THE EXPECTED NUMBER OF EXTREME POINTS
OF A RANDOM LINEAR PROGRAM

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Abstract

There has been increasing attention recently on average case algo-
rithmic performance measures since worst case measures can be qualitatively
quite different. An important characteristic of a linear program, relating
to Simplex Method performance, is the number of vertices of the feasible
region, we show 2^n to be an upper bound on the mean number of extreme
points of a randomly generated feasible region with arbitrary probability
distributions on the constraint matrix and right hand side vector. The
only assumption made is that inequality directions are chosen in accordance
with a series of independent fair coin tosses.

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Attempts to establish the mean number of extreme points and other topological properties of random linear constraint sets have included distribution dependent approaches (see, for example, Efron [1965], Liebling [1972], Schmidt and Mattheiss [1977], [1980]) as well as essentially distribution free approaches (May and Smith [1982], Adler and Berenguer [1981], [1983]). Apart from the purely mathematical interest of these models and results, they have proved useful in establishing average case performance of variants of the Simplex Method (Borgwardt [1982], Adler [1983], Saigal [1984], Smale [1983], Todd [1983]), Adler, Karp and Shamir [1982], Adler and Meggido [1983], and Haimovich [1983]).

McMullen [1970] demonstrated that the worst case number of extreme points goes up factorially fast in problem dimension for the class of convex polytopes. Independently, May and Smith [1982] and Adler and Berenguer [1981], using the theory of arrangements of hyperplanes partitioning Euclidean space (Grunbaum [1967], Buck [1943]), established distribution invariant topological results for randomly generated linear constraint sets. They showed that the average number of extreme points for a class of distributions over convex polytopes increases only exponentially fast in problem dimension. They assumed the generating distribution to be continuous, thus excluding sparsity, integer constraint coefficients, degeneracy, and other characteristics that real world and many randomly generated linear programs are known to possess. It was also assumed that inequality directions were decided on the basis of independent tosses of a fair coin. Retaining only the latter assumption, (now referred to as sign invariance) we show in this paper that for arbitrary distributions, the expected number of extreme points is bounded from above by 2^n where n is the dimension of the constraint set. Note that this result for random linear programs does not exclude degeneracy.

1. A Model for Random Linear Programs

We express the constraint sets in the general form $Ax \gtreqless b$ where A is an mxn matrix, b is column m-vector and x is a column n-vector. The inequality directions are specified by a column m-vector e, where $e_i = 0$ or 1 according to whether the ith constraint is $\leq$ or $\geq$ respectively. A probability distribution can then be specified over the space of all possible polyhedral sets $S = (A,b,e)$ for n and m fixed. Let F1, F2 and F3 be the marginal distributions on A, b and e respectively. In particular define F3 by the m-tuple $(p_1, p_2, ..., p_m)$ where $p_i = P(e_i = 1)$ and $e_1, e_2, ... e_m$ are assumed independent. Most of the occurrences of S correspond to empty polyhedral sets. We shall, however, be concerned with conditional properties of polyhedral sets so generated given that they are non-empty. After randomly generating the constraint set, we can obtain a region of dimension 0, 1, ..., or n. Let $E_k(m, n)$ denote the expected number of extreme points of a generated region of dimension k. Then the expected number of extreme points of a generated region of dimension n,

$$E(m, n) = \sum_{k=0}^{n} E_k(m, n) \, p(k),$$

where $p(k)$ is the probability of generating a region of dimension k.

If $E_k(m, k) \leq 2^k$ we would then have

$$E(m, n) = \sum_{k=0}^{n} E_k(m, k) \, p(k) \leq \sum_{k=0}^{n} 2^k p(k) = 2^n \sum_{k=0}^{n} p(k) = 2^k.$$

It suffices to show, without loss of generality, that

$$E_n(m, n) \leq 2^n$$

(we would clearly have that $E_k(m, n) \leq 2^k$).

In particular, we are interested in

$E(n,m) =$ the expected number of extreme points of a non-empty

polyhedral set formed by m constraints in R^n according to the model above.

We shall impose the following restrictions. We require that $p_i=1/2$ for $i = 1,\ldots,$ m in the m-tuple for F3; this is the crucial symmetry assumption of the main results of May and Smith [1982] and Adler and Berenguer [1981,1983].

