THE UNIVERSITY OF MICHIGAN

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A METHOD OF STIFFNESS COEFFICIENTS FOR THE BEAM ON AN ELASTIC FOUNDATION

BY

Glen V. Berg

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SYNOPSIS

This paper presents a method of stiffness coefficients for solving the problem of the beam on an elastic foundation. It is an approximate method in which the actual applied loads and foundation forces on the beam are replaced by concentrated forces applied at discrete points. The foundation modulus need not be constant, and any end conditions on the beam (free, supported, or fixed) are admissible.

For beams of constant cross section, the method reduces to an attractively simple process, for which the necessary stiffness coefficients are given in the Appendix.

INTRODUCTION

The problem of the beam on an elastic foundation has from time to time been given considerable attention in the technical literature.[1,2,3,4,5,6] The problem is bothersome because the governing differential equation cannot be solved by direct integration, as can the elementary equations of beam flexure; and the equation is of the fourth order, in contrast with the second order equations of elementary buckling problems.

The classical problem of the beam on an elastic foundation can be expressed by the equation

$$EI \frac{d^4y}{dx^4} + ky = q(x) \quad . \tag{1}$$

In Equation (1)

E = Young's modulus of elasticity

I = moment of inertia of the beam

k = foundation modulus

q = load per unit length

x = length coordinate

y = deflection coordinate

Equation (1) implies that the intensity of the force exerted on the beam by the foundation at any point is proportional to the deflection at that point. It also implies that the foundation is capable of exerting either upward or downward forces on the beam. These are the Winkler foundation assumptions.[7] They are not truly representative of a continuous elastic medium, but have been found to be sufficiently accurate for many practical situations. In the classical problem, E, I, and k are constants.

Equation (1) has been solved for special loading conditions on infinite and semi-infinite beams. Many finite beam problems can be solved by superposition of these elementary solutions. The validity of superposition stems from the linearity of the equation. Hetenyi[2] has explored this approach extensively, and has published solutions for a wide range of finite beam problems. He has also pointed out how the classical methods can be employed in certain cases of beams of variable flexural rigidity, and foundations of variable modulus. In these cases, the resulting mathematical expressions are often hopelessly cumbersome.

Approximate methods have been developed for solving the finite beam problem under any given condition of loading. Hetenyi[2] showed in some detail a method of employing Fourier series for this purpose. Levinton[3] presented a numerical method, in which the assumption of a piecewise linear variation of foundation pressure is used to reduce the problem to a set of linear equations. In a discussion of Levinton's paper, Gold[4] presented a numerical process in which the second derivative d^2y/dx^2 in the basic equation of flexure is replaced by a finite difference expression. Popov[5] developed an iterative process in which a deflection curve is first assumed, and the corresponding foundation pressure is then used to compute a new deflection curve. The assumed and computed deflection curves are averaged to get a new approximation. In all of these approximate methods, E, I, and k need not be constants, but may be functions of x. In the Levinton and Gold methods, the presence of end restraints or interior supports would require a change of procedure. In the Popov process, end restraints or interior supports induce no significant complication, and a foundation which is capable of exerting only upward forces is admissible. Divergence of the iteration may be troublesome if the foundation is too stiff.

THE METHOD OF STIFFNESS COEFFICIENTS

In the method of stiffness coefficients, the beam, Figure 1, is divided into N panels, the panel points being numbered 0, 1, 2, ... N. The panel lengths are usually equal, but not necessarily so. The actual applied loads and the unknown foundation forces are replaced by concentrated loads and foundation forces at the panel points, as in Figure 2. The applied loads at the panel points are denoted q_0, q_1, ... q_N, and the foundation forces are denoted f_0, f_1, ... f_N. For the present, the problem in which moments are applied to the beam will not be considered; it is covered later in the paper.

The panel loads and panel foundation forces are found by treating the actual loads and distributed foundation force as though they were applied to the beam through sets of stringers spanning the beam panels, as shown in Figure 3. Each panel load or panel foundation force is the sum of the reactions from the adjacent stringers. For the applied loads, this is a standard procedure encountered in many truss problems. For the foundation panel force there is a complication due to the fact that the actual distribution of the foundation force is unknown. But the foundation force intensity at each panel point is the product of the foundation modulus and the unknown deflection at that point. One can obtain a close

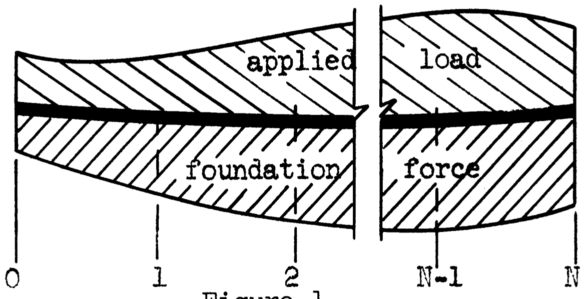

Figure 1

Actual Loading

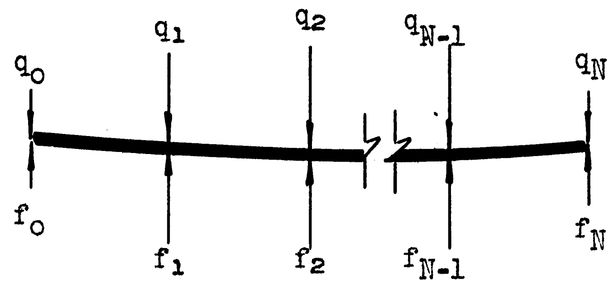

Figure 2

Panel Loading

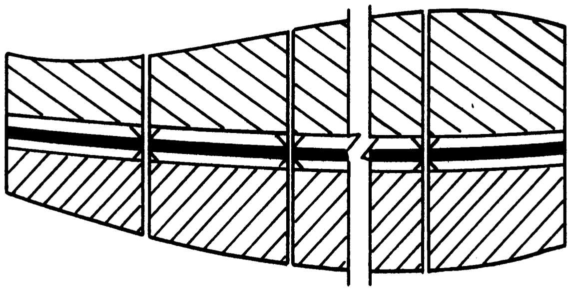

Figure 3

Loading Through Stringers

approximation to each panel force by a method due to Newmark,[8] namely, assuming that the foundation force distribution is sectionally parabolic over the two adjacent panels. Let the foundation modulus at the panel points have the values k_0, k_1, ... k_N. It is assumed that the foundation modulus is continuous (but not necessarily constant) over the length of the beam. Let the unknown deflections at the panel points be y_0, y_1, ... y_N. The deflections are best expressed in terms of the dimensionless parameters z_0, z_1, ... z_N, defined by

$$z_i = \frac{y_i}{\lambda} \tag{2}$$

where λ is the length of one panel. If the panel lengths are not equal, any panel may be arbitrarily chosen as the "basic" panel, and the length of that panel appears in Equation (2) for all values of i.

