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THE ENERGY METHOD OF ANALYSIS OF RUBBER SHELLS OF REVOLUTION

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The Energy Method of Analysis of Rubber Shells of Revolution

V. L. Biderman, B. L. Bukhin

A carcass, composed of some number, usually even, of layers of rubberized cord is the load carrying element of the shell. In every layer the filaments of cord are directed parallel to one another while the neighboring layers are laid up crosswise, symmetrical with respect to the meridian of the shell. The intervals between the filaments are filled with rubber.

The analysis of rubber-cord shells is a difficult problem in view of the strongly anisotropic elastic properties of a rubber-cord shell. It is necessary also to consider that these shells are pre-stressed, since they carry a load only in the presence of an internal pressure.

In reference 1 the general case is considered of the deformation of rubber-cord shells of revolution, on the assumption that the stresses are carried only by the filaments of cord. The equations obtained in this work are extremely complex and their direct solution is possible only for special cases. Energy methods are very powerful in this respect. The application of these to the analysis of rubber-cord shells of revolution is the object of the current work.

Let us consider a rubber-cord shell, which is fastened on two parallel circles (see figure). In the presence of internal pressure it acquires a defined (equilibrium) configuration. This configuration will be considered the initial (undeformed) condition of the shell. There are already considerable stresses on the elements of the shell in this condition, caused by the internal pressure.

The position of a point on the undeformed shell will be characterized by two coordinates. The position of the meridian will be defined by the central angle Θ, while the position on the meridian is defined by the distance s on the meridian from the parallel. Along with the coordinate s, the angle Φ between the normal to the shell and the axis of revolution is given from s by the relation

$$ds = \rho d\Phi \qquad (1)$$

where ρ is the radius of curvature of the meridian.

The distance of the point from the axis of revolution will be called r.

In the deformation of the shell every point moves some distance ξ, the normal, meridianal, and circumferential components of which will be called, respectively, w, v, u.

The strains of the middle surface of the shell of revolution are expressed in terms of the deformations in the manner of references 2 and 3. The relative elongation in the meridianal (ϵ_1) and circumferential (ϵ_2) directions and the shear γ are

$$\epsilon_1 = \frac{\partial v}{\partial s} + \frac{w}{\rho} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial s} + \frac{w}{\rho}\right)^2 + \left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial w}{\partial s} - \frac{v}{\rho}\right)^2\right];$$

$$\epsilon_2 = \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + v\cos\phi + w\sin\phi\right) + \frac{1}{2r^2}\left[\left(\frac{\partial u}{\partial \theta} + v\cos\phi + w\sin\phi\right)^2 + \left(\frac{\partial v}{\partial \theta} - u\cos\phi\right)^2 + \left(\frac{\partial w}{\partial \theta} - u\sin\phi\right)^2\right] \qquad (2)$$

$$\gamma = \frac{\partial u}{\partial s} + \frac{1}{r}\left[\frac{\partial v}{\partial \theta} - u\cos\phi + \left(\frac{\partial v}{\partial s} + \frac{w}{\rho}\right)\left(\frac{\partial v}{\partial \theta} - u\cos\phi\right) + \frac{\partial u}{\partial s}\left(\frac{\partial u}{\partial \theta} + v\cos\phi + w\sin\phi\right) + \left(\frac{\partial w}{\partial s} - \frac{v}{\rho}\right)\left(\frac{\partial w}{\partial \theta} - u\sin\phi\right)\right].$$

The expression for the curvature in the meridianal (χ_1) and circumferential (χ_2) direction and the twist τ have the form,

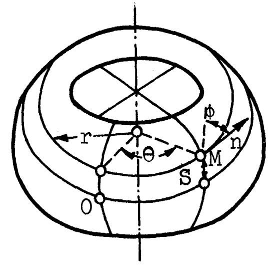
$$\chi_1 = -\frac{\partial^2 w}{\partial s^2} + \frac{\partial}{\partial s}\left(\frac{v}{\rho}\right); \quad \chi_2 = -\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial u}{\partial \theta}\sin\phi\right) - \frac{\cos\phi}{r}\left(\frac{\partial w}{\partial s} - \frac{v}{\rho}\right)$$

$$\tau = -\frac{1}{r}\left[\frac{\partial^2 w}{\partial s\,\partial \theta} - \frac{\cos\phi}{r}\frac{\partial w}{\partial \theta} - \frac{1}{\rho}\frac{\partial v}{\partial \theta} + \sin\phi\left(\frac{\cos\phi}{r}\cdot u - \frac{\partial u}{\partial s}\right)\right]. \qquad (3)$$

Nonlinear relations, as will be obvious from the following, are needed only for the calculation of the internal volume of the shell. In all the remaining cases it is possible to neglect the nonlinear terms.

