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A CONSUMPTION BASED MODEL OF ASSET PRICES: THE CASE OF PARTIALLY OBSERVABLE AGGREGATE CONSUMPTION

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A Consumption Based Model of Asset Prices: The Case of Partially Observable Aggregate Consumption

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Abstract

A consumption-based model of asset prices in a multigood economy is developed under condition of partially observable aggregate consumption. Previous studies show that when there are m consumption goods, expected excess returns on risky securities are functions of their covariances with 2m random variables - aggregate consumptions of m goods and their market prices. Without making any further assumptions, the present model shows that a similar equilibrium relationship can be presented in terms of covariances of asset returns with only m+1 random variables - aggregate consumptions of k goods and prices of (m+1-k) consumption goods. The number of variables required to measure riskiness of securities is further reduced if direct utility functions are separable. The results of the paper significantly reduce the amount of data needed to test the consumption-based asset pricing model.

*I would like to thank seminar participants at The University of Michigan for their comments. I am, of course, responsible for any remaining errors.

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This paper develops an equilibrium model of asset prices that is in principle testable even when aggregate consumption is only partially observable. We show that when there are m consumption goods, riskiness of securities can be measured through their covariances with aggregate consumptions of $k < m$ goods and prices of (m+1-k) goods. Hence, in contrast to previous models that require one to estimate the stochastic properties of 2m random variables - aggregate consumptions of m goods and their prices - the present model can be empirically tested using the information provided by only (m+1) random variables.

Furthermore, if direct utility functions one separable in bundles of consumption goods, aggregate consumptions and market prices of even a smaller number of goods will provide enough information to test the consumption-based pricing model of risky assets.

To illustrate this point we will consider three specific examples. First, if aggregate consumption of only one good is observable, observing prices of all consumption goods provides enough information for the consumption-based asset pricing model (CBAPM) to be empirically tested. Second, if aggregate consumption of all goods can be observed, the CBAPM can be empirically tested even if prices of consumption goods are unobservable (the price of the numeraire is always known). Finally, if inventors' direct utility functions display separability (e.g., see Varian [11], pp. 146-49) and aggregate consumption of only one good is observable, prices of all consumption goods need not be observed any more, and the model can be tested using the information provided by a smaller number of goods prices.

Rubinstein [10] demonstrates that an appropriate measure of risk for an uncertain cashflow is its covariance with the level of aggregate consumption.

Breeden [3], using a continuous-time model, shows that Merton's [7] multi-beta asset pricing model can be collapsed into a single-beta model where the riskiness of a security is measured through the covariance of its return with changes in aggregate consumption. Breeden [3] also extends the above model to a multi-good economy and shows that two price indices must be calculated in order to characterize the equilibrium expected returns on risky securities under a stochastic consumption opportunity set.[1]

To test the CBAPM, one would face two basic difficulties. The first difficulty, which is addressed by Cornell [4], has to do with the nonstationarity of consumption-betas of risky securities. Cornell demonstrates that though Merton's multi-beta asset pricing model collapses into a single-beta model, the resulting beta will be nonstationary, and to present the stochastic process followed by the consumption-beta of a security, the "state variables" still have to be observed. Consequently, empirical verifications of the two models give rise to the same problem--identifying the relevant 'state variables.'

The second problem, which is the focus of this paper, has to do with the amount and the type of data required to test the model; following problems arise in empirical tests of CBAPM.[2]

a) The theoretical definition of aggregate consumption requires us to account for consumption <u>expenditures</u> on <u>nondurables</u> and <u>actual</u> consumption <u>flows</u> derived from <u>durables</u>. The U.S. government, for instance, only reports consumption expenditures on <u>all</u> goods, and hence, consumption flows have to be estimated from expenditures data.

b) Consumption expenditures are reported at discrete points of time, while the theoretical model is developed within a continuous-time framework (Rubinstein [10] employs a discrete-time model).

c) The reported expenditures are the integral of consumption expenditures over the reported time interval rather than expenditures at discrete points of time.

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[1]Other relevant studies are Bhattacharya [2], Grossman and Shiller [6], and Cox, Ingersoll, and Ross [5].

[2]For empirical tests of the CBAPM see Nanisetty [9].

It is evident that there are serious difficulties in estimating the stochastic properties of aggregate consumption. On the other hand, data on prices of consumption goods are more readily available; it would be advantageous if one could use their stochastic properties instead of those of aggregate consumption expenditure to test the CBAPM. Furthermore, reported figures of prices of consumption goods are not as likely as reported aggregate consumption figures to be corrupted by measurement errors.

This paper should therefore be viewed as rather significant because it shows that data on the aggregate consumption of only a few goods and data on prices of a sufficient number of consumption goods provide enough information to test the CBAPM empirically. As an extreme example, if prices of all consumption goods can be observed, then aggregate consumption of only one good has to be observed, and it can be chosen such that the best possible data are used in the empirical test.