If the panel lengths are equal, of length λ, and if a parabolic distribution of foundation force intensity over the two adjacent panels is assumed, the foundation panel force at an interior panel point m (Figure 4) is

$$f_m = \frac{\lambda^2}{12} (k_{m-1}z_{m-1} + 10\ k_m z_m + k_{m+1}z_{m+1}) \quad . \tag{3}$$

Assuming parabolic distribution over the first two panels, the panel force at end panel point 0 (Figure 5) is

$$f_o = \frac{\lambda^2}{24} (7\ k_o z_o + 6\ k_1 z_1 - k_2 z_2) \quad . \tag{4}$$

Similarly, the force at end panel point N is

$$f_N = \frac{\lambda^2}{24} (-k_{N-2}z_{N-2} + 6\ k_{N-1}z_{N-1} + 7\ k_N z_N) \quad . \tag{5}$$

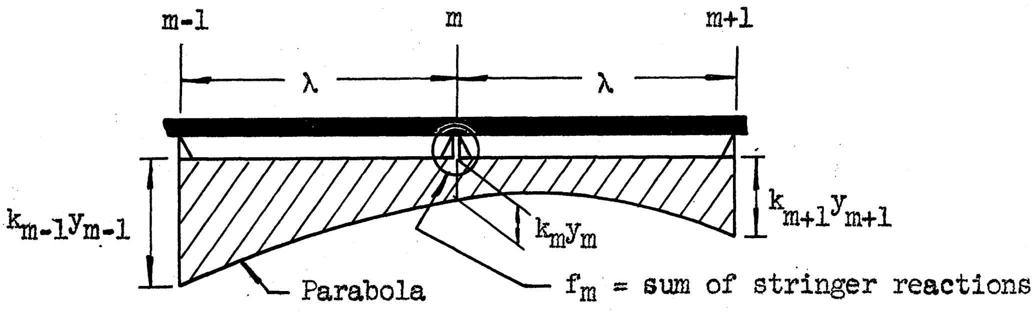

Figure 4

Foundation Force at Interior Panel Point

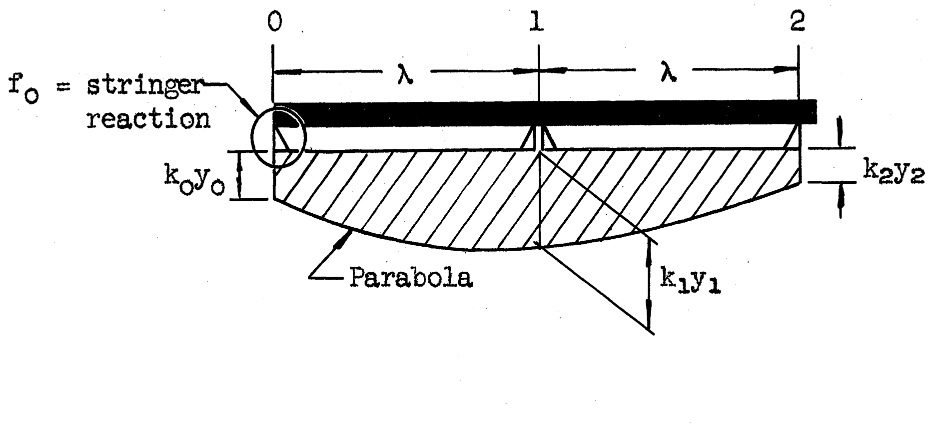

Figure 5

Foundation Force at End Panel Point

If the panel lengths were unequal, or if the foundation modulus were discontinuous, equations would have to be derived to fit the particular conditions. These cases will not be discussed here.

The assumption of a piecewise linear variation of foundation force could also be used satisfactorily, and might be advantageous if the panel lengths were unequal. However, if the panel lengths are equal, the number of panels required to achieve a given order of accuracy will generally be less with the parabolic assumption than with the linear assumption. To minimize computation, it is desirable to keep the number of panels small, as will be seen.

Let the subscripts i and j denote panel points of the beam. One can then put the complete set of equations for these panel forces in the form

$$f_i = \sum_{j=0}^{N} c_{ij} z_j \tag{6}$$

where i can take on any value from 0 to N. The coefficients c_{ij} are easily determined from Equations (3), (4), and (5). For example (if $N \geq 3$)

$$c_{00} = \frac{7\,k_0\lambda^2}{24}, \qquad c_{01} = \frac{6\,k_1\lambda^2}{24}, \qquad c_{02} = \frac{-k_2\lambda^2}{24},$$

$$c_{10} = \frac{k_0\lambda^2}{12}, \qquad c_{11} = \frac{10\,k_1\lambda^2}{12}, \qquad c_{12} = \frac{k_2\lambda^2}{12},$$

$$c_{20} = 0, \qquad c_{21} = \frac{k_1\lambda^2}{12}, \qquad c_{22} = \frac{10\,k_2\lambda^2}{12}, \text{ etc.}$$

The array of coefficients

$$[C] = \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0N} \\ c_{10} & c_{11} & \cdots & c_{1N} \\ \cdots & \cdots & \cdots & \\ c_{N0} & c_{N1} & \cdots & c_{NN} \end{bmatrix} \tag{7}$$

will be called the foundation matrix.

