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A FURTHER NOTE ON REINVESTMENT ASSUMPTIONS IN CHOOSING BETWEEN NET PRESENT VALUE AND INTERNAL RATE OF RETURN

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Introduction

The fact that, under certain circumstances, net present value (NPV) and internal rate-of-return (IRR) capital budgeting techniques rank a given set of projects in a different order has been discussed in the literature numerous times.[1] Most recently, Dudley [2] has approached the issue in a manner that is illuminating from one point of view, yet confusing from another point of view. He fails to identify the cause of present value--internal rate-of-return ranking conflicts; he only provides a reasonable means of choosing between them.

Dudley was most concerned with the notion, fostered, he said, mainly by elementary texts in corporate finance, that the source of any NPV-IRR ranking conflicts could be found in the fact that "the present value approach assumes reinvestment of intermediate cash receipts at the discounting rate, while the internal rate-of-return approach assumes reinvestment at the internal rate.[2] He showed that neither of the two discounted cash flow methods make any assumptions about the rate at which the projects' intermediate cash flows are reinvested. In addition, he identified the conditions under which a net present value ranking of projects would be identical to a net terminal value (NTV) ranking and

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[1] See, for instance, Lorie and Savage [8], Solomon [10], Renshaw [9], and Carlson, Lawrence, and Wort [1]. We should add that these authors, as well as ourselves, are well aware of the context in which the suggested methods are discussed. For instance, we know that most directly, the methods apply only to projects that fall in the same risk class as does the entire firm.

[2] See Van Horne [12; p. 81], Weston and Brigham [14; p. 292], and Lindsey and Sametz [4; p. 197].

the conditions under which an internal rate-of-return ranking would be identical to a net terminal value ranking. Since the NTV method does indeed make an explicit assumption about reinvesting a project's intermediate cash flows, Dudley prefers the method, conceptually, over the other two. However, since neither the NPV method or the IRR method yields a set of rankings identical to that of the NTV method, we can use the appropriate set of rankings, under various conditions, as a proxy for the NTV ranking.[3] This is the basis on which Dudley proposes we solve the dilemma of conflicting NPV and IRR rankings.

We find little to argue with in any of this. Dudley elucidates the issue of the reinvestment rate assumption. He's clarified a procedure for choosing between the two, sometimes conflicting sets of rankings; however, he has not identified the source of the conflict between the two procedures. What is still confusing in all of this is that the argument initially made by Solomon [10], made again by Renshaw [9], and clarified by Dudley, allows us to select between NPV and IRR rankings, but contrary to their claims and the claims of the textbooks Dudley mentions, the argument does not explain why the conflict exists.[4]

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[3]The conditions involve the relationship between the reinvestment rate and Fisher's rate (see footnote 11). If the former is greater than the latter, the IRR rankings should be used as a proxy for the NTV rankings; if less, the NPV rankings should be used as the proxy.

[4]Dudley claims ". . . the source of the difficulty [conflict in rankings] can often be traced to the question of the reinvestment of cash flow receipts . . ." [2; p. 907]. Renshaw claims that ". . . the apparent conflict between these two ranking procedures was due to differing implicit assumptions about reinvestment rates . . ." [9; p. 193]. Our claim is that neither author captures the cause of the conflict, but rather they have identified one potential procedure for choosing between the conflicting NPV and IRR rankings.

In other words, no argument has yet been offered to explain (1) why NPV, at times, conflicts with NTV and IRR, and (2) why IRR, at times, conflicts with NTV and NPV.

The purpose of this paper is to provide such an explanation, and, in so doing, to make unnecessary the arguments (initially suggested by Solomon [10]) as to implicit assumptions about the reinvestment rate, i.e., to make unnecessary Dudley's solution to the dilemma of conflicting NPV and IRR rankings.

Restatement of the Accept-Reject Decision

In order to clarify the arguments presented in the following section, this section presents the usual NPV and IRR techniques in a slightly different light.

