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THE DEMAND AND SUPPLY OF
FORWARD EXCHANGE CONTRACTS

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Abstract

This paper derives a complete set of demand and supply functions for forward exchange contracts in a multiperiod economy under uncertainty. Investors can invest in domestic and foreign risky securities; they consume foreign as well as domestic goods which have stochastic prices. Because the aggregate exposure to exchange risk may be different from zero and because investors may have heterogenous expectations, they take speculative positions. The central theme of the paper is an analysis of the role of forward exchange contracts in the optimal allocation of exchange risk.

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1. Introduction

Forward exchange contracts are important financial instruments in the sense they do not represent claims on real assets; they are created by investors to allocate exchange risk in an efficient manner. As shown elsewhere [Breeden (1984)], when there are "enough" financial securities, all individual risks can be diversified and each investor will bear only some portion of aggregate risk, resulting in a Pareto-optimal allocation of risk.

The purpose of this paper is to derive explicit demand and supply equations for forward exchange contracts using a continuous-time, investment-consumption decision approach. Investors have access to both domestic and foreign risky assets and they buy domestic as well as foreign consumption goods. Demand and supply functions for forward exchange contracts are derived by assuming that each investor maximizes his expected utility of lifetime consumption, recognizing the fact that investors' optimal positions in risky assets (including forward exchange contracts) must clear securities markets.

Forward exchange contracts, similar to all financial securities, are inside assets.[1] This means, if an individual takes a long position in the forward exchange market, another investor must simultaneously take a short position in the same market. Furthermore, contrary to real assets, returns on forward exchange contracts are completely endogenous and depend on whether exchange risk can be diversified. Exchange risk cannot be diversified on an aggregate level if after a change in the exchange rate there must be a net change in aggregate wealth in order to keep all investors as well off as before.

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[1]Obviously all positions in assets with contractually fixed returns that are mismatched in terms of currencies generate the same returns as forward contracts. Also, see the discussion of Equation (13).

We will demonstrate that an investor's optimal position in the forward exchange market depends, among other things, on his attitude toward risk and his exposure to exchange risk vis-a-vis the aggregate exposure. This paper shows that an investor may assume a particular position in the forward exchange market not because he wants to hedge an open position, but because he is more tolerant toward risk than other investors. In other words, the investor is willing to provide other market participants with cover against exchange risk for adequate compensation.

Previous studies of the forward exchange market leave a number of unanswered questions. Some models [e.g., Tsiang (1959), Stein (1962), Grubel (1966), Sohmen (1966), and McCormick (1977)] present demand and supply functions for forward exchange contracts that are not based on expected utility maximization. By not using a framework of expected utility maximization this group of models cannot address the role of forward exchange contracts in the optimal allocation of risk. In this approach, it is often assumed that market participants either hedge, arbitrage, or speculate. Such classification is rather ad hoc, for risk tolerant investors will be involved in all three activities simultaneously.

Some models [Feldstein (1968), Leland (1971), and Dalal (1979)] study the behavior of individuals in the forward exchange market using an expected utility maximizing framework. They, however, ignore the fact that investors' demand and supply functions must clear securities markets and, hence, positions held by various investors in the forward exchange market are interrelated through the market clearing mechanism.

These authors emphasize the relationship between the forward rate and the expected future spot rate. For instance, it is often argued that an investor takes a long position in the foreign currency if the forward rate is larger than the expected future spot rate (i.e., there is a risk premium in the forward rate). The risk premium is, however, an endogenous variable and, in principle, depends on the diversifiability of exchange risk and investors' attitude toward risk [Frankel (1979), Stulz (1982)]. In this paper, the risk premium does not appear in the derived demand and supply functions for forward exchange contracts. Instead, investors' optimal positions are presented in terms of their degree of risk aversion and the aggregate exposure to exchange risk. Also, these authors employ single-period investment models which cannot account for the intertemporal effects of exchange rate changes on the investment opportunity set (i.e., real exchange risk) and on prices of consumption goods (i.e., consumption exchange risk).

Finally, some papers [Krugman (1981) and Stulz (1983)] derive demand and supply functions for expected utility maximizing investors that consume foreign goods. Similar to previous studies, these papers do not require that investors' positions be consistent with market clearing prices--i.e., demand and supply functions are not required to clear securities markets. Furthermore, these two studies assume that only prices of imported goods are affected by exchange rate changes and that the exchange rate and the price of the foreign good are perfectly correlated.

Here we develop demand and supply functions for forward exchange contracts under the most general set of assumptions. In particular, we show that in a multiperiod model the optimal forward exchange position of an expected utility maximizing investor can be explained by the following terms:

a) his foreign investment position;

b) his exposure to changes in prices of consumption goods that result from variations in the foreign exchange rate;

c) his exposure to "real" exchange risk;

d) his expectation about changes in the exchange rate;

e) aggregate exposure to exchange risks (i.e., in "real" and "nominal" terms);

f) the effects of Jensen's inequality.

The plan of the paper is as follows: Section 2 presents the notation and the assumptions of the paper. Section 3 derives the demand and supply functions for forward exchange contracts. Section 4 presents some examples to illustrate the properties of the above functions, and Section 5 offers some concluding remarks.

2. Assumptions and Definitions

In developing the model we make the following assumptions:

(A1) A Two-country world: We consider a world economy consisting of two countries, say, US and ROW. The exchange rate is the price of one ROW pound in terms of US dollars and is denoted by S(t).

