2900 - 392 - T

Report of Project MICHIGAN

THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS

(Abdel Y.)

A. Y. BILAL

September 1963

Navigation and Guidance Laboratory
Institute of Science and Technology
THE UNIVERSITY OF MICHIGAN
Ann Arbor, Michigan

NOTICES

Sponsorship. The work reported herein was conducted by the Institute of Science and Technology for the U. S. Army Electronics Command under Project MICHIGAN, Contract DA-36-039 SC-78801 and for the National Science Foundation under grant GP-524. Contracts and grants to The University of Michigan for the support of sponsored research by the Institute of Science and Technology are administered through the Office of the Vice-President for Research.

Note. The views expressed herein are those of Project MICHIGAN and have not been approved by the Department of the Army.

Distribution. Initial distribution is indicated at the end of this document. Distribution control of Project MICHIGAN documents has been delegated by the U. S. Army Electronics Command to the office named below. Please address correspondence concerning distribution of reports to:

Commanding Officer
U. S. Army Liaison Group
Project MICHIGAN
The University of Michigan
P. O. Box 618
Ann Arbor, Michigan

DDC Availability. Qualified requesters may obtain copies of this document from:

Defense Documentation Center
Cameron Station
Alexandria, Virginia

Final Disposition. After this document has served its purpose, it may be destroyed. Please do not return it to the Institute of Science and Technology.

PREFACE

Project MICHIGAN is a continuing long-range research and development program for advancing the Army's combat surveillance and target-acquisition capabilities. The program is carried out by a full-time staff of specialists in physics, engineering, mathematics, and psychology at the Institute of Science and Technology, and by members of the teaching faculty and graduate students of other research groups and laboratories of The University of Michigan.

The emphasis of the Project is upon research in imaging radar, MTI radar, infrared, radio location, image processing, and special investigations. Particular attention is given to all-weather, long-range, high-resolution sensory and location techniques.

Project MICHIGAN was established by the U. S. Army Signal Corps at The University of Michigan in 1953 and has received continuing support from the U. S. Army. The Project constitutes a major portion of the diversified program of research conducted by the Institute of Science and Technology in order to make available to government and industry the resources of The University of Michigan and to broaden the educational opportunities for students in the scientific and engineering disciplines.

Progress and results described in reports are continually reassessed by Project MICHIGAN. Comments and suggestions from readers are invited.

Robert L. Hess
Director
Project MICHIGAN

ACKNOWLEDGMENTS

The assistance and encouragement received from the administration of The University of Michigan, the faculty of the Department of Electrical Engineering, and the staff of the Navigation and Guidance Laboratory of the University's Institute of Science and Technology were instrumental in the completion of this work. The interest and continual aid given by Professor L. F. Kazda and Professor W. Porter are especially acknowledged. The suggestions, comments, and criticisms of Professors L. F. Kazda, W. Brown, Kuei Chaung, W. Kaplan, and Arch Naylor are sincerely appreciated. The author also wishes to express his thanks to Mrs. Louise Snider, who edited this report, Mr. August Antones, who drew the figures, and Mrs. Linda Bennett, who typed the first and final drafts.

CONTENTS

FIGURES

TABLES

SYMBOLS

$a_i(t)$ = System time-varying parameters

A^2 = A number obtained from $\int_0^T \alpha^2(t)\,dt \leq A^2 < \infty$

A_m^2 = A number obtained from $\int_0^T \alpha_m^2(t)\,dt \leq A_m^2$

C^2 = A number obtained from $\int_0^T n^2(t)\,dt \leq C^2 < \infty$

C_m^2 = A number obtained from $\int_0^T n_m^2(t)\,dt \leq C_m^2$

$D^n = \frac{d^n}{dt^n}$, the n-th differential operator $n = 0, 1, 2, \ldots p$

$d(t)$ = An enforced bound within which the system response must remain

$E_{(i, k)_m}$ = $E_T[x_m^{(i)}(kh)]$, the truncation error in the evaluation of $x_m^{(i)}(kh)$

E_r = Round-off errors

$E_T(xt)$ = The truncation error in x(t)

$F(t, u, x)$ = The weighted nonlinear function (for the single-degree-of-freedom case) = $W(t, u)N(x)$

