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THE CONDITIONAL PROBABILITY OF MORTGAGE DEFAULT

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The Conditional Probability of Mortgage Default

This research examines the implications of contingent claims models for empirical research on default. We focus on the probability of default over a short horizon given the current state of the world, i.e., the conditional probability of default, which more closely resembles the estimates of empirical models. We highlight the differences between the conditional and unconditional approaches and provide guidance for empirical research by illuminating situations where the expected sign reverses over the shorter horizon or where the functional form is highly non-linear.

While the importance of the embedded options in a mortgage contract has been recognized for almost two decades (Findlay and Capozza, 1977; Asay, 1978), the analysis of default probabilities using contingent claims models is more recent (Kau, Keenan, and Kim 1993, 1994). Nevertheless, the link between the theoretical models of default and the empirical tests remains weak. Typically, theoretical models have focused on the unconditional probability of default. Unconditional probabilities are most suitably tested using data on the cumulative defaults on mortgage loans over the entire thirty year life of the loans. However, empirical work commonly focuses on conditional probabilities over short horizons by using annual default data for seasoned loans.

Conditional probabilities are important for three reasons. First, most empirical work analyzes conditional probabilities. Second, conditional probabilities are essential for pricing mortgage loans; and third, conditional probabilities are fundamental since unconditional probabilities can be obtained from the conditional by integration of the state variables.

In this research we explore the conditional probabilities in the context of a contingent claims model. Our primary goal is to provide guidance on functional form and specification of the independent variables for empirical research on default. Our results highlight situations where the importance of some independent variables is greatly diminished or where care must be taken in specifying functional form. The effect of transactions costs and trigger events is also examined in a precise way.

To preview the conclusions, the results show that not only is the relationship between default and several of its determinants highly non-linear, but also the magnitude and even the sign depends on the horizon. We show that some variables, like the volatility of house prices, which are important unconditionally, are not very important conditionally. For other variables, like the current interest rate, interaction terms are essential.

This paper is organized as follows. The next section presents the contingent claims model that is solved by backward induction to determine the optimal stopping boundary. The third section presents and discusses the simulation results and the final section contains the conclusion.

The Model

To capture the dual effects of prepayment and default, we employ a two factor model. House prices are assumed to follow geometric Brownian motion and interest rates are an Ornstein-Uhlenbeck mean reverting process. Once each month, just prior to a mortgage payment, the mortgagor decides whether to prepay, default or make the scheduled mortgage payment. We ignore delinquency and assume that default results in immediate loss of the house in exchange for forgiveness of the debt.

House Price Process

House prices (H) are assumed to follow the process:

$$dH = (g - \gamma)Hdt + \sigma_H H dW \qquad (1)$$

where

g = the required return on housing given its risk,

γ = the rental rate or "rent-to-price" ratio for the house, (analogous to the dividend rate on common stock.)

σ_H = the volatility of house prices, and

W = standard Brownian motion.

Hedging arguments (*e.g.*, Hull 1993) yield the risk neutral pricing process given by

$$dH = (r - \gamma)Hdt + \sigma_H H dV \qquad (2)$$

where
r = the risk free interest rate, and
V = an alternate Brownian motion.

The risk neutralized drift of the house price process depends not on the gross return to housing, but rather on the risk free interest rate. For consistency, the risk neutralized house price process is used when modeling the default decision, however, the original process is used when modeling the probability of default[1].

Interest Rate Process

Interest rates are also stochastic in the model. Interest rate changes have two influences over default decisions. First, if interest rates rise, default is less likely because the mortgagor values the low cost mortgage and will hesitate surrendering it. Second, if interest rates fall, refinance becomes likely and extinguishes the default option. We assume that interest rates follow a discrete version of the Ornstein-Uhlenbeck mean reverting process:

$$dr = \beta(\alpha - r)\, dt + \sigma_r\, dW \qquad (3)$$

where
r = the interest rate,
α = the equilibrium interest rate to which rates will tend to return,
β = non-negative reversion parameter which describes the intensity at which rates will return to the equilibrium rate, and
σ_r = the volatility of interest rates.

In equation (3), if $\beta=0$, the process is a pure random walk model (normal diffusion). When $\alpha=r$, the process is a pure random walk over the next instant. For other parameter values, the interest rate tends to α. The term structure of interest rates is implied by the difference between r and α.[2]

Transactions Costs and Trigger Events

Two "real world" considerations are typically incorporated into contingent-claims mortgage termination models. The first is the transactions costs of default; and the second is "trigger events" or exogenous[3] termination such as job relocation. Transactions costs are incorporated into the default decision by adding a term to the current payoff at default. Transaction costs include monetary moving costs, social and family costs of the move, and financial disruption from a blemished credit standing or deficiency judgments that claim other assets. In the simulations that follow, transaction costs are modeled as a fixed dollar cost.

A trigger event converts the multi-period default decision into a one period decision. There are two types of trigger events. The first type arises if the borrower must move. In this case the opportunity or transactions costs of default are minimal since the borrower incurs the costs whether he defaults or not. The second type occurs when the borrowers is unable to continue making payments. In this case the transactions costs remain relevant since a default will necessitate a move that need not occur otherwise.

Correlation between Interest Rates and House Prices

Our model for predicting mortgage defaults, therefore, is a two stochastic state variable model that incorporates transactions costs and exogenous termination. A zero correlation between house prices and interest rates is assumed. This is consistent with the empirical evidence. Studies of house prices (e.g. Haurin, Hendershott and Kim 1991) show that there is considerable variation in house price movements among metropolitan areas, even though all markets face the same interest rate environment. Kau, Keenan and Kim (1994) present reasons why house prices could be positively or negatively correlated with interest rates. To explore this issue further, we analyzed the large panel data set of 64 metropolitan areas from 1979 to 1994, described in Capozza, Kazarian, and Thomson (1995). We were unable to find any R^2 greater than .02 when house prices were regressed on interest rates in various specifications including levels, first differences, real and nominal interest rates, and house prices.

Simulating Empirical Default

Empirical analysis of mortgage default data is usually based on annual data for seasoned loans rather than on newly originated loans. The default rate is computed by dividing the number of loans that default by the number of loans at risk that year. To approximate the empirical default rate in the simulations, we compute the probability that a loan is still alive, *i.e.*, neither prepaid nor defaulted at the beginning of each 12 month period. We then sum the default probabilities for the next 12 months to determine the probability that a default will occur over the next full year. The annual default rate is the probability of default over the next 12 months divided by the probability that the loan was alive at the beginning of the period.

We initiate the analysis by requiring that the mortgage be fairly priced. The coupon rate on a loan at origination must set the present value of payments (including default and prepayment options, and the effect of exogenous terminations) equal to the principle amount. The model parameters are then varied individually from their base case values to determine the effect of changing a parameter's value when all other parameters are held constant. The results are presented graphically to facilitate assessments of the direction, the strength and the linearity of the relationships.

Measuring Current Loan to Value Ratios

Current loan to value (CLTV) is the current value of the loan divided by the current value of the house. Book CLTV (BCLTV) is the contractual balance on the note divided by the current house value. Market CLTV (MCLTV) is defined to be the market value of the loan divided by the current house value. BCLTV is easily computed as long as the initial loan parameters and current house prices are available. BCLTV explicitly allows for the amortization of the mortgage and changing house prices. However, it does not account for the effect of interest rates on current loan values. For example, when interest rates rise, the market value of a mortgage falls. The market CLTV will fall while the book CLTV is unaffected. The change in market value will enter the borrower's decision calculus. This suggests that MCLTV may be a better measure for default analysis. There are three ways to deal with the difference between BCLTV and MCLTV in empirical analysis:

Use simple ad hoc rules for computing MCLTV. Foster and Van Order (1984), recognizing that BCLTV and MCLTV can diverge when interest rates change, propose estimating the economic value of the mortgage by discounting the remaining payments at the current interest rate and assuming repayment after 40% of its remaining contractual life.

Use MCLTV in empirical analysis. An alternate approach is to use a contingent claims model to compute the MCLTV for each observed set of values of the independent variables. Given that an empirical study may have 50,000 or more combinations of independent variable values, it would be impractical to make the computations that would be required to compute the MCLTV.[4] Computing BCLTV is quicker since the only variables that affect BCLTV are mortgage age, coupon rate and house value. MCLTV complicates the interpretation since the source of the change in MCLTV may not be readily available.