(A2) The investment opportunity set: There are N risky securities (excluding forward exchange contracts). The first n securities are dollar-denominated, while the remaining N-n securities are pound-denominated. The price of security i at time t is denoted by $V_i(t)$ and its return generating process can be expressed as an Ito process [see Liptser and Shiryayev (1977), Merton (1971, 1973) for a discussion of the Ito process] of the following form

$$dV_i(t) = [V_i(t)\mu_i(t) - D_i(t)]dt + V_i(t)\sigma_i(t)dZ_i(t), \tag{1}$$

where $\mu_i(t)$ is the instantaneous expected <u>total</u> rate of return on asset i, $D_i(t)$ is the payout of asset i ($D_i(t)$ can be nagative), $\sigma_i(t)$ is the instantaneous standard deviation of the rate of return of asset i, and $Z_i(t)$ is a standard Wiener process (i.e., $EdZ_i = 0$ and $VardZ_i = dt$).

The return generating processes need not be stationary. In particular, $\mu_i(t)$ and $\sigma_i(t)$, can be functions of other Ito processes called state variables. Without loss of generality, we assume there are M state variables and the dynamics of state variable j, $X_j(t)$, can be presented by an Ito process; i.e.,

$$dX_j(t) = \gamma_j(t)dt + \delta_j(t)dY_j(t) \tag{2}$$

where $\gamma_j(t)$ is the instantaneous expected change in $X_j(t)$, $\delta_j(t)$ is its instantaneous standard deviation, and $Y_j(t)$ is a standard Wiener process.

Some of the state variables could be endogenous economic variables such as the exchange rate, while some could be totally exogenous such as weather. For the purpose of this paper we do <u>not</u> need to specify the state variables any further. If the exchange rate is one of the state variables, we will assume that it is always state variable M. We will use subscript s to denote the process that represents changes in the spot exchange rate (e.g., $\gamma_s(t)$, $\delta_s(t)$, and $dY_s(t)$).

Investors can also borrow and lend at risk free rates. The risk free rate for borrowing and lending in terms of the dollar and the pound are denoted by $r_1(t)$ and $r_2(t)$, respectively.

Since the parameters of the return generating processes are not stationary, the investment opportunity set employed in this model is said to be <u>stochastic</u>.

(A3) The consumption opportunity set: Without loss of generality we assume that each country produces one good that is traded internationally. Let $P_i(t)$ represent the price of consumption good i at time t in terms of dollars; hence, its price in terms of pounds will be $P_i(t)/S(t)$. Prices of consumption goods need not be constant over time and they are, in general, functions of state variables presented in equation (2). Because prices of consumption goods are allowed to be stochastic, the consumption opportunity set employed in this paper is said to be <u>stochastic</u>.

(A4) Investor choice: Each investor maximizes the expected value of a time-additive von Neuman-Morgenstern utility of lifetime consumption of the following form,

$$\max_{V_i, C_j} E \int_0^t U(C_1, C_2, t)dt \tag{3}$$

subject to the wealth constraint. In the above equation, E is the expectation operator conditional on information available at time zero and C_j is the amount of consumption j consumed by the investor.

Without loss of generality we consider the behavior of two representative investors--one from each country. Superscript a and b will represent variables germaine to investors from the US and ROW, respectively. While this assumption has <u>no</u> effect on the following analysis, it will simplify the mathematical expressions.

(A5) Information structure: Investors need not agree on the parameters of the Ito processes presented in equations (2) and (3). Without loss of generality, we assume that investors have different expectations only about changes in the exchange rate. That is, investor i believes that the expected change in the exchange rate is $\gamma_s^i(t)$.[2]

(A6) Perfect markets: Securities and goods markets are perfectly competitive and there are no transaction costs and taxes.

3. Demand and Supply Functions

To derive the demand and supply functions for forward exchange contracts, we need to develop first the stochastic process satisfied by the forward exchange <u>rate</u> and forward exchange <u>contracts</u>.

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[2]Investors' expectations need not be specified exogenously. We can assume that investors <u>cannot</u> observe $\gamma_s(t)$ and have to estimate it using all available information which includes a noisy private signal. Then in a noisy rational expectations model, investors will end up with different beliefs about $\gamma_s(t)$ [e.g., see Diamond and Verricchia (1981), Hellwig (1980), and Admati (1985)].

Let $F(t,T)$ be the dollar price of a forward contract at time t with maturity date T. If the contract is issued at time t, then by definition $F(t,T) = 0$. The value of the contract at maturity is

$$F(T,T) = G(t,T) - G(T,T),$$

where $G(t,T)$ is the forward rate available at time t for delivery at time T. Hence, the return from holding a short-term forward contract issued at (t-dt) with the maturity date t is

$$F(t,t) - F(t-dt,t) = G(t-dt,t) - G(t,t),$$

or $$dF(t) = -dG(t). \tag{4}$$

We use the Interest Rate Parity Theorem to derive the dynamics of the forward rate. Let $B_i(t)$ be the current price of a riskless bond; i.e., $dB_i(t) = r_i(t)B_i(t)dt$; in the context of the Interest Rate Parity Theorem, we can write:

$$dG(t) = d\left[\frac{S(t)B_2(t)}{B_1(t)}\right]. \tag{5}$$

Applying Ito's lemma [Merton (1971)] to equation (5) and using the results of equation (4), we obtain:

$$dF(t) = -G(t)[(r_2(t) + \gamma_s^i(t) - r_1(t))dt + \delta_s(t)dY_s] \tag{6}$$

Equation (6) presents the total return from buying dollars forward. Note, the expected return on the forward contract is proportional to deviations from the International Fisher effect. That is,

$$E\left[dF(t)\right] = -G(t)\left[r_2(t) + \gamma_s^i(t) - r_1(t)\right]dt. \tag{7}$$

If the Fisher effect holds, then the expected rate of change in the exchange rate, $\gamma_s^i(t)$, will be equal to the interest rates differential, $r_1(t) - r_2(t)$.

