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CAPITAL MARKET EQUILIBRIUM WITH DIVERGENT BORROWING AND LENDING RATES: COMMENTS AND CORRECTIONS

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Introduction

Brennan (1971) showed that under divergent borrowing and lending rates capital market equilibrium will be achieved in such a way that rates of return on risky assets and the portfolio composition of individual investors may be described by linear equations. This paper shows that his proof is invalid because of an unacceptable implicit assumption. It further shows that his explicit assumptions and procedure should have led to conclusions other than those he presented.

Brennan's Proof

In the article, Brennan made certain assumptions about the expected utility function, $EU_i[E(R),\sigma^2(R)]$, of individual i, i=1,...,M:

$$\frac{\partial EU_i}{\partial E(R)} > 0 \ , \tag{1}$$

$$\frac{\partial EU_i}{\partial \sigma^2(R)} < 0 \ . \tag{2}$$

This constitutes the definition of a risk averter.[1] From this, Brennan proceeded to derive the portfolios that individual investors would hold under different borrowing and lending rate conditions. To simplify the symbols carried throughout, he defined a parameter:

$$a_i = -\frac{2\,\dfrac{\partial EU_i}{\partial \sigma^2(R)}}{\dfrac{\partial EU_i}{\partial E(R)}} \ . \tag{3}$$

He then arrived at equation (9) in his article which was

$$X_i = \frac{1}{a_i}(\Omega^{-1}r-[\upsilon_i\ell+(1-\upsilon_i)b]\Omega^{-1}I) \tag{4}$$

where

X_i is the Mx 1 vector of amounts invested in each of the risky securities by investor i;

Ω is the Mx M variance-covariance matrix of rates of returns on risky securities;

I is the Mx 1 unit vector;

υ_i is a parameter which is

$$= \begin{cases} 1, \text{ if the investor is a net lender,} \\ 0, \text{ if the investor is a net borrower,} \\ \text{between 0 and 1, if neither;} \end{cases}$$

ℓ is the lending rate;

b is the borrowing rate ($b \geq \ell$);

r is the Mx 1 vector of expected returns on risky assets.

He interpreted the above equation to mean that every individual investor will hold portfolios consisting of two basic portfolios. The first is the portfolio of risky assets with the minimum variance, the second, the optimal portfolio of risky assets when borrowing and lending rates are both equal to zero. This result was then aggregated to yield the equation determining the rates of return on each risky asset.

$$r = \left(\sum_1^M \frac{1}{a_i}\right)\Omega\sum_1^M X_i + \left(\sum_1^M \frac{1}{a_i}\right)\sum_1^M \frac{1}{a_i}[\upsilon_i\ell+(1-\upsilon_i)b]I \ . \tag{5}$$

On the basis of the preceding equation Brennan concluded that rates of return on risky assets are linear functions of their covariances with the market portfolio.

It should be pointed out that Brennan's interpretations of equations (4) and (5) are correct only with the implicit assumption that a_i is constant. Otherwise, equation (4) could not be the solution for X_i, since the right side of the equation would also contain the unknown X_i's. Likewise, equation (5) would have r terms on the right side of the equation.

Constant a_i[2] may be represented by utility functions of the following types:[3]

$$EU = a + bE(R) + c\sigma^2(R) \ , \tag{6}$$

or

$$EU = K_1 e^{K_2 E(R) + \frac{K_2}{a_i} \sigma^2(R)} \ . \tag{7}$$

These indifference curves are linear in mean-variance space and parabolic in mean-standard deviation space. They describe investors who demand a constant marginal price for risk when risk is measured by the variance of return. On the other hand, they describe investors who demand increasing marginal price for risk when risk is measured by the standard deviation. It is therefore difficult to accept the assumption of constant a_i.