The unit vector of the normal to the deformed shell, $\bar{n}^*$, is defined by the formula

$$\bar{n}^* = \bar{n} - \left(\frac{\partial w}{\partial s} + \frac{v}{\rho}\right)\bar{m} - \frac{1}{r}\left(\frac{\partial w}{\partial \theta} - u\sin\phi\right)\bar{t}. \qquad (4)$$



Here, $\bar{n}$, $\bar{m}$, $\bar{t}$ are unit vectors, which are normal and tangent to the meridian and parallel to a circumference of the deformed shell.

The strain of an element of a rubber-cord shell is accompanied both by a change in the length of the filaments of cord and by a variation of the angles between them. Since the stiffness of the filaments of cord is many times greater than the stiffness of rubber, the strain connected with the elongation of the filaments of cord is essentially zero, as is verified by experiments. Therefore, most problems can be solved by assuming inextensibility of the filaments of cord in membrane deformations. In addition, it is necessary to assume the relative extensions of the filaments of cord in both directions, $\epsilon_{\ell 1}$ and $\epsilon_{\ell 2}$, equal to zero in order to write the two equations

$$\epsilon_{\ell_{1,2}} = \epsilon_1 \cos^2 \beta + \epsilon_2 \sin^2 \beta \pm \gamma \sin \beta \cos \beta \qquad (5)$$

$$\epsilon_1 \cos^2 \beta + \epsilon_2 \sin^2 \beta = 0 \qquad \gamma = 0 \qquad (6)$$

where ϵ_1, ϵ_2, γ are defined by the relations (2). β is the angle made by the filament of a cord with the meridian as a consequence of the manufacturing process

$$\sin \beta = k\gamma \qquad (7)$$

where k is a constant for a given shell size.

In some cases it appears possible to neglect the internal stresses on the rubber, in comparison with the stresses on the filaments of cord.

The potential energy π of a deformed shell in the general case, without use of simplifying assumptions, is made up of the energy of the air pressure and the energy stored in the rubber and cord in the presence of membrane and bending deformations. It is only necessary to compute the variation of the potential energy of deformation, since the quantity with the constant component does not affect the relations obtained by variational methods.

The change of potential energy of the pressure of the air is written as

$$W_1 = -p\Delta V \qquad (8)$$

where ΔV is the change of the internal volume of the shell and p is the internal pressure.

One should compute the variation of the internal volume, even for small deformations, accounting for terms of second order relative to displacement, because in the initial state the volume is maximized (if the stresses in the rubber and the elasticity of a cord are negligible). Therefore the terms of first order relative to the displacements in the expression for the variation of the volume are reduced to zero.

The increase of the shell volume may be calculated exactly up to second order in displacements and their derivatives:

$$\Delta V = \int\int \frac{1}{2}\left(d\,\bar{F}_o + d\bar{F}\right)\xi \tag{9}$$

Here $d\bar{F}_o$ and $d\bar{F}$ are vectors of an elemental area, corresponding to the undeformed and deformed states.

$$d\bar{F}_o = \bar{n} r ds d\theta \; ; \quad d\bar{F} = \bar{n}^*(1 + \epsilon_1)\, ds\, (1 + \epsilon_2)\, r d\theta \cos\gamma \quad .$$

Using the relations (2) and (4), the following is obtained.

$$d\bar{F} = \bar{n}\left[1 + \frac{\partial v}{\partial s} + \frac{w}{\rho} + \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + v\cos\phi + w\sin\phi\right)\right] - \bar{m}\left(\frac{\partial w}{\partial s} - \frac{v}{\rho}\right) - \bar{t}\,\frac{1}{r}\left(\frac{\partial w}{\partial \theta} - u\sin\phi\right) \quad .$$

The increase of the volume equals

$$\Delta V = \int\int_F \left\{w + \frac{1}{2}\left[w\left\{\frac{\partial v}{\partial s} + \frac{w}{\rho} + \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + v\cos\phi + w\sin\phi\right)\right\} - v\left(\frac{\partial w}{\partial s} - \frac{v}{\rho}\right) - \frac{u}{r}\left(\frac{\partial w}{\partial \theta} - u\sin\phi\right)\right]\right\} r ds d\theta \tag{10}$$