We are now prepared to investigate the investors' optimal portfolio decisions. A US investor will maximize (we will omit superscript a whenever the context is clear)

$$J(W,X,t) = \max_{C_i, V_i, F} E\left\{\int_t^T U(C_1, C_2, \tau)d\tau\right\} \tag{7a}$$

subject to the budget constraint,

$$dW = \sum_{i=1}^{n} dV_i + \sum_{i=n+1}^{N} d[SV_i] + dB_1 + dSB_2 + dF. \tag{7b}$$

Using the definitions of V_i, B_i, F, and S and employing Ito's lemma, the budget constraint can be expressed as follows:

$$\begin{aligned} dW = & \sum_{i=1}^{n} V_i[(\mu_i - r_1)dt + \sigma_i dZ_i] \\ & + \sum_{i=n+1}^{N} SV_i[(\mu_i - r_2 + \sigma_{is})dt + \sigma_i dZ_i] + Wr_1 dt \\ & + (\sum_{i=n+1}^{N} SV_i + SB_2 - KG)[(r_2 + \gamma_s - r_1)dt + \delta_s dY_s] \\ & - (C_1P_1 + C_2P_2)dt, \end{aligned} \tag{8}$$

where $C_1P_1 + C_2P_2$ is $(\sum_{i=1}^{n} D_i + \sum_{i=n+1}^{N} SD_i)$, K is the amount of pounds sold forward (i.e., KG is the amount of US dollars bought forward), and σ_{is} is the covariance between the rate of return on asset i and changes in the exchange rate. In deriving equation (8), we have eliminated B_1 by noting that it is equal to $(W - \sum_{i=1}^{N} V_i - \sum_{i=n+1}^{N} SV_i - SB_2)$ and that the initial market value of a forward contract is zero.

In equation (8), $(\sum_{i=n+1}^{N} SV_i + SB_2 - KG)$ represents the size of the "open" position that the investor decides to carry. If we denote this term by L, then we can write:

a) L = 0 ---> The investor has no foreign exchange open position;

b) L > 0 ---> The amount of US dollars bought by the investor in the forward exchange market is smaller than his total foreign investment position;

c) L < 0 ---> The amount of US dollars bought by the investor in the forward exchange market is greater than his foreign investment position.

The first order conditions for the maximization problem stated in equation (7) are[3]

$$U_i - J_W P_i = 0 \qquad i = 1,2 \tag{9}$$

$$T \begin{bmatrix} \mu_1 - r_1 \\ \vdots \\ \mu_N - r_2 + \sigma_{NS} \\ r_2 + \gamma_s - r_1 \end{bmatrix} - \Sigma \begin{bmatrix} V_1 \\ \vdots \\ SV_N \\ L \end{bmatrix} + \Delta \begin{bmatrix} H_1 \\ \vdots \\ H_M \end{bmatrix} = 0, \tag{10}$$

where T is the investor's degree of <u>absolute</u> risk <u>tolerance</u>, H_i is the amount of compensation that the investor should receive after an increase in state variable i if his marginal utility of wealth, J_W, is to remain constant (i.e., $H_i = \frac{\partial W}{\partial X_i}\Big|_{J_W}$), Σ is the variance-covariance matrix of asset returns; and Δ is the covariance matrix of asset returns with changes in state variables.

Equation (10) can be used to solve the domestic investor's optimal investment decision. But the investors' optimal decisions must be consistent with market clearing conditions. Thus, we must also derive the optimal investment decision of a foreign investor. Having done so, the optimal portfolio decision of a domestic investor can be written as follows (see the Appendix for the derivation).

$$\begin{bmatrix} V_1 \\ \vdots \\ SV_N \\ L \end{bmatrix} = \frac{T}{\bar{T}} \begin{bmatrix} \bar{V}_1 \\ \vdots \\ S\bar{V}_N \\ \bar{L} \end{bmatrix} + \Sigma^{-1} \left\{ \Delta \begin{bmatrix} H_1 - \frac{T}{\bar{T}} \bar{H}_1 \\ \vdots \\ H_M - \frac{T}{\bar{T}} \bar{H}_M \end{bmatrix} + T \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma_s - \bar{\gamma}_s \end{bmatrix} \right\}$$

$$+ \frac{T \; T^b}{\bar{T}} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{11}$$

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[3]For discussions of optimal stochastic control and dynamic programming see Flemming and Rishel (1975), Merton (1971, 1973), and Breeden (1979).

where $\overline{V}_i$ is the total value of a given security i in terms of its home currency, $\overline{T}$ is the measure of aggregate absolute risk tolerance, $\overline{H}_i$ is the sum of H_i over all investors, and $\overline{\gamma}_s$ is the weighted average of investors' expectations regarding changes in the exchange rate.

To gain a better understanding of equation (11), let us consider the demand for asset i--a domestic security.

$$V_i = \frac{T}{\overline{T}} \overline{V}_i + \sum_{j=1}^{M} \Theta_{ij}(H_j - \frac{T}{\overline{T}} \overline{H}_j) \qquad (12)$$

where Θ_{ij} is the (ij) element of $\Sigma^{-1}\Delta$.