Let the stiffness coefficient a_{ij} be the vertical force at panel point i, corresponding to a configuration in which the beam has unit deflection at panel point j, ($z_j = 1$) and zero deflection at all other panel points, with foundation forces absent. Both subscripts i and j can assume all values from 0 to N. This is illustrated in Figure 6. The large deflections shown are only symbolic. Linear flexural relations are used, and these are valid only if slopes and curvatures remain small. Deflections and forces are considered positive downward. Likewise the coefficient a_{ij} is positive when it represents a downward force. The stiffness coefficients depend only upon the flexural properties of the beam and the end conditions (free, supported, fixed); they are entirely independent of the foundation.

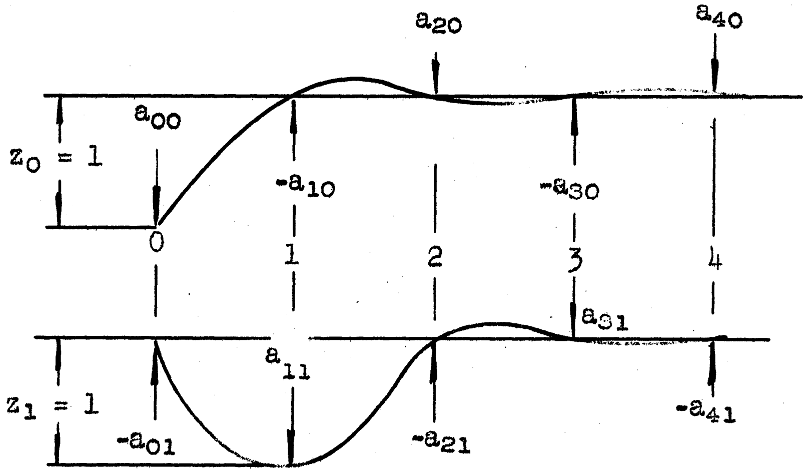

Figure 6

Stiffness Coefficients

Given any configuration of the beam, defined by the panel point deflections z_o, z_1, ... z_N, one can use the stiffness coefficients and the principle of superposition to write a set of equations for the panel forces in terms of the panel point deflections, namely:

$$p_i = a_{io}z_o + a_{i1}z_1 + \dots + a_{iN}z_N \tag{8}$$

or

$$p_i = \sum_{j=0}^{N} a_{ij} z_j \quad , \tag{9}$$

where i can assume any value from 0 to N. The forces p_i are the forces that would be necessary to hold the beam in the given configuration if the foundation forces were not acting.

The array of stiffness coefficients

$$[A] = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0N} \\ a_{10} & a_{11} & \cdots & a_{1N} \\ \cdots & \cdots & \cdots & \\ a_{N0} & a_{N1} & \cdots & a_{NN} \end{bmatrix} \tag{10}$$

is called the stiffness matrix of the beam. By virtue of Maxwell's principle, the matrix is symmetric; that is, $a_{ij} = a_{ji}$ for all values of i and j. The usual concept of a flexibility matrix, the inverse of the stiffness matrix, is not applicable in this case. The determinant of the coefficients is necessarily zero, since the beam without the foundation is unstable. The inverse of the stiffness matrix, therefore, does not exist.

For every panel point i, equilibrium requires that the applied panel load be equal to the panel force required to hold the beam and the foundation in the deflected configuration. In other words,

$$q_i = p_i + f_i \qquad (i = 0, 1, 2, \ldots N) \quad . \tag{11}$$

Combining Equations (6), (9), and (11), one gets

$$q_i = \sum_{j=0}^{N} a_{ij} z_j + \sum_{j=0}^{N} c_{ij} z_j \; . \tag{12}$$

Let

$$b_{ij} = a_{ij} + c_{ij} \tag{13}$$

for all values of i and j.

The equation for the deflection of the system can then be written:

$$q_i = \sum_{j=0}^{N} b_{ij} z_j \qquad (i = 0, 1, 2, \ldots N) \quad . \qquad (14)$$

If the beam with its foundation is stable, there is a unique solution to Equation (14). For small N, say $N \leq 5$, the equation can be solved, either by a direct process or by iteration, with the aid of a desk calculator. If $N > 5$, solution by desk calculator becomes excessively laborious and more efficient means must be sought. The system is well adapted to solution on a high speed digital computer, and can be solved for large systems, say up to $N = 30$, in a matter of a few minutes.

THE EFFECT OF END RESTRAINTS, INTERIOR SUPPORTS, AND APPLIED MOMENTS

If the beam is simply supported at the ends, the above method of solution is still applicable, with the simplification that z_0 and z_N are both zero, and the order of the system of equations is therefore reduced by two. The coefficients a_{ij} and c_{ij} are exactly the same as before, except that the coefficients for $i = 0$, $i = N$, $j = 0$, and $j = N$ are not needed. Equations (12) and (14) remain the same as before, except that the summations run from 1 to N-1, instead of from 0 to N. With its first and last rows and columns eliminated, the matrix [A] has a positive determinant, because in this case it represents the stiffness matrix of a beam which is stable even if the foundation forces are missing.

If only one end of the beam is simply supported, say the left end, one then has $z_0 = 0$, and the coefficients for $i = 0$ and $j = 0$ are not needed. In this case, the summations in Equations (12) and (14) run from 1 to N.

If there are interior supports, the panel divisions must be chosen so that a panel point occurs at each interior support. The method of solution then remains as above, except that the deflection at each interior support is zero, and the corresponding coefficients will not appear in the equations.

If the beam has one or both ends fixed, the method remains the same, but the stiffness coefficients a_{ij} must take into account the fixed-end condition and will differ from the coefficients for a free-free beam.

If the problem involves moments applied to the beam, whether at the ends or at interior points, the above procedure requires modification. First, the panel divisions are chosen so that all applied moments occur at panel points. This may demand that the panel lengths be unequal. Then additional variables are introduced into the equations, namely, the slopes of the elastic curve at the points of applied moment. Moments are taken as positive when they tend to induce clockwise rotation, and slopes are taken as positive when they correspond to clockwise rotation of the tangent to the elastic curve. Stiffness coefficients corresponding to the additional variables must also be introduced. With these modifications, the procedure is comparable to that considered earlier. This is illustrated in Example 2, below.