To test whether MCLTV is a sufficient statistic for predicting default rates, we run the model with various combinations of house prices, coupon rates and current interest rates to create a range of MCLTV values. Loans with different coupons but subject to the same spot rate will have different MCLTVs. Figure 1 illustrates the computed default rate over the next year as MCTLV is varied by adjusting the mortgage coupon rate from rates below the spot rate to coupon rates above the spot rate. Three cases are presented for spot rates of 5%, 10% and 15%. If MCTLV is a sufficient statistic for assessing default, then all lines should plot on top of each other (*i.e.*, for a given MCLTV the default rates will be equal). Instead, the default rate over most of the range of the graph increases as the coupon rate rises.

In general, if the coupon rate is high relative to the spot rate, default becomes more likely. However, near an MCLTV = 1 the default rate falls because refinancing, which extinguishes the default option, becomes increasingly likely. MCLTV changes most in response to house price changes or interest rate changes. Near an MCLTV of one, with default imminent, as spot interest rates fall, the probability of default first rises because borrowers are increasingly unhappy with the existing high coupon rate. However, if rates continue to fall and the borrower has equity in the house, the borrower will refinance and extinguish the default option. If the loan is "under water" however, an alternate response will be to default. For the same measured MCLTV, one observes in one case more defaults, and the other sharply fewer. Thus, MCLTV is not a sufficient statistic.

Use BCLTV but properly include other covariates to adjust for the changes in the model parameters. Other variables, like the coupon rate, can be included in empirical models to mitigate any shortcomings of BCLTV. The analysis in the next section confirms that using BCLTV in conjunction with other variables is an effective approach.

Results

The model evaluates conditional default probabilities for seasoned mortgages. The parameters were chosen to be realistic (*e.g.*, 360 month mortgages), and similar to those used in other studies. Table 1 presents the base case parameters. Relevant parameters include the levels and volatilities of both house prices and interest rates, the rental rate, the interest rate reversion parameter, transactions costs and trigger events. The method of solution is described in the appendix.

The results are presented for five year old mortgages, the half life of a mortgage. Since mortgage age conditional on the other parameters does not have a large effect on default probabilities, the results for these mortgages are representative of the results for other mortgage ages.

Because few defaults occur for houses with low CLTV, we assess default rates for high CLTV loans only. We present results for the stochastic house price process (house price volatility and dividend rate), the interest rate process (level of interest rate, interest rate volatility, interest rate reversion), transaction costs of default and refinance, and the exogenous termination rate. In each figure three panels depict the default rates measured over a 1 year and 10 year horizon for three CLTVs (90%, 100%, 110%). The 10 year horizon is representative of the unconditional probability of default. The 1 year horizon is intended to reflect the conditional default rates found in annual default data.

House Price Volatility

Figure 2 summarizes the effect of house price volatility on defaults. There is a sign reversal at high CLTVs that is important and that requires explanation. At low CLTVs the effect of volatility on defaults is positive; but at high CLTVs the effect is negative and large. This seems counter-intuitive since option prices increase in volatility. The total number of defaults does increase as house price volatility increases; but it does not follow that the next period's default rate increases conditional on a given CLTV. An increase in volatility has two offsetting effects. First, the stopping boundary for house prices falls; and secondly the probability of reaching this boundary increases. It is not obvious which effect will dominate *a priori*. Figure 2 shows that the effect of house price volatility depends on CLTV. When house price volatility is low and CLTV is high, optimal default occurs immediately since there is little benefit to waiting to see if house prices fall lower.[5] As house price volatility increases, however, the probability of default in the next period declines since expected house price changes make waiting more valuable (see Kau and Kim 1994).

Because most empirical data are heavily weighted with low CLTV loans, it is not surprising that empirical studies have found a positive relationship between house price volatility and mortgage default (*e.g.*, Schwartz and Torous 1993). Our results, however, imply that the effect of house price volatility interacts with CLTV and is negative and large at high CLTVs.

Rental Rate

Figure 3 plots the effect of the rental rate on default probabilities. As with house price volatility, there is a sign reversal at high CLTV. The rental rate is analogous to the dividend rate on a stock. The effect of the rental rate should be similar to the effect of dividend yield on early exercise of an American put. Again there are two offsetting effects. On the one hand, a higher rental rate implies a lower rate of appreciation of the property which increases the likelihood of hitting the default boundary sometime in the future. On the other hand, a high rental rate makes the existing mortgage payment more attractive relative to renting. When default is imminent, the borrower has an incentive to continue paying longer. The shorter the horizon and the higher the CLTV, the more likely the negative effect on defaults will dominate. Because of the offsetting effects and the small impact in all states as indicated in Figure 3, the anticipated effect of the rental rate in empirical studies is small.

Interest Rates

Figure 4 illustrates that when spot interest rates increase from the level at origination, the computed default rate falls. If interest rates fall, however, there is little impact on default. At CLTVs close to one, the default rate actually declines slightly as interest rates fall. Figure 4 demonstrates that a positive interest rate spread variable is essential in empirical models that use BCLTV as the measure of CLTV. Notice also that the size of the interest rate effect is much greater at high CLTVs. An interaction term of interest rates with CLTV will be needed to capture the effect empirically.

Interest Rate Volatility

Figure 5 shows the effect of interest rate volatility on the probability of default. The effect of volatility is ambiguous *a priori* because refinancings, which foreclose the default option, occur when rates fall. If rates rise, the value of the mortgage is reduced also leading to fewer defaults. When there is little or no equity in the house, however, default is a likely response to falling rates because the value of the high coupon loan increases and further reduces effective equity.

In panel A of Figure 5, at low CLTVs, there is minimal effect on default probability. For high CLTVs (Panel C) increasing interest rate volatility slightly decreases the likelihood of default over the next year. As a result empirical studies may have difficulty distinguishing any statistically significant effect.

Reversion Parameter

Figure 6 indicates that the affect of the interest rate reversion parameter on next year's default probability is quite small and mostly positive. The stronger the reversion, the less likely a favorable interest rate environment will present itself, reducing the value of the default option and increasing the probability of default. The effect is more pronounced for high CLTV loans but still quite small.

Transaction Costs of Default

Transactions costs are particularly interesting because they can vary in three ways. First, each individual faces different transactions costs from family and job characteristics. It is well known that single individuals are more likely to move than other household types. Second, transactions cost vary by location since the legal remedies available to lenders differ. In one-remedy states, borrowers can default with minimal consequences to their personal finances (Jones 1993). Third, this cost can vary over time for the same individual when personal circumstances change. A divorce or job change can greatly reduce the cost of default since the borrower will need to move independently of the default decision. Ambrose et al. (1996) point out that transactions costs can even be negative for borrowers who may enjoy a period of free rent before foreclosure is completed.

Figure 7 shows the effect of these transactions costs on default probabilities. The impact on unconditional default is negative and modest in size at the 90% CLTV. Transactions costs reduce conditional default probabilities at all CLTVs but have their most dramatic impact at high CLTV (Panel C). This difference between the unconditional and conditional may account for some of the disagreement on the importance of transactions costs (Kau, Keenan and Kim 1993, Lekkas, Quigley and Van Order 1993). The results suggest that for empirical models, transactions costs should be interacted with CLTV rather than included as a linear covariate.

Transactions Costs of Refinancing

The transactions costs of refinancing include origination fees, points and legal fees. Absent these costs, refinancing would be optimal whenever interest rates fall below the contract interest rate and borrower equity is positive. The model includes both fixed and variable transaction costs of refinancing. Figure 8 indicates that the short horizon default rate increases slightly as the transaction cost of refinance increases. Overall this variable has little effect on the default rate.

Trigger Events

Historically, industry analysts have assumed that exogenous events (*e.g.*, divorce or unemployment) play a major role in mortgage default. In our model, exogenous events are random events with a given probability of occurrence. The borrower realizes that exogenous events may occur in the future, and adjusts his decisions to default or prepay today appropriately. We separate the defaults into those that were due to the optimal decision at the time, and those which are in response to an exogenous event. This allows computation of the probability that an exogenous event will cause a default. In the base case, it is assumed that transaction costs are present for all default decisions. In many cases, however, an exogenous event may result in several changes for the decision maker - some of which may reduce the transactions costs of defaulting to zero. For example, if a move results from this exogenous event (perhaps a change of employment location), then the moving costs of default are no longer relevant to the decision. For this reason, a modified model was also analyzed for default rates when transaction costs are zero and the exogenous event occurs.

