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A BAYESIAN MODEL FOR EXPLAINING MONEY SUPPLY GROWTH RATES

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INTRODUCTION

For a number of years a controversy has been building over the relative importance to the economy of monetary and fiscal policies. The issues, of course, are partly illative in nature and there appears to be no exact agreement (or disagreement) between the montarists and their opponents on the measurement of the effects of various policies. The monetarist position, most notably advanced by Milton Friedman, states that, in general, money supply aggregates and their growth rates greatly influence short-run economic performance and that, therefore, monetary policy can and should be employed to add stability to that performance. This view is not shared by those economists who advocate, instead, that while money is not unimportant, fiscal policy is more useful in guiding economic growth and reducing the impact of fluctations. A good example of the two arguments is presented in Friedman and Heller [5].

This paper is addressed to the initial problem of assessing current monetary policy. While measurement alone can in no way resolve the above questions regarding the relative importance of various policies, it can, hopefully, provide insights into the relationships between money and economic performance. A problem in measurement exists, especially for those outside the Federal Reserve System, because policy directives issued by the Board of Governors are largely inaccessible to the public. Until recently, the policy record of the Federal Open Market Committee (FOMC) provided only the most general information on the Committee's decisions, and then only after a three-month period had elapsed.

The FOMC meets each month to discuss and modify the open market operations of the Federal Reserve System. Informal telephone meetings usually occur more often. The record of policy actions decided at each month's meeting is published three months later in the Federal Reserve Bulletin. Before April of this year the policy toward money supply growth rates was reported in rather brief and vague terms. Beginning in April it was decided to publish more specific policy actions with respect to the narrowly defined money stock[1] (M1) and the more broadly defined money stock[2] (M2), still with the three-month delay in disclosure.[3]

In developing the model discussed below it was assumed that Federal Reserve policy depends upon short-run and long-run objectives both with respect to the individual growth rates of money supply aggregates and the reciprocal influences between them. The day-to-day open market operations of the System Account Manager are designed to meet the immediate financing needs of the government and to counteract short-run fluctuations in the money markets. At the same time, the Account Manager must attempt to achieve intermediate-range money growth rates within tolerances specified by the FOMC. Finally, long-run objectives

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[1] Demand deposits plus currency.

[2] M1 plus time deposits and time certificates other than large certificates of deposit.

[3] For a more complete description of the new disclosure policy, see the Federal Reserve Bulletin, May 1974.

aimed at achieving desirable levels in interest rates, unemployment, and other important economic variables are pursued.

A number of factors prevent the Federal Reserve from achieving precise control of money supply growth rates, e.g., an uncertain lag between open market operations and the response of the various money stock aggregates, shifts in the demand for money to finance purchases of goods and services which cannot be anticipated exactly, and fiscal policy actions which influence the growth of the money stock in ways not precisely expected or understood. These and other variables not controllable by the Federal Reserve System cause errors of varying magnitude between FOMC goals and actual money growth rates.

A RANDOM COEFFICIENT TIME SERIES MODEL

A useful model for describing the behavior of certain discrete time series is that of Kalman and Bucy [7], developed extensively in the engineering literature. This method also has interesting applications in so-called varying-parameter econometric models, e.g., McWhorter, Spivey and Wrobleski [8] and Rosenberg [10]. In the application presented here the latter interpretation, with its assumption of a set of explanatory variables, is not considered.

In its general form the Kalman-Bucy model consists of an observation equation:

$$y(t) = X(t)\beta(t) + \varepsilon(t), \qquad t = 1,\ldots,N \tag{1}$$

where $y(t)$ is a px1 vector of observations at time t, $X(t)$ is a known pxr matrix (regressors in the econometric sense), $\beta(t)$ is the rx1 state vector, and $\varepsilon(t)$ is a px1 vector of white noise, serially uncorrelated with covariance matrix $R(t)$. The state equation describing the behavior of $\beta(t)$ is:

$$\beta(t) = T(t)\beta(t-1) + \alpha(t) + u(t). \tag{2}$$

Here $T(t)$ is a known rxr transition matrix, $\alpha(t)$ is a (possibly known) rx1 policy vector, and $u(t)$ is an rx1 white noise vector with covariance $Q(t)$.

Equations (1) and (2) are in the form presented by Duncan and Horn [2] with the exception of the $\alpha(t)$ term appearing in [2]. When the α's and the covariances $R(t)$ and $Q(t)$ are given, the well-known recursive updating extimators of $\beta(t)$ are:

$$\hat{\beta}(t) = \hat{\beta}(t|t-1) + S(t|t-1)X'(t)D(t)^{-1}(y(t)-X(t)\hat{\beta}(t|t-1)) \tag{3}$$

where

$$S(t) = S(t|t-1) - S(t|t-1)X'(t)D(t)^{-1}X(t)S(t|t-1), \tag{4}$$

and

$$\hat{\beta}(t|t-1) = T(t)\hat{\beta}(t-1) + \alpha(t), \qquad t = 2,\ldots,N \tag{5}$$

$$S(t|t-1) = T(t)S(t-1)T'(t) + Q(t), \qquad t = 2,\ldots,N \tag{6}$$

$$D(t) = X(t)S(t|t-1)X'(t) + R(t), \qquad t = 1,\ldots,N. \tag{7}$$

In addition, the initial $\hat{\beta}(1|0)$ must be known and $S(1|0) = Q(1)$. For a derivation of these estimators see Sorenson [11]. Duncan and Horn [2] discuss the conditions under which equation (3) yields minimum variance and minimum mean square linear estimators, as well as other properties related to their distribution.

Equations (3) and (4) have also been derived from Bayesian considerations, Ho and Lee [6] and maximum likelihood methods, Rauch, Tung and Striebel [9]. These approaches are used in the model extensions given below.

When the covariances $R(t)$ and $Q(t)$ are not known recursive estimation by (3) is no longer possible, although various numerical methods can be employed. Furthermore, when $\alpha(t)$ is unknown (5) cannot be employed. Methods of estimation are explored below when the covariances and policy vector are assumed to be time invariant.

GROWTH RATES OF MONEY SUPPLY AGGREGATES

To motivate our model for growth rates of money supply aggregates a very simple empirical study of the monthly growth rate movements of M1 and M2 is considered first. Recalling the above discussion of Federal Reserve short-run and long-run policy objectives, denote by $y_1(t)$ and $y_2(t)$, respectively, the actual annualized growth rates of the monthly seasonally adjusted money stock aggregates M1 and M2. The Federal Reserve long-run policy goal for the growth rate of the i-th money stock measure, assumed fixed for a given time period, will be denoted by ρ_i, $i = 1,2$; and the adjustment to meet short-run objectives will be denoted by $\delta_i(t)$, $i = 1,2$. Then the short-run growth rate goal at time t for the i-th money stock measure, denoted by $\beta_i(t)$, is the sum $\rho_i + \delta_i(t)$. The actual annualized growth rates (for each month $t = 1,\ldots,N$) are expressed in the form:

(8) $$y_1(t) = \delta_1(t) + \rho_1 + \varepsilon_1(t)$$
$$y_2(t) = \delta_2(t) + \rho_2 + \varepsilon_2(t),$$

where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ reflect random kinds of changes in the money stock aggregates not associated with policy goals nor their implementation through the market operations of the FOMC.

The accompanying graphs illustrate the results of estimating ρ and $\beta(t)$ by ad hoc methods. The values of ρ were obtained by taking twelve-month moving averages of annual changes in M1 and M2 centered at the current time period, and $\beta(t)$ was computed by taking a three-month moving average of the observed growth rates $y_1(t)$ and $y_2(t)$ centered at the current period.

In Figure 1 the values of M1 and M2, seasonally adjusted, are plotted for the past decade. Figures 2 and 3 give the annualized growth rates for these series, computed monthly, and showing a high degree of fluctuation from month-to-month. Also in these figures are plots of the long-run growth rate approximations described above, displaying, of course, far less variation but obviously not constant. In Figures 4 and 5 the short-run growth rate estimates are superimposed, reflecting a rather high amount of variation.

In Figures 6 through 9 time series and frequency distributions plots are given for the residuals between the observed and estimated policy series represented by $\varepsilon_1(t)$ and $\varepsilon_2(t)$ in equation (8). The histograms in Figures 8 and 9 reveal a white noise type of process. Similarly, Figures 10 through 13 give time series and frequency distribution graphs for the one month differences in the adjustment term estimators, $\delta_i(t) - \delta_i(t-1)$, $i = 1, 2$. Again a white noise assumption for these errors appears reasonable.

A KALMAN-BUCY MODEL FOR SHORT-RUN GROWTH RATES

One purpose of the above empirical discussion is to develop a strategy for detecting patterns in, and identifying the components of long-run and short-run Federal Reserve policy with regard to the growth rates of M1 and M2. On the basis of these ad hoc empirical analyses, it appears plausible that the long-term growth rate ρ varies gradually but is far more stable than the accompanying short-term growth rate β. Thus it may be reasonable to assume that ρ is constant over short periods of time. Based on this assumption an application of the Kalman-Bucy model is proposed for estimating the short-term and long-term growth rates of the money stock aggregates M1 and M2.

For estimation purposes, the proposed model is stated in the form,

$$\begin{aligned} y_i(t) &= \beta_i(t) + \varepsilon_i(t) && i = 1,2 \\ \beta_i(t) &= \beta_i(t-1) + \alpha_i + u_i(t) && t = 1,2,\ldots,N \end{aligned} \tag{9}$$

Again, $y_i(t)$ is the seasonally adjusted, annualized rate of growth of the i-th money measure for month t. Equations (9) can be written in matrix form corresponding to (1) and (2) as:

$$\begin{aligned} y(t) &= X\beta(t) + \varepsilon(t) \\ \beta(t) &= \beta(t-1) + \alpha + u(t) \end{aligned} \qquad t = 1,\ldots,N \tag{10}$$

where

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\alpha' = (0,\alpha_1,\ 0,\alpha_2),\ \hat{\beta}'(1|0) = (\hat{\beta}_1,0,\hat{\beta}_2,0),$$

and the covariance of u(t) is taken to have the fixed, speical form:

$$Q = \begin{bmatrix} q_{11} & 0 & q_{12} & 0 \\ 0 & 0 & 0 & 0 \\ q_{21} & 0 & q_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The 2x2 covariance matrix R for $\varepsilon(t)$ is also assumed constant over time. Referring to (2), we see that in (10), T(t) = I, the 4x4 identity matrix.