COMPUTING THE STIFFNESS COEFFICIENTS

Evaluating the stiffness coefficients is an elementary, but sometimes laborious, continuous beam problem. For hand computation, the method of moment distribution is suitable. If N is large, computing the stiffness coefficients by hand or by desk calculator is tedious, and a high speed computer is desirable. If each panel of the beam is of constant cross section, the three moment equation is a convenient tool for machine computation, since it enables one to write sets of linear equations for the bending moments at the panel points in terms of the appropriate unit deflections at the panel points. Solving such sets of linear equations is a basic problem in high speed computation.

The Appendix gives stiffness matrices for beams of constant cross section and panels of equal length. The cases given are beams of 2, 3, 4, and 6 panels for the following end conditions:

(a) both ends free;

(b) left end restrained, right end free;

(c) both ends restrained.

The matrices for some other conditions of interest, such as simple supports at one or both ends, with or without applied end moments, or fixed end supports at one or both ends, are obtained from the above matrices simply by deleting appropriate rows and columns of coefficients.

EXAMPLES

<u>Example 1</u>. A 10" WF x 60 lb free-free beam 16'-0" long, shown in Figure 7, supports concentrated vertical loads of 40 kips at the left end and 50 kips at the right end. The beam rests on an elastic foundation having a modulus of 3,000 lbs. per sq. in.; that is, 3,000 lbs. per inch of deflection per inch length of beam. Find the deflection curve and the bending moment at the center.

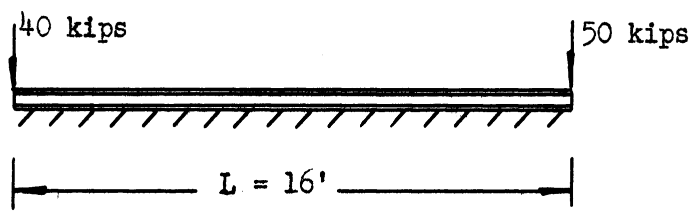

Figure 7

Beam for Example 1.

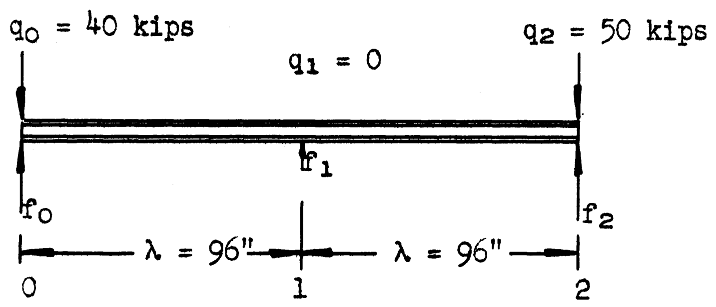

Figure 8

Replacement System for Example 1.

Solution. Divide the beam into two equal panels, as shown in Figure 8. By the definition of stiffness coefficients given previously, the coefficients are the forces shown in Figure A1 of the Appendix. Elementary deflection formulas give the coefficients:

$$[A] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \frac{3EI}{2\lambda^2} \qquad (15)$$

Although Figure A1 pictures large deflections, it should not be inferred that large deflections are admissible. The use of linear theory requires that deflections remain small.

The foundation force on the beam is treated as though it were applied through a pair of stringers, as shown in Figure 3. Using the assumption of parabolic distribution, the foundation coefficients are found from Equations (3), (4), and (5).

They are,

$$[C] = \begin{bmatrix} 7 & 6 & -1 \\ 2 & 20 & 2 \\ -1 & 6 & 7 \end{bmatrix} \frac{k\lambda^2}{24} \quad . \tag{16}$$

Evaluating the above coefficients, and taking E = 30,000,000 psi, one gets

$$[A] = \begin{bmatrix} 1,678.2 & -3,356.4 & 1,678.2 \\ -3,356.4 & 6,712.9 & -3,356.4 \\ 1,678.2 & -3,356.4 & 1,678.2 \end{bmatrix} \text{kips} \ , \tag{17}$$

and

$$[C] = \begin{bmatrix} 8,064 & 6,912 & -1,152 \\ 2,304 & 23,040 & 2,304 \\ -1,152 & 6,912 & 8,064 \end{bmatrix} \text{kips} \ , \tag{18}$$

Equations (14), written out in full, are

$$\begin{aligned} 9,742.2\, z_o + 3,555.6\, z_1 + 526.2\, z_2 &= 40 \\ -1,052.4\, z_o + 29,752.9\, z_1 - 1,052.4\, z_2 &= 0 \\ 526.2\, z_o + 3,555.6\, z_1 + 9,742.2\, z_2 &= 50 \end{aligned} \tag{19}$$

The solution is

$$\begin{aligned} z_o &= .003735 \\ z_1 &= .000303 \\ z_2 &= .004820 \end{aligned} \tag{20}$$

or

$$\begin{aligned} y_o &= .359 \text{ in.} \\ y_1 &= .029 \text{ in.} \\ y_2 &= .463 \text{ in.} \end{aligned} \tag{21}$$

One can then use Equations (6) and (18) to find the panel foundation forces on the beam, obtaining,

$$f_o = 26.66 \text{ kips}$$

$$f_1 = 26.68 \text{ kips} \qquad (22)$$

$$f_2 = 36.66 \text{ kips}$$

The bending moment at the middle is then

$$M_1 = -1{,}281 \text{ inch-kips}. \qquad (23)$$

The solution can be improved by taking finer subdivisions. Table I indicates how the solution converges toward the exact solution as N is increased.

TABLE I

SOLUTION OF EXAMPLE 1 FOR DIFFERENT VALUES OF N

N	Deflection, inches			Moment at Midpoint
	Left end	Midpoint	Right end	inch-kips
2	.359	.029	.463	-1281
3	.384	--	.494	--
4	.387	.002	.503	-1120
Exact	.397	-.001	.513	-1086

Example 2. A 10" concrete slab 20 ft. long, shown in Figure 9, is simply supported at the left end and rests on an elastic foundation having a modulus of 83.3 lb. in.$^{-3}$. The slab is loaded with a total uniform load of 4,000 lbs. per sq. ft. and a moment of 20 ft.-kips per foot width at the left end, as shown. Find the deflection curve and the left end reaction.