$F_{ik}(t, u, x)$ = $W_{ii}(t, u)N_{ik}(x)$ = the weighted nonlinear functions (for multiloop case)

$F^{(m)}(t, u, x)$ = The m-th partial derivative of F(t, u, x) with respect to t

$F_{ik}^{(\mu)}(t, u, z)$ = The μ-th derivative of $F_{ik}(t, u, z)$ with respect to time

$g_1, \ldots, g_m$ = System inputs for the multiloop system

$g(t)$ = System input for the single-degree-of-freedom case

h = Subinterval of time $= \frac{\text{total interval}}{\text{number of intervals}}$

$K(t, u)$ = An upper bound of $\frac{\partial F(t, u, x)}{\partial x}$ in a given domain D

$K_{ik_\mu}(t, u)$ = An upper bound of $\frac{\partial F_{ik}^{(\mu)}(t, u, z)}{\partial z}$ in a given domain D

$K_m(t, u)$ = An upper bound of $\frac{\partial F^{(m)}(t, u, x)}{\partial x}$ in a given domain D

$L_2[0, T]$ = L_2 space in the closed interval [0, T]

$L(D, t)$ = Time-varying linear differential operator (single-degree-of-freedom case)

$L_{ik}(D, t)$ = Time-varying linear differential operator (multiloop case)

Lim = Limit in the ordinary sense

L.i.m. = Limit in the mean square sense

$n(t)$ = A function obtained from $\int_0^T F(t, u, x_0)\,dt \le n(t)$

$n_{ik}(t)$ = A function of time obtained from $\int_0^T F_{ik}^{(\mu)}(t, u, x_{k0})\,du \le n_{ik}(t)$

$n_m(t)$ = A function obtained from $\int_0^T F^{(m)}(t, u, x_0)\,du \le n_m(t)$

$N(x)$ = Nonlinear functions with memory (single-degree-of-freedom case)

$N_{ik}(x)$ = Nonlinear functions with memory (multiloop case)

q^2 = Mean square error = $\int_0^T (x - x_n)^2\,dt$

t = Time

T = Total interval of time

$\overline{v}(t)$ = State vector with component $v_1(t), \ldots, v_n(t)$ representing the state variables of the system

$W(t, u)$ = Impulse response of $L(D, t)$

$W_{ii}(t, u)$ = Impulse response of $L_{ii}(D, t)$

$||x||$ = The L_2 norm of $x = \int_0^T x^2(t)\,dt$

$x_1, x_2, \ldots, x_n$ = System outputs for the multiloop system

$x(t)$ = System output (for the single-degree-of-freedom case)

$x_{i0}(t)$ = Solution of $L_{ii}(D, t)x(t) = g_i(t)$

$x_{mk}^{(i)}$ = An approximate value of $x_m^{(i)}(t)$ at a time $t = kh$

$x^{(i)}(t)$ = The i-th derivative of $x(t)$

$x_m(t)$ = The m-th iterate of $x(t)$

$\{x_n(t)\}$ = An infinite sequence of functions $x_n(t)$, $n = 1, 2, \ldots, \infty$

$x_0(t)$ = Solution of $L(D, t)x(t) = g(t)$

$\alpha^2(t)$ = A function obtained from $\int_0^t K^2(t, u)\,du \le \alpha^2(t)$

$\alpha_m^2(t)$ = A function obtained from $\int_0^T K_m^2(t, u)\,du \le \alpha^2(t)$

$\alpha_r^2(t)$ = A function of time obtained from $\int_0^T K_{ik_\mu}^2(t, u)\,du \le \alpha_r^2(t)$

$\beta(t)$ = An upper bound for $x(t)$ inside $d(t)$ which belongs to $L_2[0, T]$

$\beta_i(t)$ = Upper bounds for the state variables $v_i(t)$. $\beta_i(t)$ belongs to $L_2[0, T]$.

$\delta(t - u)$ = The unit impulse response applied at $t = u$

ϵ = Belongs to

ε_n = The error in x(t) when approximated by $x_n(t) = x - x_n(t)$

$\varepsilon_{(i, k)_m}$ = The discretization error in $x_{mk}^{(i)} = x_{mk}^{(i)} - x_m^{(i)}(kh)$

THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS

ABSTRACT

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{x_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters.

Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov. The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy. For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis. The method lends itself easily to digital computer calculations.

1 INTRODUCTION

In literature dealing with the general area of automatic control, information on the use of the partitioning technique for the analysis and synthesis of a class of nonlinear systems has appeared only recently. This introduction is intended to provide adequate background information on this subject for the reader. The origin of the partitioning technique is discussed from the author's point of view, and this is followed by a brief review of the theory and its limitations as presented by A. A. Wolf [41 through 47]. The author also considers questions and arguments that have been raised recently as a result of comparison of the partition method with other methods of analysis. Finally, the objectives of this study are summarized and discussed.

1.1. HISTORICAL ORIGIN AND REVIEW

During the past two decades, many papers have been written about techniques for analyzing and synthesizing nonlinear systems. But the techniques thus far developed are restrictive because each applies to a particular class of problems, each has its own particular value and usefulness, and each lacks an underlying or unifying theory that would join it to others. However, there are some exceptions. One is the general theory of stability via Lyapunov's Second Method [14]. This has recently attracted the attention of the engineering community of this country. Another exception is the technique of optimum control analysis. Generally, some of the techniques for analysis of nonlinear systems are linearization techniques and others are approximation techniques, sometimes quite restricted. However, the purpose of this report is not to discuss different possible approaches to the solution of nonlinear systems (since these approaches are adequately treated elsewhere [21, 44]), but to present a detailed study of one approach, the partitioning technique. This technique is applicable to a class of nonlinear systems which contain some linear terms, some nonlinear terms, and a forcing function. However, the idea of separating the original nonlinear equation into linear and nonlinear terms is not original. It was first introduced by Henri Poincaré [33] at the end of the nineteenth century. In his classical researches on celestial mechanics, Poincaré developed quantitative methods of approximations by expansion in terms of small parameters. During the past thirty years or so, the methods of Poincaré have been adapted to the problems of applied science in general. This was done as a result of the work of Mandelstom, Papalexi, Andronov, Krylov, Bogolyubov, and other Russian scientists working jointly in this field [15, 19, 28, 33].

In a recent book [10], Cunningham mentioned the idea of partitioning in an explanation of procedures for finding the approximate analytical solution for a nonlinear equation. The procedures are shown in Figure 1. In this diagram the original nonlinear differential equation is first separated into two parts: one part is a linear equation simple enough to permit exact solution; the other part contains any terms that are difficult to handle and usually involves the nonlinear terms, but there may be other terms as well. Next, the linear equation is solved to give the zero-order or generating solution. This generating solution is then employed in some way with the nonlinear terms of the original equation to produce first-order correction terms. Finally, these correction terms are combined with a generating solution to yield a first-order corrected solution which is an approximate solution of the original equation. The exact form of the correction terms depends upon the particular details of the method being employed. In some cases, the correction terms are merely added to the generating solution. In other cases, the correction terms produce changes in amplitude and phase of the generating solution. If the degree of nonlinearity is sufficiently small, a single application of this method will probably yield a first-order correction solution of sufficient accuracy. If the degree of nonlinearity is large, it is sometimes possible to improve the accuracy by applying the method a second time to obtain a second-order corrected solution (as shown in Figure 1). In theory, continued increases in accuracy are possible by further repeated applications of the method, but this is