Numerical results were examined to evaluate the amount of default that is due to exogenous versus optimizing decisions. For the numerical results that follow, the PSA rate is set to 150%, and the default rates are those over the next year. When CLTV is low (0.90), the default rate is low (1/2%). Of the defaults that are observed, 6.5% are due to exogenous events. Given that an exogenous event has occurred, there is about a 1/2% chance it will result in default. This conditional rate is about equal to the overall default rate. When the CLTV is high (1.1), the default rate is also high (69%). Of those defaults, about 1.5% are due to exogenous events.

The default rate, given an exogenous event, is about 35% which is much lower than the overall default rate.

Model results were also examined for a zero transactions costs case. One result that may seem counter-intuitive is the slightly negative slope of the default curve for some parameters. That is, there are fewer defaults over the next year as the exogenous event probability increases. This occurs because the borrower delays default today anticipating that an exogenous event may occur in the future. If the exogenous event does occur the borrower can default without incurring transactions costs. A numerical analysis shows that when CLTV is low (0.9), the default rate remains low (3/4%). Of the defaults that do occur, 35% are due to exogenous events. Given that an exogenous event has occurred, there is about a 4% likelihood of default. When the CLTV is high (1.1), the default rate is high (66%), though not as high as when there are transaction costs (because borrowers wait to see if transactions costs can be avoided when an exogenous event occurs). Of those defaults, about 4% are due to exogenous events. The default rate, given an exogenous event, is about 75%. So while default, given an exogenous event is high, the default rate is high even without exogenous events. The default rate, given an exogenous event, is higher in this case than when there are transactions costs. Therefore, default is a more profitable strategy, given that an exogenous event has occurred, if there are no transaction costs to default.

Optimal default (and prepayment) is the first choice available to the borrower. If the borrower does not default or prepay; then, an exogenous event may present itself (at the PSA rate). When an outside observer observes both a default and an exogenous event, he may assume that the "trigger event" caused the default, rather than the default occurring as an optimal decision in the same period a trigger event occurs. The default should not be credited to the trigger event, unless the event occurs only because of the trigger event. Our overall conclusion, is that trigger events play a minor role. If house prices are low (precipitating high CLTVs) defaults will be high, regardless of whether trigger events occur or not. Figure 9 demonstrates the minor impact on default as the exogenous termination rate is varied.

Conclusions and Empirical Implications

This paper has developed a contingent claims model of mortgage default and applied the model to obtain insights for empirical modeling. There are four types of results. First, some variables suggested by options pricing matter unconditionally but are secondary or tertiary in importance conditionally. These include the rental rate, interest rate volatility and interest rate reversion.

Second, some effects reverse signs under certain conditions. These include most notably house price volatility, but also the rental rate and interest rate volatility. The sign reversal for volatility from positive to negative occurs when the default option is in the money (high CLTV). In this range, volatility causes borrowers to delay default in anticipation of the possibility of more favorable exercise conditions in the future. This result contrasts with the unconditional probability which is positively affected by volatility.

Third, some parameters have so little effect that they can be safely ignored in empirical analysis. The interest rate volatility and reversion, as well as the transactions costs of refinancing, are in this category. Trigger events, while having traditionally been considered important precipitators of default, have relatively little influence.

Fourth, functional form is complicated for some independent variables. Interest rate declines, for example, have little effect on defaults. However, increases from the origination rate reduce defaults and further reduce them when the option is in the money. Some parameters like volatility and the rental rate, as noted earlier, reverse sign at high CLTVs. Some, like transactions costs and trigger events, have more impact at high CLTVs.

The key variable in predicting default is the CLTV. BCLTV and MCLTV, however, are not equivalent. Some empirical studies have adjusted the BCLTV for changes in spot interest rates. A contingent claims model could be used to compute an MCLTV. The computer resources required, however, prevent extensive use of this approach. In addition MCLTV is not sufficient because it does not fully explain observed defaults. Our results support using BCLTV because it is easy to compute and because adding appropriate covariates with a suitable functional form adjusts for the shortcomings of BCLTV.

Trigger events do have an effect, albeit small, on defaults. The increase in defaults from trigger events is most pronounced when transactions costs are low and the default option is only slightly out of the money. In other cases the default rate conditional on an exogenous events is similar to the rate without the trigger event.

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Appendix
Numerical Solution

The model is solved using backward recursion of a stochastic dynamic programming model. Dynamic programming requires discretizing the two stochastic processes. The house price process of equation (2) is discretized using the standard Cox, Ross and Rubinstein (1979) equations of:

$u = exp(\sigma_H \sqrt{\Delta t}),$

$d = 1/u$, and

$p_u = \dfrac{exp[(r-\gamma)\Delta t] - d}{u-d}.$

where:

u = the next period house price multiplier if an upstate occurs

d = the next period house price multiplier if a down state occurs

p_u = probability of an upstate

The interest rate process of equation (3) is discretized using the Nelson and Ramaswamy (1990) equations of:

$r_u = r + \sigma_r \sqrt{\mathrm{Dt}}$,

$r_d = r - \sigma_r \sqrt{\Delta t}$,

$$q_u = \begin{cases} 0.5 + \sqrt{\Delta t}\,\beta(\alpha - r)/2\sigma_r & \text{if } 0 \le 0.5 + \sqrt{\Delta t}\,\beta(\alpha - r)/2\sigma_r \le 1 \\ 0 & \text{if } \quad 0.5 + \sqrt{\Delta t}\,\beta(\alpha - r)/2\sigma_r \le 0 \\ 1 & \text{otherwise} \end{cases}$$

where:

r_u = the interest rate next period if the up state occurs,

r_d = the interest rate next period if the down state occurs, and

q_u = the probability of the upstate occurring.

The borrower's problem at each time is to determine which action provides the lowest discounted housing cost. In other words, just prior to a mortgage payment due date, the mortgagor assesses whether it less costly to default, to refinance, or to make the scheduled mortgage payment. This decision can be written as:

(A1)

where

$$P_t^d(H_t, r_t) = H_t + TC \qquad \text{(A2)}$$

$$= Pmt + (1+X)\, MB_t + F \qquad \text{(A3)}$$

and

$$P_t^w(H_t, r_t) = [p_u q_u P_{t+1}(u H_t, r_u) + (1-p_u) q_u P_{t+1}(d H_t, r_u) + p_u(1-q_u) P_{t+1}(u H_t, r_d) + (1-p_u)(1-q_u) P_{t+1}(d H_t, r_d)]\, \delta_t + Pmt. \qquad \text{(A4)}$$

where

$P_t^d(H_t, r_t)$ = the value of the house plus the transaction costs, which is the value of the loan (to the mortgagor) if he chooses to default on the loan now,

TC = transactions costs of default,

$P_t^r(H_t, r_t)$ = is the cost to obtain a fair value new mortgage, which is the value if one refinances the mortgage now. The lender receives a fair exchange mortgage for providing the funds to refinance the mortgage balance. The nature of the new loan is not specified. The new loan could be a non-prepayable, or prepayable, a fixed or variable rate loan, etc. The only requirement here is that it is fairly priced so that it is equal to the mortgage balance that must be refinanced. The deadweight cost of a new mortgage includes any excess points (*X*) that are not priced in the mortgage instrument itself, and a fixed fee (*F*). Dunn and Spatt (1986) emphasize the importance of the mortgage refinancing costs on refinancing decisions, and on the value of the mortgage. In particular they note that prepayable mortgages may trade at values above par when it is not advantageous for a mortgagor to refinance in response to a small interest rate decrease due to the transaction costs of refinance. In the absence of refinance costs, any decrease in interest rates would lead to refinance.

X = deadweight refinancing costs that are a function of the loan size (e.g. the commonly used one point origination fee for a mortgage),

F = deadweight fixed refinancing cost (*e.g.* cost of a property survey or credit check),

MB_t = the mortgage balance at time t,

Pmt = the monthly mortgage payment,

$P_t^W(H_t,r_t)$ = the present value of the loan assuming the decision maker makes the contractual mortgage payment and thus continues the mortgage at least one more period. This is the expected discounted future value of the loan given that the house price and interest rate changes next period according to the processes noted.