Equation (12) suggests that the US investor will buy $T/\overline{T}$ percentage of the total supply of asset i and hold Θ_{ij} $(H_j - \frac{T}{\overline{T}} \overline{H}_j)$ dollars of the same asset through his investment in "hedge portfolio" j. These "hedge portfolios" are constructed so that the rate of return on portfolio j has maximum correlation with changes in state variable j.[4]

Hence, to analyze an investor's optimal investment in asset i, we need to know (a) the type of hedge that asset i provides with regard to changes in all state variables (i.e., $\Sigma^{-1}\Delta$), (b) the investor's attitude towards changes in all state variables, H_j, and (c) the attitude of the whole market towards such changes, $\overline{H}_j$. The analysis can be made much simpler by assuming that markets are effectively complete.[5] Only when markets are complete, can investors use all available securities to create portfolios completely hedged against unexpected changes in state variables.

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[4]Original discussions of hedge portfolios is due to Merton (1973); further investigations are made in Breeden (1979, 1984).

[5]Complete markets are investigated in Debreu (1959) and Arrow (1964).

In the present model, securities markets are effectively complete if there are M securities such that their returns are perfectly correlated with changes in M state variables. As demonstrated by Breeden (1984), when markets are complete in the sense used above, allocations will be Pareto-optimal. On the other hand, when there are not enough securities to complete the markets, investors will not be able to diversify all individual risks and they have to bear risks that--in principle--could be diversified away. In the absence of transactions and information costs, risk averse investors will create the necessary financial securities to diversify all individual risks.[6] There are, of course, numerous examples of financial securities that have been created by investors over the past few decades (e.g., forward and futures contracts for commodities, foreign exchange rates, and interest rates and options).[7]

At this point we introduce an additional assumption.

(A7) Complete securities markets: Assume that the number of available securities (excluding forward exchange contracts) is equal to M-1 (note that state variable M was assumed to be the foreign exchange rate), and that the return on security i is perfectly correlated with changes in state variable i.

Given the above assumption, $\Sigma^{-1}\Delta$ will be a diagonal matrix, and with suitable normalizations of state variables, the diagonal elements can all be set to unity. We are now prepared to present the US investor's optimal foreign exchange position; from equation (11) we have

$$L = \frac{T}{\bar{T}}\bar{L} + (H_s - \frac{T}{\bar{T}}\bar{H}_s) + Te(\gamma_s - \bar{\gamma}_s) + \frac{T\,T^b}{\bar{T}}, \qquad (13)$$

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[6]For a discussion of this and related issues see Radner (1982).

[7]For a discussion on the process of such innovation in international financial markets see Dufey and Giddy (1981).

where e is the last element of $\Sigma^{-1}\Delta$, and from the properties of variance-covariance matrices we know that it is strictly positive (assuming that Σ is nonsingular).[8]

From our discussion of the budget constraint [see equation (8)], we know that L is equal to the net foreign currency open position optimally assumed by the US investor, i.e.,

$$L = \sum_{i=n+1}^{N} SV_i + SB_2 - KG. \tag{14}$$

In the above expression, only L and SV_i, i=n+1,..., N, are choice variables. This means that the optimal mix of foreign bonds, SB_2, and dollars bought forward, KG, is indeterminant. It is well known that in the absence of market imperfections forward contracts and foreign currency denominated riskless bonds provide equal protection against exchange rate variations. Hence, it is not surprising that in our model the optimal mix of the two securities is indeterminant. For the remainder of the paper we will look at investors' total demands for securities that can provide a perfect hedge against unexpected changes in the exchange rate. We will denote this total demand by F(t); i.e.,

$$F(t) = KG - SB_2. \tag{15}$$

If F(t) is positive, then the investor has assumed a long position in the dollar in the forward exchange market. We will continue to call F(t) the investor's optimal demand for forward contracts.

Given the above discussion, equation (13) can be rewritten as follows:

$$F(t) = \sum_{i=n+1}^{N} SV_i - \frac{T}{\bar{T}}\bar{L} - T(\gamma_s - \bar{\gamma}_s)e - (H_s - \frac{T}{\bar{T}}\bar{H}_s) - \frac{T\,T^b}{\bar{T}}. \quad 16)$$

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[8]Anderson (1958).

The remainder of this paper is devoted to the interpretation of the terms appearing on the right hand side of the above expression.

To begin our discussion we need the following definitions.

a) The foreign investment position (FIP) risk: if an investor has invested in foreign currency denominated securities, a change in the exchange rate would affect the value of his portfolio and hence, his welfare.

b) The investment opportunity set (IOS) risk: if the exchange rate is one of the state variables, exchange rate changes would alter the return generating processes of securities (i.e., the parameters appearing in equation (1) are affected). Obviously, a change in the investment opportunity set affects the welfare of investors.

c) The consumption opportunity set (COS) risk: a change in the exchange rate often leads to further changes in prices of domestic as well as foreign consumption goods, which, of course, affects the welfare of investors.

The first term on the right hand side of equation (16), $\sum_{i=n+1}^{N} SV_i$, represents the US investor's exposure to the FIP risk. The investor buys US dollars in the forward exchange market to protect himself against the FIP risk. This component of investors' demands for forward contracts has been widely discussed in the literature and is often referred to as a hedging demand.

Now consider the next term that appears in equation (16).