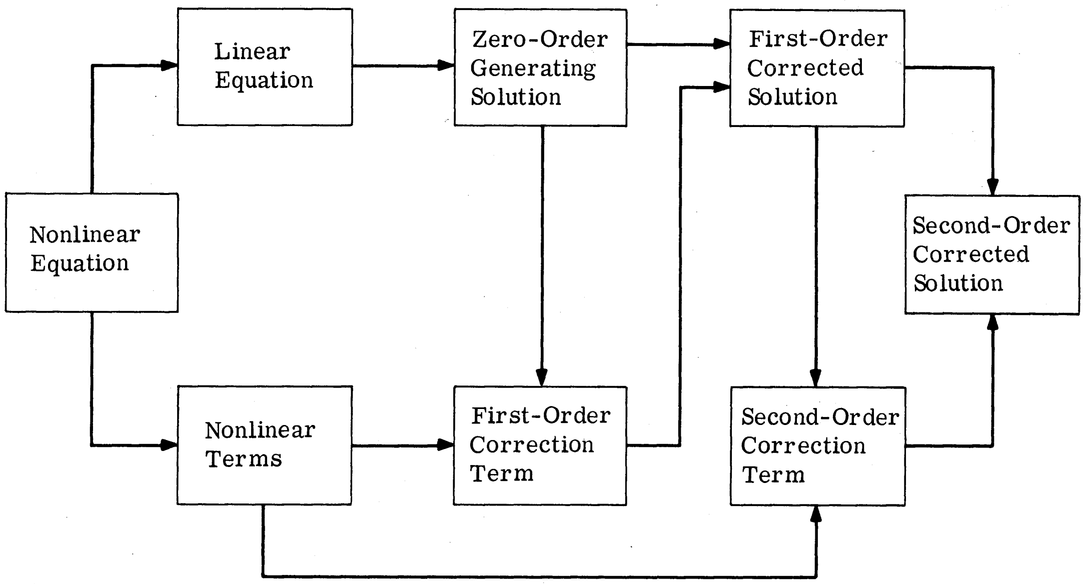

FIGURE 1. TYPICAL OUTLINE OF STEPS FOLLOWED IN FINDING AN APPROXIMATE SOLUTION FOR A NONLINEAR EQUATION

usually not practical because any small increase in accuracy is accompanied by a large increase in mathematical complications. Actually, an important uncertainty inherent in this kind of method is the error in the solution it yields, since this error is often difficult to calculate.

Some well-known methods of analysis depend on the above procedure, such as the perturbation, reversion, variation of parameter, and averaging techniques by Galerkin and Ritz. Although details of the different methods vary, most of them are rather similar and follow the steps diagrammed in Figure 1. However, it should be noted that uniqueness of solution was assumed in the previous discussion. In his partition theory, A. A. Wolf studied conditions which must be satisfied by the linear, nonlinear, and forcing function terms of the system equation if the system is to yield a unique solution. Although nonlinear differential equations generally are known not to possess unique solutions [47, Appendix A, B, C, E], it is apparent from physical conditions that, for a particular initial state, physical systems must possess unique solutions which exist in a domain to the right of that initial state. This means that the solution of the given nonlinear differential equation corresponding to a given initial state should be unique; otherwise the analysis given does not adequately describe the behavior of the system [29, p. 36].

Since the publication of Wolf's dissertation [47], which gave a detailed account of the partition theory, many articles have appeared which expound the deterministic process of analysis of a class of nonlinear systems. With the aid of the statistical transform theorem, the theory of systems subjected to stochastic processes is extended [43, 47].

Let us now consider a typical class of physical nonlinear systems whose behavior can be described by the following nonlinear equation:[1]

$$Z(D)x(t) + N(x, x, \ldots, x^{(m)}) = g(t) \tag{1}$$

where the following assumptions and restrictions, which are sufficient but not necessary for the analysis, are made:

(1) The operator Z(D) is linear with constant coefficients and is restricted so that it has an impulse response y(t) defined as

$$y(t) = \frac{1}{2\pi j}\int_{\sigma - J\infty}^{\sigma + J\infty} \frac{e^{st}}{Z(s)}\,ds \tag{2}$$

which is of exponential type and order at most[2] one.

(2) The forcing function g(t) is of exponential type and order at most one. (3)

(3) The nonlinear function N(s) is single valued, continuous, and satisfies the following condition:

Given a pair of positive constants M and a and two continuous functions v(t) and u(t), which are asymptotically like e^{-at}, then N must satisfy the modified Lipschitz condition:

$$|N[v(t)] - N[u(t)]| \leq Me^{-at}|v(t) - u(t)| \tag{4}$$

It should be noted that condition 4 is true only when N is a function of x alone,[3] and does not depend on $\dot{x}$, $\ddot{x}$, or $x^{(m)}$.

As a result of imposing these three conditions on the system described in Equation 1 and applying the partitioning techniques described later in this section, a system solution x(t) is found which is analytic and unique. The exact solution is represented as a limit of a sequence

[1]The method of partitioning discussed in this section is applicable only to systems described by equations of the same type as Equation 1.

[2]A function f(t) is of exponential type if it satisfies the inequality $|f(t)| \leq Me^{at}$, where M and a are two arbitrary finite numbers. See Reference 37, page 248, for a precise definition of the order of an analytic function.