From (10) it is apparent that the error of estimation of y(t) is attributed to two factors: a transient source, $\varepsilon(t)$; and a permanent effect, u(t). This would seem to be a plausible explanation of the behavior of the growth rates of M1 and M2. For example, a sharp change in government deposit levels can cause a temporary swelling in private demand deposits before they are invested in other assets. On the other hand, sudden and unexpected shifts in the demand for goods and services can have long-lasting effects on interest rates, influencing, in turn, money growth rates.

The Kalman-Bucy model has the further advantage of accounting for the obviously strong correlation between the changes in M1 and M2 through their joint estimation. The problem of assessing this correlation is discussed below.

INTREPRETATIONS OF THE KALMAN-BUCY GROWTH RATE MODEL

In the following discussion, the subscript i has been omitted. The second of the equations (9) can be written:

$$\beta(t) = \beta(0) + t\alpha + \sum_{s=1}^{t} u(s). \tag{11}$$

Now suppose that, beginning at time $t = 0$, it is desired to achieve an average long-run growth rate in the money supply of ρ_o over a period of m months. Let $\bar{\beta}_m$ be the average of the growth rates representing short-run policy over these m months.

$$\bar{\beta}_m = \frac{1}{m}\sum_{t=1}^{m}\beta(t) = \beta(0) + \frac{m+1}{2}\alpha + \bar{u}_m, \tag{12}$$

where $\bar{u}_m$ is the mean of the various $\sum_{s=1}^{t} u(s)$ terms appearing in (11). Then the expected value of $\bar{\beta}_m$ satisfies:

$$E[\bar{\beta}_m] = \beta_o + \frac{m+1}{2}\alpha.. \tag{13}$$

Equating $E[\bar{\beta}_m]$ to ρ_o yields $\alpha_o = 2(\rho_o - \beta_o)/(m+1)$. Therefore, from (11) it is seen that a short-run growth rate policy in which $E[\beta(t)] = \beta_o + \alpha_o t$ satisfies the desired long-run policy goal of $E[\bar{\beta}_m] = \rho_o$. Furthermore, from the first equation in (9) we have,

$$E[\bar{y}_m] = \rho_o,$$

where $\bar{y}_m$ is the average of the actual realized short-run growth rates for the m month period.

It should be noted that the short-run growth rate policy for which $E\,[\beta(t)] = \beta_o + \alpha_o t$ represents a policy of constant growth α_o over the m month period of interest. This short-run policy aimed at achieving a given long-run growth rate goal is, of course, implicitly expressed by the equation system:

$$\begin{aligned} y(t) &= \beta(t) + \varepsilon(t) \\ \beta(t) &= \beta(t-1) + \alpha_o + u(t) \end{aligned} \tag{14}$$

where the forcing policy α_o is chosen as $\alpha_o = 2(\rho_o - \beta_o)/(m+1)$. The equation system (14) is of the same form as the Kalman-Bucy model for short-run growth rates given in (9).

Returning to the adjustments model suggested earlier, write equation (8) as:

$$\begin{aligned} y(t) &= \beta^*(t) + \varepsilon(t) \\ \beta^*(t) &= \delta(t) + \rho + u(t). \end{aligned} \tag{15}$$

Suppose the same long-run average growth rate goal ρ_o is desired. Then the two models (14) and (15) have identical expected short-run growth rates if:

$$\beta_o + \alpha_o t = \rho_o + \delta_o\,(t)\ ,$$

or, equivalently, if the adjustment policy is given by:

$$\delta_o(t) = (\rho_o - \beta_o)\left(\frac{2t}{m+1} - 1\right). \tag{16}$$

It is readily shown that:

$$\delta_o(t) = \delta_o(t-1) + \frac{2(\rho_o - \beta_o)}{m+1}\ ,$$

which was suggested by the ad hoc empirical analysis previously discussed.

Thus, when the adjustment policy (16) is selected in (15), the growth rate models represented by (14) and (15) have identical average long-run growth rates ρ_o and the expected short-run growth rates $E[\beta(t)]$ and $E[\beta^*(t)]$ associated with the two models are also identical.

RECURSIVE BAYESIAN ESTIMATION

Now consider the situation in which the long-term policy vector α in (2) is assumed constant but unknown. An algorithm for recursive, Bayesian estimation of α is presented below. As mentioned above, while long-run Federal Reserve policy is likely to shift over time, it may be reasonable to regard it as fixed during a give time interval. To the outsider interested in estimating this fixed policy and updating these estimates as new data become available, the following model provides useful insights.

Suppose that at time $t = 0$, α has a proper normal prior distribution with known moments:

(17) $$\alpha \sim N(\hat{\alpha}(o), S_\alpha(0)),$$

and that, given $Y'(t-1) = (y'(1),\ldots,y'(t-1))$, the posterior distribution of α is normal

(18) $$(\alpha|Y(t-1)) \sim N(\hat{\alpha}(t-1), S_\alpha(t-1)).$$

Suppose, further, that the errors $\varepsilon(t)$ and $u(t)$ in (1) and (2) are normally distributed with known covariances R and Q, respectively. Then by Bayes theorem:

$$P(\alpha|Y(t)) = \frac{P(\alpha,y(t)|Y(t-1))}{P(y(t)|Y(t-1))} \tag{19}$$

$$\propto P(y(t)|\alpha,Y(t-1))P(\alpha|Y(t-1))$$

$$\propto |D(t)|^{-\frac{1}{2}}\exp[-\tfrac{1}{2}||y(t) - X\hat{\beta}(t)||_{D(t)^{-1}}]$$

$$|S_\alpha(t-1)|^{-\frac{1}{2}}\exp[-\tfrac{1}{2}||\alpha - \hat{\alpha}(t-1)||_{S_\alpha(t-1)^{-1}}]$$

where $\hat{\beta}(t)$ is the mean of the marginal distribution of $\hat{\beta}(t)$ given $Y(t)$.

It is shown by Enns [4] that:

$$\hat{\hat{\beta}}(t) = \hat{\beta}(t|t-1) + S_\beta(t|t-1)X'C(t)^{-1}(y(t) - X\hat{\beta}(t|t-1)), \tag{20}$$

where

$$\hat{\beta}(t|t-1) = T\hat{\beta}(t-1) + \hat{\alpha}(t-1) \tag{21}$$

$$S_\beta(t|t-1) = S(t|t-1) + S_\alpha(t-1) \tag{22}$$

$$C(t) = XS_\beta(t|t-1)X' + R \tag{23}$$

and $S(t|t-1)$ is given in (6).

By completing the square in α in the exponent of (19) it can then be shown that, given $Y(t)$, α has normal distribution with mean:

$$\hat{\alpha}(t) = \hat{\alpha}(t-1) + S_\alpha(t-1)X'(D(t) + XS_\alpha(t-1)X')^{-1} (y(t) - X\hat{\beta}(t|t-1)), \tag{24}$$

and covariance matrix:

$$S_\alpha(t) = S_\alpha(t-1) - S_\alpha(t-1)X'(D(t) + XS_\alpha(t-1)X')^{-1}XS_\alpha(t-1), \tag{25}$$

where $D(t)$ is given in (7).

In addition, $\beta(t)$ can be shown to have a posterior normal distribution with mean $\hat{\beta}(t)$ and covariance:

$$S(t) = S_\beta(t|t-1) - S_\beta(t|t-1)X'C(t)^{-1}XS_\beta(t|t-1). \tag{26}$$

The updated Bayesian estimator for α in (24) is similar in form to $\hat{\beta}(t)$ in (3). The difference $y(t) - X\hat{\beta}(t|t-1)$ is the error in the one-step ahead estimation of y(t) and its coefficient resembles the Kalman gain $K(t) = S(t|t-1)X'(t)D(t)^{-1}$ in (3). Hence, $\hat{\alpha}(t)$ consists of the previous estimate $\hat{\alpha}(t-1)$ with a correction for the latest observation. Furthermore, the estimator $\hat{\beta}(t)$ in (20) differs from (3) because of the presence of the one-step ahead forecast covariance $S_\beta(t|t-1)$ which includes a measure of the uncertainty about α through $S_\alpha(t-1)$.

It is interesting to note that, under rather mild assumptions, the difference $S_\alpha(t-1) - S_\alpha(t)$ is positive definite. A similar statement about S(t) is not generally true. This reflects the fact that α is fixed and additional observations can only increse the precision of its posterior distribution. On the other hand, $\beta(t)$ is a time-varying parameter whose posterior distribution may have progressively decreasing precision.

These observations lead to conjectures about the limiting distribution of α which are currently being studied. For the application at hand, small sample properties are likely to be of equal interest, however. Hence, the choice of the prior distribution moments for α may be crucial and offer a means of introducing a subjective opinion about Federal Reserve policy into the estimation procedure.

MAXIMUM LIKELIHOOD ESTIMATION OF THE COVARIANCES R AND Q

The above analysis assumed exact knowledge of the error covariances R and Q. When these matrices are not known the "reproducing" feature of the distributions of $\beta(t)$ and α is, in general, lost. To obtain (suboptimal) estimators numerical methods can be employed. For the special form of the money supply growth model, an approximate maximum likelihood method has been explored by Enns [4]. This approach is an extension of a model developed by Cooley and Prescott [1].

The basic estimation problem is the allocation of the error of estimation between the transient and permanent factors represented by R and Q, respectively. From (10) the state equation can be written:

(27) $$\beta(t) = \beta(N+1) - (N-t+1)\alpha - \sum_{s=t+1}^{N+1} u(s)$$

and

(28) $$y(t) = X\beta(N+1) - (N-t+1)X\alpha - X\sum_{s=t+1}^{N+1} u(s) + \varepsilon(t).$$

From (28) it can be seen that, given α, the vector $z(t) = y(t) + (N-t+1)X\alpha$ has a normal distribution:

(29) $$z(t) \sim N(X\beta(N+1),\ (N-t+1)XQX'+R)$$

Therefore, the vector $Z' = (z'(1),\ldots,z'(N))$ has a joint normal distribution:

(30) $$Z \sim N(\bar{X}\beta(N+1),\ W\otimes\Sigma + (I-W)\otimes R)$$

where W is the symmetric NxN matrix with elements

(31) $$w_{ij} = \min(N-i+1, N-j+1),\ i,j = 1,\ldots,N$$

(32) $$\Sigma = XQX' + R.$$

In (30), $\bar{X}' = (X',\ldots,X')$ has dimension rxNp and $\otimes$ is the Kronecker product operator.