The plan of the paper is as follows. Part I presents the notation and the assumptions of the paper. Part II gives a brief derivation of the CBAPM. Part III presents the main results, and Part IV summerizes the paper.

I. Assumptions and Definitions

We use the following notation: t = time subscript, X(t) = a vector of state variables describing the state of the economy, E_t = expectation operator conditional on all information available at t, $C_j(t)$ = amount of consumption good $j=1,\ldots,m$ consumed at t, $w_i(t)$ = amount of wealth invested in asset $i=0,\ldots,n$ at t, $W(t) = \sum_{i=0}^{n} w_i(t)$, $U(C_1(t),\ldots,C_m(t),t)$ = a time-additive von Neuman-Morgenstern utility of consumption at t, and $D_j(t)$ = price of consumption good j at t.

We also use the following assumptions:

(A.1) Perfect markets.

(A.2) State independent utility. Each investor's <u>direct</u> utility of consumption does not explicitly depend upon the state of the economy.

(A.3) Consumption opportunities. There are m consumption goods available at all times. Prices of consumption goods are stochastic and their dynamics can be presented by a system of Ito processes (e.g., Arnold [1]) of the following form:

$$dD_j(t) = \gamma_j(t)dt + \delta_j(t)dz_j(t) \qquad j=1,\ldots,m$$

where, $\gamma_j(t)$ = expected change in $D_j(t)$;

$\delta_j(t)$ = standard deviation of unexpected changes in $D_j(t)$;

$z_j(t)$ = standard Wiener process.

Consumption good j=1 is chosen as the numeraire and its price, $D_1(t)$, is by definition equal to one. The parameters of the above system of Ito processes can be functions of other variables (e.g., the state variables).

(A.4) Investment opportunities. There are n risky securities available to investors (n need not be constant). Their price dynamics can be presented by a system of Ito processes of the following form:

$$dP_i(t) = P_i(t)\mu_i(t)dt + P_i(t)\sigma_i(t)dy_i(t) \qquad i=1,\ldots,n$$

where, $P_i(t)$ = price of security i at t;

$\mu_i(t)$ = expected rate of return on asset i;

$\sigma_i(t)$ = standard deviation of unexpected changes in $P_i(t)$;

$y_i(t)$ = standard Wiener process.

Security i=0 is chosen as the riskless asset and its rate of return is denoted by r(t). $\mu_i(t)$ and $\sigma_i(t)$ can be functions of other stochastic variables.

(A.5) Investor's choice. Each investor maximizes a utility function of the following form (investors are <u>not</u> identical) over his/her choice variables, $w_i(t)$ i=0,...,n and $C_j(t)$ j=1,...,m.

$$J(W(t), X(t),t) = \max E_t \int_t^T U(C_1,\ldots,C_m,s)ds$$

subject to[3]

$$dW(t) = \sum_{i=1}^{n} w_i(t)[(\mu_i(t) - r(t))dt + \sigma_i(t)dy_i(t)]$$

$$+ W(t)r(t)dt - \sum_{j=1}^{m} C_j(t)D_j(t)dt.$$

II. The Consumption-Based Asset Pricing Model

Here, we present a brief derivation of the CBAPM to provide a better perspective on the contributions of this paper.

The first order conditions for the above optimization problem are:

$$J_w(W(t), X(t),t) - \frac{U_j'(t)}{D_j(t)} = 0 \qquad j=1,\ldots,m \tag{1}$$

$$\mu_i(t) - r(t) + \mathrm{Cov}(d\ln J_w(\cdot,t),d\ln P_i(t)) = 0 \qquad i=1,\ldots,n \tag{2}$$

where, $J_w(t) = \frac{\partial J(t)}{\partial W(t)}$ and $U_j'(\ ,t) = \frac{\partial U(t)}{\partial C_j(t)}$.

Equation (1) represents the familiar envelop condition and equation (2), which is equivalent to equations (15) of Merton [7], (10) of Breeden [3], and (27) of Cox, Ingersoll, and Ross [5], can be used to represent the risk premium on asset i.

Let $V_c(t)$ represent the marginal utility of consumption expenditure; i.e.,

$$V(\ \ t) = \max_{C_j} U(C_1,\ldots,C_m,t) + \lambda[C - \sum_{j=1}^{m} C_jD_j].$$

Then from the envelop condition we have:

$$J_w(t) = V_c(t).$$

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[3]For a detailed derivation of the budget constraint see Merton [7] and [8].