[3]For a discussion of situations in which N contains derivatives of x, see Reference 47, page 22.

of iterates $\{x_n\}$ as n tends to infinity. Each of these iterates $\{x_n\}$ (for all finite n) is of exponential type and order one, and is given by the following recurrent equation:

$$x_{n+1} = x_0(t) - \int_0^t N[x_n(\tau)]\, y(t - \tau)\, d\tau, \quad (n = 0, 1, 2, \ldots) \tag{5}$$

where, for the case of zero initial condition, $x_0(t)$ is defined by

$$x_0(t) = \int_0^t g(\tau)\, y(t - \tau)\, d\tau \tag{6}$$

Therefore,

$$x(t) = \lim_{n\to\infty} x_n(t), \; n \geq 1 \tag{7}$$

This limit has been shown [47] to be a uniquely convergent procedure which gives the required solution. In other words, there is a certain class of linear systems (not necessarily physically realizable) in which the response $x_n(t)$, for n = 0, 1, 2, . . . of each of these systems forms a point set, where the limit as $n \to \infty$ is the response of the nonlinear differential Equation 1. The members of the set are generated according to Equations 5 and 6. Thus, among the sequence $\{x_n\}$, there is a particular x_n whose behavior approximates that of the actual response as closely as possible. This means that the iterates x_n for all $n \geq q$ approximate the solution x(t) in such a way that the error ε_n as given by

$$|x(t) - x_n(t)| = \varepsilon_n(t) \tag{8}$$

satisfies the inequality

$$\varepsilon_n(t) \leq \delta \tag{9}$$

where δ and q are positive numbers. The physical meaning of the generation of the iterates is explained in Figures 2 through 5.

From these figures one can consider $x_0(t)$ to be the response of a linear network having zero feedback, as shown in Figure 2. One may also consider the sequence of functions x_n as being generated by a collection of feedback systems, the response of one member of which depends on the response of the one just preceding it. The networks $M_n(s)$ have the property that if $x_n(t)$ excites $M_{n-1}(s)$, the response is $N[x_{n-1}(t)]$. In this sense one may consider the networks $M_n(s)$ as linear, although they may not be physically realizable. It should be noted that the method of analysis introduced above is similar in spirit to the techniques used in the so-called fixed point theorems in functional analysis. Detailed discussions of this appear in References 3 and 16.

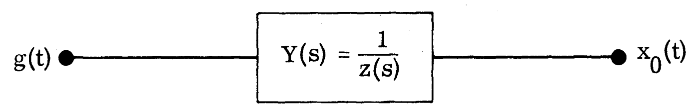

FIGURE 2. THE PHYSICAL MEANING OF $x_0(t)$

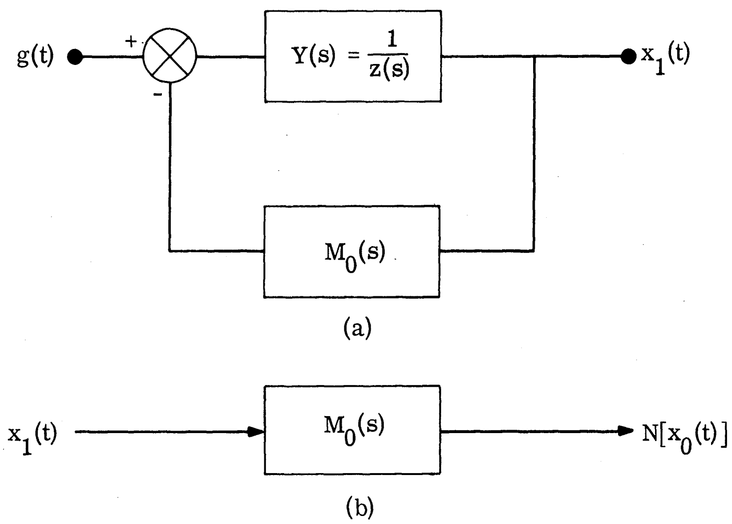

FIGURE 3. THE PHYSICAL MEANINGS OF $x_1(t)$ AND $M_0(s)$.
(a) $x_1(t)$. (b) $M_0(s)$.