Now consider the symmetric matrix G with elements g_{ij}

satisfying $r_{ij} = (1-g_{ij})\sigma_{ij}$ and $0 \leq g_{ij} \leq 1$, where r_{ij} and σ_{ij} are the individual elements of R and Σ, respectively. The purpose of G is to allocate Σ between R and XQX'. Then the covariance of Z in (30) can be written as:

(33) $$\mathrm{Cov}(Z) = V = ((1-g)I - gW) \otimes \Sigma + (I - W) \otimes A.$$

Here, g is an arbitrary value satisfying $0 \leq g \leq 1$ and A is the symmetric pxp matrix with elements $a_{ij} = (g-g_{ij})\sigma_{ij}$.

The likelihood function for Z is:

(34) $$L(\beta(N+1),\Sigma|Z) \propto |V|^{-\frac{1}{2}} \exp[-\tfrac{1}{2}||Z - \bar{X}\beta(N+1)||_{V^{-1}}]$$

where for the application presented here,

(35) $$X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ 0 \\ \beta_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

Now write $\tilde{\beta}' = (\beta_1,\beta_2)$ and consider the special case where $g_{ij} = g$ for all i and j. Then V reduces to

(36) $$V = F \otimes \Sigma$$

where $F = (1-g)I + gW$. This simplification causes allocation of Σ between R and XQX' to be the same for all elements of the covariance and reduces the computational effort of the estiamtion procedure.

The log likelihood function becomes:

(37) $$\begin{aligned} \ln L(\beta,\Sigma;Z) &= -\frac{p}{2}\ln|F| - \frac{N}{2}\ln|\Sigma| - \tfrac{1}{2}||Z - \bar{X}\beta||_{V^{-1}} \\ &= -\frac{p}{2}\ln|F| - \frac{N}{2}\ln|\Sigma| - \tfrac{1}{2}\sum_{s,t=1}^{N} f^{st}(z(s)-\tilde{\beta})'\Sigma^{-1}(z(t)-\tilde{\beta}) \end{aligned}$$

where f^{st} is the (s,t) element of F^{-1}. Using matrix differentiation methods, see Dwyer [3], the maximum likelihood estimators, for a given value of g, become:

$$(38)\qquad \hat{b}(g) = (\bar{X}'(F\otimes\hat{\Sigma})^{-1}\bar{X})^{-1}\bar{X}'(F\otimes\hat{\Sigma})^{-1}Z$$

$$= \left(\sum_{s,t=1}^{N} f^{st}\hat{\Sigma}^{-1}\right)^{-1} \sum_{s,t=1}^{N} f^{st}\hat{\Sigma}^{-1}z(t)$$

and

$$(39)\qquad \hat{\Sigma}(g) = \frac{1}{N}\sum_{s,t=1}^{N} f^{st}(z(s)-\hat{b}(g))(z(t)-\hat{b}(g))'$$

By substituting $\hat{b}(g)$ and $\hat{\Sigma}(g)$ for $\tilde{\beta}$ and Σ in (37) we obtain the concentrated likelihood function (except for a constant):

$$(40)\qquad l_c(g;Z) = -\frac{p}{2}\ln|F| - \frac{N}{2}\ln|\hat{\Sigma}(g)|.$$

Following Cooley and Prescott [1], the strategy is to search the interval $0 \leq g \leq 1$ for the value g* which maximizes (40).

As those authors observe, this method is computationally impractical if the NxN matrix F must be inverted for each value of g. However, they demonstrate that the variables can be transformed, independent of g, to produce a diagonal covariance matrix. In this extension the covariance of Z is block diagonal. Suppose D(g) = PFP' is the Cooley-Prescott transformation. Then

$$(41)\qquad Z^* \sim N(X_o^* b,\ D(g)\otimes\Sigma)$$

where $Z^* = (P\otimes I)Z$ and $X_o^{*\prime} = (I,\ldots,I)$ is the pxNp matrix consisting of pxp identity matrices. Also, partition Z* into N px1 vectors z*(1), ...,z*(N), and partition X_o^* into N pxp matrices X*(1),...,X*(N).

Now (38) becomes:

$$\hat{b}(g) = (X_o^{*\prime}(D(g)\otimes\Sigma)^{-1}X_o^*)^{-1}X_o^{*\prime}(D(g)\otimes\Sigma)^{-1}Z^* \tag{42}$$
$$= (\sum_{t=1}^{N} \frac{1}{d_t(g)} X^*(t)'\hat{\Sigma}^{-1}X^*(t))^{-1} \sum_{t=1}^{N} \frac{1}{d_t(g)} X^*(t)'\hat{\Sigma}^{-1}z^*(t)$$

and (39) becomes:

$$\hat{\Sigma}(g) = \frac{1}{N} \sum_{s,t=1}^{N} \frac{1}{d_t(g)} (z(s)-\hat{b}(g))(z(t)-\hat{b}(g))'. \tag{43}$$

Equation (41) requires knowledge of Σ which is not known. One possible method is to obtain a preliminary estimate $\hat{b}_{OLS}$ using ordinary least squares and use the residuals to form an estimate of Σ:

$$\hat{\Sigma}_{OLS} = \frac{1}{N} \sum_{s,t=1}^{N} (z(s)-\hat{b}_{OLS})(z(t)-\hat{b}_{OLS})' \tag{44}$$

This results in a two-stage generalized least squares estimation procedure resembling the seemingly unrelated regression model of Zellner [12].

When the assumption regarding the allocation matrix G, namely $g_{ij} = g$ for all i and j, is not reasonable, the log likelihood function (37) assumes a more complicated form because of the presence in (33) of the term $(I-W)\otimes A$. The maximum likelihood estimators of β and Σ, conditional on g and G, are more difficult to develop analytically. Furthermore, the search strategy must be extended to a multiple search over g and the $\frac{1}{2}p(p+1)$ unique elements of G. One possible advantage in expressing V as in (33) is that V^{-1} can be approximated by a truncated Taylor series not involving F^{-1} where $F = (1-g)I - gW$.

In general, the maximum likelihood method described above estimates XQX' but does not provide a unique estimate for Q. Since Q is required in the recursive Kalman-Bucy model a generalized inverse approach might be attempted since X generally is a singular matrix. However, for the special case studied in this paper it is seen that:

$$XQX' = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} . \tag{45}$$

Therefore, the identification problem is avoided.

To obtain the series $z(t)$, $t = 1,\ldots,N$, an estimate of α is required. It can be seen from (28) that the first differences in the $y(t)$ series satisfy:

$$\Delta(t) = y(t) - y(t-1) = X\alpha + Xu(t) + \varepsilon(t) - \varepsilon(t-1), \tag{46}$$

for $t = 2,\ldots,N$. In the example considered here, where $\alpha' = (0,\alpha_1,0,\alpha_2)$, we have

$$X\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} . \tag{47}$$

Therefore, ordinary least squares provides the unbiased estimator

$$X\hat{\alpha} = \frac{1}{N-1} \sum_{t=2}^{N} \Delta(t), \tag{48}$$

which is simply the mean of the differences. This estimator is not, in general minimum variance. The series $z(t) = y(t) + (N-t-1)X\hat{\alpha}$ can then be used in the above estimation procedure.

For the recursive Bayesian estimation phase the maximum likelihood estimators can be used to obtain the initial prior parameters. First note that the covariance $S_\alpha(0)$ has the special form:

$$S_\alpha(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & s_{11} & 0 & s_{12} \\ 0 & 0 & 0 & 0 \\ 0 & s_{21} & 0 & s_{22} \end{bmatrix}. \tag{49}$$

The s_{ij} might be estimated from the residuals $\Delta(t) - \hat{\alpha}$. On the other hand, they might be assigned arbitrarily large values to reflect an initial "vague" prior feeling about α. The estimate $\hat{\alpha}$ could be used to assign a prior mean $\alpha(0)$.

Similarly, the non-zero elements of $\hat{\beta}(1|0)$ might be set equal to the maximum likelihood estimator $\hat{\beta}(N+1)$, where time N+1 corresponds to time $t = 0$ in the Kalman-Bucy model.

SUMMARY

Computer algorithms are being developed by the authors for implementing both the step-wise maximum likelihood estimation procedure for R and Q described above and the recursive Bayesian procedure for estimating α and, subsequently, β which employs the former maximum likelihood estimates.

When the model interpretation regarding short-run and long-run growth rates of the money stock aggregates M1 and M2 is considered, these theoretical results developed for the general Kalman-Bucy model provide more adequate statistical methods than existing ad hoc ones for empirically examining and estimating the Federal Reserve policy pertaining to these growth rates. Using these new statistical methods, Bayesian inference and forecasting with respect to monetary policy can be pursued, and these Bayesian methods permit prior opinion about monetary policy to be used in such analyses. In addition, using these statistical methods, conditions under which patterns in monetary policy are recognizable can be investigated and Bayesian comparison of hypotheses regarding policy undertaken.

