z) \tag{1.2}$$

Another CE, suggested by expected utility, is the **u-mean CE** $M_u(\cdot)$ defined implicitly by

$$E\,u(Z - M_u(Z)) = 0 \quad \text{for all RV's } Z \tag{1.3}$$

M_u does not induce the same order[2] as the classical CE C_u, i.e. the two CE's are not equivalent.

Expected utility theory is "the major paradigm in decision making ..., It has been used **prescriptively** in management science (especially decision analysis), **predictively** in finance and economics, **descriptively** by psychologists .... The EU model has consequently been the focus of much theoretical and empirical research ... ", [32].

Empirical tests (e.g. [1], [2] and [17]) revealed systematic violations (also called "paradoxes") of the EU model axioms which were traced to the **linearity in probabilities** of the expected utility. Alternative theories of decisions under risky choices were proposed which avoid the said paradoxes, e.g. the **prospect theory** of Kahneman and Tversky [17], the **local utility theory** of Machina [22] and Yaari's **dual theory** [38]. In particular, Yaari's risk aversion is compatible with **linearity in payments**[3].

Given a monotone function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, **Yaari's certainty equivalent** $Y_f(\cdot)$ is

$$Y_f(Z) = \int f(1 - F_Z(t))dt \tag{1.4}$$

where F_Z is the cumulative distribution function of the RV Z. In particular, both Yaari's CE (1.4) and the u-mean CE (1.3) are **shift additive** in the sense that

$$\text{CE}(Z + c) = \text{CE}(Z) + c \quad \text{for all RV } Z \text{ and constant } c \tag{1.5}$$

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[2]There are a utility $u(\cdot)$ and RV's X, Y such that $E\,u(X) \geq E\,u(Y)$ but $M_u(X) < M_u(Y)$.

[3]"In studying the behavior of *firms*, linearity in payments may in fact be an appealing feature", [38, p. 96]. Indeed, a firm which divides the last dollar of its income as dividends, cannot be equated with the proverbial rich who value the marginal dollar at less than that. Yet both the firm, and the rich, can be risk averse.

In the EU model a **risk-averse decision maker**, i.e. one for whom a RV X is less desirable than a sure reward of EX, is characterized by a concave utility function u. The concavity of u also expresses the attitude towards wealth (decreasing marginal utility). Thus the DM's **attitude towards wealth** and his **attitude towards risk** are "forever bonded together", [38, p. 95]. Certain difficulties with the EU model are due to this fact. In Yaari's dual theory [38], and in the RCE model proposed here, the attitude towards wealth and the attitude towards risk are effectively separated.

The above mentioned alternative theories, which lost much of the elegance, simplicity and tractability of EU, address the discrepancies between the EU model axioms and actual choices under risk as observed in psychological tests. In this paper we focus on the **predictive** usage of the EU model, which is dominant in economics and finance. Here too the EU model has a mixed record, giving valid predictions, as well as implausible ones.

To be specific, we consider two models of economic behavior under uncertainty, a **competitive firm under price uncertainty** [31],[21] and **investment in safe and in risky assets**, [3],[9],[15].

For the competitive firm, the EU model yields the fundamental result, that optimal production under uncertainty is less than that under (comparable) certainty. It also gives a sensible condition (necessary and sufficient) for production to start, [31].

One would however expect that an increase in the selling price will result in increased production, but the EU model claims the opposite for certain risk-averse utilities. The dependence[4] of the optimal output on the fixed cost is another source of difficulty.

In the investment model, diversification is prescribed by the EU model under a natural condition. An impressive illustration of the predictive power of the EU model is the following result of Tobin, [36], which holds for all risk-averse utilities:

> "If a is the demand for risky investment when the return is a random variable X, then $a/1+h$ is the demand when the return is the random variable $(1+h)X$".

In the investment model, when the rate of return of the safe asset increases, one would expect part of the investment capital to switch from the risky asset to the safe asset. However, the EU model allows the opposite behavior

[4]Called "paradoxical" in [21] and "seemingly paradoxical" in [31].

for certain risk-averse utilities[5]. Also it was established empirically[6] that the elasticity of demand for cash balance is ≥ 1[7], but here again the EU model leaves open the possibility of elasticity < 1 for certain risk-averse utility functions.

To avoid these pathologies (of the EU model), additional hypotheses are customarily imposed on the utility function $u(\cdot)$. These hypotheses are stated in terms of the **Arrow-Pratt absolute risk-aversion index**

$$r(z) = -\frac{u''(z)}{u'(z)} \tag{1.6}$$

and the **Arrow-Pratt relative risk-aversion index**

$$R(z) = zr(z) \tag{1.7}$$

In the investment model, a typical postulate of Arrow [3] is:

$$\text{“}r(\cdot) \text{ is non-increasing”} \tag{1.8}$$

The Arrow investment model [3] does not consider the effects of the rate of return ρ of the safe asset. This question was addressed in [15] where (p. 1068) it was shown that, even with r non-increasing, it is possible for ρ to increase and for investment money to shift from safe to risky assets! A sufficient condition to exclude this possibility is

$$\text{“}r(\cdot) \text{ is non-decreasing or } R(\cdot) \leq 1\text{”} \tag{1.9}$$

Both (1.8) and (1.9) are conditions on the 3rd derivative of the utility u, which may be difficult to check. Moreover, the only risk-averse utility function with $r(\cdot)$ both non-decreasing and non-increasing is the exponential utility.

The CE advocated here is the **recourse certainty equivalent** (RCE)

$$S_v(Z) := \sup_z \{z + E_Z v(Z - z)\} \tag{1.10}$$

where $v(\cdot)$ is the **value-risk function** of the DM, mapping the possible (yet to be realized) values x of a RV, into their values $v(x)$ to the DM, at the time

[5] To quote from [15]: "such optimal behavior appears to be unlikely".

[6] See references in [3, p. 103].

[7] "Thus, the notion that security, in the particular form of cash balances, has a wealth elasticity of at least one, seems to be the only remaining explanation of the historical course of money holdings", [3, p. 104].

of decision (before realization). We propose the RCE S_v as a criterion for decision making (DM) under uncertainty i.e. for ranking RV's. The given value-risk function v induces a complete order $\succeq$ on RV's,

$$X \succeq Y \iff S_v(X) \geq S_v(Y) \tag{1.11}$$

in which case X is preferred over Y by a DM with a value-risk function v.

Any new approach to DM under uncertainty should be measured against the standards of the classical expected utility (EU) approach. "What matters is whether the model offers higher predictive accuracy than competing models of similar complexity .... What counts is whether the theory ... predicts behavior not used in the construction of the model", [32].

The advantages of the RCE approach, for predictive purposes, are demonstrated here by reexamining the above classical models of production and investment, and by studying a classical problem of optimal insurance coverage. In particular, the RCE approach (i) retains the successful predictions of the EU model (as listed above), (ii) does not require restrictive (third-derivative) conditions on u (thus the conclusions are valid for the whole class of risk-averse utilities), and (iii) is mathematically tractable, comparable in simplicity and elegance to the EU model.

We derive the RCE, using considerations of **stochastic programming with recourse**, in § 2. The main properties of the RCE are collected in Theorem 2.1. One such property is **shift additivity**, which holds for arbitrary value-risk functions v,

$$S_v(Z + c) = S_v(Z) + c \quad \text{for all RV } Z \text{ and constant } c. \tag{1.12}$$

Thus the RCE separates deterministic changes in wealth from the random variable which it evaluates. As mentioned above, shift additivity holds also in Yaari's **dual theory of choice** [38], and in the u-mean CE (1.3). In contrast, the classical CE (1.2) is shift additive only for **linear** and **exponential** utilities. For this reason, certain results (discussed in [4]), which in the EU model hold only for the exponential utility, hold in the RCE model for arbitrary utilities. Examples are the bridging of the gap between the buying and selling values of information, and the well known separation theorem in portfolio selection[8].

In Theorem 2.1 it is also shown that **risk aversion** in the sense of

$$S_v(Z) \leq EZ \quad \text{for all RV} \quad Z$$

[8]The proofs in [4] use only shift additivity.

is equivalent to[9]

$$v(x) \leq x \quad \text{for all} \quad x$$

which, for normalized value-risk functions v (i.e. $v(0) = 0$ and $v'(0) = 1$) is a weaker requirement than the concavity of $v(\cdot)$.

In Section 3 we discuss the recoverability of the value-risk function $v(\cdot)$ from the RCE S_v. It is shown that $v(\cdot)$ measures the **rate of change** in the RCE, when moving from a sure situation to a risky one (Theorem 3.1). Besides rendering a precise meaning of the value-risk function, this also suggests an empirical way to construct it from observed behavior.

It is natural to ask, in the RCE model, what notion of risk-aversion corresponds to the concavity of $v(\cdot)$. The answer is given in Section 4, where we show that $v(\cdot)$ is concave iff the RCE $S_v(\cdot)$ exhibits risk-aversion in the sense of Rothschild and Stiglitz [28], i.e. aversion to **mean preserving increase in risk.**

Certain functionals, associated with the RCE and useful in applications, are studied in Section 5. Section 6 is devoted to production under price uncertainty. The next two sections deal with investment in safe and in risky assets: The Arrow model [3] in Section 7, and a slight generalization in Section 8. An application of the RCE to the problem of optimal insurance coverage is discussed in Section 9.

In the last section, § 10, we attempt to explain the success of the RCE theory in making plausible predictions with fewer assumptions than the EU theory.

2 The Recourse Certainty Equivalent

A **decision under uncertainty,** as the name implies, is a decision made before the realization of the random variable in question. The consequences of this (apriori) decision depend on the (posteriori) realization. A rational decision maker must weigh these consequences according to their likelihood and value.

This is the rationale for **stochatic programming with recourse** (SPwR), or **two-stage stochastic programming,** proposed in 1955 by G. Dantzig [10], see also Dantzig and Madansky [11] and Beale [5]. For illustration, consider the problem

$$\max \{f(z) : g(z) \leq Z \} \tag{2.1}$$

[9]We acknowledge the help of the referees in clarifying this point.

Here:

z - decision variable,

Z - budget,

$g(z)$ - the budget consumed by z,

$f(z)$ - the profit resulting from z.

If Z is random then the apriori decision z may violate the (stochastic) constraint

$$g(z) \leq Z \tag{2.2}$$

for some realizations of Z. In SPwR the optimal decision z^* is determined by considering for each realization of Z a **second stage decision** y, consuming $h(y)$ budget units and contributing $v(y)$ to the profit. Thus z^* is the optimal solution of

$$\max_z \{f(z) + E_Z \left(\max_y \{v(y) : \; g(z) + h(y) \leq Z\} \right)\} \tag{2.3}$$

The success of SPwR stems from the fact that it takes into account the trade off between **greed** (profit maximization) and **caution** (honoring the budget constraint).