Proposition 1. $\bar{L}$ is equal to the net foreign investment position of the US.

Proof.

$$\bar{L} = L^a + L^b$$

$$= \left(\sum_{i=n+1}^{N} SV_i^a + SB_2^a - K^a G\right) + \left(-\sum_{i=1}^{n} V_i^b - B_1^b - K^b G\right)$$

$$= \left(\sum_{i=n+1}^{N} SV_i^a + SB_2^a\right) - \left(\sum_{i=1}^{n} V_i^b + B_1^b\right). \tag{17}$$

The last line is derived by noting that forward exchange contracts are inside assets and hence,

$$K^bG + K^aG = 0.$$

When $\bar{L}$ is positive, the total hedging demand of US investors for dollars in the forward exchange market arising from their holdings of pound-denominated assets exceeds the total hedging demand of ROW investors for pounds which arises, in turn, from their investments in dollars. In other words, certeris paribus, when $\bar{L}$ is positive, there is a net excess demand for dollars.

Equation (16) suggests that when the US is a net foreign investor (i.e., $\bar{L} > 0$), a risk tolerant US investor will assume a speculative position by <u>buying</u> a smaller amount of dollars than his exposure to the FIP risk.[9] In some sense then, each risk tolerant investor assumes an open short position in dollars such that the above mentioned excess demand for dollars disappears. Note, an investor who has a relatively high degree of risk tolerance assumes a relatively large open position (i.e., he holds $T/\bar{T}$ percentage of $\bar{L}$).

Ignoring the other terms that appear in equation (16), the US investor will completely hedge against the FIP risk if $\bar{L}$ is equal to zero. When $\bar{L}$ is zero, the FIP risk can be diversified on an aggregate level because US investors' hedging demands for dollars are completely matched by ROW investors' hedging demands for pounds.[10]

The next term appearing in equation (16) represents that part of the investor's demand that arises because his expectation about the future spot rate differs from that of the market.

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[9] A ROW investor also speculates by buying an extra amount of pounds.

[10] Solnik (1973) shows that the net foreign investment position is one of the determinants of the risk premium in the forward rate.

Without imposing further restrictions on the variance-covariance matrix of asset returns, we cannot determine the size of e. For example, if Σ is diagonal, then e^{-1} will be equal to the variance of the exchange rate. However, since we know that e is positive, the investor will reduce his purchase of dollars in the forward market if $\gamma_s > \bar{\gamma}_s$. This means, if the investor believes that the value of the pound will increase more than the market believes, then he will reduce his holdings of dollars.

Two points must be brought up here. First, $\bar{\gamma}_s$ is a weighted average of the expectations of all investors where each investor's degree of absolute risk tolerance is used to adjust his expectations; hence, the expectations of more risk tolerant investors have larger weights in the determination of the aggregate expectation, $\bar{\gamma}_s$. Second, ceteris paribus, the more risk tolerant an investor is, the larger the speculative position he takes.

The next term that appears in equation (16) results from the effect of Jensen's inequality. Several articles have discussed Jensen's inequality in the context of the forward exchange market [Siegel (1972), Krugman (1981) and Beenstock (1985)], which arises because

$$E\left(\frac{dS}{S}\right) > -E\left(\frac{dS^{-1}}{S^{-1}}\right).$$

That is, the expected percentage <u>increase</u> in the value of the pound in terms of dollars is <u>greater</u> than the expected percentage <u>decrease</u> in the value of the dollar in terms of pounds. Hence, everything else being constant, a US investor prefers to take a long position in the pound and a ROW investor prefers to take a short position in the dollar. There is, however, nothing special about this effect because the sign of $\frac{T\,T^b}{\bar{T}}$ would change if we were to redefine the exchange rate as the price of one dollar in terms of pounds.

We start our discussion of the next term on the right hand side of equation (16) by presenting the determinants H_s, the investor's exposure to other types of exchange risks.

Proposition 2.

$$H_s = \alpha_s + T\,[\beta_1\,\pi_1\,(R - \eta_1) + \beta_2\,\pi_2\,(R - \eta_2)] \tag{18}$$

where α_s is $\frac{\partial W}{\partial \ln S}\Big|_C$; C is $(C_1P_1 + C_2P_2)$, the domestic investor's consumption expenditure; β_i is the budget share of good i in the investor's consumption expenditure; π_i is the exchange rate elasticity of the price of good i, $\frac{\partial \ln P_i}{\partial \ln S}$; η_i is the consumption expenditure elasticity of good i, $\frac{\partial \ln C_i}{\partial \ln C}$, and R is the investor's measure of relative risk aversion, $-\frac{\partial \ln V_c}{\partial \ln C}$.

Proof. Let $V(C, P_1, P_2, t)$ denote the investor's indirect utility function; i.e.,

$$V(C, P_1, P_2, t) = \max \{U(C_1, C_2, t) + \lambda(C - C_1P_1 - C_2P_2)\}.$$

We stated that H_s measures the effect of a change in the exchange rate on the investor's marginal utility of wealth, i.e.,

$$H_s = \frac{\partial W}{\partial \ln S}\Big|_{J_W} = -\frac{SJ_{WS}}{J_{WW}}. \tag{19}$$

We know from equation (9) that

$$U_iP_i = J_W = V_c. \tag{20}$$

Hence, using the above expression we can write H_s as follows:

$$H_s = -S\left(V_{c1}\frac{\partial P1}{\partial S} + V_{c2}\frac{\partial P2}{\partial S} + V_{cc}\frac{\partial C}{\partial S}\right)\left(V_{cc}\frac{\partial C}{\partial W}\right)^{-1}. \tag{21}$$

Equation (18) will be obtained by applying Roy's identity [Varian (1985)] to V_{c1} and V_{c2} and rearranging the terms.