FIGURE 1: MONEY STOCK MEASURES
MONTHLY FIGURES, IN $BILLION, SEASONALLY ADJUSTED

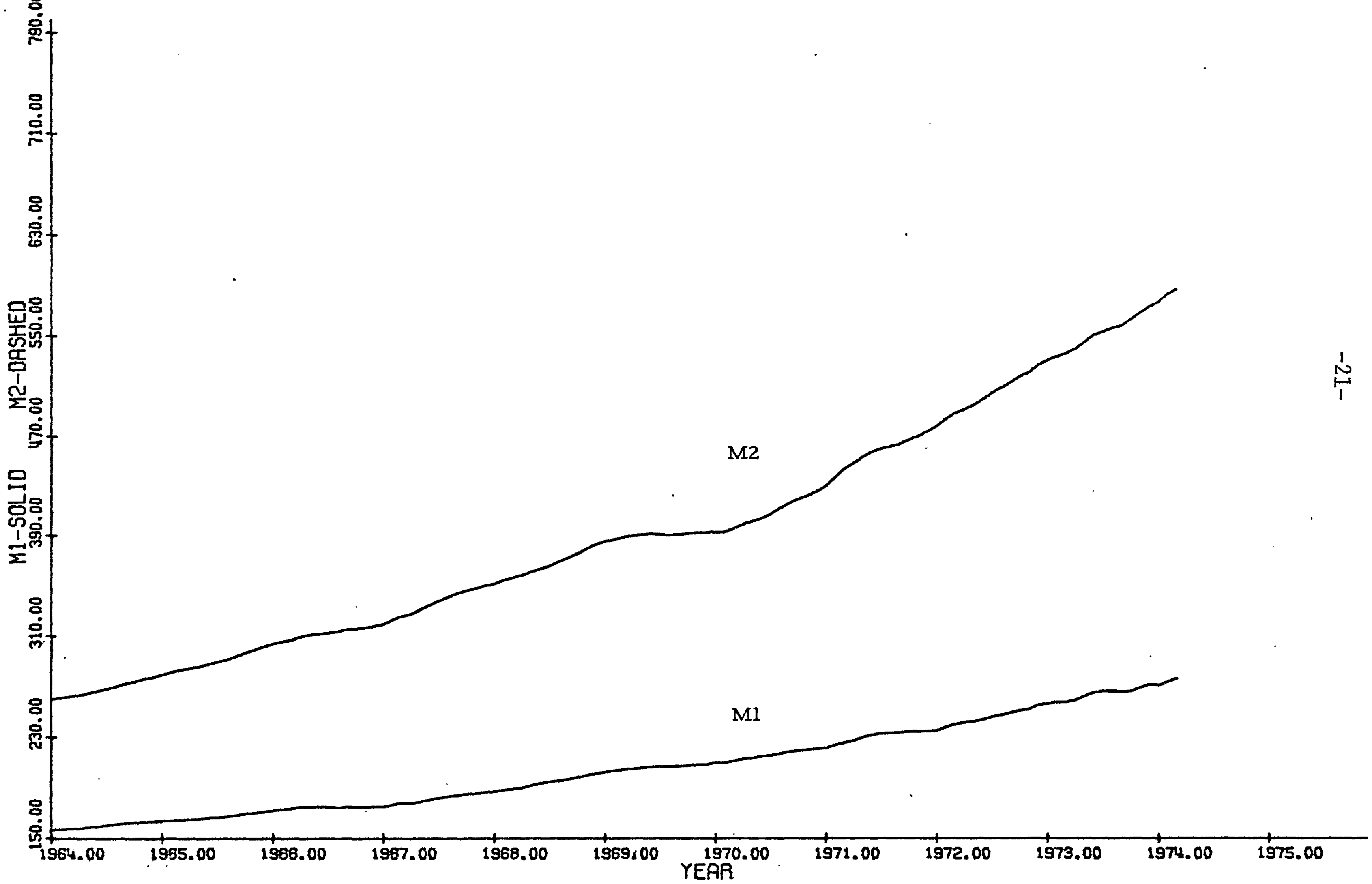


FIGURE 2: GROWTH RATE OF M1
ACTUAL MONTHLY AND LONG-RUN POLICY ESTIMATE

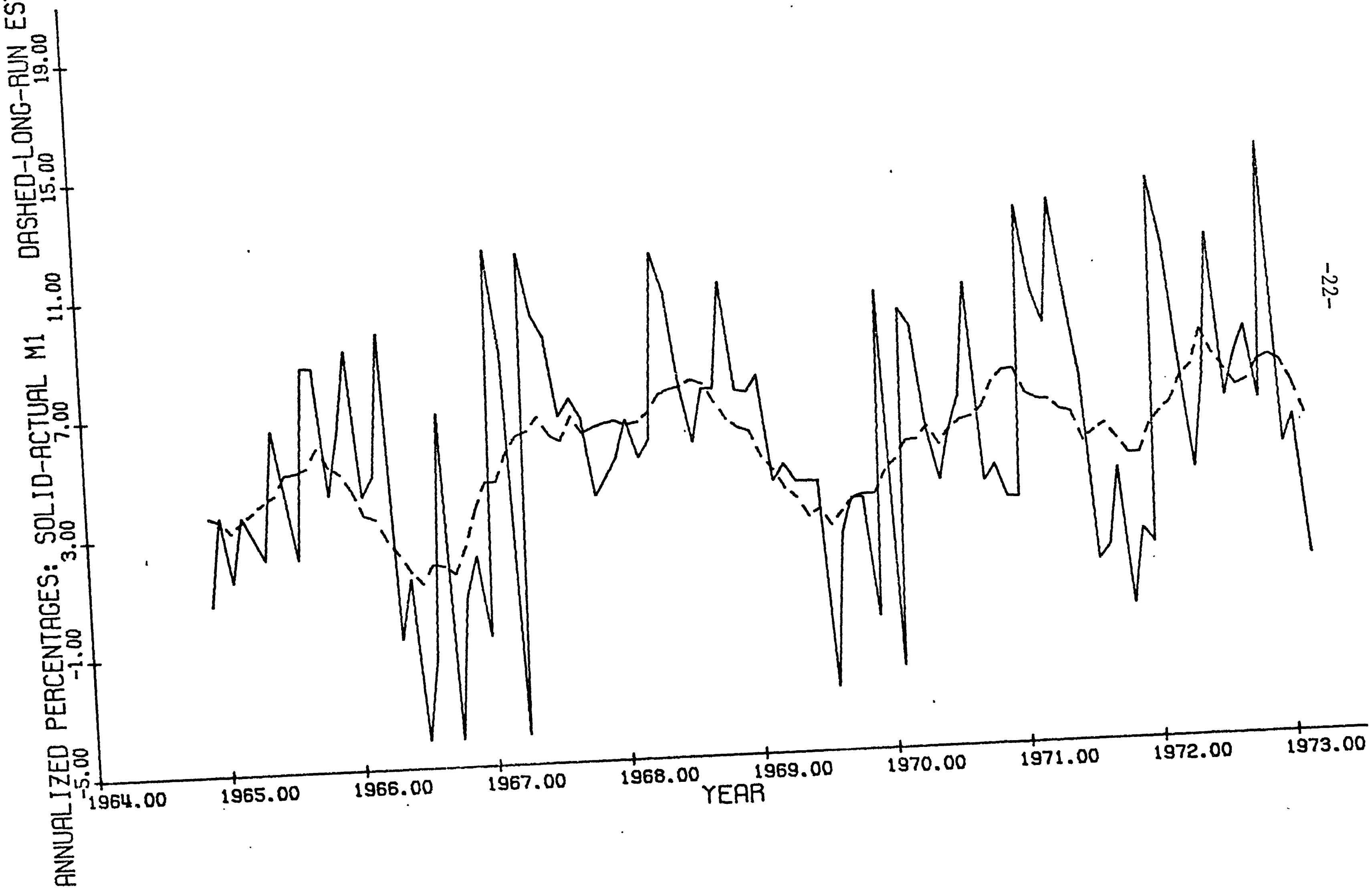


FIGURE 3: GROWTH RATE OF M2
ACTUAL MONTHLY AND LONG-RUN POLICY ESTIMATE

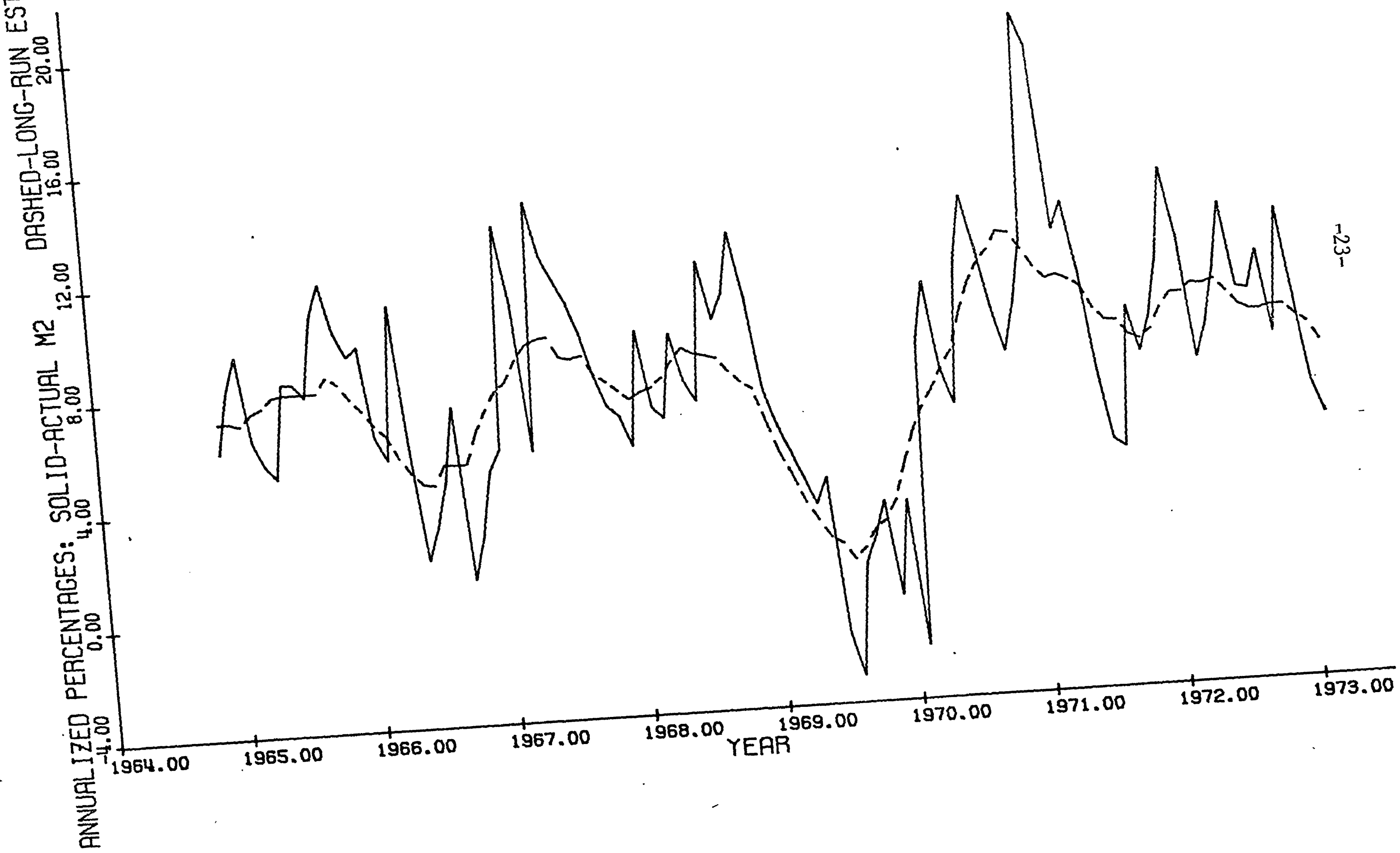