In those cases where z, y are scalars (e.g. levels of production), $h(\cdot)$ is monotone increasing ("more costs more") and $v(\cdot)$ is monotone increasing ("more is better"), we can rewrite (2.3) as

$$\max_z \{f(z) + E_Z \left(\max_y \{v(y) : \; g(z) + y \leq Z\} \right)\} \tag{2.4}$$

where y, v correspond in (2.3) to $h(y)$, $v \circ h^{-1}$ respectively. If v is monotonely increasing then (2.4) is equivalent to:

$$\max_z \{f(z) + E_Z \, v(Z - g(z))\} \tag{2.5}$$

in which y has been eliminated. The optimal value in (2.5) is the "SPwR value" of the SP (2.1).

In this paper we use the SPwR paradigm to "evaluate" RV's. Our thesis is that **assigning a value to a RV** is in itself a **decision problem**. Thus, the "value" of a RV Z to a DM is the "most that he can make of it", i.e.

$$\text{value of } Z = \max \{z : z \leq Z\} \tag{2.6}$$

and we interpret (2.6) as the "SPwR value" which, by analogy with (2.5), is the RCE (1.10)

$$\sup_z \{z + E_Z v(Z - z)\}$$

Here $v(x)$ is the current (before realization) value of the realized value x. We call $v(\cdot)$ the **value-risk function.**

Remark 2.1 Another possible interpretation of (2.6) is

$$\max\{z: \; u(z) \le E_Z u(Z)\} \tag{2.7}$$

where $u(\cdot)$ is a utility function. For monotonely increasing $u(\cdot)$, the optimal value of (2.7) is then the classical certainty equivalent (1.2). The difficulties of modelling stochastic constraints by utility surrogates, such as the above or others, have been noted elsewhere, see [20]. In particular, the formulation (2.7) does not allow trade off between "greed" and "caution" as in (1.10).

Remark 2.2 Note that $S_v(Z)$ can be viewed as a **temporal induced preference functional** in the sense of Kreps and Porteus [19], see also [24], but unlike the **multiperiod** setting in the above references we have here a **single period.** However, in this single period there are two "periods", or time instants, induced by risk: The instant **before** the realization of the RV and the instant **after.** The time separation between these instants is irrelevant for our purposes.

We list now several assumptions on $v(\cdot)$ which are reasonable, and useful for our purpose.

Assumption 2.1
(v1) $v(0) = 0$
(v2) $v(\cdot)$ *is strictly increasing*
(v3) $v(x) \le x$ *for all* x
(v4) $v(\cdot)$ *is strictly concave*
(v5) v *is continuously differentiable*

Remark 2.3 By Assumption 2.1(v1),(v2)

$$v(x) < 0 \;\text{ for }\; x < 0,$$

thus $v(\cdot)$ can also be viewed as a **penalty function**, penalizing violations of the constraint

$$z \le Z$$

Of particular interest is the following class of value-risk functions

$$\mathcal{U} = \left\{ v : \begin{array}{c} v \text{ strictly increasing, strictly concave, continuously} \\ \text{differentiable}, \ \ v(0) = 0, \ \ v'(0) = 1 \end{array} \right\} \tag{2.8}$$

which, for the purpose of comparison with the EU model, can be thought of as **normalized utility functions**[10].

The question of the attainment of the supremum in (1.10) is settled, for any $v \in \mathcal{U}$, in the following:

Lemma 2.1 *Let the RV Z have support* $[z_{\min}, z_{\max}]$, *with finite* $z_{\min}$ *and* $z_{\max}$. *Then for any* $v \in \mathcal{U}$ *the supremum in (1.10) is attained uniquely at some* z_S,

$$z_{\min} \leq z_S \leq z_{\max}, \tag{2.9}$$

which is the solution of

$$E\, v'(Z - z_S) = 1, \tag{2.10}$$

so that

$$S_v(Z) = z_S + E\, v(Z - z_S) \tag{2.11}$$

Proof. Note that $Z - z_{\min} \geq 0$ with probability 1. Also $v'(\cdot)$ is decreasing since v is concave. Therefore

$$E\, v'(Z - z_{\min}) \leq E\, v'(0) = 1$$

Similarly

$$E\, v'(Z - z_{\max}) \geq E\, v'(0) = 1 .$$

Since v' is continuous, the equation[11]

$$E\, v'(Z - z) = 1$$

has a solution z_S in $[z_{\min}, z_{\max}]$, which is unique by the strict monotonicity of v'. Now z_S is a stationary point of the function

$$f(z) = z + E\, v(Z - z) \tag{2.12}$$

which is concave since $v \in \mathcal{U}$, see (2.8). Therefore the supremum of (2.12) is attained at z_S. □

[10] For concave v the gradient inequality

$$v(x) \leq v(0) + v'(0)x$$

shows that all $v \in \mathcal{U}$ satisfy (v3) of Assumption 2.1.

[11] This equation is the necessary condition for maximum in (1.10). Differentiation "inside the expectation" is valid if e.g. v' is continuous and $E\, v'(\cdot) < \infty$, see [8, p. 99].

Theorem 2.1 (Properties of the RCE)

(a) **Shift additivity**. *For any $v : \mathbb{R} \to \mathbb{R}$, any RV Z and any constant c*

$$S_v(Z + c) = S_v(Z) + c$$

(b) **Consistency**. *If v satisfies (v1), (v3) then, for any constant c* [12],

$$S_v(c) = c \tag{2.13}$$

(c) **Subhomogeneity**. *If v satisfies (v1) and (v4) then, for any RV Z,*

$$\frac{1}{\lambda} S_v(\lambda Z) \quad \text{is decreasing in } \lambda, \quad \lambda > 0$$

(d) **Monotonicity**. *If v satisfies (v2) then, for any RV X and any nonnegative RV Y,*

$$S_v(X + Y) \geq S_v(X)$$

(e) **Risk aversion**. *v satisfies (v3) iff*

$$S_v(Z) \leq EZ \quad \text{for all RV's } Z \tag{2.14}$$

(f) **Concavity**. *If $v \in \mathcal{U}$ then for any RV's X_0 , X_1 and $0 < \alpha < 1$,*

$$S_v(\alpha X_1 + (1 - \alpha)X_0) \geq \alpha S_v(X_1) + (1 - \alpha)S_v(X_0) \tag{2.15}$$

(g) **2nd order stochastic dominance**. *Let X, Y be RV's with compact supports. Then*

$$S_v(X) \geq S_v(Y) \quad \text{for all } v \in \mathcal{U} \tag{2.16}$$

if and only if

$$E\,v(X) \geq E\,v(Y) \quad \text{for all } v \in \mathcal{U} \tag{2.17}$$

Proof. (a) For any function $v : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} S_v(Z + c) &= \sup_z \{z + E\,v(Z + c - z)\} \\ &= c + \sup_z \{(z - c) + E\,v(Z - (z - c))\} = c + S_v(Z) \end{aligned}$$

[12] Considered as a degenerate RV.

(b) For any constant c,

$$\begin{aligned} S_v(c) &= \sup_z \{z + v(c - z)\} \\ &\le \sup_z \{z + (c - z)\} \quad \text{by (v3)} \\ &= c \end{aligned}$$

Conversely,

$$\begin{aligned} S_v(c) &\ge \{c + v(c - c)\} \\ &= c \quad \text{by (v1)} \end{aligned}$$

(c) For any $v : \mathbb{R} \to \mathbb{R}$ and $\lambda > 0$ define v_λ by

$$v_\lambda(x) := \frac{1}{\lambda} v(\lambda x), \quad \forall\, x \tag{2.18}$$

Then

$$S_{v_\lambda}(Z) = \frac{1}{\lambda} S_v(\lambda Z), \quad \text{for all RV } Z, \tag{2.19}$$

as follows from,

$$\begin{aligned} S_{v_\lambda}(Z) &= \sup_z \{z + \frac{1}{\lambda} E_Z v(\lambda(Z - z))\} \\ &= \frac{1}{\lambda} \sup_{\bar{z}} \{\bar{z} + E_Z v(\lambda Z - \bar{z})\} \quad (\bar{z} = \lambda z) \\ &= \frac{1}{\lambda} S_v(\lambda Z) \end{aligned}$$

It therefore suffices to show that

$$v_\lambda(z) \quad \text{is decreasing in } \lambda, \quad \lambda > 0$$

Indeed, let

$$0 < \lambda_1 < \lambda_2$$

By the concavity of v it follows, for all z,

$$\frac{v(\lambda_2 z) - v(\lambda_1 z)}{\lambda_2 - \lambda_1} \le \frac{v(\lambda_1 z) - v(0)}{\lambda_1}$$

and by (v1)

$$\frac{v(\lambda_2 z)}{\lambda_2} \le \frac{v(\lambda_1 z)}{\lambda_1}$$

(d)

$$\begin{aligned} S_v(X+Y) &= \sup_z \{z + Ev(X+Y-z)\} \\ &\geq \sup_z \{z + Ev(X-z)\} \quad \text{by (v2)} \end{aligned}$$

(e) If v satisfies (v3) then for any RV Z,

$$\begin{aligned} S_v(Z) &= \sup_z \{z + Ev(Z-z)\} \\ &\leq \sup_z \{z + E(Z-z)\} = EZ \end{aligned}$$

Conversely, if for all RV's Z

$$S_v(Z) \leq EZ$$

then, for any RV Z and any constant z,

$$\begin{aligned} z + Ev(Z-z) &\leq EZ \\ \therefore\ Ev(Z-z) &\leq E(Z-z) \\ \therefore\ Ev(Z) &\leq EZ \end{aligned}$$

proving (v3).
(f) Let $0 < \alpha < 1$, and $X_\alpha = \alpha X_1 + (1-\alpha)X_0$. Then by the concavity of v, for all z_0, z_1,

$$Ev(X_\alpha - \alpha z_1 - (1-\alpha)z_0) \geq \alpha Ev(X_1 - z_1) + (1-\alpha)Ev(X_0 - z_0)$$

Adding $\alpha z_1 + (1-\alpha)z_0$ to both sides, and supremizing jointly with respect to z_1, z_0, we get

$$\begin{aligned} S_v(X_\alpha) &\geq \sup_{z_1,\, z_0} \{\alpha[z_1 + Ev(X_1 - z_1)] + (1-\alpha)[z_0 + Ev(X_0 - z_0)]\} \\ &= \alpha S_v(X_1) + (1-\alpha)S_v(X_0) \end{aligned}$$

(g) (2.17) $\Longrightarrow$ (2.16). Since each $v \in \mathcal{U}$ is increasing, (2.17) implies

$$z + Ev(X-z) \geq z + Ev(Y-z) \quad \forall\ z, \text{ and } \forall\ v \in \mathcal{U}$$

and (2.16) follows by taking suprema.
(2.16) $\Longrightarrow$ (2.17). Let z_X, z_Y be points where the suprema defining $S_v(X)$ and $S_v(Y)$ are attained, see Lemma 2.1. Then, for any $v \in \mathcal{U}$,

$$\begin{aligned} S_v(X) &= z_X + Ev(X - z_X) \geq z_Y + Ev(Y - z_Y), \quad \text{by (2.16)} \\ &\geq z_X + Ev(Y - z_X) \end{aligned}$$

Therefore

$$Ev(X - z_X) \geq Ev(Y - z_X) \quad \text{for all } v \in \mathcal{U}, \quad \text{implying (2.17).} \quad \square$$

Remark 2.4 Theorem 2.1 lists properties which seem reasonable for any certainty equivalent. Property (b) is natural and requires no justification. The remaining properties will now be discussed one by one.