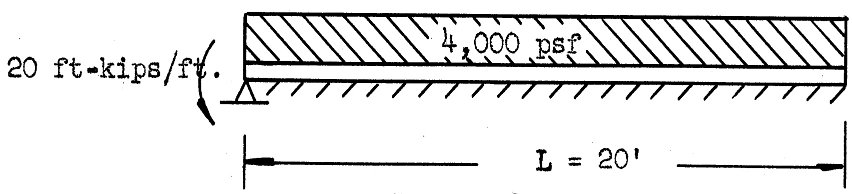

Figure 9
Slab for Example 2.

Solution. A one-foot width of slab is analyzed as a beam, and is divided into two equal panels. The moment applied at the left end is taken as a load parameter, defined as

$$q_{-1} = \frac{M}{\lambda} \quad . \tag{24}$$

The slope of the beam at the left end, measured in radians, is defined by the deflection parameter z_{-1}. These parameters are dimensionally consistent with the rest, and the deflection equation can be written

$$q_i = \sum_{j=-1}^{2} a_{ij} z_j + \sum_{j=-1}^{2} c_{ij} z_j \quad (i = -1, 0, 1, 2) \quad . \tag{25}$$

The stiffness coefficients are the forces and moments shown in Figure A2 of the Appendix, for the restrained-free beam. The stiffness matrix, from the Appendix, is

$$[A] = \begin{bmatrix} 4 & 5 & -6 & 1 \\ 5 & 8 & -11 & 3 \\ -6 & -11 & 16 & -5 \\ 1 & 3 & -5 & 2 \end{bmatrix} \frac{6EI}{7\lambda^2} \tag{26}$$

The foundation coefficients are again computed from Equations (3), (4), and (5). They are

$$[C] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 7 & 6 & -1 \\ 0 & 2 & 20 & 2 \\ 0 & -1 & 6 & 7 \end{bmatrix} \frac{k\lambda^2}{24} \tag{27}$$

Inserting numerical values, and taking the modulus of elasticity of concrete to be 3,000,000 psi, one gets

$$
[A] = \begin{bmatrix} 714.3 & 892.9 & -1,071.4 & 178.6 \\ 892.9 & 1,428.6 & -1,964.3 & 535.7 \\ -1,071.4 & -1,964.3 & 2,857.1 & -892.9 \\ 178.6 & 535.7 & -892.9 & 357.1 \end{bmatrix} \text{ kips} \qquad (28)
$$

and

$$
[C] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4,200 & 3,600 & -600 \\ 0 & 1,200 & 12,000 & 1,200 \\ 0 & -600 & 3,600 & 4,200 \end{bmatrix} \text{ kips} \qquad (29)
$$

Because the left end is simply supported, $z_o = 0$, and Equations (14) can be written out as follows:

$$
\begin{aligned}
714.3\, z_{-1} - 1,071.4\, z_1 + 178.6\, z_2 &= -2 \\
892.9\, z_{-1} + 1,635.7\, z_1 - 64.3\, z_2 &= q_o \\
-1,071.4\, z_{-1} + 14,857.1\, z_1 + 307.1\, z_2 &= 40 \\
178.6\, z_{-1} + 2,707.1\, z_1 + 4,557.1\, z_2 &= 20 \ .
\end{aligned} \qquad (30)
$$

The solution is

$$
\begin{aligned}
z_{-1} &= .00051 \\
z_1 &= .00267 \\
z_2 &= .00278 \\
q_o &= 4.65 \text{ kips}
\end{aligned} \qquad (31)
$$

from which

$$
\begin{aligned}
y_1 &= 0.321 \text{ in.} \\
y_2 &= 0.334 \text{ in.}
\end{aligned} \qquad (32)
$$

The parameter q_o represents the panel force at the left end due to the applied load and the reaction exerted by the support. The panel force is 20 kips, and the reaction is therefore

$$R_o = 20 - q_o = 15.35 \text{ kips} . \qquad (33)$$

Increasing the number of panels improves the accuracy of the solution. Table II shows the results for various values of N.

TABLE II

SOLUTION OF EXAMPLE 2 FOR VARIOUS VALUES OF N

N	Left end reaction, kips	Center deflection, inches	Right end deflection, inches
2	15.35	.321	.334
3	14.36	-	.327
4	14.16	.3380	.3342
6	14.04	.3378	.3366
8	13.99	.3376	.3370
Exact	13.88	.3372	.3375

If, in either of the above examples, the foundation modulus varied along the length of the beam, certain obvious modifications would be necessary in computing the foundation coefficients, as indicated by Equations (3), (4), and (5). In such cases, "exact" solutions are rarely feasible.

CHOICE OF PANEL LENGTH

The choice of panel length, λ, determines both the accuracy that can be obtained and the amount of effort necessary to obtain it. If the solution is to be carried out by hand, the time required to solve the equations will vary approximately as the cube of the number of equations. It is therefore desirable to keep the number of panels as small as is consistent with the accuracy needed in the solution and the accuracy with which the foundation modulus is known. If the solution is carried out by an automatic digital computer, the amount of input data required may vary approximately as the square of the number of equations, if a standard matrix routine is used for the solution; or it may even be independent of the number of equations, if the entire problem is programmed for automatic computation. In any event, if the panel length is chosen small enough so that the accuracy of the solution is consistent with that of the input parameters, there is nothing to be gained by further refinement.

The diagonal stiffness coefficients (a_{00}, a_{11}, a_{22}, ... a_{NN}) increase in magnitude as N increases. On the other hand, the diagonal foundation coefficients decrease with increasing N. The result is that as N is increased, the equations become less well conditioned for solution. This means that if they are solved by iteration, the rate of convergence becomes slower as N increases, and if they are solved by elimination, error accumulation becomes more important as N increases. This merely emphasizes the desirability of keeping N small.