Substituting the above result in equation (2) and applying Ito's lemma (e.g., Arnold [1]) to the marginal utility of consumption expenditure, we obtain the CBAPM reported by Breeden [3]; i.e., we have:

$$\mu_i(t) - r(t) = R\,\mathrm{Cov}(d\ell nC, d\ell nP_i) + \sum_j^m R_j \mathrm{Cov}(d\ell nD_j, d\ell nP_i) \qquad i=1,\ldots,n \quad (3)$$

where R = measure of relative risk aversion, $-\frac{\partial \ell n V_c(t)}{\partial \ell nC}$; $R_j = -\frac{\partial \ell n V_c(t)}{\partial \ell nD_j}$.

Breeden [3] rearranges the last term on the right-hand side of equation (3) and drives two price indices which are used to determine real changes in aggregate consumption expenditure and real returns on risky securities. Note that we have not aggregated equation (3) across investors but the aggregated equation would still have the same structure.[4]

According to equation (3), the expected excess rate of return on asset i, $\mu_i(t) - r(t)$, is a function of its covariances with changes in aggregate consumption, $C = \sum_{j=1}^m C_j D_j$, and changes in prices of consumption goods, D_j $j=1,\ldots,m$. Market prices of risks for the sources of uncertainity are dentoed by R and R_j $j=1,\ldots,m$.

To test equation (3), one has to determine the stochastic properties of aggregate consumption and prices of consumption goods. But to determine the stochastic properties of aggregate consumption one has to observe aggregate consumptions of all goods, C_j $j=1,\ldots,m$, which include aggregate consumption flows derived from durables. These variables, however, are unobservable and hence, must be estimated from data on aggregate expenditures on durable goods. Consequently, errors will be introduced into data on aggregate consumption leading to weaker empirical tests of the CBAPM.

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[4]For the aggregation procedure see Breeden [3].

The purpose of the next part is to present a consumption-oriented model of asset prices under condition of partially observable aggregate consumptions and/or prices of goods.

III. The Model

If the marginal utility of wealth, $J_w(t)$, is eliminated from equation (1), we will have a system of (m-1) equations; i.e.,

$$\frac{U_i'(t)}{D_i(t)} = \frac{U_j'(t)}{D_j(t)} \qquad i,j=1,\ldots,m \tag{4}$$

There are 2m arguments in equations (4) and they include optimal consumption of m goods and their prices. Because utility functions are assumed to be strictly concave, we can solve for any (m-1) variables in terms of the remaining (m+1) variables. To formalize this result, we use the following notation:

$\hat{C}_k$ = a vector of optimal quantities of k consumption goods;

$\hat{D}_g$ = a vector of prices of g consumption goods (excluding the price of the numeraire, D_1, which is assumed to be always equal to one).

Using equations (4), we can solve for optimal consumption of good j as follows:

$$C_j \equiv C_j(\hat{C}_k, \hat{D}_{m-k}, t) \qquad j=1,\ldots,m, \qquad C_j \notin \hat{C}_k,\ k > 1. \tag{5}$$

That is to say, optimal quantity of consumption good j, $C_j \notin \hat{C}_k$, is solved for as a function of optimal consumptions of k goods and price of (m-k) goods. Substituting the above results into the marginal utility of good i, we will have:

$$U_i'(C_1,\ldots,C_m,t) \equiv U_i'(C_j(\hat{C}_k,\hat{D}_{m-k},t),\hat{C}_k,t) \qquad i,j=1,\ldots,m \qquad C_j \notin \hat{C}_k.$$

Or to simplify the notation, we will write:

$$U_i'(C_1,\ldots,C_m,t) \equiv F_i(\hat{C}_k,\hat{D}_{m-k},t). \tag{6}$$

In other words, the marginal utility of consumption good i can be written as a function of optimal quantities of k consumption goods and prices of m-k goods (note that the price of the numeraire is always known).

Since the marginal utility of the numeraire, $U_1'(t)$, is equal to the marginal utility of wealth, $J_W(t)$, we can rewrite equation (2) as follows:

$$\mu_i(t) - r(t) = -\text{Cov}(d\ell nU_1'(t),d\ell nP_i(t)) \qquad i=1,\ldots,n. \tag{7}$$

Applying Ito's lemma to $U_1'(t)$ and using the results of equation (6), we obtain:

$$\mu_i - r = \sum_j H_j \text{COV}(d\ell nC_j,d\ell nP_i) + \sum_g K_g \text{Cov}(d\ell nD_g,d\ell nP_i) \tag{8}$$

where $C_j \in \hat{C}_k$ and $D_g \in \hat{D}_{m-k}$;

$$H_j = -\frac{\partial \ell nF_1(\cdot,t)}{\partial \ell nC_j(t)};$$

$$K_g = -\frac{\partial \ell nF_1(\cdot,t)}{\partial \ell nD_g}.$$

Assuming that investors have homogeneous expectations, equation (7) can be aggregated across individuals using the same approach employed by Merton [7] and Breeden [3].[5] For brevity, we assume that equation (8) is already in aggregate form.