From this expression it is possible to exclude a term of the first order in the displacements. For the case of inextensible filaments of cord, studying the first of the relations (6) and the equation of equilibrium of the configuration of a rubber-cord shell,[1] one obtains

$$\Delta V = \frac{1}{2}\int\int_F \left\{w\left[\frac{\partial v}{\partial s} + \frac{w}{\rho} + \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + v\cos\phi + w\sin\phi\right)\right] - v\left(\frac{\partial w}{\partial s} - \frac{v}{\rho}\right) - \frac{u}{r}\left(\frac{\partial w}{\partial \theta} - u\sin\phi\right) - \frac{\rho_o\, r_o \cos\beta_o}{r\cos\beta}\left[\left(\frac{\partial v}{\partial s} + \frac{w}{\rho}\right)^2 + \left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial w}{\partial s} - \frac{r}{\rho}\right)^2\right] + \frac{\rho_o r_o \cos\beta_o + \tan^2\beta}{r^3\cos\beta}\left[\left(\frac{\partial u}{\partial \theta} + v\cos\phi + w\sin\phi\right)^2 + \left(\frac{\partial v}{\partial \theta} - u\cos\phi\right)^2 + \left(\frac{\partial w}{\partial \theta} - u\sin\phi\right)^2\right]\right\} r ds d\theta \tag{11}$$

Here ρ_o, ν_o, β_o are the values of the corresponding quantities at the point of zero Gaussian curvature.

We will pass to the computation of the energy of deformation of filaments of cord and rubber. The over-all length of filaments of one direction in an elemental section is equal to

$$\frac{1}{2}\, i\ r\ ds\ d\theta$$

i is the end count of filaments, that is, the number of filaments in a 1 cm. section, normal to the filaments, and n is the over-all number of layers.

The expression for the energy of the membrane deformation of filaments of cord has the form

$$W_2 = \iint \frac{1}{2} E_c \left[\left(\epsilon_{\ell_0} + \epsilon_{\ell_1}\right)^2 + \left(\epsilon_{\ell_0} + \epsilon_{\ell_2}\right)^2\right] \frac{n}{2}\, ir\ ds\ d\theta$$

where ϵ_{ℓ_0} is the initial deformation of the filaments, and E_c is the modulus of elasticity of a cord.

Using (5) and disregarding the constant component, we get

$$W_2 = \iint \left\{ N_o(\epsilon_1 \cos^2\beta + \epsilon_2 \sin^2\beta) + \frac{1}{2} E_c \left[(\epsilon_1 \cos^2\beta + \epsilon_2 \sin^2\beta)^2 + (\gamma \sin\beta \cos\beta)^2\right]\right\} ni\ rds\ d\theta \qquad N_o = E_c(\epsilon_{10}\cos^2\beta + \epsilon_{20}\sin^2\beta) \quad (12)$$

where N_o is the initial stress in the filaments of cord, and ϵ_{10}, ϵ_{20} are the initial deformations.

Because the initial state is axisymmetric, the shear γ_o vanishes. It is obvious, that in the case of the assumption of inextensible cords, the term W_2 is equal to zero.

The energy of the rubber due to membrane deformation is given by the expression

$$W_3 = \iint 2G_p h^* [(\epsilon_{10} + \epsilon_1)^2 + (\epsilon_{20} + \epsilon_2)^2 + (\epsilon_{10} + \epsilon_1)(\epsilon_{20} + \epsilon_2) + \frac{1}{4}\gamma^2] rdsd\theta$$

or

$$W_3 = \iint 2G_p h^* \,[\epsilon_1(2\epsilon_{10} + \epsilon_{20}) + \epsilon_2(\epsilon_{10} + 2\epsilon_{20})]\ rdsd\theta + \iint 2G_p h^*(\epsilon_1^2 + \epsilon_2^2 + \epsilon_1\epsilon_2 + \frac{1}{4}\gamma^2)\ rdsd\theta \quad (13)$$

where G_p is the shear modulus of rubber and h^* is the reduced thickness of the shell. The introduction of reduced thickness[4] takes into consideration that case where the rubber in the layer between the filaments of cord is deformed more severely than in the insulation rubber between the layers. The reduced thickness is calculated using the ratio of the filament diameter, d, to the spacing $t = 1/i$ by the following formula, where h is the actual thickness of the shell:

$$h^* = h + nd(k - 1) \qquad k = \frac{t}{d}\left[\frac{2}{\sqrt{1-(d/t)^2}} \tan^{-1}\sqrt{\frac{1+d/t}{1-d/t}} - \frac{\pi}{2}\right]$$