(a) Note that shift additivity holds for all functions v, i.e. it is a **generic property** of the RCE.

To explain shift additivity consider a decision-maker indifferent between a lottery Z and a sure amount S. If 1 Dollar is added to all the possible outcomes of the lottery, then an addition of 1 Dollar to S will keep the decision maker indifferent.

(c) An important consequence (and the reason for the name "subhomogeneity") is

$$S_v(\lambda Z) \leq \lambda S_v(Z), \quad \text{for all RV } Z \text{ and } \lambda > 1$$

Thus indifference between the RV Z and its CE $S_v(Z)$ goes together with preference for $\lambda S_v(Z)$ over the RV λZ, for $\lambda > 1$. This is explained by

$$E(\lambda Z) = \lambda EZ$$

$$\text{Var}(\lambda Z) = \lambda^2 \text{Var}(Z) > \lambda \text{Var}(Z) \quad \text{if } \lambda > 1$$

An interesting result, in view of (c) and (e), is that for $v \in \mathcal{U}$,

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} S_v(\lambda Z) = EZ$$

(d) If v satisfies (v1) and (v2), and if the RV Z satisfies $Z \geq z_{\min}$ with probability 1, then

$$S_v(Z) \geq z_{\min} \tag{2.20}$$

This follows from part (d) by taking $X = z_{\min}$ (degenerate RV) and $Y = Z - z_{\min}$.

(e) In the EU model, risk aversion is characterized by the concavity of the utility function. In the RCE model risk aversion is carried by the weaker property $v(x) \leq x$, $\forall x$. We show in § 4 that concavity of v corresponds to strong risk aversion in the sense of Rothschild and Stiglitz, [28].

(f) The concavity of $S_u(\cdot)$, for all $u \in \mathcal{U}$, expresses risk-aversion as aversion to variability. To gain insight consider the case of two independent RV's X_1 and X_0 with the same mean and variance. The mixed RV $X_\alpha = \alpha X_1 + (1-\alpha) X_0$

has the same mean, but a smaller variance. Concavity of S_u means that the more centered RV X_α is preferred.

The risk-aversion inequality (2.14) is implied by (f): Let $Z,\ Z_1,\ Z_2,\ \ldots$ be independent, identically distributed RV's. Then by (f),

$$\begin{aligned} S_u\Big(\frac{1}{n}\sum_{i=1}^{n} Z_i\Big) &\geq \frac{1}{n}\sum_{i=1}^{n} S_u(Z_i) \\ &= S_u(Z) \end{aligned}$$

As $n \to \infty$, (2.14) follows by the strong law of large numbers.

In contrast, the classical CE $u^{-1}Eu(\cdot)$ is not necessarily concave for all concave u.

(g) In general, for a given $u \in \mathcal{U}$,

$$E\,u(X) \geq E\,u(Y) \tag{2.21}$$

does not imply

$$S_u(X) \geq S_u(Y) \tag{2.22}$$

i.e. (2.21) and (2.22) may induce different orders on RV's, see [7]. Note however that in (2.16) and (2.17) the inequality holds for all $u \in \mathcal{U}$ [13]. This defines a partial order on RV's, the (2nd order) stochastic dominance, [16].

Example 2.1 (Exponential value-risk function) *Here*

$$u(z) = 1 - e^{-z}\ ,\quad \forall\, z, \tag{2.23}$$

and equation (2.10) becomes $Ee^{-Z+z} = 1$, *giving* $z_S = -\log\ Ee^{-Z}$ *and the same value for the RCE*

$$S_u(Z) = -\log\ Ee^{-Z} \tag{2.24}$$

A special feature of the exponential utility function (2.23) is that the classical CE (1.2) becomes

$$u^{-1}Eu(Z) = -\log\ Ee^{-Z}$$

showing that for the exponential function, the certainty equivalents (1.10) and (1.2) coincide.

[13] In which case Y is called riskier than X.

Example 2.2 (Quadratic value-risk function) *Here*[14]

$$u(z) = z - \frac{1}{2}z^2 , \quad z \leq 1 \tag{2.25}$$

and for a RV Z with $z_{\max} \leq 1$, $EZ = \mu$ and variance σ^2, equation (2.10) gives $z_S = \mu$, and by (2.11)

$$S_u(Z) = \mu - \frac{1}{2}\sigma^2 \tag{2.26}$$

Corollary 2.1 *In both the exponential and quadratic value-risk functions*

$$S_u(\sum_{i=1}^{n} Z_i) = \sum_{i=1}^{n} S_u(Z_i) \tag{2.27}$$

for independent RV's $\{ Z_1, Z_2, \dots , Z_n\}$[15] □

Example 2.3 *For the so-called* **hybrid model** *([4],[33]) with exponential utility u and a normally distributed RV $Z \sim N(\mu, \sigma^2)$,*

$$S_u(Z) = \mu - \frac{1}{2}\sigma^2$$

Example 2.4 (Piecewise linear value-risk function) *Let*

$$v(t) = \begin{cases} \beta t, & t \leq 0 \\ \alpha t, & t > 0 \end{cases} \quad 0 < \alpha < 1 < \beta \tag{2.28}$$

If F is the cumulative distribution function of the RV Z, then the maximizing z in (1.10) is the $\frac{1-\alpha}{\beta-\alpha}$-percentile of the distribution F of Z:

$$z^* = F^{-1}(\frac{1-\alpha}{\beta-\alpha})$$

and the RCE associated with (2.28) is

$$S_v(Z) = \beta \int^{z^*} t\,dF(t) + \alpha \int_{z^*} t\,dF(t).$$

[14] The restriction $z \leq 1$ in (2.25) guarantees that u is increasing throughout its domain.

[15] The classical CE (1.2) is additive, for independent RV's, if u is exponential but not if u is quadratic.

The following result is stated for discrete RV's. Let X be a RV assuming finitely many values,

$$\text{Prob}\{X = x_i\} = p_i \tag{2.29}$$

We denote X by

$$X = [\mathbf{x}, \mathbf{p}], \ \mathbf{x} = (x_1, x_2, \dots, x_n), \ \mathbf{p} = (p_1, p_2, \dots, p_n) \tag{2.30}$$

The RCE of $[\mathbf{x}, \mathbf{p}]$ is

$$S_v([\mathbf{x}, \mathbf{p}]) = \max_z \{z + \sum_{i=1}^n v(x_i - z)p_i\} \tag{2.31}$$

We consider $S_v([\mathbf{x}, \mathbf{p}])$ as a function of the arguments $\mathbf{x}$ and $\mathbf{p}$.

Theorem 2.2

(a) For any function $v : \mathbb{R} \to \mathbb{R}$, and any $\mathbf{x} = (x_1, x_2, \dots, x_n)$, the RCE $S_v([\mathbf{x}, \mathbf{p}])$ is convex in $\mathbf{p}$.
(b) For v concave, and any probability vector $\mathbf{p}$, the RCE $S_v([\mathbf{x}, \mathbf{p}])$ is concave in $\mathbf{x}$.

Proof. (a) A pointwise supremum of affine functions, see (2.31), is convex.
(b) The supremand

$$z + \sum_{i=1}^n p_i v(x_i - z)$$

is **jointly concave** in z and x. The supremum over z is concave in x, [27]. □

We summarize, for a RV $[\mathbf{x}, \mathbf{p}]$, the dependence on $\mathbf{p}$ and $\mathbf{x}$, of the expected utility $Eu(\cdot)$ and 3 certainty equivalents.

	As a function of $\mathbf{p}$	As a function of $\mathbf{x}$
Eu, u concave	linear	concave
$u^{-1}Eu$, u concave	convex	?
Y_f (1.4)	convex	linear
S_v	convex	concave (if v is concave)

3 Recoverability and the Meaning of $v(\cdot)$

In § 2 we studied properties of S_v induced by v. This section is devoted to the inverse problem, of **recovering** v from a **given** S_v.

The discussion is restricted to RCE's S_v defined by $v \in \mathcal{U}$. For these RCE's, we find $v \in \mathcal{U}$ satisfying (1.10).

Our results are stated in terms of an elementary RV X

$$X = \begin{cases} x, & \text{with probability } p \\ 0, & \text{with probability } \bar{p} = 1 - p \end{cases} \tag{3.1}$$

which we denote (x, p). For this RV,

$$S_v((x,p)) = \sup_z \{z + p\,v(x - z) + \bar{p}\,v(-z)\} \tag{3.2}$$

which we abbreviate $S_v(x,p)$.

Theorem 3.1 *If* $v \in \mathcal{U}$ *then*

$$v(x) = \frac{\partial}{\partial p} S_v(x,p)\,|_{p=0} \tag{3.3}$$

Proof. For $v \in \mathcal{U}$ the supremum in (3.2) is attained at $z = z(x,p)$ satisfying the optimality condition (2.10)

$$p\,v'(x - z(x,p)) + \bar{p}\,v'(-z(x,p)) = 1 \tag{3.4}$$

which, for $p = 0$ gives

$$v'(-z(x,0)) = 1$$

and since $v \in \mathcal{U}$,

$$z(x,0) = 0 \tag{3.5}$$

Now, by the envelope theorem (appendix A),

$$\frac{\partial S_v(x,p)}{\partial p} = v(x - z(x,p)) - v(-z(x,p)) \tag{3.6}$$

and (3.3) follows by substituting (3.5) and $v(0) = 0$ in (3.6). □

To interpret this result consider an RCE maximizing individual who currently owns 0 \$, and is offered the sum x with probability p. The resulting change in his RCE is

$$\Delta(x,p) = S_v(x,p) - S_v(x,0)$$

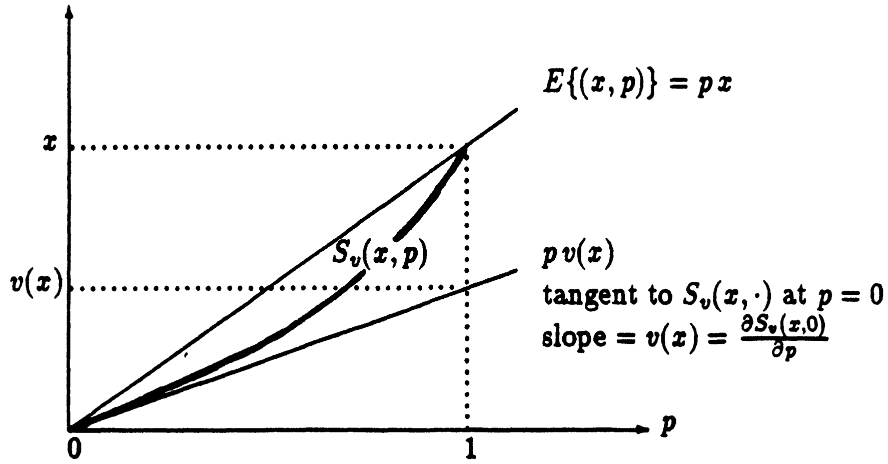

Figure 3.1: Recovering $v(x)$ from $S_v(x,p)$

and the rate of change is $\frac{\Delta(x,p)}{p}$. Theorem 3.1 says that this rate of change, for an infinitesimal change in risk ($p \to 0$) is precisely $v(x)$, the value-risk function evaluated at x.