FIGURE 4: GROWTH RATE OF M1-ANNUALIZED PERCENTAGE
ACTUAL MONTHLY, LONG-RUN AND SHORT-RUN POLICY ESTIMATES

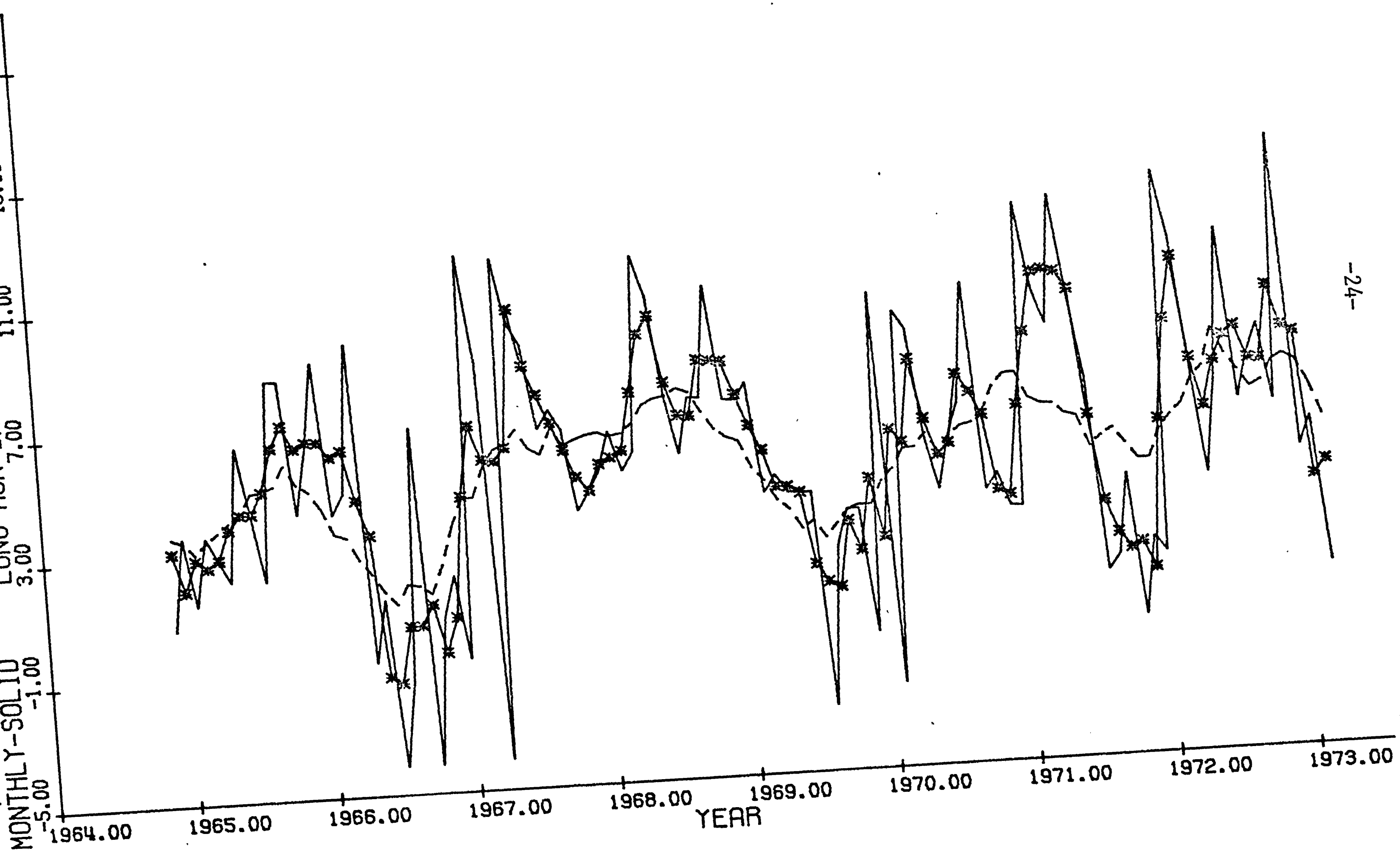


FIGURE 5: GROWTH RATE OF M2-ANNUALIZED PERCENTAGE
ACTUAL MONTHLY, LONG-RUN AND SHORT-RUN POLICY ESTIMATES

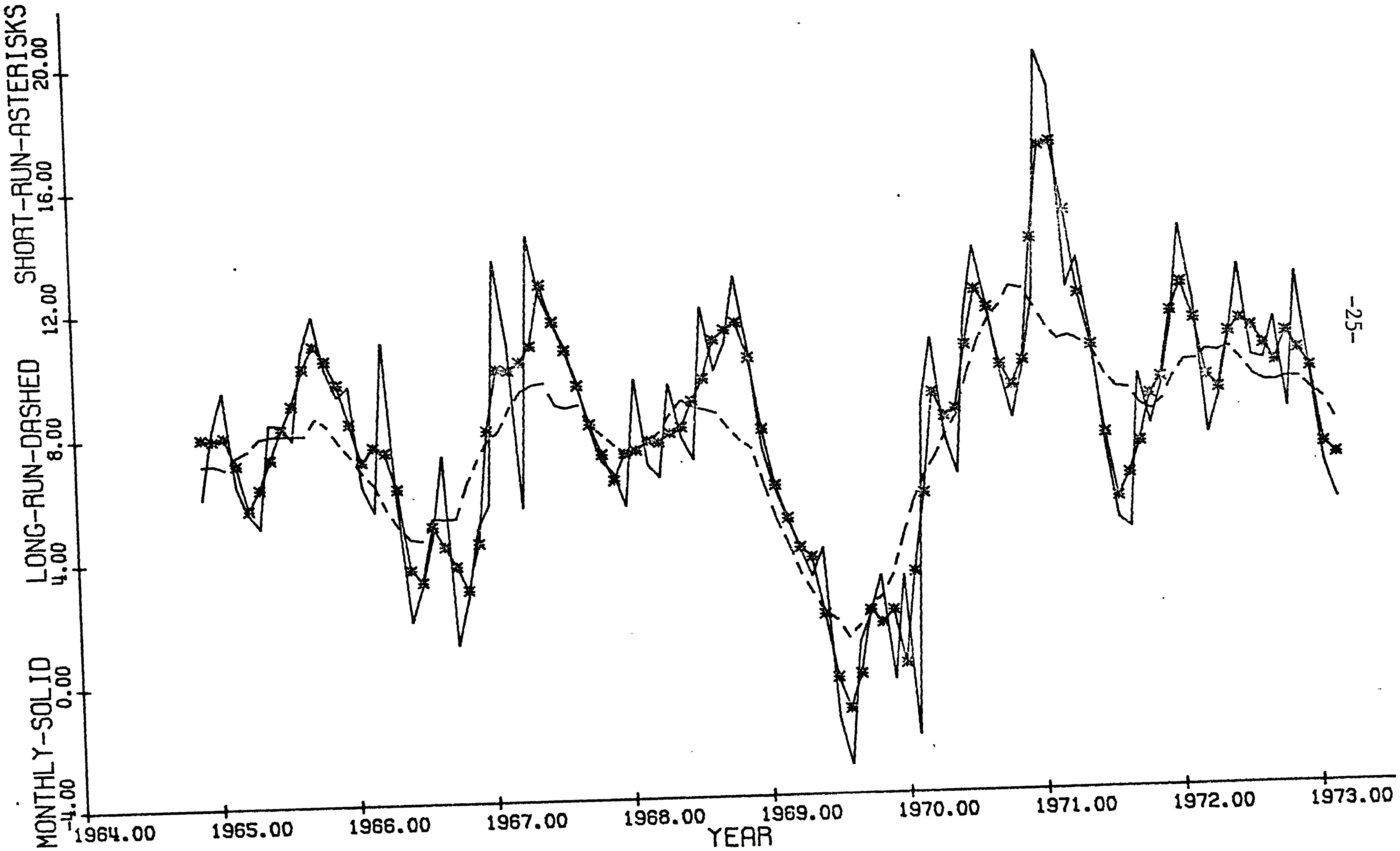


FIGURE 6