δ_t = the one period discount factor for the current spot interest rate.

The model is revised to allow for events which are exogenous to the optimizing decision as described thus far. These events cause premature termination of the mortgage, in the sense that when it is not otherwise optimal to choose to default or prepay, an exogenous event may precipitate one of these decisions. The exogenous event is assumed to follow the pattern established by the PSA[6] function though it may be scaled by a chosen multiple. When an exogenous event occurs, the decision maker decides whether it is better to prepay or to default by simply assessing what is best at this instant as there is no future to be considered. The best choice is the minimum cost choice - either to pay off the mortgage balance, or to default if the house price plus transaction costs of default are lower than the mortgage balance. Equation (A1) is modified to allow for exogenous events as:

$$P_t(H_t,r_t) = Min\{P_t^d(H_t,r_t)\,,\, P_t^r(H_t,r_t),\, (1-\lambda_t)\,P_t^W(H_t,r_t) + \lambda_t\, Min[MB_t\,,\, H_t + TC]\} \quad (A5)$$

where:

λ_t = the exogenous event rate at period t.

To complete the model one must specify the appropriate boundary condition at maturity. In the final period, the wait choice vanishes. The value of the mortgage, if default is chosen, is the value of the house plus transactions costs. The alternative is to payoff the mortgage by making the final payment. The mortgagor chooses the lower cost alternative, which in general is to make the final mortgage payment. The mortgagor will make the final mortgage payment if the transaction cost of default exceeds the mortgage payment. The boundary equation at maturity is:

$$P_T(H_T, r_T) = Min(H_T + TC\ ,\ Pmt). \tag{A6}$$

Solving for the optimal decision sequence and values is by backward induction starting with equation (A6) and working to the present. It is not appropriate to prepay a mortgage the instant it is taken out (one would wait at least one month in this discrete time model) and one does not make a payment on a mortgage until the first month. The value of a mortgage at initiation is thus the present value of the mortgage one period from now less a payment which need not be made at initiation:

$$P_0(H_0, r_0) = P_t^W(H_0, r_0) - Pmt \tag{A7}$$

The model is appropriate for determining the optimal decision at each stage (time period) and for computing the value of the loan. The primary purpose of this modeling, however, is to learn about default probabilities, which requires the use of lattice process probabilities to compute projected default rates. Because default at any stage is conditional on what has happened in the past, this computation is made as a forward recursion on the lattice using the optimal default and prepayment boundaries as stopping points for the process. While hedging arguments provide that the default option, and optimal stopping boundary are determined using the risk neutralized house price process of equation (2), determining the probability of default is done using the actual house price process of equation (1). To implement the computation of the default probabilities, the probability of an upstate in house prices is computed using the same CRR equation, but with the gross return to housing used in place of the risk free interest rate, that is:

$$\pi_u = \text{actual probability of an upstate} = \frac{exp[(g-\gamma)\Delta t] - d}{u-d}.$$

Operationally, the model is run; and the optimal default and prepayment hitting boundaries are stored. Then starting with a probability of 1 from the initial node, the probability of reaching each interest rate and house price node in the next period is computed. If that node involves a prepayment, the probability of reaching that node is credited to the probability of a prepayment, and then the probability of that node is set to zero. A similar approach is used for default nodes. For nodes for which neither a prepayment or default is chosen, their probability is reduced by the exogenous termination rate. The exogenous event probability is credited to either prepayment or default, depending on whether the mortgage balance exceeds the house value plus transaction cost of default. The process is then continued for another stage. The correct conditional probabilities are computed because nodes where a default or prepayment has occurred have their probability set to zero, so forward movements from these nodes are made with zero probability. The probability of default at each stage is computed by summing the probabilities of all default nodes at that stage.

Table 1. Base case parameters for numerical modeling.

Initial House Price	$H_0 = \$100{,}000$
Initial Loan Amount	$MB_0 = \$90{,}000 = P_0(H_0, r_0)$
Contract Mortgage Rate	0.850% monthly (10.2% annual rate)
Monthly Mortgage Payment	$Pmt = \$803.15$
Initial Spot Interest Rate	r_0 =0.797% monthly (10% effective annual yield, 9.57% annual rate)
Gross Return To Housing	$g = 0.11$
House Rental Rate (dividend)	$\gamma = 0.05$
House Price Volatility	$\sigma_H = 0.1$
Reversion Parameter	$\beta = 0.1$
Interest Rate Equilibrium	$\alpha = r_0 = 0.10$
Interest Rate Volatility	$\sigma_r = 0.01$
Deadweight Refinance Costs	$F = \$500$
	X = 0.5% of loan balance
Transaction Cost Of Default	$TC = \$5{,}000$
Prepayment Rate	λ_t = 50% PSA

Figure 1. MCLTV and Default

The figure plots the default probability over the next year for three spot interest rates as market current loan to value varies for a 5 year old loan and for a book current loan to value ratio = 1.0. Model parameters are as given in Table 1 with the following exceptions: i) House value = loan balance = \$100,000; ii) Mortgage coupon rate varied from about 2% below indicated spot rate to 2% above indicated spot rate to vary the market current loan to value. The figure illustrates that MCLTV does not fully capture the effect of interest rate changes on default.

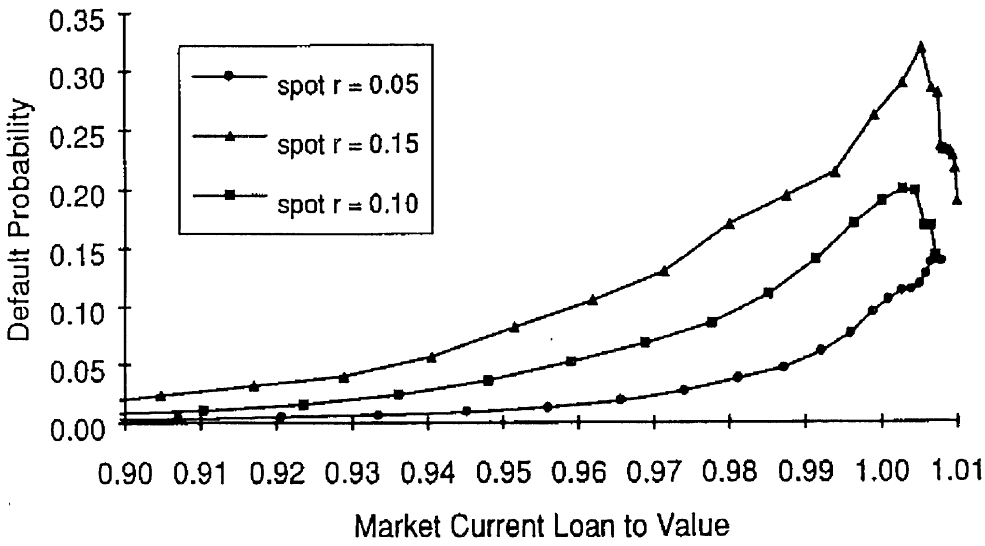


Figure 2. The effect of house price volatility.
The three panels of this figure depict the effect of house price volatility on default for a 5 year old mortgage and three BCLTVs. Base case parameters are given in Table 1. Solid line is the default probability for the next year and the dashed line for the next 10 years.