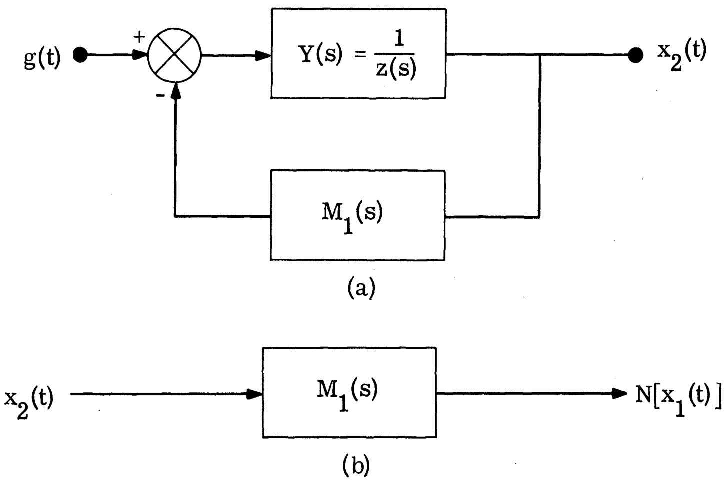

FIGURE 4. THE PHYSICAL MEANINGS OF $x_2(t)$ AND $M_1(s)$.
(a) $x_2(t)$. (b) $M_1(s)$.

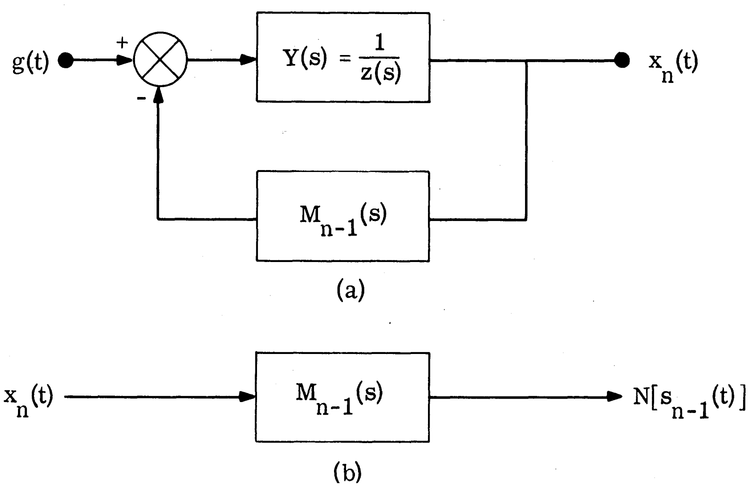

FIGURE 5. THE PHYSICAL MEANINGS OF $x_n(t)$ AND $M_{n-1}(s)$. (a) $x_n(t)$. (b) $M_{n-1}(s)$.

The actual desired solution of the system equation can be obtained in the series form, to any degree of accuracy, by applying the rules of partitioning as follows:

(1) The dynamic equation representing the system is partitioned so that the linear terms involving the highest order derivative are contained in one member of the equation.

(2) A modified forcing function A(t) is then obtained which consists of the actual forcing term, the nonlinear term, and possibly some linear terms.

(3) Next, the expansion of A(t) is determined by the particular properties of the linear terms, nonlinear terms, and driving function of the system, and the required form of the system solution. (This may be clarified by a consideration of the different forms of expansion of A(t) given in Equations 48 through 50.)

(4) The properly partitioned differential equation is then solved.

(5) Certain coefficients which arise from the auxiliary equation can then be calculated.

(6) Finally, these coefficients, which appear in the partitioned equations, are eliminated to obtain the exact solution.

Applying the partitioning rules to Equation 1 gives

$$Z(D)\, x(t) = A(t) \tag{10}$$

which is the partition equation, and

$$A(t) = g(t) - N[x(t)] \tag{11}$$

which is the auxiliary equation. The partitioned nonlinear system represents Equations 10 and 11 and is shown diagrammatically in Figure 6.[4] To effectively use Equations 10 and 11 we must take advantage of the a priori properties of A(t) which were deduced by means of the conditions required of the linear and nonlinear parts of Equation 1 and its forcing function.