According to this equation, the risk premium on risky asset i=1,...,n can be represented as a function of market prices of risks, H_j and K_g, and covariances of return on asset i with m sources of uncertainty, $\hat{C}_k$ and $\hat{D}_{m-k}$.

[5]Homogeneity of expectations can be somewhat relaxed; e.g., see Grossman and Shiller [6].

The following examples illustrate this point more clearly.

Example 1. Aggregate consumptions of all goods are completely observable. In this case we can set k=m, and present a consumption-oriented model of asset prices in a multigood setting with unobservable prices of consumption goods (the price of the numeraire is known). If there are m portfolios and their returns are perfectly correlated with changes in consumptions of m goods, we can obtain a multibeta asset pricing model. In contrast to Merton [7], it is less difficult to construct these portfolios because the relevant state variables, aggregate consumptions of m goods, are observable. In particular we have:

$$\mu_i(t) - r(t) = \sum_{j=1}^{m} \hat{\beta}_{ij}[\hat{\mu}_j(t) - r(t)] \tag{9}$$

where $\hat{\beta}_{ij}$ = beta of risky asset i with respect to changes in aggregate consumption of good j;

$\hat{\mu}_j$ = the expected rate of return on portfolio j.

Example 2. Aggregate consumption of only one good, say, the numeraire, and prices of all goods are observable. In this case we can set k=1 and thus present a consumption-oriented model of asset prices in a multigood setting with partially observable aggregate consumption (i.e., data on aggregate consumption of only one good is available). We are therefore able to present a pricing relationship similar to the one presented in equation (9), except that the portfolios are constructed such that the return on the first one is perfectly correlated with changes in aggregate consumption of the numeraire and returns on the remaining m-1 portfolios are perfectly correlated with changes in prices of consumption goods (note that the price of the numeraire is constant).

Example 2 is of particular interest because it allows one to present testable hypotheses even when aggregate consumption expenditure is not completely observable, provided that we can observe prices of a sufficient number of consumption goods. Hence, one can overcome the problem of estimating the stochastic properties of aggregate consumption flows derived from durables by using the information provided by aggregate consumptions of a few nondurables and prices of a 'sufficient' number of consumption goods.

Example 3. Separable direct utility functions. Let there be $s < m$ groups, written $m = m_1+\ldots+m_s$ in such a way that each consumption good belongs to exactly one group (there are m_i goods in group i). Let the utility function of an investor be weakly separable in the following sense,

$$U(C_1,\ldots,C_m,t) = U_1(Q_1,t) +\ldots+ U_s(Q_s,t), \tag{10}$$

where Q_i = a vector of consumption goods belonging to group i. Under equation (10), the marginal utility of each good depends only on the quantities consumed of the goods that belong to the same group. Using this property, equation (6) can be written as follows:

$$U_i'(C_1,\ldots,C_m,t) \equiv F_i(\hat{C}_k, \hat{D}_{m_j+1-k},t)$$

where, $C_i \in Q_j$, $\hat{C}_k \in Q_j$, and $k < m_j$.

That is to say, the marginal utility of consumption good i which belongs to group Q_j can be solved for as a function of quantities of k consumption goods which also belong to group Q_j and prices of m_j+1-k goods from the same group. Note that the subscript of $\hat{D}$ is m_j+1-k because the numeraire may not belong to this group. Hence, if direct utility functions display weak separability, the amount of data required to test CBAPM is reduced even further. For example, if we can observe aggregate consumption of the numeraire, we will need data on prices of only those goods that belong to the same group that the numeraire belongs to.

It is obvious that separability is a strict assumption. Hence, the results of the last example are rather weak and may not be supported by data on consumer behavior. On the other hand, other results of the paper (e.g., Examples 1 and 2) do not require separability and still allow one to test the asset pricing model even if aggregate consumption is unobservable.

IV. Summary

This paper develops a consumption oriented model of asset prices in a multigood setting when aggregate consumption expenditure is partially observable, which significantly reduces the amount of data required to test or apply the CBAPM.

We show that when there are m consumption goods, one need not observe all 2m variables--aggregate consumptions of m goods and their market prices. Using a continuous-time framework, we demonstrate that observing m+1 of the above 2m variables provides enough information to characterize equilibrium returns on risky assets. Hence, the consumption-based asset pricing model can be empirically tested by estimating the stochastic properties of aggregate consumptions of a few nondurables and those of prices of a "sufficient" number of consumption goods. In particular, if prices of all consumption goods can be observed, then data on aggregate consumption of only one good are enough to test the CBAPM.

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