The present value of a project is the amount, PV, that would have to be invested today at the investors' opportunity rate, K, so as to generate a sequence of benefits, $A_1, \ldots, A_n$, identical to those generated by the project. Algebraically, we identify the present value of a project defined in this way by solving the following equation for PV:[5]

$$\left\{[PV(1 + K) - A_1][1 + K] - A_2\right\}\left\{1 + K\right\} - \ldots - A_n = 0. \tag{1}$$

[5] This can, of course, be rewritten in the traditional way as

$$PV = \sum_{j=1}^{n} \frac{A_j}{(1 + K)^j}.$$

Dudley reminded us, and it is clear from this algebraic statement of our definition, that the PV method makes no assumption whatsoever about reinvesting the benefits $A_1, \ldots, A_n$. Whether or not to accept the project becomes a simple matter of selecting the cheapest way of producing the sequence of benefits $A_1, \ldots, A_n$. They can be generated by undertaking the project and spending an amount equal to the cost of the project, C, or by investing an amount, PV, at the opportunity rate. If $C < PV$, then the project represents the cheapest way of generating $A_1, \ldots, A_n$ and should, therefore, be accepted. The acceptance criteria of $C < PV$ is, of course, identical to $NPV > 0$.

<u>The internal rate of return</u> of a project is the rate, r, at which an amount equal to the cost of the project, C, would have to be invested so as to generate a sequence of benefits, $A_1, \ldots, A_n$ identical to those generated by the project. Algebraically, we identify the internal rate of return of a project defined in this way by solving the following equation for r:[6]

$$\left\{[C(1 + r) - A_1][1 + r] - A_2\right\}\left\{1 + r\right\} - \ldots - A_n = 0. \tag{2}$$

Dudley again reminded us, and it is clear from this statement of our definition, that the IRR method makes no assumption about reinvesting the benefits $A_1, \ldots, A_n$. Whether to accept the project or not becomes a simple matter of whether or not there is an opportunity available to

[6]For "simple" projects (see footnote 7) this can, of course, be rewritten in the traditional way as the discount rate that sets the

$$NPV = 0: \sum_{j=1}^{n} \frac{A_j}{(1 + r)^j} - C = 0$$

invest an amount equal to C at the rate that is necessary to generate the sequence of benefits $A_1, \ldots, A_n$. If the rate available, K, is less than the rate that must be earned to replicate the project's benefits, r, then the project should be accepted. In other words, the project should be accepted if $K < r$. This is, of course, the traditional IRR acceptance criteria.

It is easy to prove that whenever $C < PV$, $K < r$, and that whenever $C > PV$, $K > r$. Therefore, there are no conflicts between NPV and IRR methods for accept-reject decisions.[7]

The Methods under Ranking Situations

Using the same general approach of the preceeding section, we now describe PV and IRR methods when project ranking is required. In the following section we explain why the traditional NPV and IRR techniques can yield conflicting rankings of projects even though we've indicated above that they never yield conflicting accept-reject decisions. With the material from this section, that explanation becomes trivial.

Project ranking becomes becomes necessary under two general conditions: (1) when the projects are mutually exclusive and (2) when the firm constrains

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[7] Such a proof can be found in any elementary text; see Watson and Brigham [14; p. 272]. Actually, it can only be proven in general for "simple" projects in which an initial cash outflow is followed by a sequence of cash inflows. "Nonsimple" projects potentially can run into the well-known multiple internal rate of return problems. In these latter cases, Teichroew, Robichek, and Montalbano [11] have shown how to respecify a project's rate of return so that no conflicts exist in accept-reject situations between rate-of-return procedures and PV procedures.

itself by rationing capital. In the latter case, a capital constraint could occur only in the current decision period, or the firm may be rationing capital in future years, also. If the firm has no plans to ration capital in future periods, then the opportunity rate needed for PV, IRR, and NTV decision making should be set at the firm's cost of capital. If such future rationing is planned, the level at which the opportunity rate is set depends on conditions in the capital markets (see Elton [3] and Haley-Schall [7]). For relatively frictionless markets, the opportunity rate is still the firm's cost of capital. Under other market conditions, however, the opportunity rate should be set at a level higher than the cost of capital. This higher rate has come to be known as the reinvestment rate, although, conceptually, opportunity rate and reinvestment rate are identical.[8]

Ranking project A with an initial cost C_A and benefits $A_1, \ldots, A_n$, and project B with an initial cost C_B and benefits $B_1, \ldots, B_n$ by means of a present value approach involves comparing them in much the

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[8]Under conditions of future capital rationing, the practical difficulties of identifying a value at which to set the reinvestment rate are not to be underestimated. In relatively simple situations, the reinvestment rate for a given period is the return expected on the marginal project during that period. For more complex environments, Weingartner's [13] programming approach may be necessary.