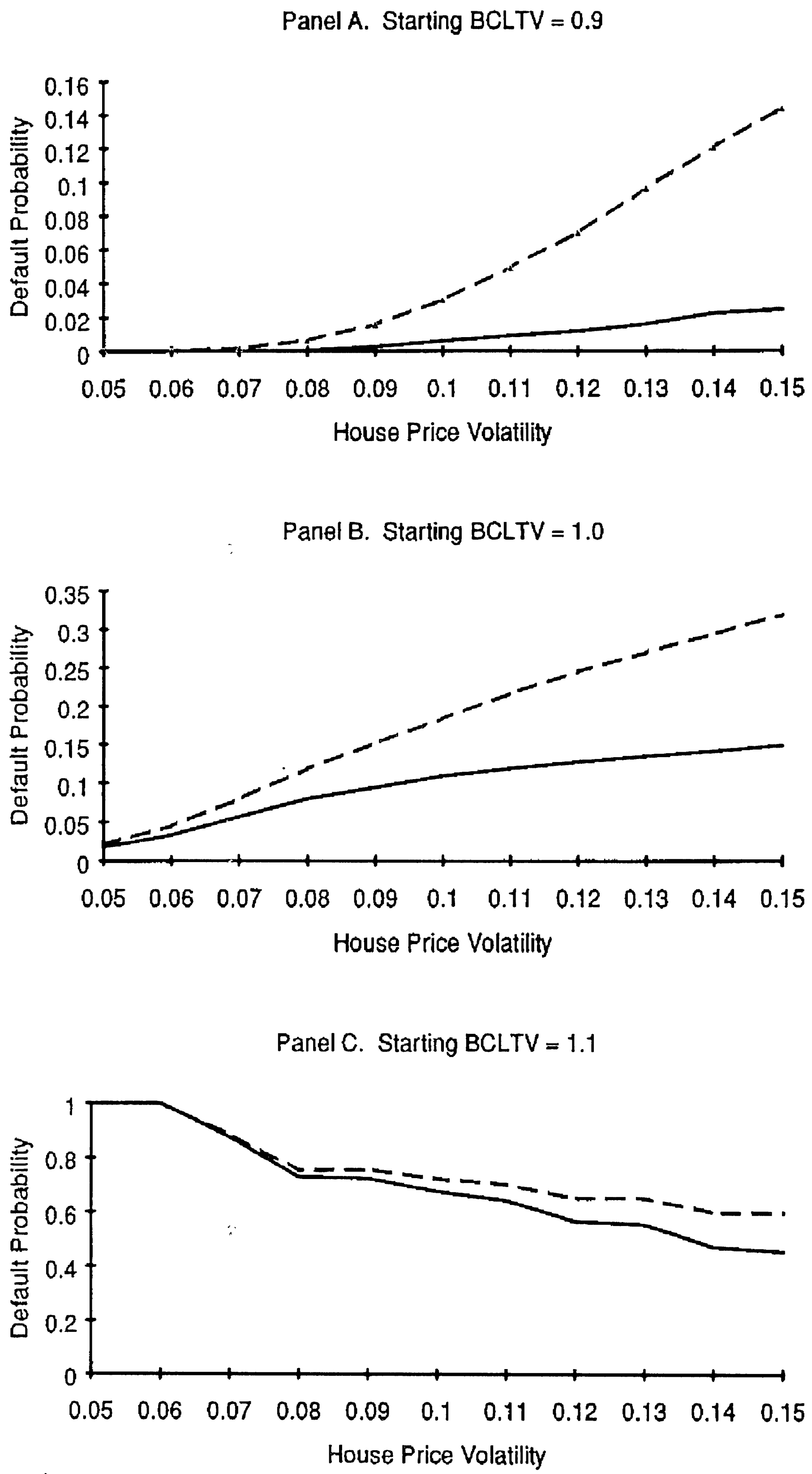


Figure 3. The effect of the rent to price ratio.
The three panels of this figure show the effect of the rent-to-price ratio on default for a 5 year old mortgage and three BCLTVs. Base case parameters are given in Table 1. Solid line is the default probability for the next year and the dashed line for the next 10 years..

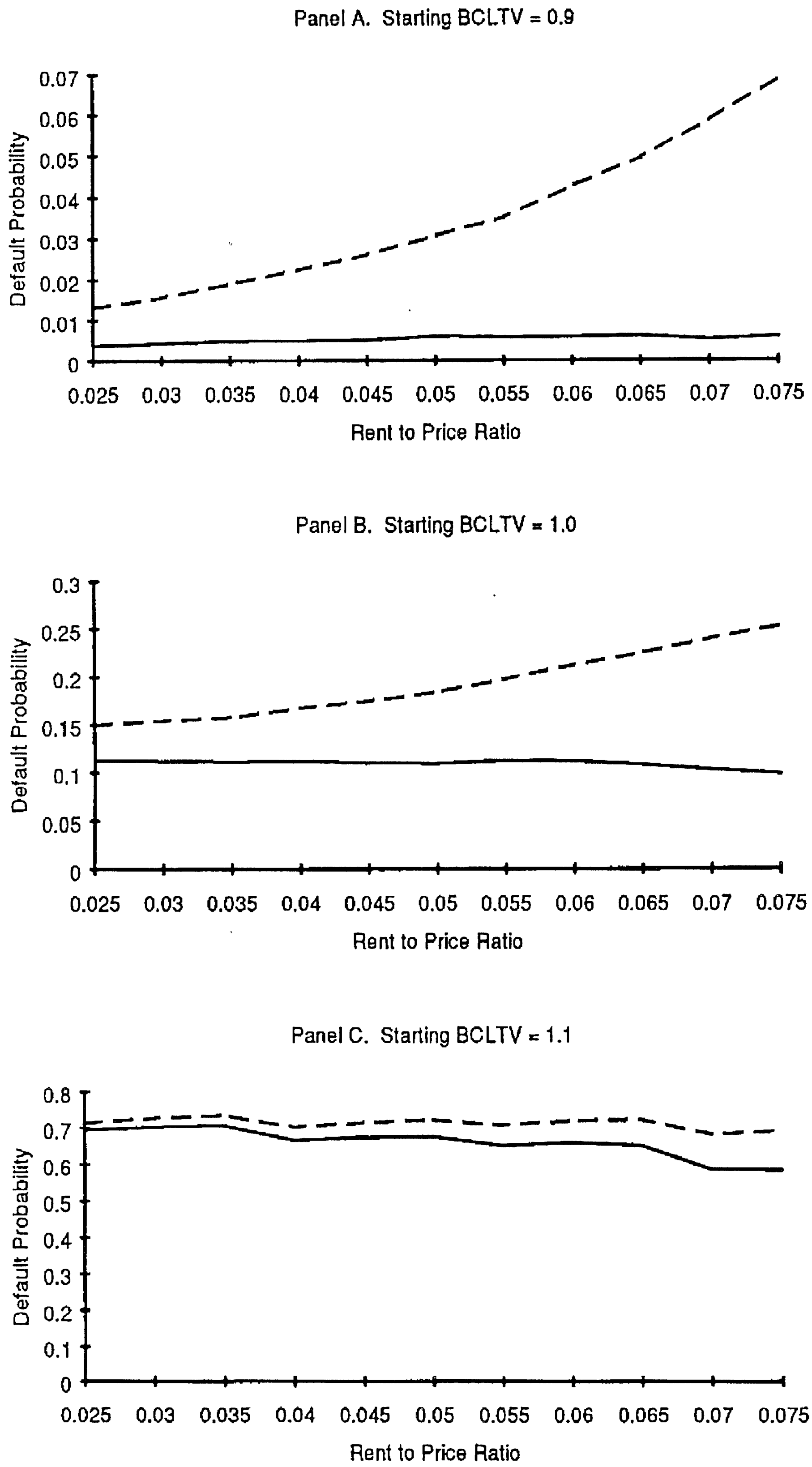


Figure 4. The effect of interest rate changes.
The three panels of this figure illustrate the effect of interest rate changes since origination on default for a 5 year old mortgage and three BCLTVs. Base case parameters are given in Table 1. Solid line is the default probability for the next year and the dashed line for the next 10 years.