Note that for a risk-neutral DM the added value Δ is $E\{(x,p)\} = px$. We illustrate this, for fixed x, in Fig. 3.1.

The following theorem is a companion of Theorem 3.1. It says that the limiting rate of change $\frac{\Delta(x,p)}{x}$ is exactly the probability p of obtaining x.

Theorem 3.2 *If $v \in \mathcal{U}$ then*

$$p = \frac{\partial}{\partial x} S_v(x,p)\mid_{x=0} \tag{3.7}$$

Proof. Substituting $x = 0$ in (3.4) gives

$$v'(-z(0,p)) = 1 \tag{3.8}$$

By the envelope theorem (Appendix A) we get

$$\frac{\partial S_v(x,p)}{\partial x} = p\,v'(x - z(x,p))$$

which, substituting $x = 0$ and (3.8) gives,

$$\frac{\partial S_v(0,p)}{\partial x} = p \quad \Box$$

It is natural to ask, for any certainty equivalent $\mathrm{CE}(x,p)$, for the values

$\frac{\partial}{\partial p}\mathrm{CE}(x,0)$	the **value risk function** of CE
$\frac{\partial}{\partial x}\mathrm{CE}(0,p)$	the **probability risk function** of CE

We summarize the results, in the following table, for the classical CE, the u-mean CE and the RCE.

Certainty equivalent $\mathrm{CE}(x,p)$	$\frac{\partial}{\partial p}\mathrm{CE}(x,0)$	$\frac{\partial}{\partial x}\mathrm{CE}(0,p)$
$C_u = u^{-1}Eu$	$\frac{u(x)-u(0)}{u'(0)}$	p
M_u	$\frac{u(x)-u(0)}{u'(0)}$	p
S_v	$v(x)$	p

For the Yaari CE (1.4)

$$Y_f(x,p) = \begin{cases} x f(p), & x \geq 0 \\ x\,[1 - f(\bar{p})], & x \leq 0 \end{cases} \tag{3.9}$$

We get:

$$\lim_{p\to 0^+} \frac{Y_f(x,p) - Y_f(x,0)}{p} = \begin{cases} x f'(0), & x \geq 0 \\ x f'(1), & x \leq 0 \end{cases}$$

and the two-sided derivatives

$$\begin{aligned} \lim_{x\to 0^+} \frac{Y_f(x,p) - Y_f(0,p)}{x} &= f(p) \\ \lim_{x\to 0^-} \frac{Y_f(x,p) - Y_f(0,p)}{x} &= 1 - f(\bar{p}) \end{aligned}$$

Remark 3.1

(a) The value-risk function for the EU model is thus precisely the **normalized utility function**

$$u_N(x) := \frac{u(x) - u(0)}{u'(0)} \quad (u_N(0) = 0,\ u'_N(0) = 1)$$

This result suggests a new way to recover the utility function in the EU theory.

(b) The probability-risk function (for a nonnegative RV) in Yaari's theory is thus precisely the function $f(p)$ in terms of which Y_f is uniquely defined. This is a new interpretation of f.

(c) Note that the value-risk function corresponding to Yaari's CE is of the form

$$v(x) = \begin{cases} \alpha x, & x \leq 0 \\ \beta x, & x \geq 0 \end{cases} \tag{3.10}$$

with $\alpha = f'(0),\ \beta = f'(1)$. The convexity of f plus its normalization $f(0) = 0,\ f(1) = 1$ imply

$$\alpha < 1 < \beta$$

The function v in(3.10) is the source of the piecewise linear value-risk function in Exmaple 2.4.

4 Strong Risk Aversion

In the EU model risk aversion is characterized by the concavity of the utility function, while in the RCE model it is equivalent to the weaker property (Theorem 2.1(e))

$$v(x) \leq x, \quad \forall\, x. \tag{4.1}$$

It is natural tö ask what corresponds, in the RCE model, to the **concavity** of v, i.e.

$$v \in \mathcal{U} \tag{4.2}$$

The answer is given here in terms of a classical notion of risk-aversion due to Rothschild and Stiglitz [28], see also [12].

Definition 4.1 *Let F_X, F_Y be the c.d.f. of the RV's X, Y with support $[a,\ b]$.*

(a) If there is a $c \in [a,\ b]$ such that

$$\begin{aligned} F_Y(t) &\geq F_X(t), \quad a \leq t \leq c \\ F_X(t) &\geq F_Y(t), \quad c \leq t \leq b \end{aligned}$$

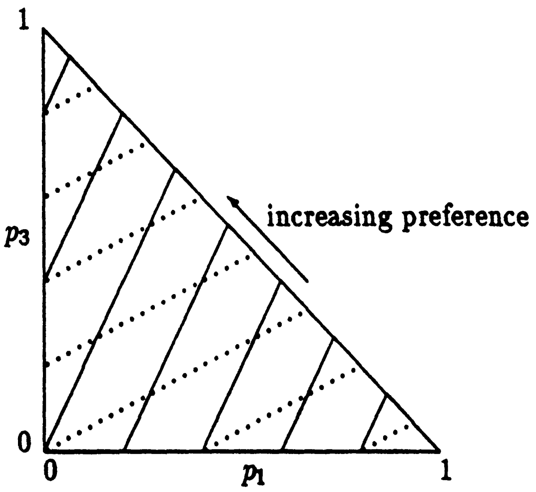

Figure 4.1: Iso-EU and iso-mean lines in $D\{x_1, x_2, x_3\}$

then F_Y *is said to* **differ** *from* F_X *by a* **mean preserving simple increase in risk** *(MPSIR).*
(b) F_Y *is said to* **differ** *from* F_X *by a* **mean preserving increase in risk** *(MPIR) if it differs from* F_X *by a sequence of MPSIR's.*

Definition 4.2 *An RCE maximizing DM with a value-risk function* v *exhibits* **strong risk-aversion** *if*

$$\left\{ \begin{array}{l} F_Y \text{ differs from } F_X \\ \text{by a MPIR} \end{array} \right\} \implies S_v(Y) \leq S_v(X)$$

This concept is best illustrated graphically as in [25]. Let

$$x_1 < x_2 < x_3$$

be fixed, and let $D\{x_1, x_2, x_3\}$ denote the **probability distributions** over the values x_1, x_2, x_3. Each $\mathbf{p} = (p_1, p_2, p_3) \in D\{x_1, x_2, x_3\}$ can be represented by a point in the unit triangle in the (p_1, p_3)-plane as in Fig. 4.1, where p_2 is determined by $p_2 = 1 - p_1 - p_3$. The dotted lines are loci of distributions with same expectation (**iso-mean lines**) i.e. points (p_1, p_3) such that

$$p_1 x_1 + (1 - p_1 - p_3) x_2 + p_3 x_3 = \text{constant} \tag{4.3}$$

As one moves in the unit triangle across the iso-mean lines, from the southeast (SE) corner to the northwest (NW) corner, the values of the mean (4.3) increase. Thus movement from the SE to the NW is in the preferred direction.

The iso-mean lines are parallel with slope (i.e. $\Delta p_3/\Delta p_1$)

$$\text{slope of iso-mean lines} = \frac{x_2 - x_1}{x_3 - x_2} > 0 \tag{4.4}$$

A movement along the iso-mean lines, in the NE direction corresponds to an MPIR as in Definition 4.1(b).

Similarly, the solid lines in Fig. 4.1 represent **iso expected utility curves** which are parallel straight lines (due to the "linearity in probabilities" of the *EU* functional) with

$$\text{slope of iso-}EU\text{ lines} = \frac{u(x_2) - u(x_1)}{u(x_3) - u(x_2)} > 0 \tag{4.5}$$

The slope (4.5) is positive because of the monotonicity of u.

For EU-maximizers **strong risk-aversion** corresponds to the iso-EU lines being steeper than the iso-mean lines, i.e.

$$\frac{u(x_2) - u(x_1)}{u(x_3) - u(x_2)} > \frac{x_2 - x_1}{x_3 - x_2} \tag{4.6}$$

which holds for all $x_1 < x_2 < x_3$ iff u is concave.

Turning to the RCE functional, the iso-RCE curves are not straight lines (since the RCE functional is convex in the probabilities). For RCE-maximizers **strong risk-aversion** means that at each point (p_1, p_3), the **slope of the iso-RCE curve** (through that point) **is steeper than the slope of the iso-mean line** (given by (4.4), see Fig. 4.2. Let now $p_3 = p_3(p_1)$ be the representation of an iso-RCE curve. By the definition (1.10), p_3 is solved from

$$\sup_z \{z + p_1 v(x_1 - z) + (1 - p_1 - p_3) v(x_3 - z) + p_3 v(x_3 - z)\} = \text{constant} \tag{4.7}$$

Then

$$\text{strong risk-aversion} \iff \frac{dp_3}{dp_1} > \frac{x_2 - x_1}{x_3 - x_2} \tag{4.8}$$

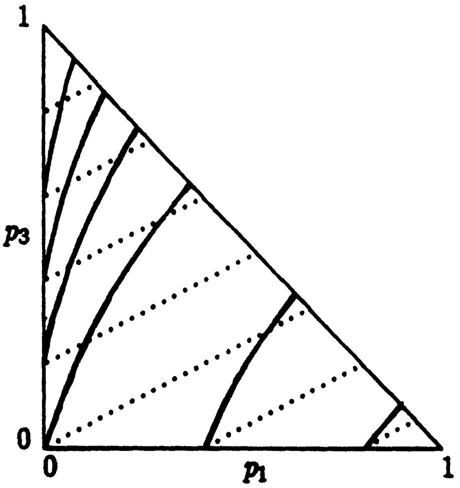

Figure 4.2: Iso-RCE curves and iso-mean lines in $D\{x_1, x_2, x_3\}$

Let the left side of (4.7) be written as a function

$$s(p_1, p_2, p_3)$$

of the probabilities p_i. Differentiating (4.7) with respect to p_1 we get

$$s_1 - s_2 - (s_2 - s_3)p_3' = 0, \quad \text{where } s_i = \frac{\partial s}{\partial p_i} \tag{4.9}$$

By the envelope theorem (Appendix A)

$$s_i = \frac{\partial s}{\partial p_i} = u(x_i - z^*) \tag{4.10}$$

where $z^* = z^*(p_1, p_3)$ is uniquely determined by (2.10)

$$p_1 v'(x_1 - z^*) + (1 - p_1 - p_3) v'(x_2 - z^*) + p_3 v'(x_3 - z^*) = 1$$

Combining (4.9) and (4.10) we thus get

$$p_3'(p_1) = \frac{v(x_2 - z^*) - v(x_1 - z^*)}{v(x_3 - z^*) - v(x_2 - z^*)}$$

and the Diamond-Stiglitz risk-aversion is, by (4.8)

$$\frac{v(x_2 - z^*) - v(x_1 - z^*)}{v(x_3 - z^*) - v(x_2 - z^*)} > \frac{x_2 - x_1}{x_3 - x_2} = \frac{(x_2 - z^*) - (x_1 - z^*)}{(x_3 - z^*) - (x_2 - z^*)} \tag{4.11}$$

which holds for all $x_1 < x_2 < x_3$ iff v is concave.