Dudley follows the tradition in most textbooks by computing PV's at the cost of capital, no matter what the opportunity rate, yet he allows the rate used to compute NTVs to vary depending on the capital rationing environment. We use the opportunity rate to compute PVs, be it the cost of capital or some other rate. If this is done for NTVs, also, we know that NTV and PV techniques will <u>always</u> yield identical capital budgeting decisions. These statements should not be construed as an argument against the cost of capital as the discount rate, but rather as an argument for consistency.

same way we compared the single project to investing at the opportunity rate. More specifically, the firm can invest in project B generating benefits $B_1, \ldots, B_n$ at a cost of C_B, or it can replicate those benefits by investing in project A at a cost of C_A (generating benefits $A_1, \ldots, A_n$) along with investing an additional amount at the investors' opportunity rate to generate benefits $B_1 - A_1, \ldots, B_n - A_n$. So, analogous to our accept-reject situation, we have two ways of generating an identical set of benefits $B_1, \ldots, B_n$; the question is which one costs the least. We know accepting project B has a cost C_B. We don't know the cost of accepting project A plus investing at the opportunity rate, because we don't know how much has to be invested at that rate to bring A's benefits up to B's. So, the question becomes how much must be invested at the opportunity rate, K, to generate the sequence of benefits $B_1 - A_1, \ldots, B_n - A_n$. Representing this amount by $PV_{B/A}$, we can, as in equation (1), solve the following for $PV_{B/A}$:

$$\left\{[PV_{B/A}(1 + K) - (B_1 - A_1)][1 + K] - (B_2 - A_2)\right\}\left\{1 + K\right\} - \ldots - (B_n - A_n) = 0 \qquad (3)$$

It follows that we can generate benefits $B_1, \ldots, B_n$ by spending C_B on project B or by spending $C_A + PV_{B/A}$ on a combination of project A plus investing at the opportunity rate. If $C_A + PV_{B/A} > C_B$, then project B should be ranked above project A because it represents the least expensive way to generate $B_1, \ldots, B_n$. This is the criterion, then, for ranking projects A and B: if $C_A + PV_{B/A} > C_B$, project B is preferred to project A; if $C_A + PV_{B/A} < C_B$, project A is preferred to project B.

The criterion also could be rewritten as:

Project B is preferred to project A if

$$C_A + PV_{B/A} > C_B$$

$$PV_{B/A} - (C_B - C_A) > 0$$

$$NPV_{B/A} > 0.$$

In other words, if the incremental net present value of project B over project A is positive, then B is preferred to A. However, the criterion is also identical to ranking projects from highest to lowest individual NPVs. To see this, note that manipulation of the original statement of the criterion yields the more traditional ranking criterion:

Project B is preferred to project A if

$$C_A + PV_{B/A} > C_B$$

$$C_A + \sum_{j=1}^{n} \frac{B_j - A_j}{(1 + K)^j} > C_B \text{[9]}$$

$$C_A + \sum_{j=1}^{n} \frac{B_j}{(1 + K)^j} - \sum_{j=1}^{n} \frac{A_j}{(1 + K)^j} > C_B$$

$$NPV_B > NPV_A.$$

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[9]This follows because equation (3) can be rewritten as:

$$PV_{B/A} = \sum_{j=1}^{n} \frac{B_j - A_j}{(1 + K)^j} .$$

We now turn to the internal rate-of-return procedure. If we again set up the procedure analogous to the set up of the IRR method for accept-reject decisions, the statement becomes: the firm can invest in project B generating benefits $B_1, \ldots, B_n$ at a cost C_B, or it can replicate those benefits by investing in project A at a cost C_A (generating benefits $A_1, \ldots, A_n$) along with investing an amount equal to the difference in the cost of the two projects, $C_B - C_A$, at a rate that will generate benefits $B_1 - A_1, \ldots, B_n - A_n$. So, again we have two ways of generating an identical set of benefits $B_1, \ldots, B_n$; the question is whether the rate that must be earned as part of the second alternative is possible in the sense that it is less than or equal to the opportunity rate, K. Representing this rate by $r_{B/A}$, we can, as in equation (2), solve the following for $r_{B/A}$:

$$\left\{[(C_B - C_A)(1 + r_{B/A}) - (B_1 - A_1)][1 + r_{B/A}] - (B_2 - A_2)\right\}$$

$$- \ldots - (B_n - A_n) = 0. \tag{4}$$

If the rate available, i.e., the opportunity rate K, is less than the rate that must be earned to bring A's benefits up to B's, $r_{B/A}$, then project B is preferred to project A. This is the criterion, then, for ranking projects A and B: if $K < r_{B/A}$, project B is preferred to project A; if $K > r_{B/A}$, project A is preferred to project B. If we rewrite equation (4) as

$$\sum_{j=1}^{n} \frac{B_j - A_j}{(1 + r_{B/A})^j} - (C_B - C_A) = 0, \tag{5}$$

it becomes obvious that our criterion can be stated: if the incremental internal rate of return of project B over project A is greater than the opportunity rate, then project B is preferred to project A.[10]

In summary, then, we've shown in this section that project B is preferred to project A

(1) if the incremental net present value of project B over project A is positive when computed at the firm's opportunity rate;

(2) if the incremental internal rate of return of project B over project A is greater than the firm's opportunity rate.

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[10] Equation (4) also can be rewritten as

$$\sum_{j=1}^{n} \frac{B_j}{(1 + r_{B/A})^j} - C_B = \sum_{j=1}^{n} \frac{A_j}{(1 + r_{B/A})^j} - C_A. \tag{6}$$

From this equation, we see that $r_{B/A}$ can be computed as the discount rate that sets the NPV of one project equal to that of the other--this is well-known as Fisher's rate of return.

Actually, if the incremental project is a "nonsimple" project, as it well may be, then we once again must rely on the procedure suggested by Teichroew, Robichek, and Montalbano (11) to compute $r_{B/A}$, the incremental rate of return. In order to avoid some of these problems, it may be desirable to compute $r_{A/B}$ rather than $r_{B/A}$.

Lorie and Savage present, without derivation, this incremental IRR approach in their original work. They saw its applicability only in terms of mutually exclusive alternatives, however, not for capital rationing situations. The same can be said of Haley and Schall [6; pp. 62-64] and Van Horne [12; p. 82]. Fleischer [4] also shows by example the consistency between the incremental IRR ranking method and the PV ranking method. His analysis is less general than ours and, as with the others, no derivation is given. Without the derivation the incremental IRR approach appears to be an ad hoc solution.

Resolving Criteria Conflicts

An immediate concern whenever two or more criteria are proposed for the purposes of making any decision is whether or not the criteria are consistent. In the second section, we indicated that NPV and IRR methods always yield consistent accept-reject decisions. Section three was concerned with whether or not incremental NPV and incremental IRR ranking criteria always rank the projects in the same order.[11] From the incremental IRR technique we know that

$$\sum_{j=1}^{n} \frac{B_j - A_j}{(1 + r_{B/A})} = C_B - C_A. \qquad (7)$$

We know from the mathematical characteristics of the present value polynomial that if $K < r_{B/A}$ then

$$\sum_{j=1}^{n} \frac{B_j - A_j}{(1 - K)^j} > \sum_{j=1}^{n} \frac{B_j - A_j}{(1 + r_{B/A})^j} . \qquad (8)$$ [12]

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[11] We should point out once again that as long as the same rates are used in the NPV and NTV methods, they will <u>never</u> yield conflicting capital budgeting decisions. Dudley and many of the textbooks he references are one rate for discounting and another rate for compounding; for this reason alone they get conflicts between NPV and NTV.

