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Final Report

THE PROBLEM OF GEODETIC DATUM AND ACCURACY

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1. SIZE AND SHAPE OF THE EARTH

Since the time of Eratosthenes (ca. 300 B.C.) the classical approach to the solution of this problem has been to measure long arcs on the surface of the earth and combine these with astronomically-observed positional data to compute a best-fitting mathematical surface. This method has, with the American continental triangulation and the Europe-Asia-Africa complex of triangulation, probably reached its practical ultimate in application in the last ten years. There are two basic circumstances that place practical limitations on this classical approach:

a. The accuracy, and even significance, of astronomical determinations of position are dependent on the assumption that a "true" vertical direction is available at the point of observation. For several generations it has been known that the only available fundamental vertical reference, the direction of gravity, is not absolute and is deflected by random gravitational forces caused by the fact that the earth's crust is composed of matter that is not uniformly distributed. This effect results in a "deflection of the vertical."

b. It has not been feasible with currently-operational measuring techniques to measure across large water spans. Thus, satisfactory connections between the separate continental land masses have not been made.

2. DEFLECTION OF THE VERTICAL

It has been demonstrated by Stokes, Clarke, Helmert, Hayford, and more recently, by Vening-Meinesz and Heiskanen, among others, that it is possible to calculate the effects of the deflections of the vertical if sufficient systematic observations of the acceleration of gravity be made on a world-wide basis. The magnitude of such an operation is tremendous, and, even now, with a tremendous program of observing and correlating gravity data the goal has not been reached, although serious attempts are being made to derive valid conclusions based on the present incomplete data. Obviously, extensive observations cannot be made in the extensive areas of the world that are not friendly to the U.S. Also, it has been difficult to obtain sufficient observations in the ocean areas. Some relief for these situations is now provided by the probability of deriving extensive conclusions about the gravity field by analysis of observed orbits of artifical earth satellites.

3. ARC MEASUREMENT

The direct measurement of a distance on the earth and the coordinates of its termini resolves itself into the procedure usually designated "arc measurement." Although, except for base-lines, the method seldom involves a direct length measurement, it is, in effect, the measurement of a series of comparatively short chords (5-50 miles) computed from what is, in effect, a series of plane triangles observed by triangulation, usually a comparatively short distance above the sea-level surface. By appropriate computation, involving knowledge of the height of each line above sea-level, these series of chords are reduced to equivalent arc-lengths at the theoretical sea-level surface. As stated in Para. 1, almost all the basic triangulation in all the civilized nations of the world has been of quality appropriate for this use, and, when inter-connected across national boundaries, has been computed into several continent-wide networks with many astronomic stations, which provide good solutions for each isolated continental mass. It is not likely that the parameters so derived, and now in common use for the reduction of triangulation data for engineering operations, can be significantly improved by re-computation with additional measurements of the same kind. A very substantial improvement is possible, however, if the continental masses can themselves be connected by direct measurement. This would not only improve the accuracy of the parameters of the size and shape of the earth, but would, more important, make it possible to relate all geographic positions, throughout the world, on a common datum.

4. REFERENCE SPHEROID

If positions of points are to be stated in terms of geographic coordinates, i.e., latitude and longitude, a mathematically-expressed surface must be assumed for the sea-level surface of the earth. This figure may be a sphere or some more-complicated surface. It is the type of surface to be assumed, the numerical parameters by which it is expressed, and the extent to which the actual non-mathematical shape of the earth conforms to this idealized figure, which, together, constitute the problem of geodesy. Since very early times, the sphere was assumed to be the ideal shape to represent the earth. What might be called the beginnings of modern geodesy was the effort of Jean Louis Picard in 1669-70, where triangulation with telescopes was used to measure a meridional arc in France. This work was based on the assumption that the earth figure was spherical. Sir Isaac Newton, utilizing the results of this work, proved (1687) that, in theory, the earth should be an oblate spheroid of revolution. This was demonstrated by the French in 1735-37 when they made their famous arc measurements in Lapland and in Peru. This ideal shape for the basis of computations has generally been used for almost all geodetic work since then, although there has been a multitude of sets of parameters determined and used by many distinguished observers and organizations. The work of Hayford in 1910 led to the computation of a figure that was adopted in 1924 as the International Ellipsoid by the International Geodetic Association. Subsequent work, particularly involving artificial earth satellites has given reason to suspect that this figure can be improved slightly.