Proposition 2 can be interpreted as follows: J(W,X,t) represents the investor's indirect utility of lifetime consumption (see equation (7a)), which is a function of time t, his wealth W, and the state variables describing the state of the world economy, X_i, i=1, ..., M. In our model various types of exchange risks are measured through the effects of a change in the exchange rate on our investor's marginal utility of wealth.

First, a change in the exchange rate affects the value of foreign-currency denominated holdings of our investor. We referred to this type of exchange risk as the FIP risk.

Second, a change in the exchange rate can affect the marginal utility of wealth directly through (a) changing the parameters of the return generating process of securities (see equation (1)) and (b) through changing prices of domestic as well as foreign consumption goods. We refer to these two types of exchange risks as the IOS and the COS risks, respectively.

The IOS risk is presented by α_s in equation (18); it is equal to the amount of compensation that an investor must receive after an increase in the exchange rate if his consumption expenditure is to remain constant (note, we are holding prices of consumption goods constant). To understand the nature of the IOS risk, consider the "envelope" condition of the optimization problem presented in equation (7) (see also equations (9) and (20)),

$$J_W(W,X,t) = V_c(C,P_1,P_2,t). \tag{22}$$

In our model, investors have "rational expectations" in the sense that in making optimal investment-consumption decisions they take into account all the information available about the current investment and consumption opportunity sets. This information, because of the Markov property of Ito processes [Liptser and Shiryayev (1977)], is represented by the current values of the investor's wealth and the state variables.

Assume that after a decrease in the exchange rate the investment opportunity set deteriorates; i.e., the marginal utility of wealth increases.

Hence, from equation (22), there also will be an increase in the marginal utility of consumption expenditure, V_c, implying a reduction in the optimal consumption expenditure. Therefore, if the investor were to keep his consumption expenditure constant after a decrease in the exchange, there must be an increase in his wealth.

Let us assume that α_s is negative. Then the investor purchases an extra amount of dollars forward; that is, he hedges by assuming a short position in the pound. The reason is that when α_s is negative, a decrease in the value of the pound makes the investor worse off. Unfortunately, without making further assumptions about the investment opportunity and developing a complete general equilibrium model, we cannot determine the sign of α_s.

We now discuss the determinants of the COS risk. As we mentioned earlier in the paper, prices of consumption goods are functions of state variables and are hence stochastic. That part of the investor's demand for forward contracts which arises from his exposure to the COS risk is:[11]

$$T[\beta_1 \Pi_1 (R - \eta_1) + \beta_2 \Pi_2 (R - \eta_2)]. \tag{23}$$

This expression represents the investor's demand for an asset that provides a perfect hedge against changes in prices of consumption goods (i.e., those changes that are caused by exchange rate variations) and provides an expected return equal to the domestic riskless rate.[12]

In equation (23),

$$T[\beta_1 \Pi_1 + \beta_2 \Pi_2]R$$

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[11] Stulz (1983) derives rather similar results; but he assumes that $\Pi_1 = 0$ and $\Pi_2 = 1$.

[12] Of course, if there are financial assets that can provide perfect hedges against total changes in prices of consumption goods, then forward exchange contracts will not be used as hedges against the COS risk.

can be interpreted as the amount of compensation that the US investor must receive after a change in the exchange rate if he is to continue with his planned consumption (note, we are now concentrating on the effects of exchange rate changes on prices of consumption goods). The objective of the investor is not, however, to keep his nominal consumption expenditure constant but to keep the shadow price of one dollar spent on consumption goods [i.e., $V_c(\cdot)$] from changing. Expression (23) presents the compensation that an investor must receive after a change in the exchange rate if his marginal utility of consumption is to remain constant.

Stulz (1983) attributes

$$T[\beta_1 \Pi_1 \eta_1 + \beta_2 \Pi_2 \eta_2]$$

to the effects of Jensen's inequality. The following simple example shows that this term would appear even if all variables are nonstochastic.

Let $V(C, P_1, P_2)$ denote an investor's indirect utility function. Now consider the total (nonrandom) change in his marginal utility of consumption: i.e.,

$$dV_c = V_{cc}dC + V_{c1}dP_1 + V_{c2}dP_2.$$

The required percentage change in income if V_c is to remain constant (i.e., $dV_c = 0$) is

$$\frac{dC}{C} = -\frac{V_{c1}}{CV_{cc}}dP_1 - \frac{V_{c2}}{CV_{cc}}dP_2 .$$

Using Roy's identity we can write

$$\frac{dY}{Y} = \beta_1 \frac{dP_1}{P_1} + \beta_2 \frac{dP_2}{P_2} - T\left(\beta_1 \eta_1 \frac{dP_1}{P_1} + \beta_2 \eta_2 \frac{dP_2}{P_2}\right),$$

where β_i and η_i continue to denote the budget share and the income elasticity of consumption good i, respectively; T is $-\frac{V_c}{V_{cc}}$, the measure of risk tolerance.

The last term in the above expression is attributed by Stulz (1983) to be due to Jensen's inequality. But we have not assumed that prices are stochastic and Jensen's inequality comes into play when we deal with concave or convex functions of random variables.