The above discussion can be generalized to a general RV X with distribution function $F \in D(T)$, where

$$D(T) := \{\text{distribution functions with compact support } T\} \tag{4.12}$$

We note that the RCE $S_v(X)$ can be written as

$$S_v(X) = \int U(x, F)\, dF(x) \tag{4.13}$$

where

$$U(x, F) = z(F) + v(x - z(F)) \tag{4.14}$$

and the maximizing $z(F)$ is obtained implicitly from (2.10)

$$\int v'(x - z(F))\, dF(x) = 1$$

Thus S_v, regarded as a function of F,

$$S_v(X) = V(F) \tag{4.15}$$

is a **generalized expected utility preference functional** in the sense of Machina, [22]. By (4.13), $U(x, F)$ is then the **local utility function** of Machina. We recall [22, Theorem 2] that for $V(F)$ Fréchet differentiable on $D(T)$, the preference order induced by V is strongly risk-averse iff $U(x, F)$ is concave in x for all $F \in D(T)$. Finally, by (4.14), the local utility $U(\cdot, F)$ is concave for all F iff the risk-value function $v(\cdot)$ is concave.

Remark 4.1 In the EU theory concavity of the utility u characterizes both **risk aversion** ($\text{CE}(X) \leq E\,X$) and **strong risk aversion**, hence the two are equivalent in EU theory. In the RCE theory, risk aversion requires that $v(x) \leq x$ while strong risk aversion requires the stronger property that v is concave.

For non-EU theories this divergence between the two notions of risk-aversion is not surprising. We note that in Yaari's dual theory

$$Y_f(X) \leq E\,X \quad \text{requires} \quad f(t) \leq t,\ \forall t$$

whereas strong risk-aversion requires the convexity of f, [38, Theorem 2][16].

[16]The convexity of f, plus Yaari's normalization $f(0) = 0$, $f(1) = 1$ implies $f(t) \leq t$.

5 Functionals and Approximations

Let $\mathbf{Z} = (Z_i)$ be a RV in $\mathbb{R}^n$, with expectation μ (vector) and covariance matrix Σ (if $n = 1$ then as above $\Sigma = \sigma^2$). For any vector $\mathbf{y} \in \mathbb{R}^n$, the **inner product,**

$$\mathbf{y} \cdot \mathbf{Z} = \sum_{i=1}^{n} y_i Z_i$$

is a scalar RV. Given $u \in \mathcal{U}$, the corresponding RCE of $\mathbf{y} \cdot \mathbf{Z}$ are taken as functionals in $\mathbf{y}$, the **RCE functional**

$$s_u(\mathbf{y}) := S_u(\mathbf{y} \cdot \mathbf{Z}), \tag{5.1}$$

We collect properties of the RCE functional in the following theorem, whose proof appears in Appendix B.

Theorem 5.1 *Let $u \in \mathcal{U}$ be twice continuously differentiable, and let $\mathbf{Z}$ and $s_u(\cdot)$ be as above. Then:*
(a) The functional s_u is concave, and given by

$$s_u(\mathbf{y}) = z_S(\mathbf{y}) + Eu(\mathbf{y} \cdot \mathbf{Z} - z_S(\mathbf{y})) \tag{5.2}$$

where $z_S(\mathbf{y})$ is the unique solution z of

$$E\, u'(\mathbf{y} \cdot \mathbf{Z} - z) = 1 \tag{5.3}$$

(b) Moreover,

$$s_u(\mathbf{0}) = 0,\ \nabla s_u(\mathbf{0}) = \mu,\ \ \nabla^2 s_u(\mathbf{0}) = u''(0)\Sigma \tag{5.4}$$

$$z_S(\mathbf{0}) = 0,\ \nabla z_S(\mathbf{0}) = \mu \tag{5.5}$$

and if u is three times continuously differentiable,

$$\nabla^2 z_S(\mathbf{0}) = \frac{u'''(0)}{u''(0)}\Sigma \quad \Box \tag{5.6}$$

Theorem 5.1 can be used to obtain the following approximation of the functional $s_u(\cdot)$ based on its Taylor expansion around $\mathbf{y} = \mathbf{0}$.

Corollary 5.1 *If u is three times continuously differentiable then*

$$s_u(\mathbf{y}) = \mu \cdot \mathbf{y} + \frac{1}{2} u''(0)\mathbf{y} \cdot \Sigma \mathbf{y} + o(\|\, \mathbf{y} \,\|^2) \quad \Box \tag{5.7}$$

Remark 5.1

(a) In particular, for $n = 1$ and $y = 1$, it follows from (5.7) that the RCE has the following second-order approximation

$$\begin{aligned} S_u(Z) &\approx \mu + \frac{1}{2}u''(0)\sigma^2 \\ &= \mu - \frac{1}{2}r(0)\sigma^2 \end{aligned} \tag{5.8}$$

where $r(\cdot)$ is the Arrow-Pratt risk-aversion index (1.6).
(b) We also note that the approximation (5.7) is exact if

(i) u is quadratic, or
(ii) u is exponential, Z is normal.

(c) By differentiating, and calculating the Taylor expansion of the classical CE (1.2) of $y \cdot Z$,

$$c_u(y) = u^{-1}Eu(y \cdot Z) \tag{5.9}$$

it follows that $c_u(y)$ is approximated by the right-hand side of (5.7). Thus we have

$$c_u(y) - s_u(y) = o(\| y \|^2) \tag{5.10}$$

showing that the CE functionals (5.1) and (5.9) are close for small y.

6 Competitive Firm under Uncertainty

The first application of the RCE is to the classical model studied by Sandmo [31], see also [21, §5.2]. A firm sells its **output** q at a **price** P, which is a RV with a known distribution function and expected value $EP = \mu$. Let $C(q)$ be the **total cost** of producing q, which consists of a **fixed cost** B and a **variable cost** $c(q)$,

$$C(q) = c(q) + B$$

The function $c(\cdot)$ is assumed normalized, increasing and strictly convex,

$$c(0) = 0, \quad c'(q) > 0, \quad c''(q) > 0 \quad \forall q \geq 0 \tag{6.1}$$

The firm has a strictly concave utility function u, i.e.

$$u' > 0, \quad u'' < 0$$

which is normalized so that $u(0) = 0,\ u'(0) = 1$. The objective is to maximize **profit**

$$\pi(q) = qP - c(q) - B$$

which is a RV. The classical CE (1.2) is used is Sandmo's analysis, so that the model studied is

$$\max_{q\geq 0} u^{-1} Eu(\pi(q))$$

or equivalently,

$$\max_{q\geq 0} Eu(\pi(q)) \tag{6.2}$$

Here we analyze the same model using the RCE. For the sake of comparison with the EU model, we assume that the firm's value-risk function is $u \in \mathcal{U}$, i.e. is a utility. The objective of the firm is therefore

$$\max_{q\geq 0} S_u(\pi(q)) \tag{6.3}$$

Now

$$\begin{aligned} \max_{q\geq 0} S_u(\pi(q)) &= \max_{q\geq 0} S_u(qP - c(q) - B) \\ &= \max_{q\geq 0}\{S_u(qP) - c(q)\} - B \end{aligned}$$

by (1.12). We conclude:

Proposition 6.1 *The optimal production output q^* is independent of the fixed cost B.* □

This result is in sharp contrast to the expected utility model (6.2) where the optimal output $\bar{q}$ depends on the fixed cost B: $\bar{q}$ increases [decreases] with B if the Arrow-Pratt index $r(\cdot)$ is an increasing [decreasing] function; the dependence is ambigious for utilities for which $r(\cdot)$ is not monotone.

Note that the objective function in (6.3) is

$$f(q) = s_u(q) - c(q) \tag{6.4}$$

where $s_u(\cdot)$ is the RCE functional (5.1). The function f is concave by Theorem 5.1 and the assumptions on c. Therefore, the optimal solution q^* of (6.3) is positive if and only if $f'(0) > 0$. By (5.4) $s'(0) = \mu$, so

$$q^* > 0 \quad \text{if and only if} \quad \mu > c'(0) \tag{6.5}$$

in agreement with the expected utility model (6.2). We assume from now on that

$$\mu > c'(0)$$

A central result in the theory of production under uncertainty is that, for the risk-averse firm (i.e. concave utility function), the optimal production under uncertainty is less than the corresponding optimal production q_{cer} under certainty, that is for P a degenerate RV with value μ. We will prove now that the same result holds for the model (6.3). First recall that the optimality condition for q_{cer} is that marginal cost equals marginal revenue

$$c'(q_{\text{cer}}) = \mu \tag{6.6}$$

Proposition 6.2 *$q^* < q_{\text{cer}}$ for all $u \in \mathcal{U}$.*

Proof. The optimality condition for q^* is

$$0 = f'(q^*) = s_u'(q^*) - c'(q^*) \tag{6.7}$$

By Theorem 5.1

$$s_u(q) = z(q) + Eu(qP - z(q)) \tag{6.8}$$

where $z(q)$ is a differentiable function, uniquely determined by the equation

$$Eu'(qP - z(q)) = 1 \tag{6.9}$$

By the envelope theorem (Appendix A),

$$s_u'(q) = E\{Pu'(qP - z(q))\} \tag{6.10}$$

and the optimality condition (6.7) becomes

$$EPu'(q^*P - z(q^*)) = c'(q^*) \tag{6.11}$$

Multiplying (6.9) by μ and subtracting from (6.11) we get

$$E(P - \mu)u'(q^*P - z(q^*)) = c'(q^*) - \mu \tag{6.12}$$

or

$$E\{Zh(Z)\} = c'(q^*) - \mu \tag{6.13}$$

where we denote

$$Z := P - \mu \,, \; h(Z) := u'(q^*Z + q^*\mu - z(q^*))$$

Since $u \in \mathcal{U}$, it follows that h is positive and decreasing, and it can then be shown (see e.g. [21, p. 249]) that

$$E\{Zh(Z)\} < h(0)EZ$$

but $EZ = E\{P - \mu\} = 0$, and so, by (6.13),

$$c'(q^*) < \mu$$

and by using (6.6)

$$c'(q^*) < c'(q_{\text{cer}})$$

and since c' is increasing,

$$q^* < q_{\text{cer}} \quad \square$$

6.1 Effect of Profits Tax

Suppose there is a proportional **profits tax** at rate $0 < t < 1$, so that the profit after tax is

$$\pi(q) = (1 - t)(qP - C(q))$$

As before, the firm seeks the optimal solution q^* of (6.3), which here becomes

$$\begin{aligned} \max_{q \geq 0} S_u(\pi(q)) &= \max_{q \geq 0} S_u((1-t)(qP - c(q) - B)) \\ &= \max_{q \geq 0} \{S_u((1-t)qP) - (1-t)c(q)\} - (1-t)B \end{aligned}$$

which can be rewritten, using the RCE functional $s_u(\cdot)$ and omitting the constant $(1 - t)B$,

$$\max_{q \geq 0} s_u((1-t)q) - (1-t)c(q)$$

Let the optimal solution be $\bar{q} = \bar{q}(t)$. The optimality condition here is

$$(1-t)s_u'((1-t)q) - (1-t)c'(q) = 0$$

giving the identity (in t),

$$s_u'((1-t)\bar{q}(t)) \equiv c'(\bar{q}(t))$$

which, after differentiating (with respect to t),

$$\left[(1-t)\bar{q}'(t) - \bar{q}(t)\right] s''((1-t)\bar{q}) = \bar{q}'(t)c''(\bar{q})$$

and rearranging terms, gives

$$\bar{q}'(t)\{c''(\bar{q}) - (1-t)s_u''((1-t)\bar{q})\} = -\bar{q}(t)s_u''((1-t)\bar{q}) \tag{6.14}$$