5. SPHEROIDS IN USE

The U.S. Coast and Geodetic Survey has adopted the Clarke Spheroid of 1866 for its work in the United States. Based on this precedent, Canada and the Central American countries have also adopted this spheroid. Most of the newly-adjusted work in Europe and Africa as well as in Asia has been computed on the International Ellipsoid. To indicate the range of the parameters, following are the data for several ellipsoids (of revolution):

Figure	Semi-Major Axis (a), Meters	Flattening (a-b)/a
Everest 1830	6,377,276	1/300.8
Bessel 1841	6,377,397	1/299.15
Clarke 1866	6,378,206	1/295.0
Clarke 1880	6,378,249	1/293.5
Helmert (1907)	6,378,200	1/298.3
International (1910)	6,378,388	1/297.0
Krassowski (1938)	6,378,245	1/298.3
Hough (1957)	6,378,270	1/297.0
Army Map Service (1960)	6,378,160	1/298.3
NASA (Kaula) (1961)	6,378,163	1/298.24

6. GEODETIC DATUM

When a reference figure has been selected, there still remains the problem of datum. Some particular point on the reference figure must be selected as an initial point from which all other points in the system are computed. The latitude and longitude of this point must be assigned and held fixed and also an azimuth to some adjacent station must be held fixed. Thus, it is possible for many separate surveys to be computed on the same reference spheroid but they will still not be based on the same geodetic datum, unless they are all directly connected by some measurement to the same initial point. Thus, it is seen that, at present, there is no way of expressing geodetic latitudes and longitudes in Europe and in America referred to the same datum (neglecting the rather weak connection across the North Atlantic Ocean by HIRAN measurements). For various scientific reasons, it is highly desirable to effect connections between the continental land masses in order to have all geodetic positions expressed in a single world-wide datum.

7. INTER-CONTINENTAL TIES

Obviously, standard triangulation techniques cannot be utilized in effecting inter-continental ties, because of the long single measurements required in the ties. It seems that one of the longest lines of triangulation ever observed was 237.7 kilometers long, observed in California by the U.S. Coast and Geodetic Survey shortly before 1900. Present limits on the HIRAN procedure seem to be of the order of 600-800 kilometers. However, some very promising techniques for extremely long-range distance measurements have recently been suggested. The most promising appears to be one that is an outgrowth of the use of ballistic cameras for tracking test missiles at the Atlantic Missile Range. Extremely precise and sophisticated systems have been conceived, designed, constructed, and used under many test conditions for the determination of precise positions of high-speed missiles at long distances from the observing sites. It seems quite practicable to utilize an extension of this system to make the observations for specially-organized operations for the making of inter-continental ties.

8. USE OF BALLISTIC CAMERAS

The use of ballistic cameras for inter-continental ties involves some bold thinking, simple principles (although complex in application), a revival of traditional astrometric principles in use for a couple of generations by astronomers, and the utilization of one new piece of hardware—the long-range rocket or an artificial earth-satellite. The basic operation, as presently considered, would involve sending a rocket into such a trajectory (and altitude) that it would be substantially above the horizon for observers on both continents to be connected. The rocket would be put into trajectory at night (for both sets of observers) and would be equipped with equipment capable of putting forth a series of high-intensity, short-duration flashes, either on command or in accordance with a pre-determined schedule. Two or more ballistic cameras would be established at located points on each continent to be connected. Shutters would be open for a short time before, during, and after the flashes, thus recording the positions of the flashes photographically, and simultaneously recording a number of stars on the same plate. Scaling the plate coordinates of the flashes, and also of the stars, would provide data which, combined with the known Right Ascensions and declinations of stars in the same vicinity on the plate, would permit computation of a direction and angle of elevation from each camera station to each flash. Obviously, if two or more cameras on each continent are accurately related to the same geodetic datum, the position of each flash can be computed, relative to that same datum. If such positions can be computed from datums on two continents, the inter-relationship of the datums can be determined. The instrumentation and computation requirements are sophisticated and expensive but the system is solidly based on theory and on extrapolation of reliable tests under conditions which need only to be expanded in scope in order to be fully operative for geodetic measurements.

9. ACCURACY OF BALLISTIC CAMERAS

The system of position determination, essentially a large-scale triangulation process, just described, is of practical application only if its inherent accuracy approaches that of conventional geodetic techniques. It has been repeatedly demonstrated in many different photogrammetric applications, ranging from astrometry to mapping and military photo-interpretation, that, with first-order instrumentation, plate coordinates can be scaled with an internal accuracy of about 2 microns. External influences, such as emulsion instability and instrument calibration, probably increase this to an absolute value of approximately 3 microns. This means that, using a ballistic camera of 300 millimeter focal length, the direction to a single flash-point can be determined to approximately 2 seconds of arc, with respect to the reference of the plate. The positions of the stars, visible on the same plate and scaled in the same manner as the flash-point, are usually known to less than one second of arc, although there are systematic errors in the star catalogs which may increase this value slightly. It should be noted that one of the basic assumptions on which the confidence of accuracy is founded is the assumption of practical elimination of the refraction problem. It is assumed that the flash point will be high enough above the earth so as to be essentially beyond any effects of the atmosphere. Thus, it is assumed that the flash position as recorded is subject to the same refraction effects as the neighboring star images to which it is referred. Thus, in the data reduction process using the absolute positions of the stars, the effects of refraction are eliminated without need for evaluation.