Our interpretation of this term is that changes in prices of consumption goods have an "income effect" (i.e., real income is reduced). Lower real income reduces the investor's demand for consumption goods by $\beta_i \eta_i$. Hence, to restore the marginal utility of consumption to its original level, income does not have to rise by $(\beta_1 d\ln P_1 + \beta_2 d\ln P_2)$, and a smaller percentage change in income will be sufficient to keep V_c constant (note, $T > 0$).

We can make several conjectures about the results of proposition 2.

(i) Consumption of imported goods tends to reduce the US investor's demand for dollars in the forward market, provided that (a) $R > \eta_2$ and (b) $\Pi_2 > 0$. The first requirement is likely to be satisfied if the imported product is a "necessary" good (i.e., η_2 is small).[13] The second requirement is also likely to be satisfied because an increase in the value of the pound usually leads to an increase in the dollar price of the imported good.

(ii) Consumption of domestic goods tends to increase the US investor's demand for dollars in the forward market, provided that (a) $R > \eta_1$ and (b) $\Pi_1 < 0$. Here, contrary to the previous case we are assuming that an increase in the value of the pound is likely to lead to a <u>decline</u> in the price of the <u>domestic</u> good.

(iii) The US investor's demand for dollars in the forward market would be reduced by a larger amount if the imported good is a "luxury" (i.e., η_2 is relatively small).

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[13] Blume and Friend (1975) estimate that R is rather close to 2.0.

Now that we have presented the determinants of an investor's exposure to the IOS and the COS risks, we can discuss the most important argument that appears in equation (16), $(H_s - \frac{T}{\bar{T}} \bar{H}_s)$.

It is obvious that whether this term leads to an increase or a decrease in demand for dollars in the forward exchange market depends on the relative sizes of H_s, $\bar{H}_s$, T, and $\bar{T}$. To illustrate, let us assume that both H_s and $\bar{H}_s$ are positive. Then an increase in the value of the pound would make everybody worse off. Now consider the following two cases.

Case 1 US and ROW investors have the same degree of risk tolerance (i.e., $\bar{T} = 2T$). We can, therefore, write:

(i) $H_s > \bar{H}_s/2$ ----------> The US investor will buy a smaller amount of dollars forward:

(ii) $H_s < \bar{H}_s/2$ ----------> The investor will buy a larger amount of dollars forward.

Because $H_s > 0$, the investor is exposed to unexpected increases in the exchange rate. However, in (ii), he is willing to sell pounds forward because his exposure is smaller than the average.

Case 2 US and ROW investors have the same degree of exposure to the IOS and the COS risks (i.e., $\bar{H}_s = 2 H_s$). We can, therefore, write:

(i) $\bar{T} > 2 T$ ----------> The US investor will buy a smaller amount of dollars forward;

(ii) $\bar{T} < 2 T$ ----------> The investor will buy a larger amount of dollars forward.

Note that though the investor is adversely affected by an increase in the exchange rate, he is willing to supply the market with pounds, provided that he is more risk tolerant than the average.

These two cases clearly demonstrate that an investor's optimal position in the forward market depends on his <u>relative</u> exposure and <u>relative</u> tolerance toward risk. An investor may hedge against exchange risk ((i) in cases 1 and 2) if other market participants are in a better position to provide coverage. On the other hand, an investor may increase his exposure to exchange risk ((ii) in cases 1 and 2) and provide the rest of the market with needed forward contracts if he is better prepared to handle exchange risk.

When exchange risk can be diversified away (i.e., $\bar{H}_s = 0$) investors' demands will not depend on the aggregate exposure to exchange risk. In this case, the investor will respond only to his own exposure and attitude toward risk.[14]

4. Examples

We now present four specific examples to illustrate the implications of the demand and supply functions presented in equation (16).

<u>Example 1</u>. The domestic investor has no tolerance for risk (i.e., $T^a \longrightarrow 0$). The investor's demand for forward exchange contracts will be (note that $TR = \frac{C}{C_w}$)

$$F = \sum_{i=n+1}^{N} SV_i - \alpha_s - \frac{C}{C_w} [\beta_1 \pi_1 + \beta_2 \pi_2].$$

His demand in this case is such that he completely hedges against all types of exchange risk. Furthermore, the investor is not influenced by the aggregate exposure to exchange risk because the loss of welfare of assuming any open position is extremely high for him. On the other hand, assuming that other market participants have some tolerance toward risk, the cost (i.e., the risk premium) of using the forward exchange market is finite and this leads him to have a completely hedged position.

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[14] Kazemi (1985) shows that when $\bar{H}_s$ and $\bar{L}$ are equal to zero, there will be no risk premium in the forward rate.

Example 2. One of the other market participants is risk neutral (i.e., $T^b \longrightarrow \infty$). A domestic investor's demand for forward exchange contracts is

$$F = \sum_{i=n+1}^{N} SV_i - \alpha_s - T\,[\beta_1\pi_1(R - \eta_1) + \beta_2\pi_2(R - \eta_2)].$$

The above demand equation is, of course, very similar to the one presented in example 1.[15] Since other market participants have a very high tolerance for risk, it is not optimal for our domestic investor to assume any risk at all. In this example, contrary to example 1, the cost of using the forward exchange market is zero and hence, the investor will completely hedge his position, although he has some tolerance toward risk.