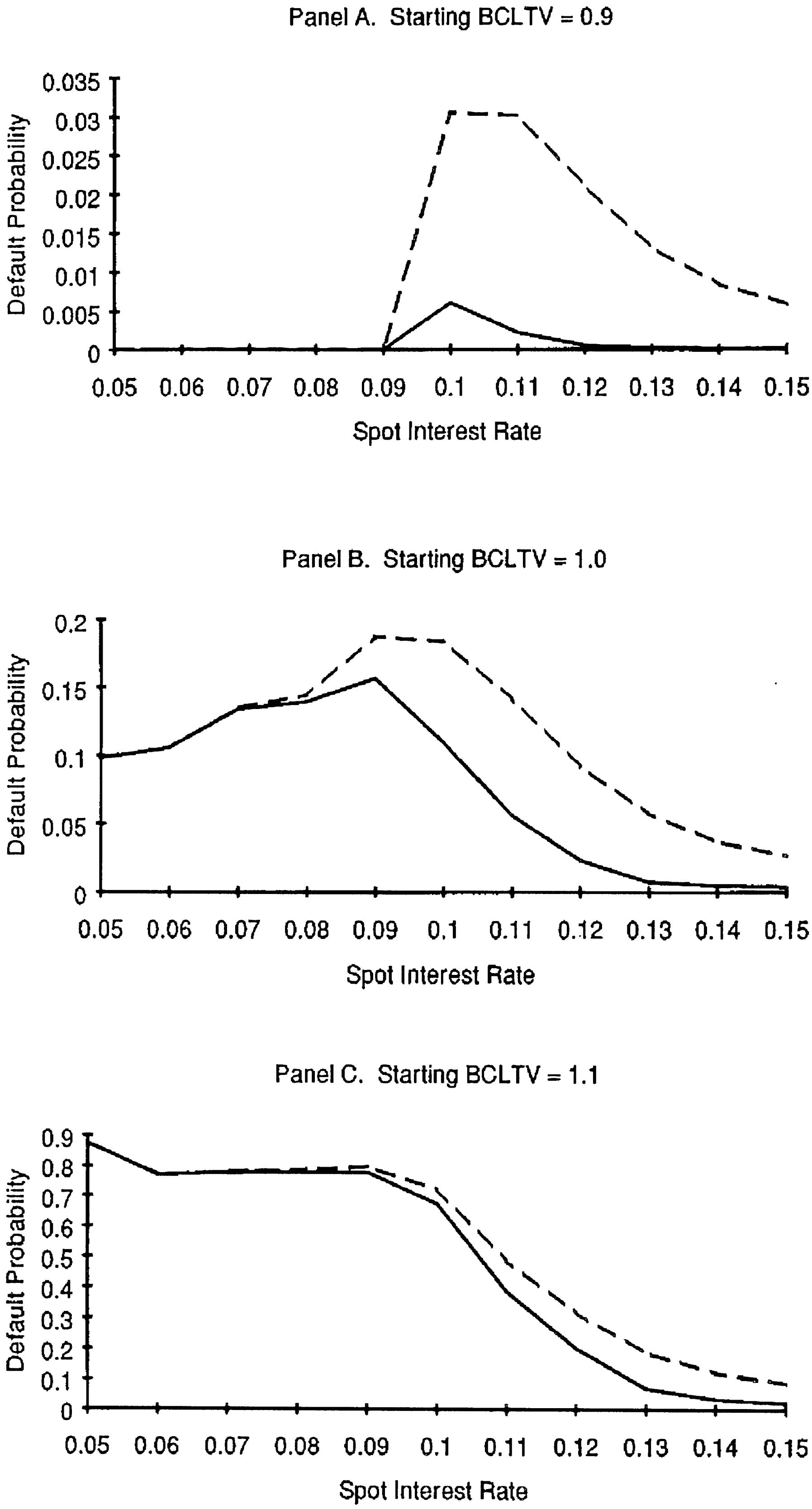


Figure 5. The effect of interest rate volatility.
The three panels of this figure present the effect of interest rate volatility on default for a 5 year old mortgage and three BCLTVs. Base case parameters are given in Table 1. Solid line is the default probability for the next year and the dashed line for the next 10 years.

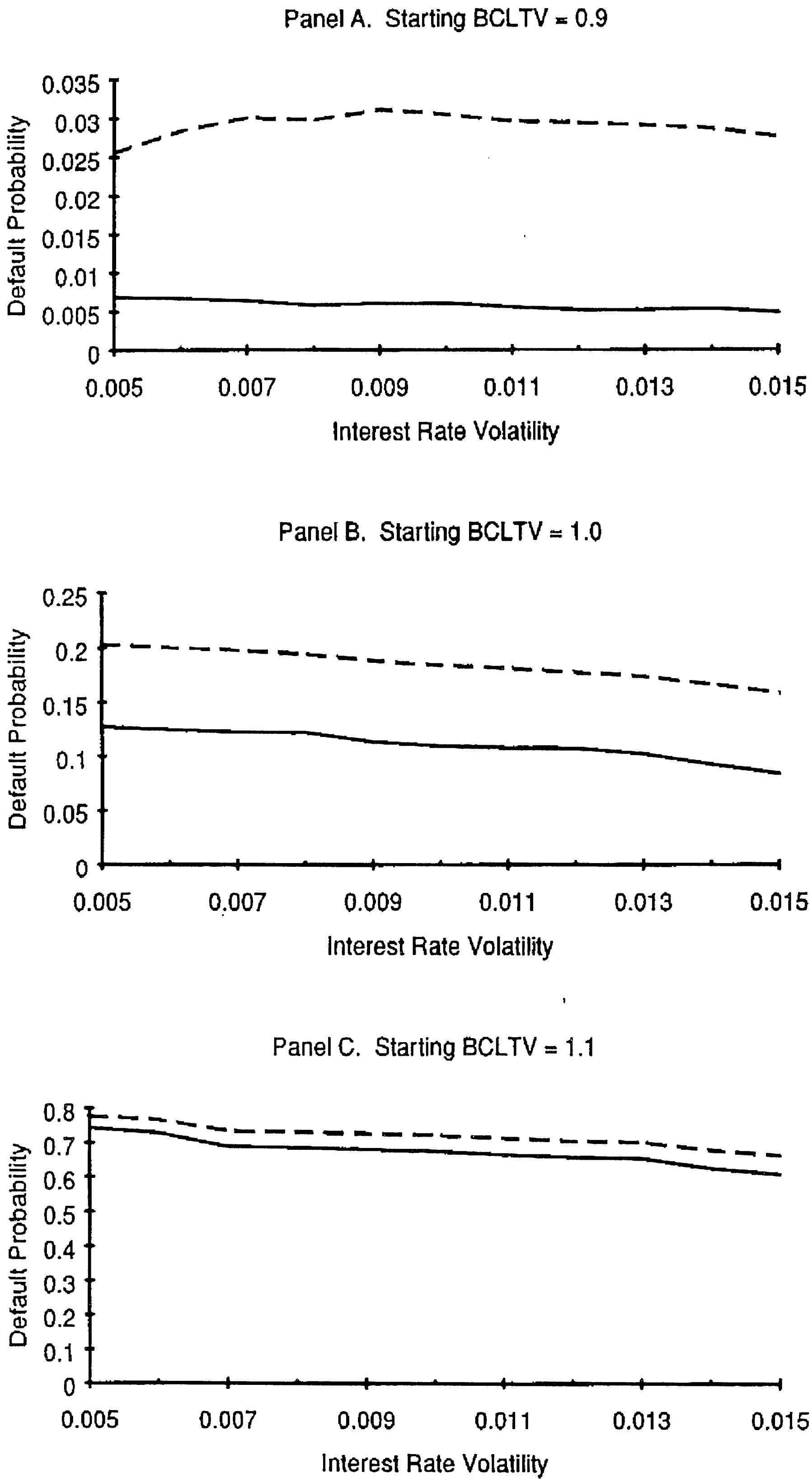


Figure 6. The effect of interest rate reversion.
The three panels of this figure show the effect of interest rate reversion on default for a 5 year old mortgage and three BCLTVs. Base case parameters are given in Table 1. Solid line is the default probability for the next year and the dashed line for the next 10 years.