ERROR BETWEEN SHORT-RUN POLICY ESTIMATE AND ACTUAL GROWTH RATE OF M1

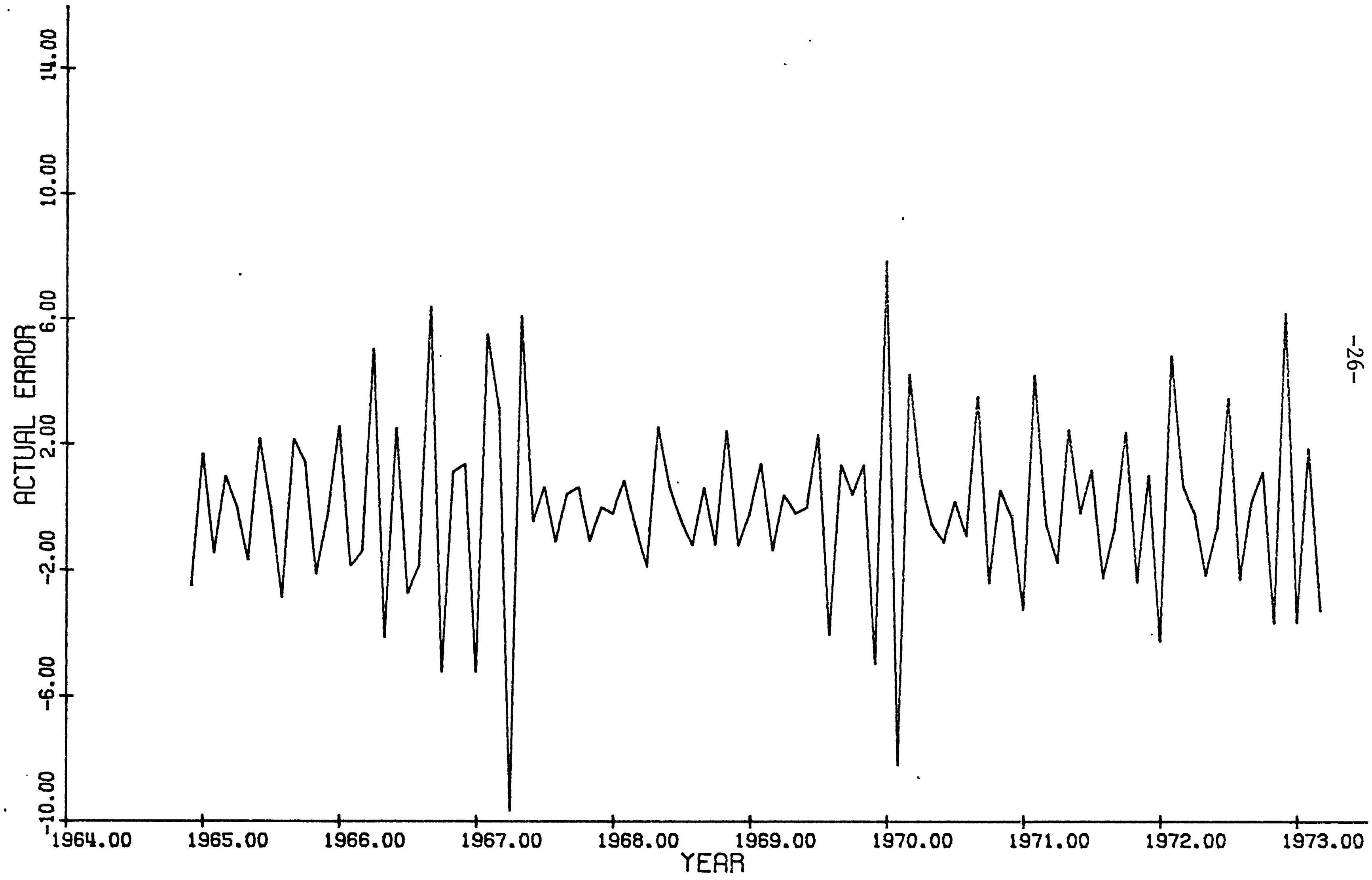


FIGURE 7
ERROR BETWEEN SHORT-RUN POLICY ESTIMATE AND ACTUAL GROWTH RATE OF M2

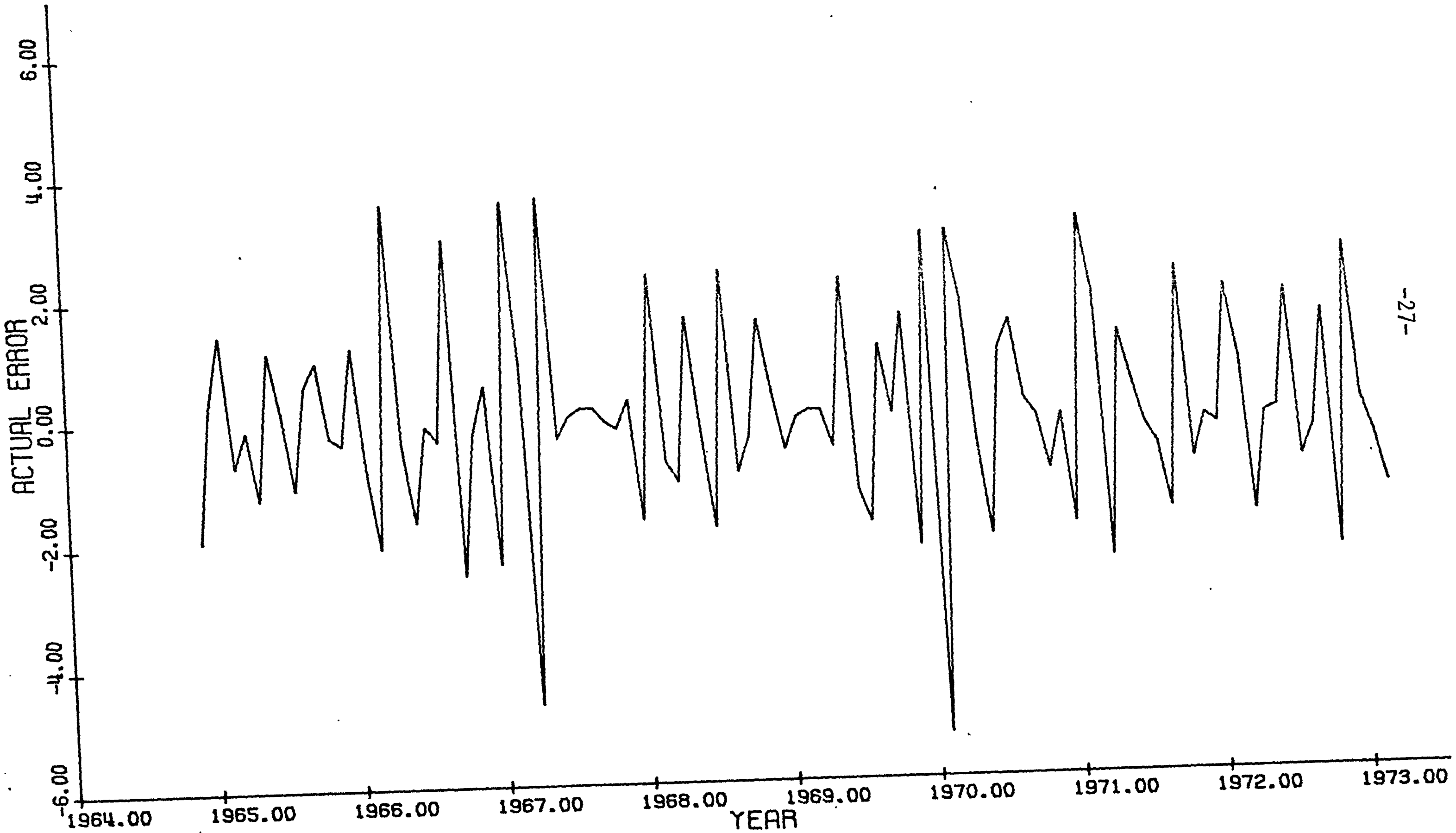


FIGURE 8

FREQUENCY DISTRIBUTION FOR ERRORS BETWEEN ACTUAL M1 AND SHORT-RUN POLICY

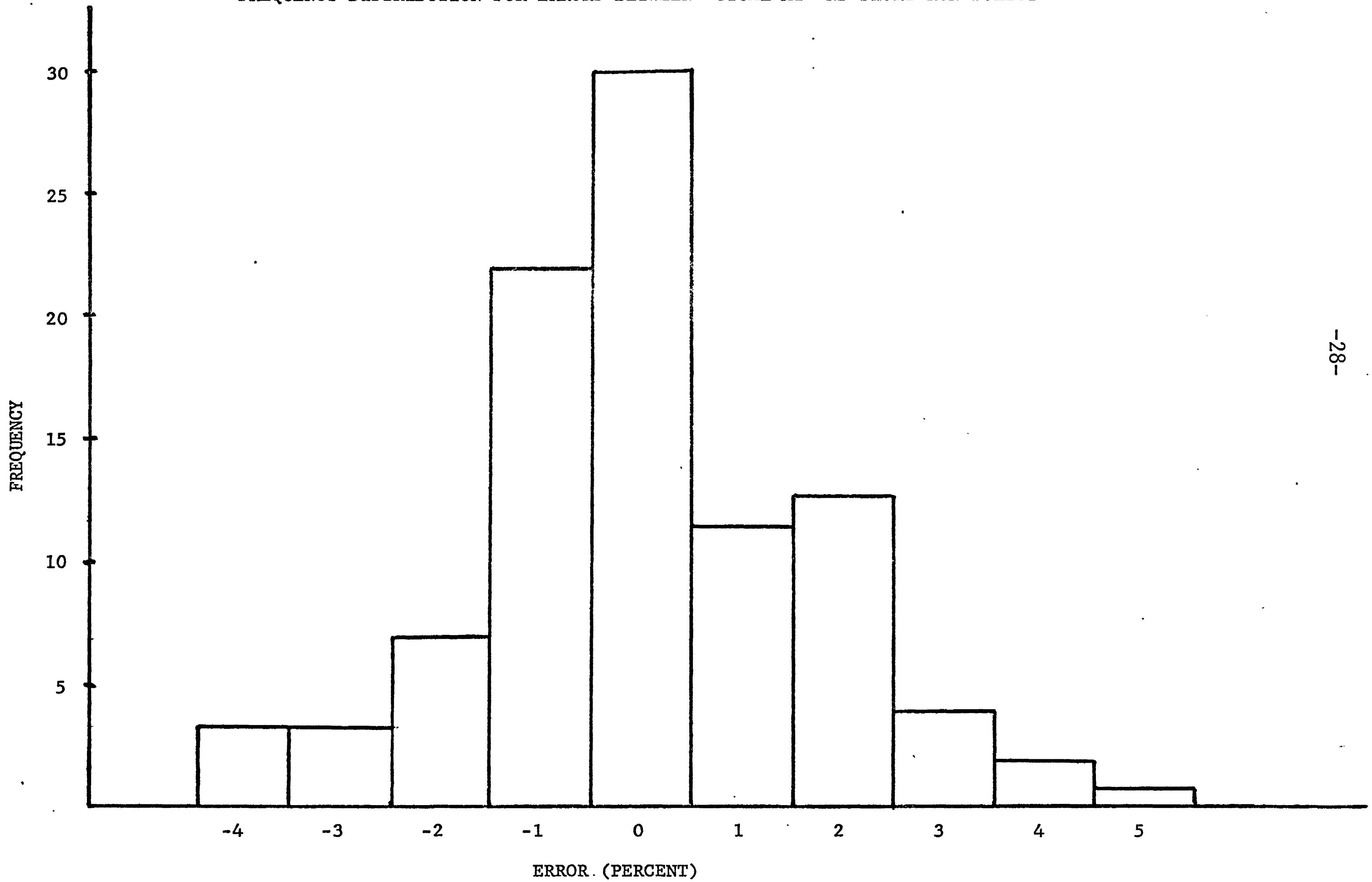