The coefficient of $\bar{q}'(t)$ is positive since $c'' > 0$ and $s_u(\cdot)$ is concave (Theorem 5.1(a)). The right-hand side of (6.14) is positive since $\bar{q} > 0$, $s'' < 0$. Therefore, by (6.14),

$$\bar{q}'(t) > 0$$

and we proved:

Proposition 6.3 *A marginal increase in profit tax causes the firm to increase production.* □

In the classical expected utility case the effect of taxation depends on third-derivative assumptions; it can be predicted unambigiously[17] only in one of the following cases:

(a) r constant and R increasing,
(b) r decreasing and R increasing,
(c) r decreasing and R constant.

In all these cases, the EU prediction agrees with our prediction in Proposition 6.3.

6.2 Effect of Price Increase

If price were to increase from P to $P + \epsilon$ (ϵ fixed), then the corresponding optimal output $\bar{q}(\epsilon)$ is the solution of

$$\max_{q\geq 0}\{\ S_u((P+\epsilon)q) - c(q)\} = \max_{q\geq 0}\{\ s_u(q) + \epsilon q - c(q)\}$$

The optimality condition for $\bar{q}(\epsilon)$ is

$$s_u'(\bar{q}(\epsilon)) + \epsilon = c'(\bar{q}(\epsilon))$$

Differentiating with respect to ϵ we get

$$\bar{q}'(\epsilon)s_u''(\bar{q}(\epsilon)) + 1 = \bar{q}'(\epsilon)c''(\bar{q}(\epsilon))$$

hence

$$\bar{q}'(\epsilon) = \frac{1}{c''(\bar{q}) - s''(\bar{q})} > 0$$

by the convexity of c and the concavity of s_u. We have so proved:

[17]See Katz's correction [18] to [31].

Proposition 6.4 *A marginal increase in selling price causes the firm to increase production.* □

This highly intuitive result is proved in the expected utility case only under the assumption that $r(\cdot)$ is non-increasing.

6.3 Effect of Futures Price Increase

The RCE criterion was also applied to an extension [14] of Sandmo's model [31], dealing with a firm under price uncertainty and where a futures market exists for the firm's product. In [14, Proposition 5] it is shown that an increase in the current futures price causes a **speculator** or a **hedger** to increase sales, but not so for a **partial hedger**, unless constant absolute risk-aversion is assumed. This pathology is avoided in the RCE model, where the above three types of producers will all increase sales, [34].

7 Investment in One Risky and in One Safe Assets: The Arrow Model

Recall the classical model [3] of investment in a risky/safe pair of assets, concerning an individual with utility $u \in \mathcal{U}$ and **initial wealth** A. The **decision variable** is the amount a to be invested in the risky asset, so that $m = A - a$ is the amount invested in the safe asset (cash).

The **rate of return** in the risky asset is a RV X.

The **final wealth** of the individual is then

$$Y = A - a + (1 + X)a = A + aX$$

In [3] the model is analyzed via the maximal EU principle, so the optimal investment a^* is the solution of

$$\max_{0 \leq a \leq A} Eu(A + aX) \tag{7.1}$$

or equivalently

$$\max_{0 \leq a \leq A} u^{-1} Eu(A + aX)$$

Some of the important results in [3] are:

(I1) $a^* > 0$ if and only if $EX > 0$.

(I2) a^* increases with wealth (i.e. $\frac{da^*}{dA} \geq 0$) if the absolute risk aversion index $r(\cdot)$ is decreasing.

(I3) The wealth elasticity of the demand for cash balance (investment in the safe asset)

$$\frac{Em}{EA} := \frac{dm/dA}{m/A} \quad \text{is at least one} \tag{7.2}$$

if the relative risk-aversion index

$$R(z) = -z\frac{u''(z)}{u'(z)} \quad \text{is increasing} \tag{7.3}$$

Arrow [3] postulated that reasonable utility functions **should** satisfy (7.3), since the empirical evidence for (7.2) is strong, see the references in [3, p. 103].

We analyze this investment problem using the RCE criterion, i.e.

$$\max_{0 \leq a \leq A} S_u(A + aX) \tag{7.4}$$

where again we assume that the investor's value-risk function is $u \in \mathcal{U}$. The optimization problem (7.4) is, by (1.12), equivalent to

$$\max_{0 \leq a \leq A} S_u(aX) + A$$

Let a^* be the optimal solution. Using the RCE functional $s_u(\cdot)$, a^* is in fact the solution of

$$\max_{0 \leq a \leq A} s_u(a) \tag{7.5}$$

Now, since $s_u(\cdot)$ is concave

$$a^* > 0 \quad \text{if and only if} \quad s_u'(0) > 0$$

but by (5.4) $s'(0) = EX$, and we recover the result (I1).

Assuming (as in [3]) an inner optimal solution (**diversification**)

$$0 < a^* < A \tag{7.6}$$

we conclude here, in contrast to (I2), that

$$\frac{da^*}{dA} = 0 \tag{7.7}$$

i.e. the optimal investment is independent of wealth[18].

An immediate consequence of (7.7) is

$$\frac{Em}{EA} > 1 \quad \forall\, u \in \mathcal{U}$$

indeed

$$\frac{Em}{EA} = \frac{A}{m}\frac{dm}{dA} = \frac{A}{A-a^*}\frac{d(A-a^*)}{dA} = \frac{A}{A-a^*}(1-\frac{da^*}{dA}) = \frac{A}{A-a^*} > 1$$

proving (7.2) for **all** risk-averse investors. Thus, in the RCE model, there is no need for the controversial postulate (7.3).

The quadratic utility (2.33)

$$u(z) = z - \frac{1}{2}z^2 \quad z \leq 1$$

violates both of Arrow's postulates (r decreasing, R increasing), and is consequently "banned" from the EU model. In the RCE model, on the other hand, a quadratic value-risk function is acceptable[19]. For this function the optimal investment a^* is the optimal solution of

$$\max_{0 \leq a \leq A} \{s_u(a) = \mu a - \frac{1}{2}\sigma^2 a^2\}$$

where $\mu = EX$, $\sigma^2 = \mathrm{Var}(X)$. Therefore

$$a^* = \begin{cases} \mu/\sigma^2 & \text{if } 0 < \mu/\sigma^2 < A \\ A & \text{if } \mu/\sigma^2 \geq A \end{cases}$$

showing that, for the full range of A values, $a^*(A)$ is non-decreasing, in agreement with **(I2)**. Moreover, if diversification is optimal, then

$$\frac{Em}{EA} = \frac{A}{A-\mu/\sigma^2} > 1$$

Following [3] we consider the effects on optimal investment, of shifts in the RV X. Let h be the **shift parameter**, and assume that the **shifted** RV $X(h)$ is a differentiable function of h, with $X(0) = X$. Examples are:

[18] However, initial wealth will in general determine when divesification will be optimal, i.e. when (7.6) will hold.

[19] Assuming $0 \leq X \leq 1$.

$$X(h) = X + h \qquad \textbf{(additive shift)},$$
$$X(h) = (1+h)X \qquad \textbf{(multiplicative shift)}.$$

For the shifted problem, the objective is

$$\max_{0 \le a \le A} S_u(aX(h)) \tag{7.8}$$

Let $a(h)$ be the optimal solution of (7.8), in particular $a(0) = a^*$. Now

$$S_u(aX(h)) = \xi(a) + Eu(aX(h) - \xi(a)) \tag{7.9}$$

where $\xi(a)$ is the unique solution of

$$Eu'(aX(h) - \xi(a)) = 1 \tag{7.10}$$

The optimality condition for $a(h)$ is

$$\frac{d}{da}\{\xi(a) + Eu(aX(h) - \xi(a))\} = 0$$

which gives (using (7.10)) the following identities in h

$$E\{X(h)u'(a(h)X(h) - \xi(a(h)))\} \equiv 0 \tag{7.11}$$
$$E\{u'(a(h)X(h) - \xi(a(h)))\} \equiv 1 \tag{7.12}$$

Differentiating (7.11) with respect to h we get, denoting $Z = aX(h) - \xi(a(h))$,

$$\dot{a}(h)E\{u''(Z)X(X - \xi'(a(h))\} = E\{\dot{X}(h)\left[u'(Z) + a(h)X(h)u''(Z)\right]\} \tag{7.13}$$

where $\dot{a}(h) = \frac{d}{dh}a(h)$ and similarly for $\dot{X}(h)$.

The second order optimality condition for $a(h)$, $\frac{d^2}{da^2}S_u(aX(h)) \le 0$, is here

$$Eu''(Z)X(X - \xi'(a(h))) \ge 0$$

hence, by (7.13),

$$\text{sign of } \dot{a}(h) = \text{sign of } E\{\dot{X}(h)\left[u'(Z) + aXu''(Z)\right]\}$$

exactly the same condition for the sign of $\frac{d}{dh}a(h)$ as in [3, p. 105, eq. (18)]. Therefore, the conclusions of the EU model are also valid for the RCE model. In particular:

Proposition 7.1 *As a function of the shift parameter h,*
$a(h)$ increases for additive shift,
$a(h)$ decreases for multiplicative shift.

These results are illustrated for the quadratic value-risk function. There

$$a^* = \frac{EX}{\mathrm{Var}(X)}$$

and

$$\begin{aligned} a(h) &= a^* + \frac{h}{\mathrm{Var}(X)} \quad \text{for an additive shift} \\ a(h) &= \frac{1}{1+h} a^* \quad \text{for a multiplicative shift} \end{aligned} \tag{7.14}$$

In fact, (7.14) holds for arbitrary $u \in \mathcal{U}$, a result proved in [36] for the EU model.