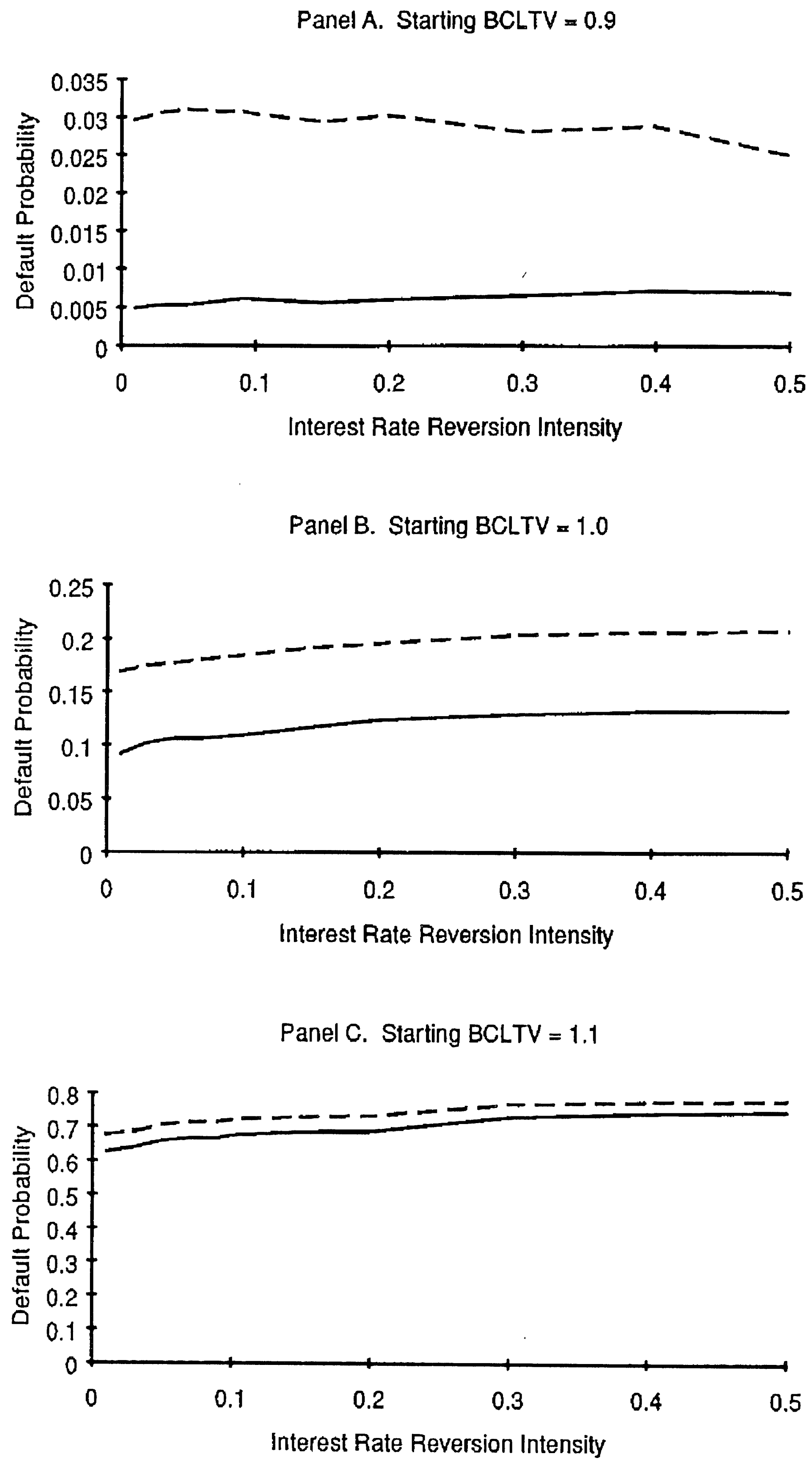


Figure 7. The effect of transactions costs of default.
The three panels of this figure illustrate the effect of default-related transactions costs on default probabilities for a 5 year old mortgage and three BCLTVs. Base case parameters are given in Table 1. Solid line is the default probability for the next year and the dashed line for the next 10 years.

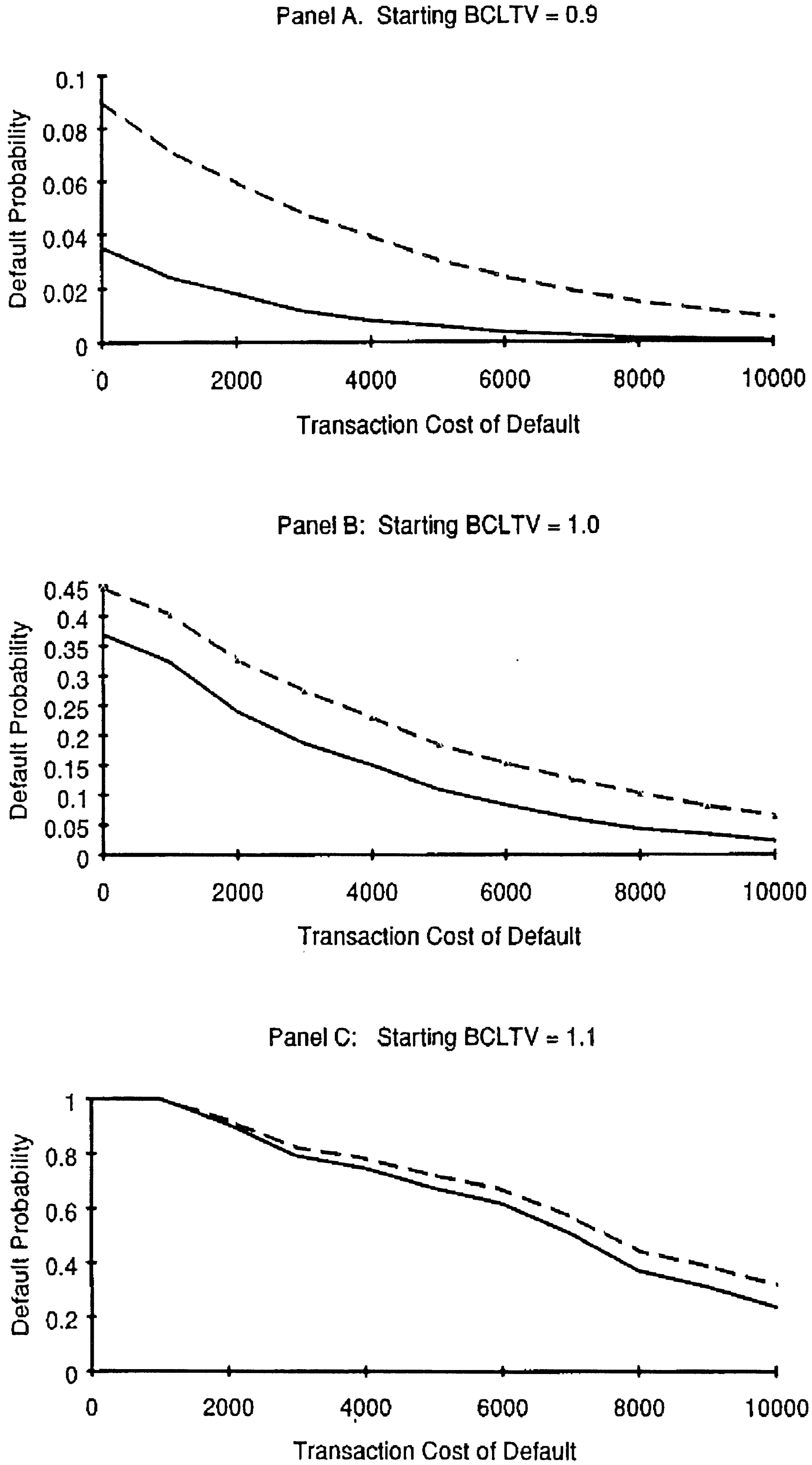


Figure 8. The effect of deadweight refinancing costs.
The three panels of this figure depict the effect of deadweight refinancing costs on default for a 5 year old mortgage and three BCLTVs. Base case parameters are given in Table 1. Solid line is the default probability for the next year and the dashed line for the next 10 years.