We shall here extend the exponential bound of 2^n for E(m,n) obtained in May and Smith [1982] and Adler and Berenguer [1981, 1983] by weakening the assumptions of F1 and F2 to arbitrary distributions over the constraint set matrix and right hand side vector respectively. Incidently their joint distribution is also allowed to be arbitrary. In keeping the one half symmetry on F3 we retain the assumption that each polyhedral set of the partition of R^n formed by the m hyperplanes is equally likely to occur as the chosen feasible region.

2. A Bound on the Expected Number of Extreme Points for an Arbitrary Number of Constraints

Theorem 1:   Let F1 over A and F2 over b be arbitrary and possibly dependent distributions. Let $e_i, i = 1,2,\ldots,m$ be i.i.d. Bernoulli random variables with $p_i = 1/2$ for all $i = 1,\ldots,$ m where e is independent of A and b. Then the expected number of extreme points for the corresponding randomly generated non-empty constraint set of m constraints and n variables, $E(n,m) \leq 2^n$ for all m and n.

Before proving Theorem 1 we need the following results.

Definitions (Winder [1966]): A set of k hyperplanes in R^n all passing through a common point is even-degenerate if the dimension of their intersection is of different parity than n - k. A set of hyperplanes that

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is not even-degenerate is termed <u>odd-degenerate</u>.

<u>Lemma</u> <u>1</u> (Winder [1966]): The number of regions in which m hyperplanes, all passing through some common point, divide R^n is equal to the number of distinct even-degenerate subsets of the given m hyperplanes minus the number of distinct odd-degenerate subsets (the empty subset is included and counted as even-degenerate).

<u>Lemma</u> <u>2</u>: Suppose a point p is determined by (that is, coincides with the intersection of) the distinct hyperplanes $H_1^p, \ldots, H^p$ in R_ℓ^n. Consider the hyperplane $H \notin \{H_1^p, \ldots, H_\ell^p\}$, also passing through p and with the property that it does not contain any linear flat of dimension greater than zero which is the intersection of two or more of the $H_j^p, j=1, \ldots, \ell$. Let $N_H(p)$ be the number of regions (n-dimensional polyhedral sets) formed by the $H_j^p, j=1, \ldots, \ell$ and lying entirely in a distinguished half-space of H. Then $N_H(p)$ is invariant over all H satisfying the property above.

<u>Proof</u>: It is sufficient to show that the number of even-degenerate and odd-degenerate sets of the partition of R^n by $H, H_1^p, \ldots, H_\ell^p$ is invariant to the choice of H, so long as it satisfies the property stated above. Then, from Lemma 1, the number of regions in the partition formed by $H_1^p, \ldots, H_\ell^p$ and cut by H is invariant to H. The remaining uncut regions are then invariant in number, so that the half of these uncut regions lying wholly in a distinguished half-space is invariant in number.

Let $S \equiv H_{i_1}^p \cap H_{i_2}^p \cap \ldots \cap H_{i_q}^p$, where $H_{i_j}^p \in \{H_1^p, \ldots, H_\ell^p\}$. It suffices to show that the parity of $H \cap S$ is invariant with respect to H. Let dim (S) $= k \geq n-q$. Clearly, dim $(H \cap S) = k-1$, for $k > 0$, by the property above that H satisfies. Also, dim $(H \cap S) = 0$ for $k = 0$; obviously, in this case $H \cap S \equiv S \equiv \{p\}$. Therefore, parity of $(H \cap S)$ is invariant with respect to H. $\blacksquare$

We <u>can</u> <u>therefore</u> <u>simplify</u> <u>the</u> <u>notation</u> <u>substituting</u> <u>N(p)</u> <u>for</u> <u>$N_H(p)$</u>.

<u>Lemma</u> <u>3</u>:   Consider a point p determined in R^n by hyperplanes $H_1^p,\ldots,H_m^p$.

Let $C(p)$ be the total number of regions formed by $H_1^p,\ldots,H_m^p$ around p.   Then

$C(p) \leq 2^n N(p)$.