Neglecting the elongation of the filaments of cord, the second term in the expression (13) we will write as

$$\iint 2G_p h^* \,(1 - \tan^2\beta + \tan^4\beta)\,\epsilon_2^2\ rdsd\theta$$

Let us consider the bending energy, assuming that initially the shell is not acted upon by moments. Since the cord has a stiffness much greater than the rubber, the position of the neutral axis is defined with sufficient accuracy by the position of the cord layers. The energy of deformation of cord filaments of one layer in the section $rdsd\theta$ is defined by the following formula

$$dW_4^i = \frac{1}{2} E_c\, \epsilon_\ell^2\ ir\ ds\ d\theta$$

The relative extension of a filament during bending is found to be

$$\epsilon_\ell = (\chi_1 \cos^2\beta + \chi_2 \sin^2\beta \pm 2\tau \ \sin\beta\cos\beta)\, y$$

where y is the distance of the layer from the neutral axis and the two signs refer to the filaments in various directions. Summing the energy of bending of all layers and integrating over the surface of the shell, one obtains

$$W_4 = \iint \frac{1}{2} \Big\{[(\chi_1 \cos^2\beta + \chi_2 \sin^2\beta)^2 + 4\tau^2 \sin^2\rho\cos^2\beta]\,\Sigma y^2 +$$

$$+ 4\,\tau \sin\beta\cos\beta\,(\chi_1\cos^2\beta + \chi_2\sin^2\beta)\,(\Sigma_1 y^2 - \Sigma_2 y^2)\Big\}\ E_c irdsd\theta \quad (14)$$

In this formula Σy^2 is the sum of the squares of the distances of all the layers from the neutral surface, while $\Sigma_1 y^2$ and $\Sigma_2 y^2$ are the sums of the squares of the distances of the layers with the various alignments. Since in real constructions asymmetry of the carcass is not very large, it is always possible to neglect the second term of the formula, proportional to $\Sigma_1 y^2 - \Sigma_2 y^2$.

The bending energy of the rubber is defined, neglecting the influence of the cord on the deformation of the rubber, by the following relation

$$W_5 = \iint \frac{2}{3} G_p\ (h_1^3 + h_2^3)(\chi_1^2 + \chi_2^2 + \chi_1\chi_2 + \tau^2)\ rdsd\,\theta \qquad (15)$$

where h_1 and h_2 are the distances of the external and internal surfaces of the shell from the neutral layer.

The potential of the external forces is given by the formula

$$U = -\iint (\bar{Q}\bar{\xi})\, r ds d\theta = -\iint (Q_n w + Q_m v + Q_t u)\, r ds d\theta \tag{16}$$

where Q_n, Q_m, Q_t are normal, meridianal, and circumferential components of the intensity of the external load.

The complete potential energy of the rubber-cord shell is calculated as the sum,

$$\Pi = U + W_1 + W_2 + W_3 + W_4 + W_5 \tag{17}$$

The minimum conditions of this functional (the equations of Ostrogradsky) are the equations of equilibrium of an infinitesimal element of the shell,

$$\begin{aligned} &\frac{\partial \phi}{\partial z} - \frac{\partial}{\partial s}\left(\frac{\partial \phi}{\partial z_s'}\right) - \frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial z_o'}\right) + \frac{\partial^2}{\partial s^2}\left(\frac{\partial \phi}{\partial z_{ss}''}\right) + \\ &+ \frac{\partial^2}{\partial s \partial \theta}\left(\frac{\partial \phi}{\partial z_{s\theta}''}\right) + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial^2 \phi}{\partial z_{\theta\theta}''}\right) = 0 \qquad (z = u, v, w) \end{aligned} \tag{18}$$

Here ϕ is the function in the relation (18). These equations may be used for determination of both the initial and the deformed states of the shell. For determination of the initial state $Q = 0$ and $u = v = w = 0$ are assumed. The equations thus obtained, if extensibility of cord is neglected, agree with those derived earlier in 1. These define the initial configuration of the shell and the law of variation of the stress N_o on the meridian of the shell.