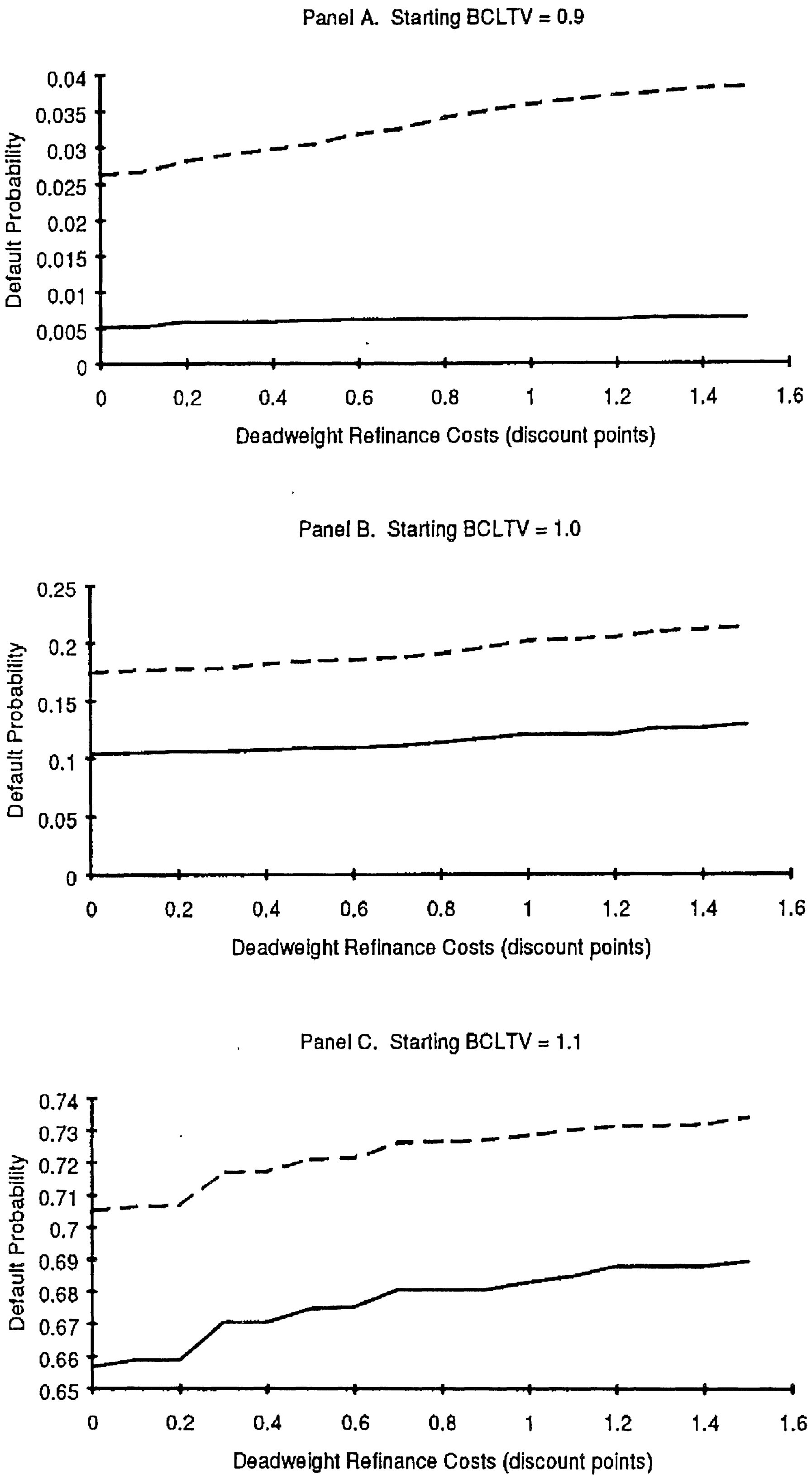
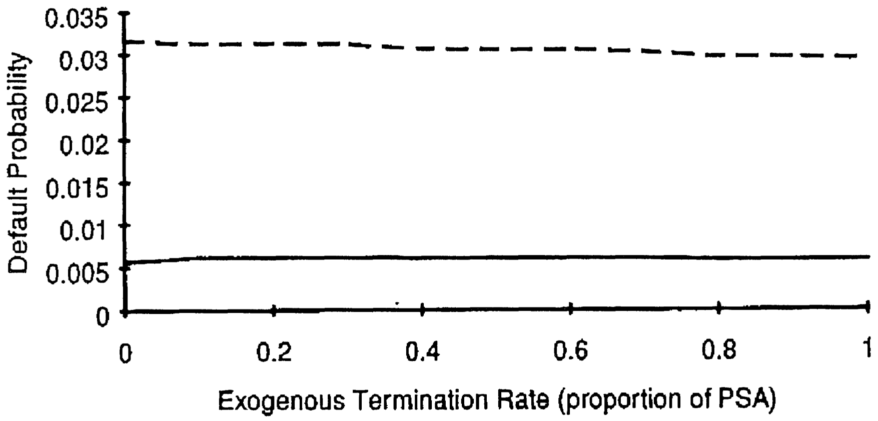
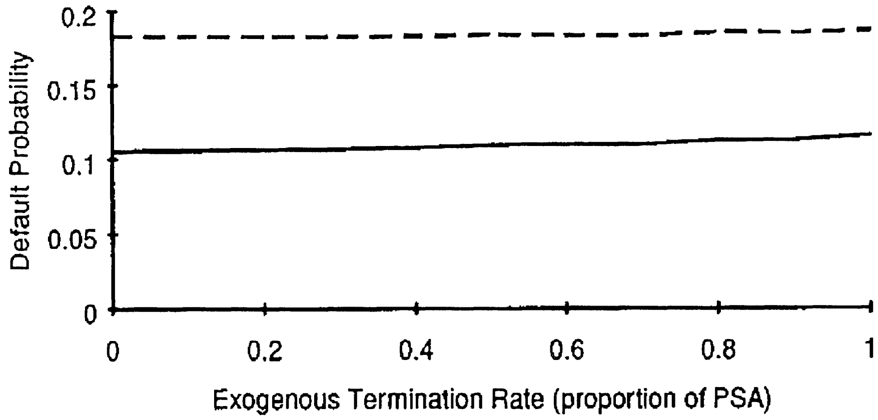


Figure 9. The effect of exogenous terminations.
The three panels of this figure show the effect of exogenous terminations on default for a 5 year old mortgage and three BCLTVs. Base case parameters are given in Table 1. Solid line is the default probability for the next year and the dashed line for the next 10 years.

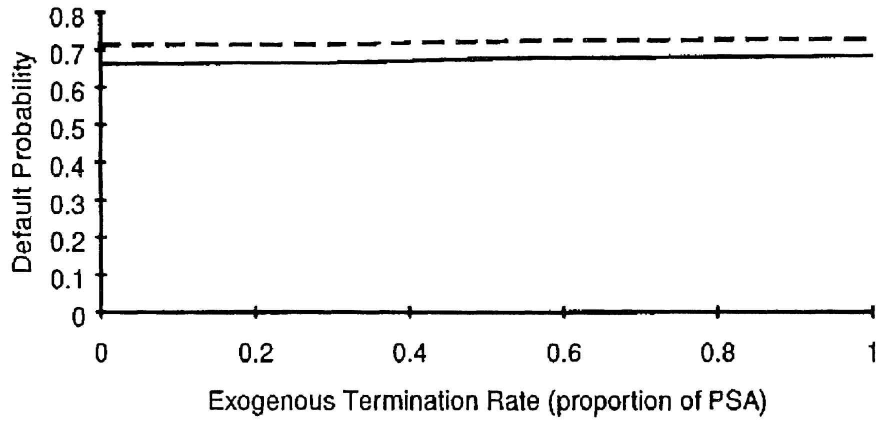
Panel A. Starting BCLTV = 0.9



Panel B. Starting BCLTV = 1.0



Panel C. Starting BCLTV = 1.1



Endnotes

[1] See Kau, Keenan and Kim 1994, for an elaboration.

[2] Negative interest rates are permitted in this model; but they are not important since the optimal refinance boundary, except when the loan is almost fully repaid, occurs at positive interest rates for realistic parameter values.

[3] As Kau, Keenan and Kim (1994) point out, these terminations may be suboptimal from the point of view of the model, but be the optimal decision for the mortgagor who chooses to terminate for reasons exogenous to the model.

[4] The computation uses more than five minutes of CPU time of on a Sun 670MP. 50,000 computations would require about six months of CPU time.

[5] Kau and Kim (1994) show that the reason one delays a current default, even if it is "in the money," is that house prices may fall in the future so that the present value of defaulting in the future may exceed the value of immediate default.

[6] The PSA model is the prepayment model produced by the Public Securities Association which projects the prepayment rate as rising linearly from 0 to 6% per annum over the first 30 months of the loan and then remaining at 6% per annum for the remainder of the loan. The monthly PSA model of prepayments can be written as:

$\lambda_t = (0.005/30)\ t \qquad t = 1, 2, 3, \ldots, 29$

$\lambda_t = 0.005 \qquad t = 30, 31, 32, \ldots, 360.$