<u>Proof</u>:   It can be easily seen that there must exist n linearly independent

hyperplanes out of $\{H_1^p,\ldots,H_m^p\}$ which suffice to determine p.   Without loss

of generality denote these n hperplanes $H_1^p,\ldots,H_n^p$.   Now consider the 2^n

orthants determined by $H_1^p,\ldots,H_n^p$ and number them 1 through 2^n.   Let the

number of regions in orthant i be $C_i (i=1,\ldots,2^n)$.   Then there clearly

exists a hyperplane H_i, satisfying the property of Lemma 2, a half-space of

which wholly contains the $i\underline{th}$ orthant.   By Lemma 2, $C_i \leq N(p)$ $(i=1,\ldots,2^n)$

since every region in the orthant entirely lies in this distinguished half-

space of H_i.   Hence it follows that

$$C(p) = \sum_{i=1}^{2^n} C_i \leq \sum_{i=1}^{2^n} N(p) = 2^n N(p). \qquad \blacksquare$$

Now, consider the partition of R^n by the m hyperplanes of the

constraint set $Ax \lesseqgtr b$.   We shall assume throughout that the m hyperplanes

are distinct, for otherwise we simply embed the problem in the

corresponding lower dimensional subspace.   Let p_r, $r=1,2,\ldots,k$ be the

vertices of the partition (k is the number of vertices of the partition, $k \leq$

$\binom{m}{n}$).   Let H be a hyperplane in R^n, H_+ a distinguished half-space of H and

H^{p_r} and $H_+^{p_r}$ the hyperplane parallel to H passing through p_r and the half-

space corresponding to H^{p_r} respectively.   Since the number of hyperplanes

in the partition is finite, we can find H such that H^{p_r} together with the

hyperplanes $H_{j_r}^{p_r}$ that determine p_r on the partition $(j_r = 1,\ldots,\ell_r)$ have

the property of Lemma 2.[*]   By Lemma 2, exactly $N(p_r)$ regions around p_r lie

entirely in $H_+^{p_r}$, and we associate these regions to p_r. Also clearly no

region R of the partition can be associated to two distinct points, say p_s

and p_t where $s \neq t$.[**]   We can then state the following lemma whose proof

is straightforward.

<u>Lemma 4</u>: Consider the partition of space formed by m arbitrary hyperplanes

in R^n. Then the total number of regions in the partition $N \geq \sum_{r=1}^{k} N(p_r)$

where $p_1, \ldots, p_k$ are the points formed by the partition (the bound on the

right is understood to be zero if there are no points).

We can now easily prove Theorem 1, using the lemmas above.

<u>Proof of Theorem 1</u>:   E(m,n) = the expected number of extreme points for a

randomly selected region of the partition = average of number of extreme

points of regions of the partition $= \sum_{r=1}^{k} C(p_r)/N \leq \sum_{r=1}^{k} 2^n N(p_r) / \sum_{r=1}^{k} N(p_r) = 2^n$. ■

(*) Let a^i be the ith row of A corresponding to a normal of the ith

hyperplane for $i = 1, \ldots, m$. Since the union of the spans of all subsets of

$a^1, \ldots, a^m$ with n-1 vectors or less cannot be all of R^n, there must exist a

vector $a \in R^n$ that is not in the span of any subset of $a^1, \ldots, a^m$ of n-1

vectors or less. Hence the solution set of ax = 0 (representing hyperplane

H) cannot contain the intersection of the solution sets of any n-1 or less

of the equations $a^1 x = 0, \ldots, a^m x = 0$.

(**) Without loss of generality, $H_+^{p_s} \subseteq H_+^{p_t}$ and $H_+^{p_t} \nsubseteq H_+^{p_s}$, so that $p_t \notin H_+^{p_s}$.

But $p_t \in R$ and $R \subseteq H_+^{p}$.   Contradiction.

3. Concluding Remarks

The bound of 2^n for the expected number of extreme points established in Theorem 1 is approached in the limit as the number of constraints increases for absolutely continuous independent distributions F1 and F2 as shown in May and Smith [1982], Adler and Berenguer [1981]. We suspect although we have not shown that all other topological properties also converge in the continuous case to those characteristic of an n-dimensional hypercube. Also since the proofs do not directly exploit the linearity of the varieties partitioning space, the same results can be expected for a wide class of nonlinear programs. The more interesting direction would be a weakening of the symmetry assumption on the inequality directions. Unfortunately, such a move would introduce strong distributional dependence of average topological properties on the particular choice for F1 and F2 as well as on their particular form of dependency.

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