10. COMPARISON WITH THEODOLITE MEASUREMENTS

With modern first-order theodolites, angle measurements may be made wherein there is significance in the first decimal-place of seconds. This might lead to conclusions that conventional measurements might have an advantage over ballistic camera methods in the range of a decimal order. However, there are three conditions that modify this apparent advantage:

1. It takes many theodolite angle measurements to cover the same distance extent that can be reached with a single ballistic camera shot.

2. Each ballistic camera observation is referred to a practically absolute <u>direction</u> reference, the star background itself, while direction is carried through a series of theodolite <u>angle</u> measurements by calculation.

3. The ballistic camera technique involves multiple closely-spaced flashes at each "shot," all recorded on one plate at the camera. Thus, each plate provides a series of directions, each independent in itself, but all referred to the same set of reference stars. Thus, multiple flashes and multiple reference stars provide redundancies which, subjected to proper statistical treatment, give increased precision and accuracy.

It is concluded, therefore, that results comparable with standard first-order triangulation may be accomplished by use of ballistic cameras and high-altitude flashes recorded against a star background.

11. USE OF ASTRONOMICAL OBSERVATORIES

One of the principal uses of equipment at the various astronomical observatories has been in the field of astrometry. Photographic techniques have been used to determine the positions of stars, planets, comets, asteroids, etc. Except for final computations these procedures are practically identical with those outlined for high-altitude flash triangulation. Obviously, many of the excellent astronomic facilities are suitable for use in flash triangulation, using their own standard data reduction techniques. In fact, they have one major advantage over ballistic cameras in that their focal lengths are usually several times longer than ballistic camera objectives. Since plate-scaling accuracy is quite independent of focal length, the error in angular direction is an inverse function of focal length. Hence directions determined to flash points by use of astronomical facilities can be expected to be several times more accurate than those determined by ballistic cameras. A major difficulty lies in the fact that such observatory equipment is not portable, but is permanently fixed in position. Hence, such facilities are useful only when the individual installations happen to be favorably situated in geographic position to provide high-strength geometry for a particular flash location in space. However, in such internationally important endeavors as inter-continental ties, every effort should be made to accomplish simultaneous utilization of as many facilities as possible not only to provide a high order or redundancy in the determination but also to insure at least the probability of satisfactory conclusion of an expensive operation in face of the many external factors that may cause failures at individual stations. It should also be pointed out that the field of view of long focal-length equipment is quite narrow, which increases the difficulty of application to flash triangulation.

12. THE PROBLEM OF POSITIONAL ACCURACY

Since it is anticipated that flash triangulation for inter-continental ties will be undertaken only with sufficient instrumentation to provide abundant redundancies in each solution, an important phase of the data reduction will be to provide figures indicating the magnitude of the error in the solution. This will involve a sophisticated rigid least-squares adjustment to be accomplished by electronic computer. However, when the computation is accomplished, the quality and accuracy of the positional data obtained must be assessed if any utility is to be obtained from the operation. In connection with the appraisal of the errors and in placing an estimate on the accuracy of the positional data, every effort must be made to include an assessment of all factors which might influence the results. Since the problem to be solved is primarily one of datum-relationships, and since any set of observing sites must be inter-connected on one datum, presently accomplished only by conventional survey methods, two questions arise which involve quantities that may enter into the magnitudes of the solution errors:

1. What is the magnitude of the accumulated error in the basic control surveys connecting a set of observing sites in a single common datum?

2. What is the magnitude of the errors in the surveys connecting the individual instruments to the basic control surveys, since it is improbable that, in general, the instruments will be positioned directly on basic control points?

13. POSITIONAL ACCURACY OF BALLISTIC CAMERA SITE

At first thought, it might seem that the camera-site must be related to geodetic datum by first-order measurements. Actually, since the distances from high-order control points to the observing sites are usually comparatively short (less than seven miles) lower-order measurements will determine the positions of the sites relative to datum with adequate accuracy. In view of the long distances between sites, the largest part of the error in the observing net is actually in the geodetic net itself.