FIGURE 9

FREQUENCY DISTRIBUTION FOR ERRORS BETWEEN ACTUAL M2 AND SHORT-RUN POLICY

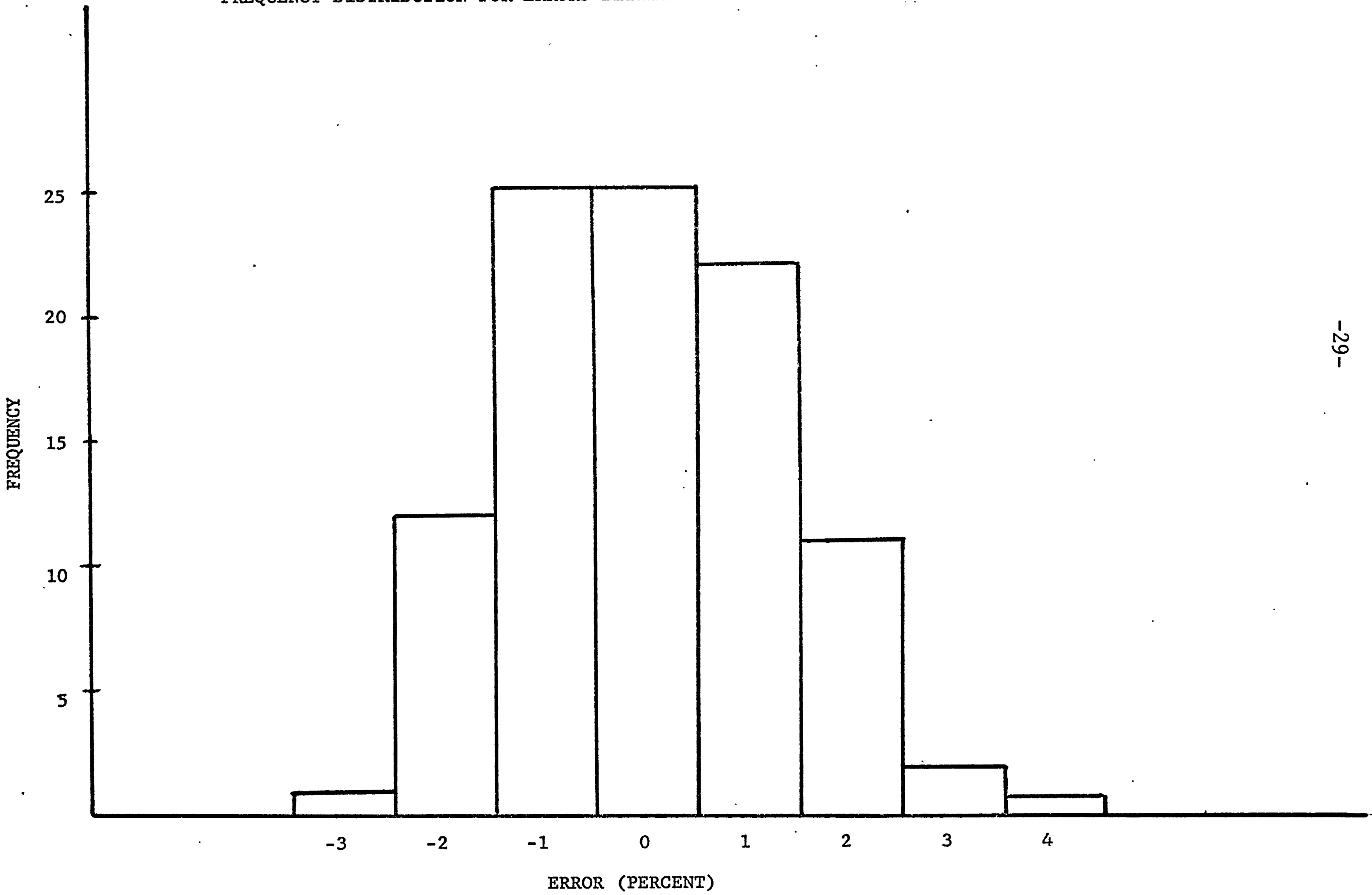


FIGURE 10
MONTH-TO-MONTH CHANGE IN SHORT-RUN POLICY ESTIMATE FOR M1

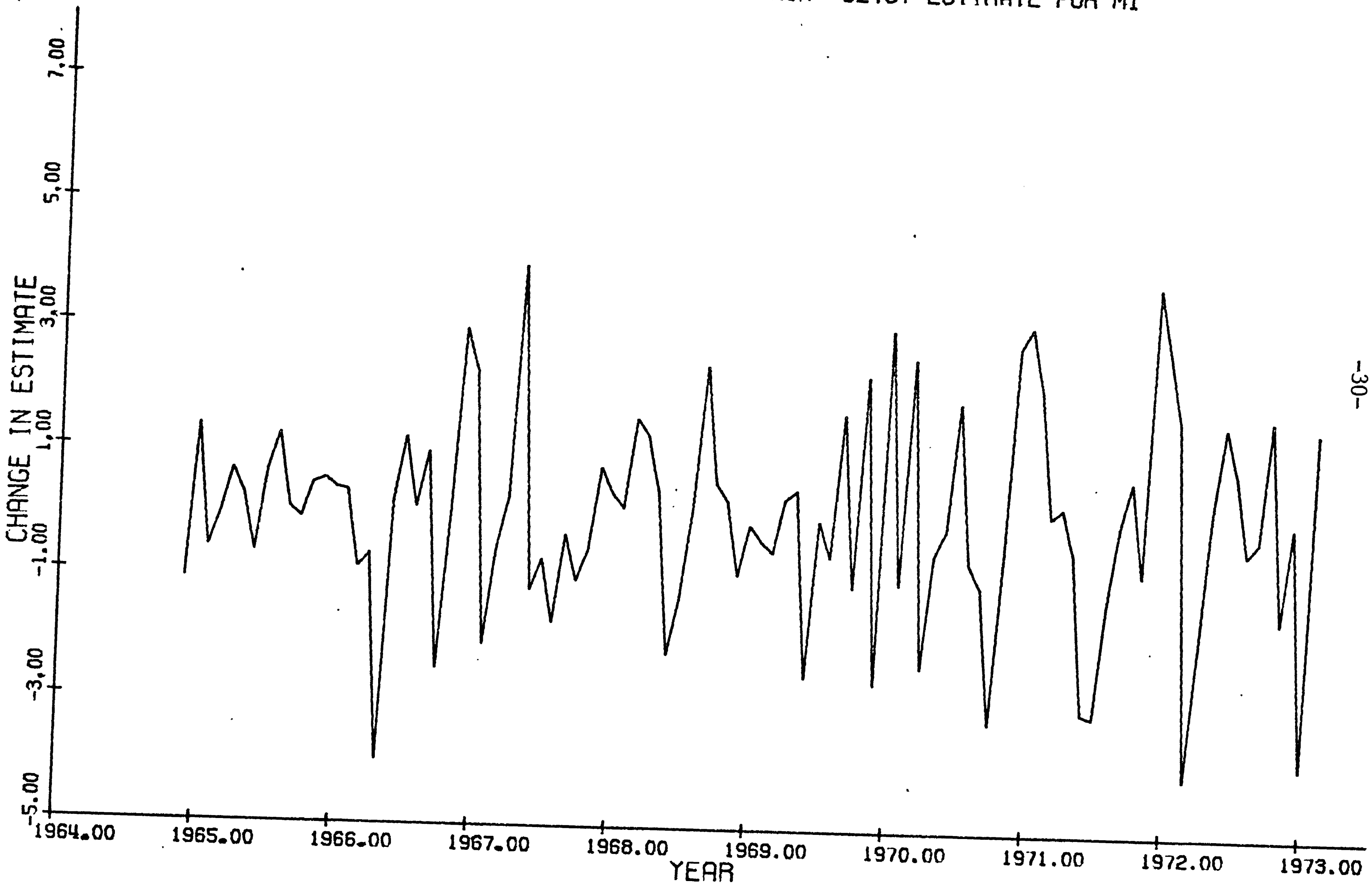


FIGURE 11
MONTH-TO MONTH CHANGE IN SHORT-RUN POLICY ESTIMATE FOR M2