Proposition 7.2 *If a^* is the demand for the risky asset when the return is the RV X, then $a(h) = a^*/1+h$ is the demand when the return is $(1+h)X$.*

Proof. The optimality condition for a^* is

$$E\{u'(a^*X - \xi^*)X\} = 0 \tag{7.15}$$

where ξ^* is the unique solution of

$$Eu'(a^*X - \xi^*) = 1 \tag{7.16}$$

The optimality conditions for $a(h)$ are given by (7.11), (7.12). Now, for $a(h) = \frac{1}{1+h}a^*$,

$$a(h)X(h) = a^*X \tag{7.17}$$

and it follows, by comparing (7.12) with (7.16), that

$$\xi(a(h)) = \xi^*$$

Substituting this in (7.11) and using (7.17), we see that (7.11) is equivalent to (7.16), and that $a(h) = a^*/1+h$ indeed satisfies the optimality conditions (7.11), (7.12). □

8 Investment in a Risky/Safe Pair of Assets: An Extension

We study the model discussed in [9] and [15], which is an extension of the model in Section 7. The analysis applies to a fixed time interval, say a year. An investor allocates a proportion $0 \leq k \leq 1$ of his investment capital W_0 to a risky asset, and proportion $1 - k$ of W_0 to a safe asset where the total annual return per dollar invested is $\tau \geq 1$. The total annual return t per dollar invested in the risky asset, is a nonnegative RV. The investor's **total annual return** is

$$kW_0t + (1 - k)W_0\tau$$

and for a utility function u, the optimal allocation k^* is the solution of

$$\max_{0 \leq k \leq 1} Eu(kW_0t + (1 - k)W_0\tau) \tag{8.1}$$

The model of §5, is a special case with $W_0 = A,\ t = 1 + X,\ kW_0 = a,\ \tau = 1$.

It is assumed in [9], [15] that $u' > 0$ and $u'' < 0$, thus we assume without loss of generality that $u \in \mathcal{U}$.

One of the main issues in [15] is the effect of an increase in the safe asset return τ on the optimal allocation. The following are proved:

(**F1**) An investor maximizing expected utility will diversify (invest a positive amount in each of the assets) if and only if

$$\frac{Etu'(W_0t)}{Eu'(W_0t)} < \tau < E(t) \tag{8.2}$$

(**F2**) Given (8.2) he will **increase** the proportion invested in the safe asset when τ increases if either

(a) the absolute risk aversion index $r(\cdot)$ is non-decreasing, or

(b) the relative risk aversion index $R(\cdot)$ is at most 1.

The same model is now analyzed using the RCE approach, i.e. with the objective

$$\max_{0 \leq k \leq 1} S_u(kW_0t + (1 - k)W_0\tau)$$

where u denotes the investor's value risk function, assumed in $\mathcal{U}$. Using (1.12) and the definition (5.1), the objective becomes

$$\max_{0 \leq k \leq 1} \{(1 - k)W_0\tau + s_u(W_0k)\}) \tag{8.3}$$

The following proposition, proved in Appendix C, gives the analogs of results (**F1**, (**F2**) in the RCE model.

Proposition 8.1 *(a) The RCE maximizing investor will diversify if and only if*

$$Etu'(W_0 t - \eta) < r < E(t) \tag{8.4}$$

where η is the unique solution of

$$Eu'(W_0 t - \eta) = 1 \tag{8.5}$$

(b) Given (8.4), he will **increase** *the proportion invested in the safe asset when r* **increases.** □

Comparing part (b) with (**F2**), we see that plausible behavior (k^* increases with r) holds in the RCE model for all $u \in \mathcal{U}$, but in the EU model only for a restricted class of utilities.

We illustrate Proposition 8.1 in the case of the quadratic value-risk function (2.25). Here the optimal proportion invested in the risky asset is:

$$k^* = \begin{cases} 0 & \text{if } r > E(t) \\ \frac{E(t)-r}{W_0\sigma^2} & \text{if } E(t) - W_0\sigma^2 \le r \le E(t) \\ 1 & \text{if } E(t) - W_0\sigma^2 > r \end{cases} \tag{8.6}$$

where σ^2 is the variance of t. Thus k^* is increasing in $E(t)$, decreasing with σ^2 and decreasing with r (so that, the proportion $1 - k^*$ invested in the safe asset is increasing with safe asset return r). These are reasonable reactions of a risk-averse investor.

We also see from (8.6) that k^* **decreases** when the investment capital W_0 **increases.** This result holds for arbitrary $u \in \mathcal{U}$, see the next proposition (proved in Appendix B). In the EU model, the effect of W_0 on k^* depends on the relative risk-aversion index, see [9].

Proposition 8.2 *If the investment capital increases, then the RCE-maximizing investor will increase the proportion invested in the safe asset.* □.

Following the analysis in [3] and § 7, we consider now the elasticity of cash-balance (with respect to W_0). Here the cash balance (the amount invested in the safe asset) is

$$m = (1 - k^*)W_0$$

and the elasticity in question is $\frac{Em}{EW_0}$.

Proposition 8.3 *For every RCE-maximizing investor with $u \in \mathcal{U}$,*

$$\frac{Em}{EW_0} \geq 1$$

Proof.

$$\frac{Em}{EW_0} = \frac{dm/dW_0}{m/W_0} = \frac{1 - k^*(W_0) - W_0 \frac{dk^*(W_0)}{dW_0}}{1 - k^*(W_0)}$$

hence

$$\frac{Em}{EW_0} \geq 1 \quad \text{if and only if} \quad \frac{dk^*(W_0)}{dW_0} \leq 0 \tag{8.7}$$

and the proof is completed by Proposition 8.2. □

The equivalence in (8.7) shows that the empirically observed fact that $Em/EW_0 \geq 1$ can be explained **only** by the result established in Proposition 8.2 that $dk^*/dW_0 \leq 0$, a result which is not necessarily true for many utilities in the EU analysis.

9 Optimal Insurance Coverage

Insurance models with two states of nature were studied in [13], [21] and the references therein. In this section we solve an insurance model with n states of nature, and give an **explicit formula for the optimal allocation of the insurance budget**, thus illustrating the analytic power of the RCE theory.

9.1 Description of the Model

The elements of the model are:

n	**states of nature**
$\mathbf{p}$	$= (p_1, \ldots, p_n)$ their **probabilities**
$\bar{q}_i$	= **premium** for 1$ coverage in state i, $\bar{q}_i > 0$
$\bar{B}$	= **insurance budget**
q_i	$= \bar{q}_i / \sum_{j=1}^{n} \bar{q}_j$ = **normalized premium**
B	$= \bar{B} / \sum_{j=1}^{n} \bar{q}_j$ = **normalized budget**
x_i	= **income in state i**
$\mathbf{x}$	$= (x_1, \ldots, x_n)$ the **decision variable**

The **budget constraint** is

$$\sum_{i=1}^{n} q_i x_i = B \tag{9.1}$$

We allow negative values for some x_i's, i.e. we allow a person to "insure" and "gamble" at the same time, e.g. [13, p. 627].

For the RCE maximizer with value-risk function v, the **optimal value of the insurance plan** is

$$\begin{aligned} \Gamma^* &= \max_{\mathbf{x}} \{S_v([\mathbf{x},\mathbf{p}]) : \sum_{i=1}^{n} q_i x_i = B\} \\ &= \max_{\mathbf{x},\, \sum q_i x_i = B} \ \max_z \{z + \sum_{i=1}^{n} p_i v(x_i - z)\} \\ &= S_v([\mathbf{x}^*,\mathbf{p}]) \end{aligned} \tag{9.2}$$

where $\mathbf{x}^* = (x_i^*)$ is the **optimal insurance coverage.**

9.2 The Solution

Theorem 9.1 *The optimal insurance coverage is*

$$x_i^* = B + \phi(\frac{q_i}{p_i}) - \sum_{j=1}^{n} q_j \phi(\frac{q_j}{p_j}) \tag{9.3}$$

where

$$\phi = (v')^{-1} \tag{9.4}$$

Moreover, the optimal value of the insurance plan is

$$\Gamma^* = B - \sum q_i \phi(\frac{q_i}{p_i}) + \sum p_i v(\phi(\frac{q_i}{p_i})) \tag{9.5}$$

Proof. The problem (9.2) is maximizing a concave function subject to linear constraints. Since the Kuhn-Tucker conditions are necessary and sufficient

$$\Gamma^* = \min_{\lambda} \max_{\mathbf{x}} L(\mathbf{x}, z, \lambda) \tag{9.6}$$

where L is the **Lagrangian**

$$L(\mathbf{x}, z, \lambda) = z + \sum p_i v(x_i - z) + \lambda(B - \sum q_i x_i) \tag{9.7}$$

The optimal $\mathbf{x}^*, z^*, \lambda^*$ satisfy

$$\frac{\partial L}{\partial z} = 1 - \sum p_i v'(x_i^* - z^*) = 0 \tag{9.8}$$

$$\frac{\partial L}{\partial x_i} = p_i v'(x_i^* - z^*) - \lambda^* q_i = 0, \quad (i = 1, \dots, n) \tag{9.9}$$

$$\frac{\partial L}{\partial \lambda} = B - \sum q_i x_i^* = 0 \tag{9.10}$$

From (9.9) and (9.8) we get

$$\lambda^* = \frac{1}{\sum q_i} = 1$$

and consequently

$$v'(x_i^* - z^*) = \frac{q_i}{p_i}$$

Since v' is monotone decreasing (v is strictly concave) we write, using (9.4)

$$x_i^* - z^* = \phi(\frac{q_i}{p_i}), \quad (i = 1, \dots, n) \tag{9.11}$$

Multiplying (9.11) by q_i and summing we get

$$\begin{aligned} \sum q_i (x_i^* - z^*) &= \sum q_i \phi(\frac{q_i}{p_i}) \\ \therefore B - z^* &= \sum q_i \phi(\frac{q_i}{p_i}) \\ \therefore z^* &= B - \sum q_i \phi(\frac{q_i}{p_i})) \end{aligned} \tag{9.12}$$

which is compared with (9.11) to give (9.3). Finally,

$$\Gamma^* = z^* + \sum p_i v(x_i^* - z^*)$$

and (9.5) follows by (9.11) and (9.12). □ In the above model, the price of insurance is **actuarially fair** if

$$q_i = p_i \quad (i = 1, \dots, n)$$

i.e. if the normalized premiums agree with the probabilities.

For actuarially fair premiums we get from (9.3), using that $v'(0) = 1$ implies $\phi(1) = 0$,

$$x_i^* = B \quad (i = 1, \dots, n)$$

i.e. the individual is indifferent between the occurrence of states $i = 1, \dots, n$.

9.3 Special Case: Two States of Nature

We translate the results of Theorem 9.1 to the special case of two states, as given in [13], [21, §3].