14. ERROR IN CONTROL SURVEYS

The fundamental question in the problem of inter-continental ties by flash triangulation is that of the accuracy with which the inter-relationship of the ballistic camera or observatory sites is known, as referred to a common geodetic datum. The problem of the establishment of a geodetic datum is very complex and is beyond the scope of this investigation. It is not proposed to discuss at length the correlative problem of assessing the accuracy with which a particular adjusted control net actually represents the datum on which it has been computed. This is a question to which partial answers are given when inter-continental ties have been accomplished and answers forthcoming as to the relationships between datums thus connected. However, it is important that estimates be available for the error in the lengths of the lines relating the observing sites from which flash-point positions are computed. It should be noted that lengths computed from geodetic position will not be greatly affected by the original choice of reference spheroid on which the geodetic datum is based. The greatest single source of error in such lengths will be the cumulative effect of the errors of observation which are distributed imperfectly during the course of the complicated procedure of geodetic adjustment. There has been little actual numerical investigation of the effect of this distribution and a theoretical derivation would be hopelessly complicated and, if general in application, too simplified to have great reliability. Let it be sufficient to state that the primary assumptions of the adjustment procedure: (1) that all systematic errors have been compensated before adjustment; and (2) that the remaining accidental errors have a normal distribution, are not completely met in practice and that, therefore, the Method of Least Squares is, at best, only a practical compromise to the problem of adjustment of discordant observations to a probable, mathematically-consistent configuration. The fact remains that there are some undefinable unevaluable systematic errors, and some accidental errors that do not follow a normal distribution. It is thus concluded that the most reliable, generally-applicable, empirical formula for defining the error as a function of the distance should not be in terms of the square root of the distance, as Least Squares theory might suggest; nor yet in terms of the first power of the distance, as would be suggested by the assumption that the error is completely systematic. As a compromise, it would seem reasonable to propose a formula involving the "two-thirds" power of the distance. This exponent implies that the error is largely normal and accidental, but increases the value slightly for the probability that some systematic error also remains. Extensive examination of distributed linear misclosures in the national network of the U.S. Coast and Geodetic Survey indicates that a general conformity with the misclosures will be obtained from the formula:

$$E = .06(K)^{2/3}$$

where:

E = error, in meters
K = distance between stations, in kilometers

It must be emphasized that the above formula is only applicable in a very general way, and the error computed from it should be rounded off to the nearest meter. It must also be pointed out that the formula is intended to work with the distance between the points as actually measured through the control system. In most cases, this will not be the air-line distance. It is quite possible for two stations a hundred miles apart to be connected only through a series of triangulation arcs totaling three hundred to four hundred miles in length.

15. PROPORTIONAL ACCURACY

The above formula for error in the adjusted triangulation of the United States is a modification of a formula originally offered by Simmons as a means for estimating the proportional accuracy of an arc of triangulation. The Simmons formula, also empirical and based on the general assumptions stated in the previous paragraph, states that the proportional accuracy is approximately 1 part in 20,000 $(M)^{1/3}$, where M is the length in miles. This figure is probably quite conservative. A less conservative figure is proposed by the Army Map Service, by whom it is stated that proportional accuracy is approximately 1 part in 8460 $(M)^{1/2}$.

16. APPLICATIONS OF FORMULA

It is emphasized again that error values derived from the above formulae, or any similar formulae, are approximate only and serve only as general guides. In an actual flash-triangulation operation for which accurate analysis is imperative, recourse should be had to the original computations of the actual points used. This might seem to be extremely troublesome and costly but, compared to the total cost of a missile operation, the additional cost for an accurate analysis would be negligible.

17. COORDINATE SYSTEM

In expanding the flash triangulation system to a world-wide operation serious questions arise as to the appropriateness of the conventional geodetic reference system of latitude and longitude. First, there is the problem of fitting long linear measurements to a reference ellipsoid which may not be a good fit to the actual conditions. In addition, the ellipsoidal system is convenient for computation only when measured lengths can be assumed to be approximate arc measurement, on or near the surface of the ellipsoid. These conditions do not present problems when surveys are made by conventional means where base-lines are closely identical with arcs and where triangle sides are so short (5-50 miles) that each can be considered as a chord and readily corrected to a corresponding arc length. Also, all operations are performed close to the surface of the ellipsoid (the highest point on the earth being less than six miles above sea-level) and effects of heights are almost inconsequential and easily corrected. On the other hand, flash triangulation involves extremely long lines which (except for refraction) are essentially straight. In addition, the actual positions of the flash points (which are important points in the computations) are high above the surface of earth—100 miles or more, to overcome earth curvature. Spherical computation, and reduction of heights is infinitely more complicated under these circumstances, and should not be used for the data reduction of flash triangulation.