More crucial is the fact that the utility function implied by a constant a_i does not allow preference ordering of the probability distributions of returns consistent in the von Neumann-Morgenstern sense. In the mean-variance space, a preference ordering consistent in the von Neumann-Morgenstern sense requires a utility function U(R) such that the expected utility,

$$EU[E(R),\sigma^2(R)] = \int_{-\infty}^{\infty} U(R)dF[R;E(R),\sigma^2(R)], \tag{8}$$

can be used to rank probability distributions of return. U(R) must be the same for all the probability distributions being ranked. If $EU = a + bE(R) + cE[R-E(R)]^2$, then $U[R,E(R)] = a + bR + c[R-E(R)]^2$. Ranking with this utility function requires that E(R) of each probability distribution being ranked be used in equation (8) to yield the different variances. Each probability distribution will, in effect, be measured by a different utility function. Consistent preference ordering is therefore nonexistent.

If
$$EU = e^{bE(R) + cE[R-E(R)]^2},$$

then
$$U(R) = e^{bE(R) + cE[R-E(R)]^2} \cdot \frac{d}{dF(R)} [bE(R)+c\sigma^2(R)] .$$

This, again, contains E(R) and will vary with the probability distribution being ranked.[4] Again, there will not be a consistent ranking of probability distribution of returns when the utility function is used. Brennan has therefore started with a very general utility function but concluded by proving his case for an unacceptable set of conditions.

This invalidates the proof provided by Brennan, but it does not invalidate the two main conclusions of his paper, namely: (1) individual investor will hold linear combinations of the two basic portfolios; (2) the rate of return on risky assets is a linear function of its covariance with a market portfolio. Black (1972) arrived at results similar to Brennan's when studying variations on the equal lending and borrowing rate assumption. However, he started with the formulation

$$\text{minimize} \quad X_i'\Omega X_i$$
$$\text{subject to} \quad X_i'r = E(R)$$
$$X_i'I = 1 \ .$$

This formulation[5] bypasses the utility function assumption. Results from the above formulation are generally interpreted as applicable to normally distributed returns or quadratic utility functions. The following section will show that if the problem is solved starting with the maximization of expected utility as Brennan did, but assuming the quadratic utility function explicitly, the same two conclusions will not be arrived at. The conclusion that investors will hold linear combinations of two basic portfolios still holds. However, the conclusion that the rates of return on risky assets are linear functions of their respective covariances with the market portfolio will not hold.

Brennan's Procedure with Quadratic Utility Function

As shown by Markowitz,[6] if it is assumed that investors (1) rank probability distributions of returns according to the expected utility maxim, (2) use only expected return and variance as parameters, and (3) make no assumption about the probability distributions, this is equivalent to assuming that investors have quadratic utility functions.[7] The following development will therefore assume that investors have quadratic utility functions:

$$U_i(\tilde{R}) = M_i + n_i\tilde{R} + P_i\tilde{R}^2 \qquad i = 1,\ldots, M \qquad (9)$$
$$n_i > 0$$
$$P_i < 0 \ .$$

A. Net lender

For net lender i, the expected return on his portfolio is

$$E_i(\tilde{R}) = X_i'r + (1-X_i'I)\ell \ , \tag{10}$$

and the variance on the portfolio return is

$$\sigma_i^2(R) = X_i'\Omega X_i \ . \tag{11}$$

His portfolio selection problem is

$$\text{Maximize } EU_i = M_i + n_i E_i(\tilde{R}) + P_i E_i(\tilde{R}^2) \ . \tag{12}$$

The first order conditions are therefore

$$\{n_i+2P_iE_i(\tilde{R})\}(r-\ell I) + 2P_i\Omega X_i = 0 \ . \tag{13}$$

Simplifying and solving for X_i

$$[\frac{n_i}{2P_i}+X_i'r+(1-X_i'I)\ell](r-\ell I) + \Omega X_i = 0 \ , \tag{14}$$

$$[\Omega+(r-\ell I)(r-\ell I)']X_i = -(\frac{n_i}{2P_i}+\ell)(r-\ell I) \ , \tag{15}$$

$$X_i = -(\frac{n_i}{2P_i}+\ell)[\Omega+(r-\ell I)(r-\ell I)']^{-1}(r-\ell I) \ . \tag{16}$$