The proper panel length is related both to the loading and to the beam stiffness. The process is based on the assumption that the intensity of the foundation force varies approximately parabolically over each pair of adjacent panels, and that the actual loads can be replaced by "equivalent" concentrated loads at the panel points. As far as the foundation is concerned, an estimate of the proper panel length can be gained from a consideration of the analytical solution of Equation (1). The solution[1] involves both hyperbolic and circular functions of an argument βx, where

$$\beta = \sqrt[4]{\frac{k}{4EI}} \tag{34}$$

The characteristic "wave length" is then

$$\frac{2\pi}{\beta} = 2\pi \sqrt[4]{\frac{4EI}{k}} \ . \tag{35}$$

If the panel length does not exceed 1/6 of this wave length, the assumption of sectionally parabolic variation of foundation force is reasonable. Therefore, it is suggested that the panel length be chosen such that

$$\lambda \approx \frac{1}{\beta} = \sqrt[4]{\frac{4EI}{k}} \tag{36}$$

In the two examples, the beam length corresponds approximately to a half wave length, and the suggested criterion would indicate that N = 3 would be appropriate. Tables I and II indicate that N = 3 is indeed appropriate for these examples. It must be borne in mind that the character of the applied load may require a smaller panel length.

In many practical problems, the length of the beam is much less than the half wave length of the above examples. For such problems, N = 2 will usually give a reasonable answer.

Stresses in a beam or slab will generally not be obtained to the same degree of accuracy as deflections. This follows from the fact that flexural stress is proportional to d^2y/dx^2. However, local variations in the foundation modulus, or any deviation from the assumed value, will also generally have a greater effect on stresses than on deflections. Therefore, one should not conclude that greater refinement is always warranted when stress is the parameter of interest.

SUMMARY

The method herein presented combines the use of the stiffness matrix[9,10] with a scheme of numerical integration[8] in a new process to reduce the problem of a beam on an elastic foundation to a set of linear algebraic equations.

If it be fitting to classify methods as "theoretical" and "practical," this method falls into both classes. It is "theoretical" in that by this method it is possible (but not necessarily practical) to solve any given problem of a finite beam on an elastic foundation to any desired degree of accuracy. The method is "practical" in that it enables many, perhaps even most, of the problems encountered in practice to be solved rapidly by elementary algebra with a desk calculator and a set of stiffness coefficients. For uniform beams, the necessary stiffness coefficients are given in the Appendix. In many cases, a set of three or four equations will yield a sufficiently accurate solution.

NOTATION

Symbols are defined where they first appear in the text. Those used frequently are listed here for reference.

[A]	Beam stiffness matrix
[C]	Foundation matrix
L	Length of beam
N	Number of panels
a_{ij}	Beam stiffness coefficient
b_{ij}	Combined beam stiffness and foundation coefficient
c_{ij}	Foundation coefficient
f_i	Foundation panel force
i,j	Subscripts denoting panel point
k	Foundation modulus
k_i	Foundation modulus at i-th panel point
p_i	Panel force corresponding to beam stiffness
q	Applied load per unit length
q_i	Panel force due to applied loads
y_i	Deflection at i-th panel point
z_i	Dimensionless deflection parameter
λ	Panel length

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APPENDIX

STIFFNESS MATRICES

This Appendix gives stiffness coefficients a_{ij} for uniform beams divided into N equal panels of length λ, and for end conditions as indicated.

A. Free-free end conditions. Coefficients are tabulated as

$$[A] = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0N} \\ a_{10} & a_{11} & \cdots & a_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N0} & a_{N1} & \cdots & a_{NN} \end{bmatrix}$$

1. Two panels, $N = 2$, $\lambda = L/2$

$$[A] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \frac{3EI}{2\lambda^2}$$

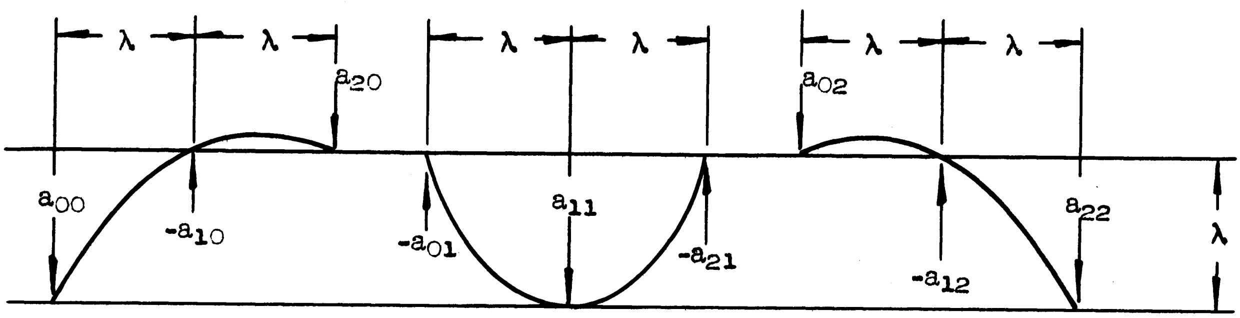

Figure A1

Stiffness Coefficients; Two-Panel Free-Free Beam

2. Three panels, $N = 3$, $\lambda = L/3$

$$[A] = \begin{bmatrix} 4 & -9 & 6 & -1 \\ -9 & 24 & -21 & 6 \\ 6 & -21 & 24 & -9 \\ -1 & 6 & -9 & 4 \end{bmatrix} \frac{2EI}{5\lambda^2}$$

3. Four panels, $N = 4$, $\lambda = L/4$

$$[A] = \begin{bmatrix} 15 & -34 & 24 & -6 & 1 \\ -34 & 92 & -88 & 36 & -6 \\ 24 & -88 & 128 & -88 & 24 \\ -6 & 36 & -88 & 92 & -34 \\ 1 & -6 & 24 & -34 & 15 \end{bmatrix} \frac{3EI}{28\lambda^2}$$