18. RECTANGULAR COORDINATES

In view of the circumstances outlined in the previous paragraph, it is recommended that flash triangulation computations be made on a system of rectangular (cartesian) coordinates. The system recommended is Geocentric Coordinates, with the origin at the center of the earth. Coordinates should be in linear units, measured from three mutually perpendicular planes. To facilitate inter-conversion with geographic coordinates (latitude and longitude) and the incorporation of astronomical data, these planes should correspond to the basic references of the geographic system. Thus the equatorial plane is the plane from which "z" coordinates are measured, positive if north, and negative if south. The second plane is the meridional plane through the prime meridian at Greenwich, from which "y" coordinates are measured, positive if east, and negative if west. The third plane is the meridional plane through 90° east and 90° west longitude, from which "x" coordinates are measured, positive for longitudes (east or west) less than 90°, negative for longitudes (east or west) greater than 90°. Coordinates are to be stated in meter units. The principal advantages of this system derive from facility of computation with straight-line measurements and intersections, and, once a basic set of coordinated points is established, subsequent position computations are absolutely independent of any reference ellipsoid.

19. CONVERSION TO GEOCENTRIC COORDINATES

If the latitude and longitude and true elevation above the reference ellipsoid are known for any point, the Geocentric Coordinates can be computed by a comparatively simple conversion, as follows:

Let:

ϕ = latitude of point
λ = longitude of point
h = elevation of point above surface of reference ellipsoid
a = semi-major axis of reference ellipsoid
b = semi-minor axis of reference ellipsoid
e = eccentricity of reference ellipsoid, defined by:

$$e^2 = (a^2 - b^2)/a^2$$

N = length of normal to the ellipsoid, at latitude "ϕ," perpendicular to the tangent to the meridional section through the point. Length is measured from the surface of the ellipsoid to the intersection with the minor (polar) axis of the ellipsoid. The elevation "h" is measured along the outward extension of the normal, from the ellipsoid surface to the point. This is the radius of curvature perpendicular to the meridian (or, in the prime vertical).

Since the ellipsoid of reference is an ellipsoid of revolution, the meridional section is an ellipse. From the properties of an ellipse:

$$N = a/(1 - e^2\sin^2\phi)^{1/2}$$

and:

$$x = [(N+h)\cos\phi]\cos\lambda$$
$$y = [(N+h)\cos\phi]\sin\lambda$$
$$z = [N(1-e^2)+h]\sin\phi.$$

20. PARAMETERS FOR CONVERSION OF CLARKE 1866 SPHEROID

For conversion of geographic coordinates based on the Clarke 1866 Spheroid, the following parameters must be used in the formulae stated above:

$$a = 6{,}378{,}206.4 \text{ meters}$$
$$e^2 = 0.00676\ 86580$$
$$1-e^2 = 0.99323\ 13420$$

Sufficient significant figures are given for computation of coordinates to millimeters. Ten decimal-place trigonometric functions must be used in the computation to attain this accuracy.

21. PARAMETERS FOR CONVERSION OF INTERNATIONAL ELLIPSOID

For conversion of geographic coordinates based on the International Ellipsoid, the following parameters must be used:

$$a = 6{,}378{,}388 \text{ meters}$$
$$e^2 = 0.00672\ 26701$$
$$1-e^2 = 0.99327\ 73299$$

22. PROBLEM OF ELEVATION

In the formulae stated in preceding paragraphs, it has been assumed that the orthometric elevation of a point above mean sea-level datum, as ordinarily determined by spirit levels, will provide the numerical value for the quantity "h." This will not theoretically be true, and, in precision operations, the accuracy of position determinations will be adversely affected. The difficulty lies in the fact that elevations determined by spirit levels are referred to the mean sea-level surface which is a gently undulating surface which is, at every point, normal to the direction of gravity. The direction of gravity at any point is the resultant of all the gravitational forces acting at that point (plus centrifugal force due to the earth's rotation), and these forces are not systematic but are distorted by the unsystematic distribution of mass in the earth. The mean sea-level surface is called the Geoid. This surface is the one actually measured and determined by the arc-measurement method, and the reference ellipsoid is thus a mathematically-defined mean surface which approximates the Geoid.

23. GEOID HEIGHTS

It is necessary, therefore, in transforming existing basic control points from geographic coordinates and elevations above the Geoid to Geocentric Rectangular Coordinates, to determine with some accuracy the relationship at each point between the Geoid and the reference ellipsoid. There is, at present, no direct method of measuring this difference. It can be inferred from analysis of extensive observations of gravity, called the gravimetric method, and it can be approximated by integration of a series of determinations of the deflections of the vertical by comparison of astronomic and geodetic measurements of latitude and longitude. This latter system is called the astro-geodetic method.