The net lender therefore holds a portfolio which is a linear combination of two basic portfolios, $[\Omega+(r-\ell I)(r-\ell I)']^{-1}r$ and $[\Omega+(r-\ell I)(r-\ell I)^1]^{-1}I$.[8]

B. Net borrower

For the jth borrower, the expected return on the portfolio is

$$E(\tilde{R}) = X_j'r + (1-X_j'I)b \ , \tag{17}$$

and the variance of the portfolio return is

$$\sigma^2(\tilde{R}) = X_j'\Omega X_j \ . \tag{18}$$

His portfolio selection problem is

$$\text{Maximize } EU_j = M_j + n_j E_j(\tilde{R}) + P_j E_j(\tilde{R}^2) \ . \tag{19}$$

The first order conditions are therefore

$$\{n_j + 2P_j E(\tilde{R})\}(r-bI) + 2P_j \Omega X_j = 0 \tag{20}$$

which yield

$$X_j = -(\frac{n_j}{2P_j} + b)[\Omega + (r-bI)(r-bI)']^{-1}(r-bI) \ . \tag{21}$$

This shows that the net borrowers also hold portfolios which are linear combinations of two basic portfolios.

C. Non-lender, non-borrower

Black (1972) has shown that for non-lender, non-borrowers, the portfolios held are linear combinations of two nonunique basic portfolios. Further, he showed that the basic portfolios may be transformed into any two other portfolios. It is therefore possible to express the portfolios of the non-lender, non-borrowers as linear combinations of the basic lending portfolio and the basic borrowing portfolio.

From this it is still correct to conclude that even with divergent lending and borrowing rates, investors still hold portfolios that are linear combinations of two basic portfolios. However, if one examines the equation for the basic lending portfolios, eq. (15), and the equation for the basic borrowing portfolios, eq. (21), one sees that both have quadratic terms in r. Therefore, r cannot, in general, be a linear function of its covariance with the market portfolio.

Implications

The linearity of the dependence of rates of return on their variances and covariances with the market portfolio has been empirically shown.[9] The results above, therefore, are not a proposal for a non-linear model. Instead, they should be taken as an addition to the list of objections against the use of the quadratic utility function assumption.

Notes

[1]A second order condition is needed to differentiate the diversifier from the plunger.

[2]Constant a_i is not the same as Lintner's constant Pratt-Arrow risk aversion measure (Lintner, 1969).

[3]The second equation is derived by solving

$$-\frac{2\dfrac{\partial EU}{\partial \sigma^2(R)}}{\dfrac{\partial EU}{\partial E(R)}} = a_i = \text{constant as a variable-separable,}$$

partial differential equation with constants as boundary conditions.

[4]Another implication of the above, which is not related to the article, is that consistent ranking by semi-variance cannot be the same as consistent ranking by one-half the variance, unless all the probability distributions being ranked have the same mean. Ranking by semi-variance and expected return may be represented by the following utility function:

$$U(R) = a + bR + c[\min(R-h,0)]^2 .$$

A consistent ranking by semi-variance requires that h be independent of the probability distributions being ranked. Since ranking by one-half the variance using the above utility function requires that h be equal to each of the means of the probability distributions being ranked, it allows consistent ranking only if all the means are equal.

[5]Note that Black's X_i is not the same as Brennan's X_i. Brennan's X_i is dollar amounts invested while Black's is the proportion of the total amount invested. Therefore, Brennan's X_i is equal to Black's X_i multiplied by the total amount invested. The next text section will use Black's X_i for simpler algebra.

[6]H. Markowitz, Portfolio Selection: Efficient Diversification of Investments, pp. 286-88.

[7]The three conditions were explicitly or implicitly assumed in Brennan's article.

[8]Assume that $[\Omega+(r-\ell I)(r-\ell I)']$ and $[\Omega+(r-bI)(r-bI)']$ are positive definite. When lending rate is zero, the first basic portfolio is the single portfolio held by the net lending investor who maximizes the quadratic utility function.

[9]cf. Black, Jensen, and Scholes, 1972.

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