4. Six panels, $N = 6$, $\lambda = L/6$

$$[A] = \begin{bmatrix} 209 & -474 & 336 & -90 & 24 & -6 & 1 \\ -474 & 1284 & -1236 & 540 & -144 & 36 & -6 \\ 336 & -1236 & 1824 & -1380 & 576 & -144 & 24 \\ -90 & 540 & -1380 & 1860 & -1380 & 540 & -90 \\ 24 & -144 & 576 & -1380 & 1824 & -1236 & 336 \\ -6 & 36 & -144 & 540 & -1236 & 1284 & -474 \\ 1 & -6 & 24 & -90 & 336 & -474 & 209 \end{bmatrix} \frac{EI}{130\lambda^2}$$

B. Restrained-free end conditions. Coefficient are tabulated as

$$[A] = \begin{bmatrix} a_{-1,-1} & a_{-1,o} & \cdots & a_{-1,N} \\ a_{o,-1} & a_{oo} & \cdots & a_{oN} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N,-1} & a_{No} & \cdots & a_{NN} \end{bmatrix}$$

1. Two panels, $N = 2$, $\lambda = L/2$

$$[A] = \begin{bmatrix} 4 & 5 & -6 & 1 \\ 5 & 8 & -11 & 3 \\ -6 & -11 & 16 & -5 \\ 1 & 3 & -5 & 2 \end{bmatrix} \frac{6EI}{7\lambda^2}$$

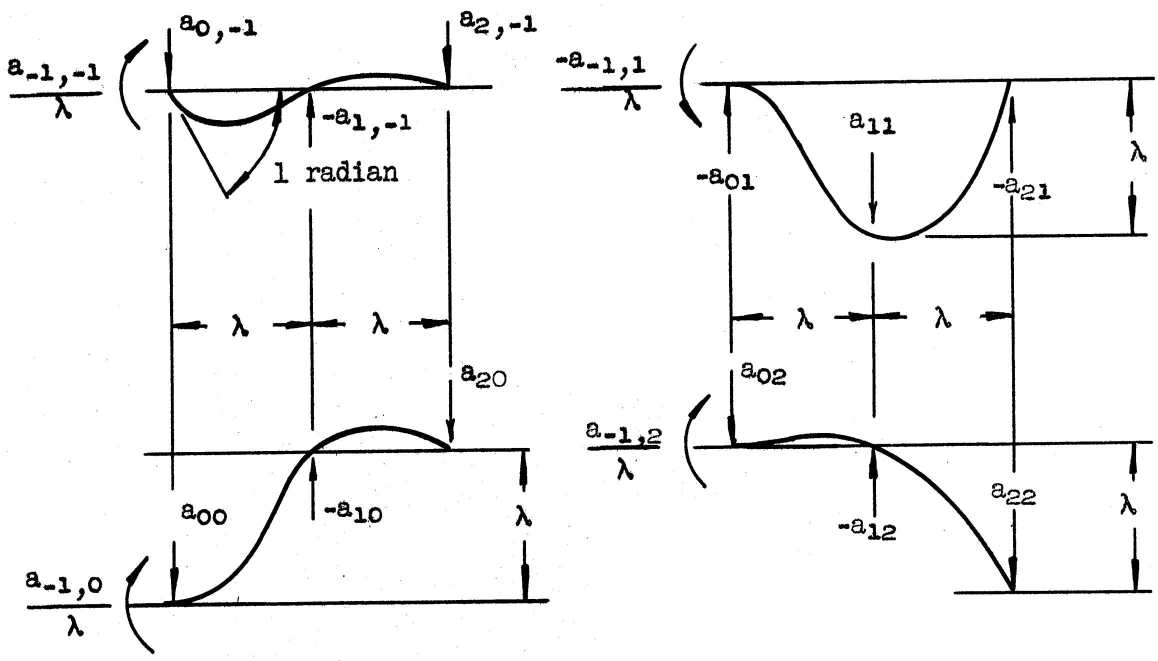

Figure A2

Stiffness Coefficients; Two-Panel Restrained-Free Beam

2. Three panels, $N = 3$, $\lambda = L/3$

$$[A] = \begin{bmatrix} 15 & 19 & -24 & 6 & -1 \\ 19 & 31 & -46 & 18 & -3 \\ -24 & -46 & 80 & -46 & 12 \\ 6 & 18 & -46 & 44 & -16 \\ -1 & -3 & 12 & -16 & 7 \end{bmatrix} \frac{3EI}{13\lambda^2}$$

3. Four panels, $N = 4$, $\lambda = L/4$

$$[A] = \begin{bmatrix} 56 & 71 & -90 & 24 & -6 & 1 \\ 71 & 116 & -173 & 72 & -18 & 3 \\ -90 & -173 & 304 & -191 & 72 & -12 \\ 24 & 72 & -191 & 232 & -155 & 42 \\ -6 & -18 & 72 & -155 & 160 & -59 \\ 1 & 3 & -12 & 42 & -59 & 26 \end{bmatrix} \frac{6EI}{97\lambda^2}$$

4. Six panels, $N = 6$, $\lambda = L/6$

$$[A] = \begin{bmatrix} 780 & 989 & -1254 & 336 & -90 & 24 & -6 & 1 \\ 989 & 1616 & -2411 & 1008 & -270 & 72 & -18 & 3 \\ -1254 & -2411 & 4240 & -2681 & 1080 & -288 & 72 & -12 \\ 336 & 1008 & -2681 & 3304 & -2429 & 1008 & -252 & 42 \\ -90 & -270 & 1080 & -2429 & 3232 & -2393 & 936 & -156 \\ 24 & 72 & -288 & 1008 & -2393 & 3160 & -2141 & 582 \\ -6 & -18 & 72 & -252 & 936 & -2141 & 2224 & -821 \\ 1 & 3 & -12 & 42 & -156 & 582 & -821 & 362 \end{bmatrix} \frac{6EI}{1351\lambda^2}$$

C. Restrained-restrained end conditions. Coefficients are tabulated as

$$[A] = \begin{bmatrix} a_{-1,-1} & a_{-1,0} & \cdots & a_{-1,N+1} \\ a_{0,-1} & a_{00} & \cdots & a_{0,N+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N+1,-1} & a_{N+1,0} & \cdots & a_{N+1,N+1} \end{bmatrix}$$