24. DEFLECTION OF THE VERTICAL

The deviation of the local direction of gravity (the normal to the Geoid) from the direction of the normal to the spheroid through the same point is the deflection of the vertical. It can be determined in terms of a particular geodetic datum by astronomic determination of latitude and longitude. Comparison of these values with the geodetic latitude and longitude computed through triangulation or other direct measurement technique will give the deflection of the vertical resolved into two components as follows:

Deflection of the vertical in the Meridian = ξ

where

$$\xi = \text{astronomic latitude—geodetic latitude}$$
$$= \phi_A - \phi_G$$

Deflection of the vertical in the Prime Vertical = η

where

$$\eta = (\text{astronomic longitude—geodetic longitude}) \text{ cosine latitude}$$
$$= (\lambda_A - \lambda_G)\cos\phi$$

NOTE: In the above expression, EAST longitudes are POSITIVE, WEST longitudes are NEGATIVE.

25. GEOID PROFILE

If the deflection components have been observed at each station in a series of control points all computed on the same geodetic datum, it is possible to calculate by numerical integration a Geoid Profile, that is to say, a series of numbers which indicate, at each station in the series, the change in vertical displacement between the actual geoid and the reference ellipsoid as represented by the geodetic datum used. It is important to realize that this method cannot, alone, give the absolute vertical displacement, but can only, starting with an assumed or estimated value, give the sequential relative relationships, in a manner analogous to a line of spirit levels which is accurate internally but is not connected to mean sea-level datum.

26. DATA REQUIRED FOR GEOID PROFILE

The computation of a geoid profile requires that the proper astronomical observations for latitude and longitude be made at each station in a connected series of control points, for each of which a geodetic latitude and longitude is available, all adjusted to a common geodetic datum. Also the height of the geoid above (or below) the ellipsoid must be known from external data, or assumed or estimated, at some one control point in the series. This point will then serve as the "datum point." As pointed out before, the absolute value of geoid height cannot be determined from the astro-geodetic profile technique.

27. SLOPE OF GEOID

If the deflection components are known at a point, the slope of the geoid at the same point can be determined because, by definition, the geoid is a surface which is everywhere perpendicular to the direction of gravity. The elemental section of the geoid at a point is thus a plane surface, on which a meridional trace is inclined to the meridional tangent to the ellipsoid by an angle equal to the meridian component (ξ) of the deflection, and, similarly, on which a line at right angles to the meridional trace is inclined to the tangent plane to the ellipse by an angle equal to the prime vertical component (η) of the deflection. The slope of this elemental geoidal section in any other azimuth can be computed by appropriate combination of these two component slopes.

28. SLOPE OF THE GEOID ON A GIVEN AZIMUTH

If the meridian and prime vertical components of the deflection are given at a point, the slope of the geoid in a particular azimuth is obtained as follows:

Given:

ξ = meridian component of the deflection of the vertical (in seconds of arc).

η = prime vertical component of the deflection of the vertical (in seconds of arc).

α = azimuth of direction in which geoid slope is wanted, stated as direction away from the given point, defined in terms of a clockwise angle from SOUTH, to conform to geodetic practice in the U.S.

χ = slope of geoid, in the direction of the azimuth α, in seconds of arc, from the tangent plane to the reference ellipsoid, at the given point. A POSITIVE sign indicates slope UPWARDS, away from the center of the ellipsoid; NEGATIVE sign indicates slope DOWNWARD, toward the center of the ellipsoid.

Then:

$$\chi = (\xi \cos \alpha + \eta \sin \alpha)$$

NOTE: The convention used here is to assign the same signs to the trigonometric functions as in the traditional convention of trigonometry, as exemplified in U.S. Coast and Geodetic Survey Special Publication No. 231, Natural Sines and Cosines.

29. MEAN GEOIDAL SLOPE BETWEEN TWO STATIONS

In calculating the difference in geoidal height between two stations, given the necessary data for computation of geoidal slope at each station, it is necessary to calculate the mean geoidal slope along the line connecting the two stations. The essential steps are as follows:

Given:

For Station A: geodetic latitude (ϕ_A), geodetic longitude (λ_A) deflection components (ξ_A) and (η_A)

For Station B: geodetic latitude (ϕ_B), geodetic longitude (λ_B) deflection components (ξ_B) and (η_B)

The compute mean geoidal slope from A to B.

1. Make geodetic inverse position computation. This will yield the geodetic azimuth from A to B (α_{AB}) and from B to A (α_{BA}) and the geodetic distance (S_{AB}), on the ellipsoid from A to B. For purposes of computing mean geoidal slope, use a "mean" value for the azimuth from A to B (which will be designated $\bar{\alpha}_{AB}$), thus:

$\bar{\alpha}_{AB}$ = $1/2[\alpha_{AB} + (\alpha_{BA} \pm 180)]$ (180° is added to or subtracted from α_{BA}, as required to keep $0° < (\alpha_{BA} \pm 180°) < 360$).