[12] Actually this statement may not be true for certain patterns of $B_j - A_j$. However, the basic notion that the two criteria are consistent in the rankings of projects can still be proven, although it would be much more involved. See Teichroew, Robichek, and Montalbano [11].

Using equation (7) we know that

$$\sum_{j=1}^{n} \frac{B_j - A_j}{(1 + K)^j} > C_B - C_A$$

$$\sum_{j=1}^{n} \frac{B_j - A_j}{(1 + K)^j} - (C_B - C_A) > 0$$

which tells us that the incremental NPV of project B over project A is positive as long as the incremental internal rate of return of project B over project A is greater than the opportunity rate. In other words, both rankings indicate that project B is preferred to project A. Since an analogous proof can be made for the case in which the inequality signs are reversed, we've just proven that the two ranking criteria are completely consistent and interchangeable.

Out of context, this conclusion may be rather surprising since the historical concern in the literature has been with conflicts in ranking between NPV and IRR. The explanation of the lack of relevance of this historical concern should be clear now that we've shown how NPV and IRR must be used to rank projects. The historical concern is essentially that ranking projects by their <u>individual</u> NPVs may yield a different order than ranking projects by their <u>individual</u> IRRs. The point is that there is no reason to expect these rankings to be identical. It is the <u>incremental</u> NPVs and <u>incremental</u> IRRs that we would expect to yield consistent rankings, and indeed they do. It is true (as we've shown above) that ranking projects by the incremental NPV technique is identical to ranking the individual NPVs, <u>however</u>, and this is the crucial point: ranking projects by the incremental IRR technique is <u>not</u> identical to ranking the individual IRRs. Thus, we should no more expect the ranking of projects by individual NPVs to yield an order identical to the ranking of individual IRRs than we should expect individual NPVs to rank projects identically to the ranking of individual payback periods or individual returns on investment (ROI).

One implication of this is that the reinvestment arguments originally proposed by Solomon and recently clarified by Dudley are unnecessary. They were concerned with finding a set of rankings that would duplicate the NTV rankings. Their preoccupation with NTV rankings is an outgrowth of the conviction that ". . . the ultimate criterion is the total wealth that the investor can expect from each alternative by the terminal date of the longer-lived project," and that ". . . an explicit and common assumption must be made regarding the rate at which funds released by either project can be reinvested up to the terminal date."[13] This concern with the reinvestment rate is handled directly in the NTV approach as part of the definition. The point that should be emphasized is that the reinvestment considerations can be handled <u>just as well</u> in the NPV and IRR approaches simply by setting the opportunity rate equal to the reinvestment rate rather than the firm's cost of capital. If this is done, NTV, NPV, and incremental IRRs all will rank any given set of projects in exactly the same order, <u>no matter what is</u>

[13] Solomon [10; p. 127].

the reinvestment rate.[14] It is clear, then, that any NPV-IRR ranking conflicts are not caused by different assumptions as to reinvestment rates, but rather by improper specification of the IRR method in ranking situations.[15]

Conclusion

In this paper we have shown

(1) that appropriately applied, NPV and IRR ranking techniques never yield conflicting decisions, and

(2) that appropriately applied NPV and IRR ranking techniques always yield decisions identical to those made using NTV rankings.

The confusion over this issue in the literature stems largely from the mistaken notion that application of the NPV and IRR techniques to the ranking of projects involves a computation and ranking of the individual projects' NPVs and IRRs. Although this is valid for the NPV method, it is not valid for the IRR method; therefore, there is no reason to expect rankings of individual NPVs and IRRs to have the same order.

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[14] Again, we do not mean to argue in favor of using a rate above the cost of capital as the opportunity rate. Our point is only that the discount rate used in calculating NPVs should equal the compound rate used to calculate NTVs.

[15] Recall that we have no basic argument with Dudley's approach to duplicating the NTV rankings. This has nothing to do with the cause of NPV-IRR conflicts.

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