1. Two panels, $N = 2$, $\lambda = L/2$

$$[A] = \begin{bmatrix} 7 & 9 & -12 & 3 & -1 \\ 9 & 15 & -24 & 9 & -3 \\ -12 & -24 & 48 & -24 & 12 \\ 3 & 9 & -24 & 15 & -9 \\ -1 & -3 & 12 & -9 & 7 \end{bmatrix} \frac{EI}{2\lambda^2}$$

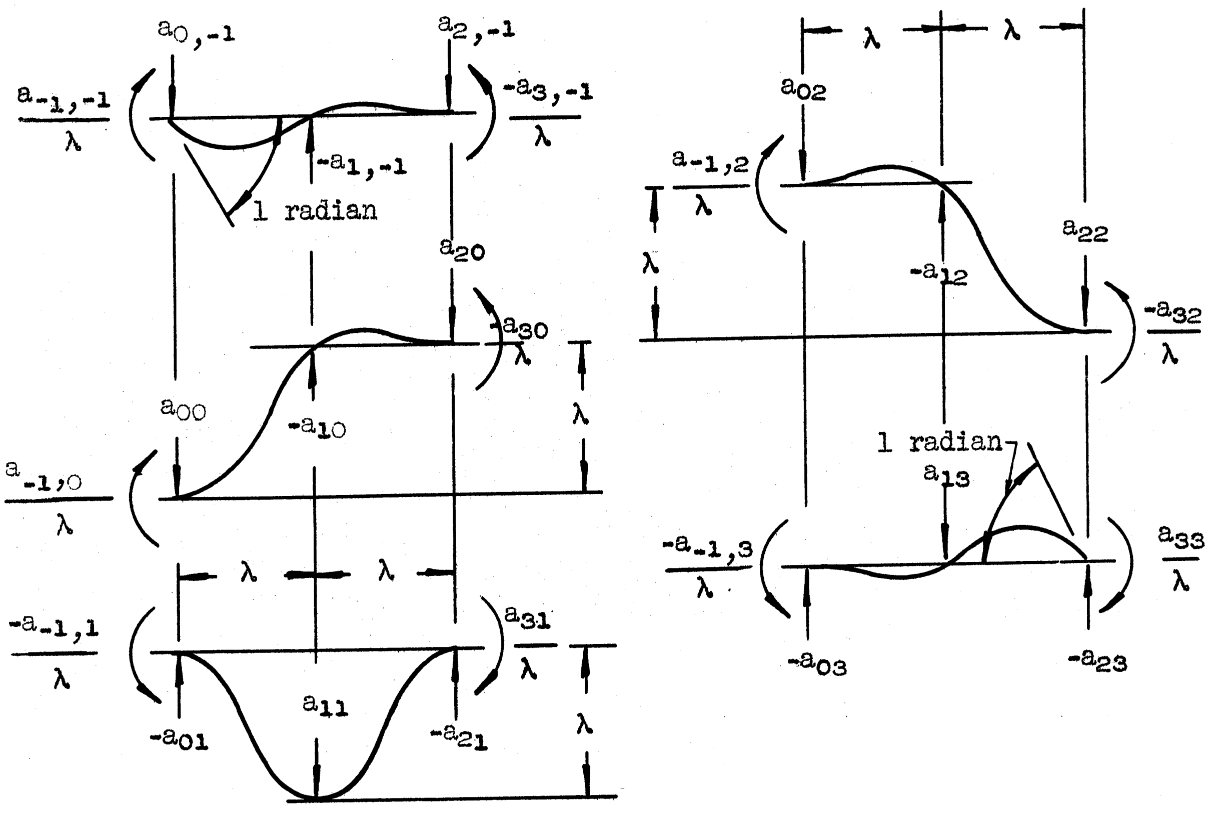

Figure A3

Stiffness Coefficients; Two-Panel Restrained-Restrained Beam

2. Three panels, $N = 3$, $\lambda = L/3$

$$[A] = \begin{bmatrix} 26 & 33 & -42 & 12 & -3 & 1 \\ 33 & 54 & -81 & 36 & -9 & 3 \\ -42 & -81 & 144 & -99 & 36 & -12 \\ 12 & 36 & -99 & 144 & -81 & 42 \\ -3 & -9 & 36 & -81 & 54 & -33 \\ 1 & 3 & -12 & 42 & -33 & 26 \end{bmatrix} \frac{2EI}{15\lambda^2}$$

3. Four panels, $N = 4$, $\lambda = L/4$

$$[A] = \begin{bmatrix} 97 & 123 & -156 & 42 & -12 & 3 & -1 \\ 123 & 201 & -300 & 126 & -36 & 9 & -3 \\ -156 & -300 & 528 & -336 & 144 & -36 & 12 \\ 42 & 126 & -336 & 420 & -336 & 126 & -42 \\ -12 & -36 & 144 & -336 & 528 & -300 & 156 \\ 3 & 9 & -36 & 126 & -300 & 201 & -123 \\ -1 & -3 & 12 & -42 & 156 & -123 & 97 \end{bmatrix} \frac{EI}{28\lambda^2}$$

4. Six panels, $N = 6$, $\lambda = L/6$

$$[A] = \begin{bmatrix} 1351 & 1713 & -2172 & 582 & -156 & 42 & -12 & 3 & -1 \\ 1713 & 2799 & -4176 & 1746 & -468 & 126 & -36 & 9 & -3 \\ -2172 & -4176 & 7344 & -4644 & 1872 & -504 & 144 & -36 & 12 \\ 582 & 1746 & -4644 & 5724 & -4212 & 1764 & -504 & 126 & -42 \\ -156 & -468 & 1872 & -4212 & 5616 & -4212 & 1872 & -468 & 156 \\ 42 & 126 & -504 & 1764 & -4212 & 5724 & -4644 & 1746 & -582 \\ -12 & -36 & 144 & -504 & 1872 & -4644 & 7344 & -4176 & 2172 \\ 3 & 9 & -36 & 126 & -468 & 1746 & -4176 & 2799 & -1713 \\ -1 & -3 & 12 & -42 & 156 & -582 & 2172 & -1713 & 1351 \end{bmatrix} \frac{EI}{390\lambda^2}$$