2. Compute: $\chi_A = (\xi_A \cos \bar{\alpha}_{AB} + \eta_A \sin \bar{\alpha}_{AB})$ (this is in seconds).

3. Compute: $\chi_B = (\xi_B \cos \bar{\alpha}_{AB} + \eta_B \sin \bar{\alpha}_{AB})$ (also in seconds).

4. Compute mean geoidal slope from A to B, the mean of the two preceding quantities:

$$\bar{\chi}_{AB} = 1/2(\chi_A + \chi_B) \qquad \text{(in seconds of arc)}$$

$$= (\chi_A + \chi_B)\ 2.424 \cdot 10^{-6} \qquad \text{(in radians)}.$$

Careful attention must be paid to the signs of all data. The value of mean geoidal slope ($\bar{\chi}_{AB}$) has a sign which indicates whether the geoid is rising (+) or falling (-) with respect to the ellipsoid in progressing from A to B. If the slope in the direction from B to A is required, then the azimuth used must be:

$$\bar{\alpha}_{BA} = 1/2[\alpha_{BA} + (\alpha_{AB} \pm 180)],$$ which will differ from $\bar{\alpha}_{AB}$ by 180° and change the signs of both trigonometric functions.

Special note must be taken of the fact that in evaluating $\eta = (\lambda_A - \lambda_G)\cos\phi$, the adopted convention assumes that longitudes are counted positively to the eastward from 0° to 360°, or, alternatively, that west longitudes are negative. This will give opposite signs to all values of η published by the U.S. Coast and Geodetic Survey, as, for example, in Hayford's classic work "Figure of the Earth and Isostasy from Measurements in the United States" or in Special Publication No. 229. It is, however, necessary to use a mathematically-consistent system in order to obtain the same sense of geoidal slope anywhere in the world.

30. GEOIDAL HEIGHT DIFFERENCES

The mean geoidal slope between two stations may be used to compute the difference in the geoidal height between the same two stations. Recalling that the geoidetic distance between the two stations is obtained by inverse computation, and assuming that the form of the geoidal profile between the stations is fairly approximated by a curve of no higher than second-degree, the difference between the geoid height at A and at B may be computed by the simple procedure of multiplying the distance AB by the mean geoidal slope in radians, thus:

$$\Delta N_{AB} = S_{AB}(\chi_A + \chi_B) \cdot 2.424 \cdot 10^{-6} \qquad \text{(units same as "S")}.$$

31. GEOIDAL PROFILE

By fixing the geoidal height (N) at some one point, it is possible to construct a geoidal profile through any number of stations along a control line or triangulation arc, provided the required astronomical observations are made at each station. The computation is routine, in the form outlined in the previous paragraphs. It is merely necessary to compute, for every consecutive pair of stations, the mean geoidal slope, and multiply by the geodetic distance. It is, of course, necessary, at each station, to calculate the geoidal slope (X) along the line coming into the station and also along the line from the station to the next point on the profile. The same deflection components are used but they are resolved separately along the two different azimuths.

The geoidal profile is thus constructed in much the same manner as a construction profile; the distance is projected into a straight representation and the height above or below the reference ellipsoid is accumulated as the continuous sum of the successive ΔN's, starting with one given or assumed value. The result of the operation is a set of "geoidal heights" corresponding to the points at which observations were made.

32. ADJUSTMENT OF GEOIDAL PROFILE

The computation method outlined above is applicable to a single series of points extending along a single route. If numerous such stations are available in an area distribution, or along a band such as in a series typified by the distribution of stations in a triangulation arc, it is possible to use the availability of several stations in proximity to provide redundant independent determinations of the geoidal height at any one station. Theoretically, geoidal heights brought into a station through several different adjacent stations should be in exact agreement. Due to accidental errors of observation, this will not be the case, in practice. Valuable information as to the accuracy of a determination will be provided by this process, and it should be utilized whenever the data are available. If such independent determinations are made, the discordance should be eliminated by meaning at the station, before proceeding to the determination at a succeeding station. This provides a rough adjustment of the data as the computation proceeds.

33. ALTERNATE COMPUTATIONAL METHODS

The method outlined above for computation of increments for geoidal profiles is completely general and can be accomplished with a minimum of tables and special equipment. Two other methods have also been used with considerable success—the "Canadian Method" and the "Hayford Method."

34. CANADIAN METHOD

The Canadian Method may be considered simpler because it does not require the azimuth between a pair of stations in the computation of the geoidal profile height increment between them. Thus, there is no need for the numerous inverse position computations between successive stations as required in the method described above. The basic data required are the same:

ϕ_1, ϕ_2 = geodetic latitudes of the two stations

λ_1, λ_2 = geodetic longitudes of the two stations

ξ_1, ξ_2 = meridian components of the deflections of the vertical at the two stations (in seconds)

η_1, η_2 = prime vertical components of the deflections of the vertical at the two stations (in seconds).