A small deformation from the initial state is now considered. It is assumed that the stresses in the filaments in the different directions are

$$N_1 = N_o + \sigma + \Delta \qquad N_2 = N_o + \sigma - \Delta \quad . \tag{19}$$

Here σ and Δ are small in comparison with N_o and are related to the strains by the equations

$$\sigma = E_c(\epsilon_1 \cos^2 \beta + \epsilon_2 \sin^2 \beta) \qquad \Delta = E_c\, \gamma \sin \rho \cos \beta \tag{20}$$

In this case, if extensibility of the filaments and stresses in the rubber are again neglected, the equations of Ostrogradsky agree with the equations of equilibrium obtained directly in reference 1. However, the main advantage of this approach is the possibility of solution of these equations by using the variational theorem $\delta\pi = 0$.

As an example the problem of the determination of the critical speed of a pneumatic tire is considered. The critical speed occurs when, on the surface of the tire at the point of tangency of contact with the road a wave exists. In addition, rupture of the tire is imminent in this state.

As a first approximation this speed is obtained by assuming inextensible filaments of cord and neglecting stresses in the rubber. As shown in reference 1, the critical rolling speed is the minimum velocity of propagation of waves of deformation on the circumference of the tire. If the energy of bending is neglected, the velocity of propagation of short waves will be minimal. Therefore, in these calculations it is necessary to retain only terms which contain the wave length to the first power.

Let the circumferential component of displacement be in the form of a sinusoidal running wave

$$u = u_o\, v(s) \sin 2\pi \frac{\theta - \omega t}{\theta_\ell}$$

where θ_ℓ is the central angle, fitting one wave and ω is the angular velocity of propagation of the wave.

v and ω are determined from Equation (7).

$$v = \frac{1}{2\pi} \theta_\ell u_o \, [v' - v \cos \phi] \cos 2\pi \left(\frac{\theta - \omega t}{\theta_\ell}\right)$$

$$w = -u_o \left[\frac{1}{\rho} + \frac{\sin \phi}{r} \tan^2 \beta\right]^{-1} \frac{2\pi}{\theta_\ell r} v \tan^2 \beta \cos 2\pi \left(\frac{\theta - \omega t}{\theta_\ell}\right)$$

The intensity of the forces of inertia is defined by the relation

$$Q = -q \frac{\partial^2 \xi}{\partial t^2}$$

where q is the mass of a unit surface area of the tire.

The potential of the inertia forces is computed in the formula (1) using the relations obtained for displacements and loading. Making a substitution, truncating the term containing θ_ℓ at the lowest power, and integrating with respect to θ over one wave, one gets

$$U = -{u_o}^2 \frac{\pi}{2} \omega^2 \left(\frac{2\pi}{\theta_\ell}\right)^3 \int q\, M^2 v^2\, r ds \qquad M = \frac{1}{r} \tan^2 \beta \left(\frac{1}{\rho} + \frac{\sin \phi}{r} \tan^2 \beta\right)^{-1}$$

Analogously the increase in volume (11) is determined

$$\Delta V = -{u_o}^2 \frac{\pi}{2} \left(\frac{2\pi}{\theta_\ell}\right)^3 \int M^3 v^2\, ds$$

These integrals are evaluated using the profile of the tire from rim to rim.

The complete energy is written

$$\Pi = U - p\Delta V = -u_o^2 \frac{\pi}{2} \left(\frac{2\pi}{\theta_\ell}\right)^3 \left[\omega^2 \int qM^2 v^2 rds - p \int M^3 v^2 ds\right]$$

From the conditions for a minimum, $\partial\pi/\partial u_o = 0$, one obtains

$$\omega^2 = p \int M^3 v^2 ds \Big/ \int q M^2 v^2 rds$$

The rolling speed, $v_* = \omega R$ where R is the external radius of the tire.

The critical rolling velocity of a racing tire 6.00-16 of the model I-141 with a cord angle at the crown of 50° under internal pressure of 4 atmospheres will be calculated as an example.

The boundary conditions expressing fixity at the rim are

$$u = v = w = \frac{\partial w}{\partial s} = 0 \qquad \text{when } r = r_*$$

The function v satisfying these has been assumed of the form

$$v = \left(\frac{r - r_*}{R - r_*}\right)^4$$

Calculation gave the following value of critical velocity

$$v_* = 191 \text{ km/hr} .$$

The actual critical rolling velocity of a 6.00-16 tire made of natural rubber was found experimentally by V. I. Novopolsky and I. A. Chizhov to be 240 km/hr.

The derivation by calculation of a low value is explained chiefly by the fact that the critical rolling velocity of a tire, owing to the presence of bending deformations, responds to a finite, and not infinitesimally small length of wave. Taking into account the energy of deformation of rubber and cord, it is possible to achieve substantial refinement of the calculations.

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