Example 3. Other market participants have no tolerance toward risk (i.e., $T^b \longrightarrow 0$). Our domestic investor's demand, assuming that he has some tolerance toward risk, is

$$F = \sum_{i=1}^{n} V_i^b + \alpha_s^b + \frac{C^b}{C_w^b}\,[\beta_1^b\,\pi_1^b + \beta_2^b\pi_2^b].$$

The right hand side of the above expression represents the foreign investor's demand for forward exchange contracts. That is, because the foreign investor has no tolerance for risk, he is willing to pay a high premium to hedge his position against exchange risks. On the other hand, the domestic investor has some tolerance for risk and hence he is willing to supply the foreign investor with needed forward exchange contracts--provided that he receives appropriate compensation. Note, here the domestic investor's demand for US dollars in the forward exchange market is completely independent of his degree of risk tolerance, because the foreign investor is not willing to assume any risk and hence

[15]The last term on the right hand side of the above expression differs from the one presented in example 1, because the hedging demand of an investor who has a concave utility function (i.e., $T > 0$) is different from that of an investor who has no tolerance for risk.

will pay the domestic investor the "right" price in order to have a completely hedged position.

Example 4. All investors have logarithmic utility functions. Under this condition the investor's demand for forward exchange contracts will be

$$F = \sum_{i=n+1}^{N} SV_i - \frac{W}{\bar{W}} \bar{L} - We\,(\gamma_s - \bar{\gamma}_s) - \frac{WW^b}{\bar{W}}.$$

The above expression is obtained by noting that for logarithmic utility functions $H_s = 0$ (i.e., investors are indifferent toward the IOS and COS risks) and $T = W$ [Merton (1971)].

Noting that $\bar{L}$ is equal to the net foreign investment position of the US, the above expression states that the speculative position of a US investor consists of two parts:

(i) $\frac{W}{\bar{W}} \bar{L}$,

and (ii) $We\,(\gamma_s - \bar{\gamma}_s)$.

Here, (i) is proportional to the net foreign investment position of the US. For instance, if the US is a net foreign investor, US investors will have long positions in pounds. On the other hand, (ii) states that the investor will take a long position in pounds if he expects a greater rise in the value of the pound than the market (i.e., $\gamma_s > \bar{\gamma}_s$). Also note that, ceteris paribus, wealthier investors take larger speculative positions.

5. Concluding Remarks

This paper presents a complete description of the demand and supply functions for forward exchange contracts under rather general assumptions. The emphasis has been on the role of forward exchange contracts as instruments employed by investors to achieve optimal allocation of exchange risk in its various manifestations. Contrary to previous studies, the derived demand and supply functions take into account an investor's exposure to various types of exchange risks and the aggregate exposure to the same risks, which is a function of the diversifiability of exchange risk.

Through several examples we show that investors who have an advantage in assuming risk (i.e., have higher tolerance toward risk) will provide more risk averse investors with a sufficient supply of forward exchange contracts, provided they receive adequate compensation. Thus, in a world of homogeneous investors there is no need for synthetic claims such as forward exchange contracts. Vice versa, the existence of differences in risk tolerences and exposures to various types of exchange risk provides the incentive to create instruments that permit efficient allocation of risk.

APPENDIX

To obtain equation (11) we need to write the first order conditions for a foreign investor--the conditions for a domestic investor appear in equation (10) (superscript b is omitted unless there may be any confusion).

$$T \begin{bmatrix} \mu_1 - r_1 - \sigma_{is} \\ \cdot \\ \cdot \\ \mu_N - r_2 \\ r_2 + \gamma_s - r_2 - \delta_s^2 \end{bmatrix} - \Sigma \begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ SV_N \\ L \end{bmatrix} + \Delta \begin{bmatrix} H_1 \\ \cdot \\ \cdot \\ \cdot \\ H_M \end{bmatrix} = 0, \qquad \text{(A1)}$$

where $T = - S \dfrac{Jw}{J_{ww}}$, the foreign investor's degree of absolute risk tolerance measured in terms of US dollars; $H_i = S \dfrac{\partial W}{\partial Xi}\Big|_{J_W}$. Note that W is in US dollars; further, $\bar{L} = - \sum_{i=1}^{n} V_i - B_1 - GK$.

Aggregating equations (10) and (A1) and rearranging the terms we obtain

$$\bar{T}\Sigma^{-1} \begin{bmatrix} \mu_1 - r_1 \\ \cdot \\ \cdot \\ \mu_2 - r_1 \\ r_2 - r_1 \end{bmatrix} + \bar{T}\Sigma^{-1} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \bar{\gamma}_s \end{bmatrix} + T^a\Sigma^{-1} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \sigma_{NS} \\ 0 \end{bmatrix} = T^b\Sigma^{-1} \begin{bmatrix} \sigma_{is} \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \delta_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{V}_1 \\ \cdot \\ \cdot \\ \cdot \\ S\bar{V}_N \\ \bar{L} \end{bmatrix} + \Sigma^{-1}\Delta \begin{bmatrix} \bar{H}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{H}_M \end{bmatrix} = 0 \qquad \text{(A2)}$$

The above expression <u>and</u> equation (10) can be used to solve

$$\bar{T}\Sigma^{-1} \begin{bmatrix} \mu_1 - r_1 \\ \cdot \\ \cdot \\ \mu_N - r_2 \\ r_2 - r_1 \end{bmatrix}.$$

Setting the results equal to each other and noting that

$$\Sigma^{-1} \begin{bmatrix} \sigma_{is} \\ \cdot \\ \cdot \\ \cdot \\ \sigma_{NS} \\ \\ \delta_s^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix},$$

equation (11) will be obtained.

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