Consider insurance against a single disaster. Specifically, let there be two states of nature:

State	Disaster	Probability
1	occurs	p
2	does not occur	$1-p$

The **final wealth** is a RV

$$X(s) = \begin{cases} y+s & \text{with probability } p \quad \text{(State 1)} \\ W-\pi s & \text{with probability } 1-p \quad \text{(State 2)} \end{cases} \tag{9.13}$$

where

W	**initial wealth**
s	**insurance coverage**
π	**premium**
y	**income in disaster state**

In [21, §3] this model is treated using the EU model

$$\max_s E\, u(X(s))$$

obtaining first order optimality conditions, comparative statics, and, in the case of exponential utility

$$u_\lambda(x) = \frac{1}{\lambda}(1 - e^{-\lambda x}), \tag{9.14}$$

the explicit solution

$$s^* = \frac{W-y}{\pi+1} - \frac{1}{\lambda(\pi+1)} \log\left(\frac{\pi}{\frac{p}{1-p}}\right) \tag{9.15}$$

To apply Theorem 9.1 here we write the incomes in the two states and their probabilities

$$x_1 = y+s, \quad p_1 = p$$
$$x_2 = W - \pi s, \quad p_2 = 1-p$$

We define the normalized premiums

$$q_1 := \frac{\pi}{1+\pi} \tag{9.16}$$

$$q_2 := 1 - q_1 = \frac{1}{1+\pi} \tag{9.17}$$

The insurance budget (9.1) is implicit in this model. The budget B can be computed by

$$\begin{aligned} q_1 x_1 + q_2 x_2 &= q_1 (y + s) + q_2 (W - \pi s) \\ &= q_1 y + q_2 W + s(q_1 - q_2 \pi) \end{aligned}$$

but $q_1 - q_2\pi = 0$ by (9.16) and (9.17), and therefore the budget is

$$B = q_1 y + q_2 W \tag{9.18}$$

Now, from (9.3),

$$\begin{aligned} x_1^* &= B + (1 - q_1)\phi(\frac{q_1}{p_1}) - q_2\phi(\frac{q_2}{p_2}) \\ &= q_1 y + q_2 W + q_2 \left[\phi(\frac{q_1}{p_1}) - \phi(\frac{q_2}{p_2})\right] \end{aligned}$$

and therefore the optimal coverage is

$$\begin{aligned} s^* &= x_1^* - y \\ &= q_2 \left[W - y + \phi(\frac{q_1}{p_1}) - \phi(\frac{q_2}{p_2})\right] \\ &= \frac{1}{1+\pi}\left[W - y + \phi\left(\frac{\pi}{(1+\pi)p}\right) - \phi\left(\frac{1}{(1+\pi)(1-p)}\right)\right] \end{aligned} \tag{9.19}$$

Note that in this two state model, actuarlially fair insurance means $\pi = \frac{p}{1-p}$, in which case $s^* = \frac{1}{1+\pi}(W - y)$

For the utility u_λ of (9.14), we get by (9.4)

$$\phi(t) = (u'_\lambda)^{-1}(t) = -\frac{1}{\lambda}\log t$$

which, substituted in (9.19), gives the formula (9.15) of s^*.

9.4 Related Work

The RCE criterion was applied in [34] for studying the existence of optimal insurance contracts. Two fundamental results of Arrow [3] concerning

- the optimality of 100% coverage (above deductibles) for a risk-averse buyer of insurance, and
- the Pareto optimality of **coinsurance** for risk-averse insurer and buyer of insurance,

were shown to hold as well in the RCE model.

10 Why Does the RCE Work ?

The models discussed above (§§ 6-9), give sufficient data for comparing the predictive powers of the RCE theory and the EU theory. We saw that the plausible predictions of EU are shared by RCE, and that the RCE criterion is a simpler and a more powerful analytical tool, e.g. § 9.2 where it gives an explicit solution for all risk-averse DM's, while in general the EU model can only provide comparative statics. Also the RCE predictions hold for all risk-averse DM's, while in the EU model risk-aversion does not suffice and, in order to avoid implausible predictions, restrictions (occassionally severe) must be imposed on the DM's subjective preference.

The simplicity of the RCE criterion can be explained at the technical level. Shift additivity makes risky choices independent of constant factors (fixed costs, initial wealth), and by using the envelope theorem, comparative statics are free of certain ungainly derivatives. Such conveniences are in general unavailable to the EU maximizer.

This however is not the whole story. The main advantage of the RCE theory, at the fundamental level of modelling choice under risk, is that its risk aversion is of the "right kind" from the start, without a need for qualifiers such as the Arrow-Pratt indices. Indeed, in the EU theory, behavior under uncertainty is analyzed in terms of the Arrow-Pratt indices $r(\cdot)$ and $R(\cdot)$. The typical postulates are

(A1) $r(w) = -\frac{u''(w)}{u'(w)}$ is a non-increasing function of w

(A2) $R(w) = -\frac{w\,u''(w)}{u'(w)}$ is a non-decreasing function of w

The economic literature contains several alternative formulations. In particular ([12, pp. 352-354] and [23, pp. 20-21]) (**A1**) is equivalent to

(**B1**) If $u(w_1 + c_1) = E\,u(w_1 + X)$ and $u(w_2 + c_2) = E\,u(w_2 + X)$ for $w_1 < w_2$, then $c_1 \leq c_2$

and (**A2**) is equivalent to

(**B2**) If $u(w_1 c_1) = E\,u(w_1 X)$ and $u(w_2 c_2) = E\,u(w_2 X)$ for $w_1 < w_2$, then $c_1 \geq c_2$

Properties (**B1**), (**B2**) can be expressed directly in terms of the classical CE

$$C_u(X) = u^{-1} E\,u(X)$$

Indeed, (**B1**) is equivalent to

(**C1**) $C_u(X + w) - w$ is a non-decreasing function of w

and (**B2**) is equivalent to

(**C2**) $\frac{1}{w} C_u(wX)$ is a non-increasing function of w

Consider now the RCE $S_v(X)$. The properties corresponding to (**C1**), (**C2**) are

(**S1**) $S_v(X + w) - w$ is a non-decreasing function of w

(**S2**) $\frac{1}{w} S_v(wX)$ is a non-increasing function of w

Now (**S1**) holds trivially, for any function $v : \mathbb{R} \to \mathbb{R}$, by the shift additivity of the RCE, Theorem 2.1(a). In fact, $S_v(X+w)-w$ is $S_v(X)$, a constant in w. Moreover, (**S2**) is the subhomogeneity property, proved in Theorem 2.1(c) for all $v \in \mathcal{U}$.

Therefore, in the RCE theory the properties (S1) and (S2) hold for all value-risk function $v \in \mathcal{U}$, i.e. for all strongly risk-averse DM's. In the EU theory, risk-aversion coincides with strong risk-aversion (see § 4), but the properties (**A1**) and (**A2**) (which correspond to (S1) and (S2)) hold only for a restricted class of utilities.

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Appendix A. The Envelope Theorem

This result is used repeatedly in this paper. For convenience we cite an elementary version here. See [35] and [30] for details and examples.

Theorem A.1 (The Envelope Theorem). *Consider the unconstrained maximization*

$$\text{maximize}_z\ y = f(z, q)$$

Let $z^*(q)$ be the maximizer, for given q, and let

$$y^* = f(z^*(q), q) = \phi(q)$$

Then

$$\phi'(q) = \frac{\partial f(z^*(q), q)}{\partial q} \quad \square$$

Appendix B. Proof of Theorem 5.1

(a) By (5.1) and (1.10), $s_u(\cdot)$ is the pointwise supremum of concave functionals, hence concave. The rest of (a) is proved as in Lemma 2.1.

(b) For $\mathbf{y} = \mathbf{0}$, (5.3) gives

$$Eu'(-z_S(\mathbf{0})) = 1$$

or $u'(-z_S(\mathbf{0})) = 1$, proving that $z_S(\mathbf{0}) = 0$. From (5.2) it follows then that $s_u(\mathbf{0}) = 0$.

Differentiating (5.3) with respect to $\mathbf{y}$ gives

$$Eu''(\mathbf{y} \cdot \mathbf{Z} - z_S(\mathbf{y}))(\mathbf{Z} - \nabla z_S(\mathbf{y})) = 0$$

which at $\mathbf{y} = \mathbf{0}$ becomes

$$u''(0)(E\mathbf{Z} - \nabla z_S(\mathbf{0})) = 0$$

proving that $\nabla z_S(\mathbf{0}) = \mu$. Then, by differentiating (5.2) at $\mathbf{y} = \mathbf{0}$ we get $\nabla s_u(\mathbf{0}) = \mathbf{0}$.

The expressions for $\nabla^2 z_S(\mathbf{0})$ and $\nabla^2 s_u(\mathbf{0})$ follow similarly by differentiating (5.3) and (5.2) twice at $\mathbf{y} = \mathbf{0}$. $\square$

Appendix C. Results from Section 8

Proof of Proposition 8.1.

(a) The objective function in (8.3)

$$h(k) = (1 - k)W_0 r + s_u(W_0 k)$$

is concave, by Theorem 5.1(a). Hence, the optimal solution x^* is an inner solution, i.e. $0 < k^* < 1$ if and only if

$$h'(0) > 0 \quad \text{and} \quad h'(1) < 0 \tag{C.1}$$

Now

$$h'(k) = -W_0\tau + W_0 s_u'(W_0 k) \tag{C.2}$$

which becomes, upon substitution of the computed expression for $s_u'(\cdot)$,

$$h'(k) = -W_0\tau + W_0 Etu'(W_0kt - \eta(W_0k)) \tag{C.3}$$

where $\eta(q)$ is the unique solution of

$$Eu'(qt - \eta) = 1 \tag{C.4}$$

Therefore

$$\begin{aligned} h'(0) &= -W_0\tau + W_0E(t) \\ h'(1) &= -W_0\tau + W_0Etu'(W_0 - \eta(W_0)) \end{aligned}$$

and (C.1) is equivalent to (8.4).
(b) Let $k(\tau)$ be the optimal solution of (8.3) for given τ, i.e. $h'(k(\tau)) = 0$, or using (C.3),

$$-\tau + E\{tu'(W_0k(\tau)t - \eta(W_0k(\tau))\} \equiv 0$$

Differentiating this identity (in τ) with respect to τ, we obtain

$$-1 + E\{tW_0(k'(\tau)t - k'(\tau)\eta'(W_0k(\tau))u''\} = 0$$

or

$$k'(\tau)W_0Et(t - \eta')u'' = 1 \tag{C.5}$$

Now, the second order condition for the maximality of $k(\tau)$ is

$$0 > h''(k) = W_0E\{tW_0(t - \eta')u''\} \tag{C.6}$$

Therefore, $k'(\tau)$ is multiplied in (C.5) by a negative number, and consequently

$$k'(\tau) < 0$$

proving that $k(\tau)$ $[1 - k(\tau)]$, the proportion invested in the risky [safe] asset, is a decreasing [increasing] function of τ, the safe asset return. □

Proof of Proposition 8.2. Let $k = k(W_0)$ be the optimal solution of (8.3), i.e. $h'(k(W_0)) = 0$, or using (C.3)

$$-\tau + E\{tu'(W_0 k(W_0)t - \eta(W_0 k(W_0))\} \equiv 0 \tag{C.7}$$

Differentiating this identity (in W_0) we get

$$Et\left[k(W_0) + W_0 k'(W_0)\right]\left[t - \eta'(W_0 k(W_0))\right] u'' = 0$$

or

$$k' W_0 Et(t - \eta')u'' = -EtkU'' \tag{C.8}$$

By the second order optimality condition (C.6) it follows that, in (C.8), k' is multiplied by a negative number. Since the right hand side of (C.8) is positive ($t, k > 0,\ u'' < 0$), it follows that

$$k'(W_0) < 0 \quad \square$$