A table of ellipsoid functions, giving the length of 1 second of latitude and 1 second of longitude with latitude argument, is also required for the particular reference ellipsoid used.

The geoidal height increment, ΔN_{1-2}, is computed by assuming the mean meridional and prime vertical components to be valid for the distance between the stations, as the distance is represented by the meridional distance and the mean latitudinal distance.

Let:

$\bar{M}$ = length, in meters, of 1 second of latitude, at latitude $(\phi_1 + \phi_2)/2$

$\bar{P}$ = length, in meters, of 1 second of longitude, at latitude $(\phi_1 + \phi_2)/2$

$\bar{\xi} = (\xi_1 + \xi_2)/2$

$\bar{\eta} = (\eta_1 + \eta_2)/2.$

Then:

$$\Delta N_{1-2} = [(\bar{M})(\bar{\xi}) + (\bar{P})(\bar{\eta})] \cdot 4.848 \cdot 10^{-6} \qquad \text{(meters)}$$

A geoidal height for a single station computed by this method from several adjacent will be subject to adjustment, as described for the previous method, to give a single value, from which to proceed to the next station.

35. HAYFORD METHOD

The Hayford Method is a semi-graphic procedure that is particularly suitable for mass-production of geoid height determinations over large areas in a wide distribution. This method requires too much preparation of maps and diagrams for economical application to single profiles but is more practicable for use in determination of geoidal contour maps where the shape of the geoid is defined in terms of contours showing its deviation from the ideal surface of the reference spheroid. The method was first applied by Hayford in his original studies of isostasy and its influence on the determination of the size and shape of the earth. It is described in detail in Hayford's classic work, "The Figure of the Earth and Isostasy from Measurements in the United States" (1909) and it is not believed to be necessary to recapitulate in detail here.

36. ACCURACY OF THE ASTRO-GEODETIC GEOID-PROFILE DETERMINATION

A method for estimating the accuracy of an astro-geodetic profile is to consider the variability of the rate of change of geoidal slope. If the change in slope along the profile proceeds in smooth transition from point to point, the method of computing geoidal height increments by utilizing the mean geoidal slope between successive stations is mathematically sound within the limits of the accuracy of the normal observing procedures by means of which the data on which the computation is based are obtained. However, if the slope changes greatly and erratically between stations, the probability of inaccurate computation of geoidal height increments is indicated. This situation can only be improved by the making of additional observations for the deflection of the vertical, thus reducing the interval between stations. The shorter interval must be selected, as necessary, to provide sufficient data to indicate smoother transition.

37. ESTIMATION OF ERROR IN GEOIDAL HEIGHT

It is possible, by analysis, to make a rough estimate of the error in a geoidal height computed by the average slope method due to method of computation wherein the mean slope over a section between two observed stations is taken to be the mean of the two values at the two stations. This is, in effect, an interpolation for the mean value between successive listed values in a table.

A reasonable approximation to this situation can be made by tabulating, for any profile, the successive values for $\overline{X}$ (in radians) for the successive profile sections between observed stations. Having made such tabulation, list the first differences and the second differences for all entries (there will, of course, be no second differences computable for the first and last entries). Bessel's formula for the interpolation of a mean value between successive entries in a table, assuming third differences constant (which they are not but their significance may be ignored), requires that the mean of two successive entries be corrected by one-eighth of the mean of the successive second differences corresponding to the two tabular entries. Thus, an indication of the error in any section may be had by merely computing one-eighth of the mean of the second differences corresponding to the two slope values (a "section" here is a portion of the profile between mid-points between three successive observed stations). The standard error of the slope of a typical section can be obtained from the standard statistical definition (the square root of [the sum of the squares of the errors of all "sections," divided by the number of sections]). This standard error will be in radians, and, when multiplied by the average length of a section, will give the standard error in geoidal height increment. The total error in geoidal height can be estimated by multiplying the standard error of geoidal height increment by the square root of the number of sections in the profile.

38. RECOMMENDATIONS

Since geoidal heights are important to the conversion of geographic positions to geocentric coordinates, it is important that the procedures be improved. It is recommended that continuing study be directed toward the development of field techniques that will increase the accuracy and reduce the time and cost involved in making the observations for the determination of the deflections of the vertical. If material progress can be made toward these objectives, not only will the utilization of existing control be simplified, but also the accuracy of computation of new control will be improved due to increase in control of azimuth by astronomical observations. The increase in computational load can easily be handled by electronic computer applications.

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