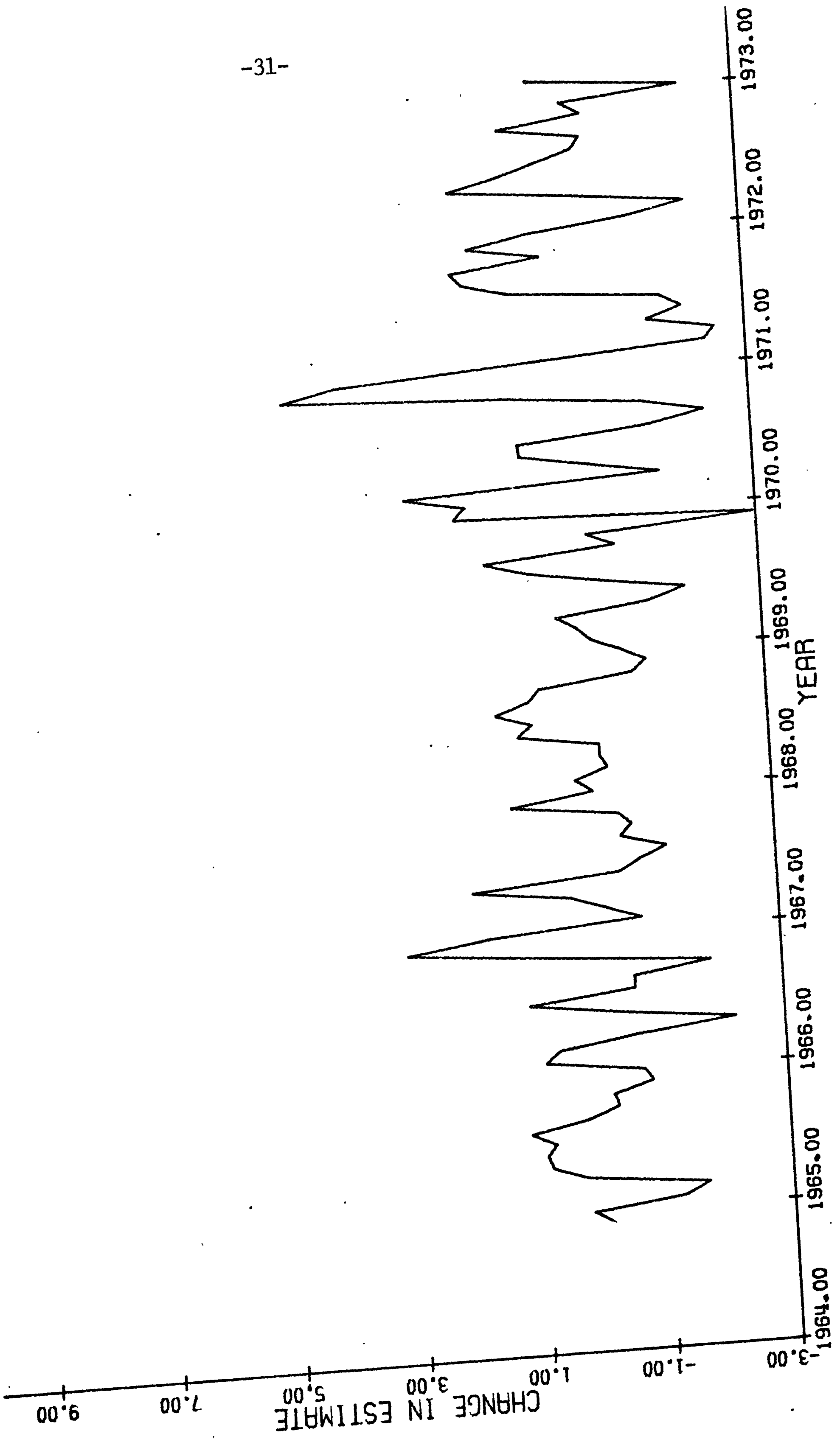


FIGURE 12

FREQUENCY DISTRIBUTION FOR MONTH-TO-MONTH CHANGES IN SHORT-RUN POLICY FOR M1

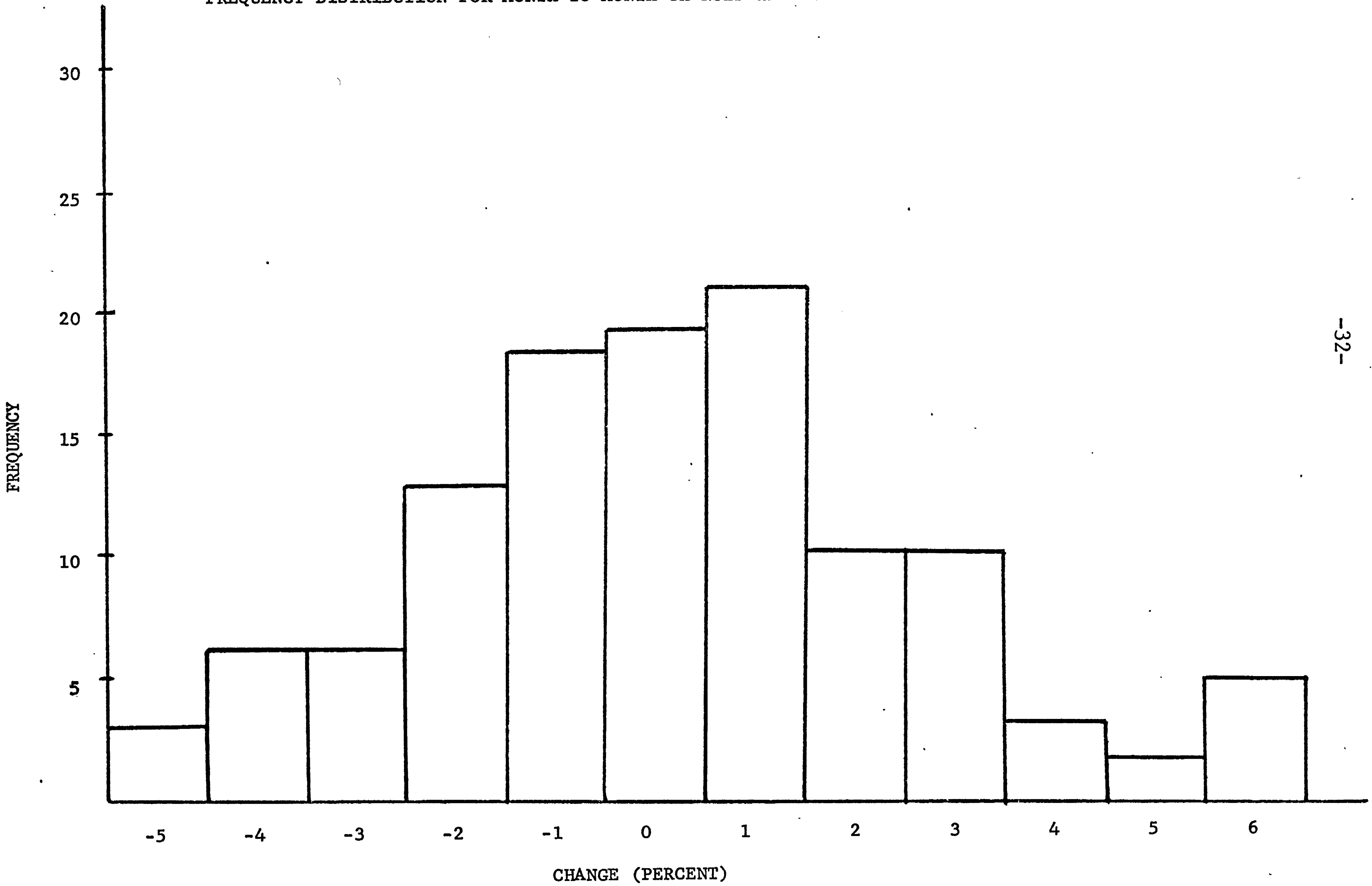
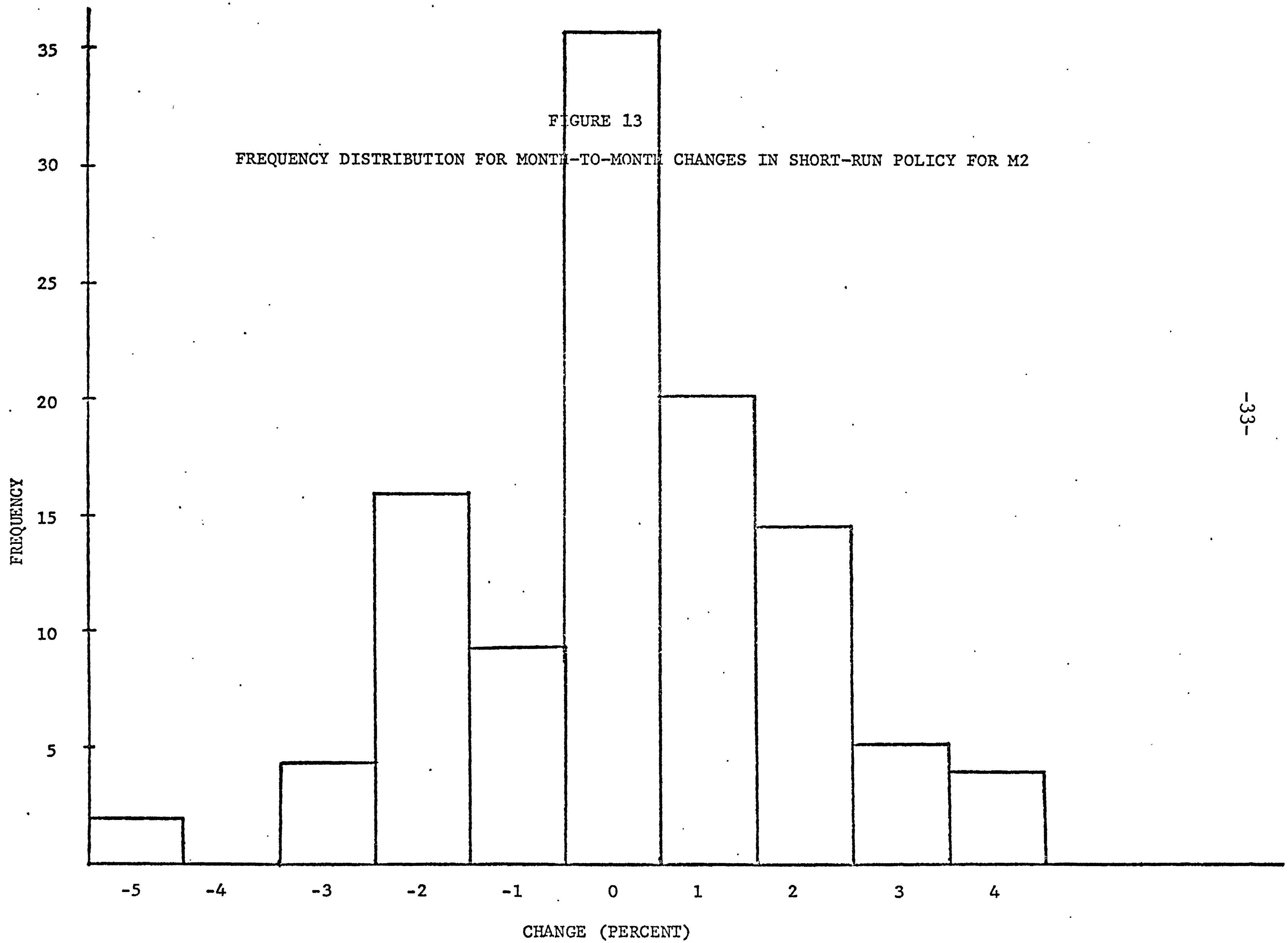


FIGURE 13

FREQUENCY DISTRIBUTION FOR MONTH-TO-MONTH CHANGES IN SHORT-RUN POLICY FOR M2



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