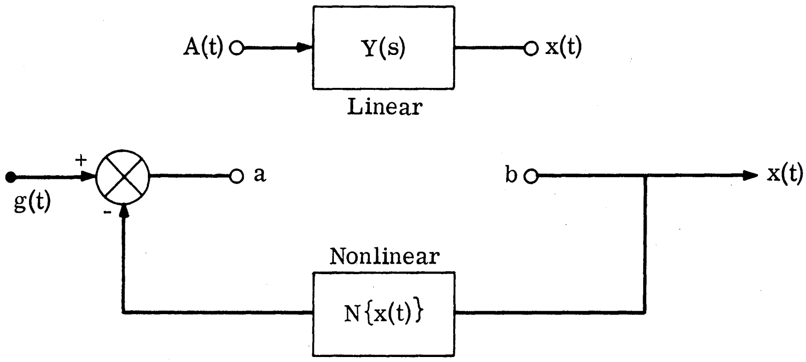

FIGURE 6. THE PARTITIONED NONLINEAR SYSTEM

From these conditions, x(t) is deduced to be analytic. If g(t) is analytic and N is analytic, then A(t) is analytic, since it is the difference between two analytic functions.

By the above means, A(t) can be expanded into a power series. In these conditions the solution of Equation 1 is given by the following:

$$A(t) = \sum_{n=0}^{\infty} C_n t^n \qquad (12)$$

Assuming zero initial conditions for the purpose of simplicity, x(t) is given by

$$x(t) = \int_0^t y(t - \tau)\, A(\tau)\, d\tau$$

$$x(t) = \int_0^t y(t - \tau) \left[\sum_{n=0}^{\infty} C_n \tau^n \right] d\tau = \sum_{n=0}^{\infty} C_n \int_0^t \tau^n\, y(t - \tau)\, d\tau \qquad (13)$$

Therefore,

$$x(t) = \sum_{n=0}^{\infty} C_n Q_n \qquad (14)$$

[4]The configuration shown in Figure 6 is sometimes known as the canonical form of nonlinear systems.

where Q_n = the n-th moment of the impulsive response of the linear part Z(D) over the half open interval (0, t), that is,

$$Q_n(t) = \int_0^t \tau^n \, y(t - \tau)\, d\tau \tag{15}$$

It should be noted that if all but the highest order derivative is transposed to the right member, a power series solution results since the resulting moment function is a power series. This can be easily shown by the following reasoning:

Partitioning Equation 1, as shown below, gives

$$\frac{dx^m}{dt^m} = g(t) - M(x) \tag{16}$$

where

$$M(x) = N(x) + \text{linear terms} \tag{17}$$

for this case, therefore, we have

$$y(t) = \frac{t^{m-1}}{(m - 1)!} \tag{18}$$

Applying Equation 15 gives

$$Q_n(t) = \int_0^t \frac{\tau^n}{(m - 1)!} (t - \tau)^{m-1} d\tau$$

Therefore,

$$Q_n(t) = \frac{n!}{(m + n)!} t^{n+m} \tag{19}$$

Equations 14 and 19 show the validity of the previous statements. Under the conditions placed on the system Equation 1 and under a special type of partitioning (at the highest-order derivative), the solution obtained is in a power series form; this fact gives rise to many questions concerning the analogy between the partition method and the classical method of analysis by Frobenius [13].

In a discussion of the partition theory [42], Professor R. L. Cosgriff (Ohio State University) stated that the partition theory could be avoided by extending the method of Frobenius to nonlinear analysis. He also stated that the present method only introduces a new facet concerning the solution of nonlinear equations, and that as each facet is exploited new insights and techniques develop, which expand our knowledge of nonlinear systems.

Also, E. V. Bohn [5], in his comments about the partition method and associated transform methods, stated that it is difficult to agree with A. A. Wolf on the usefulness of his methods for theoretical and systematic studies, and that a power series representation is generally of little use. Bohn also pointed out that conventional methods and the partitioning method are equivalent in the following manner.

The nonlinear differential Equation 1 can be put in the form

$$u_n = f_n (u_0, u_1, \ldots, u_{n-1}, t) \tag{20}$$

where the notation

$$u_n = u_n(t) = \frac{d^n x(t)}{dt^n} \tag{21}$$

is used. The solution of Equation 20 can be represented by a power series

$$x(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \tag{22}$$

From Equations 21 and 22 it follows that

$$a_k = u_k(0) \tag{23}$$

The values of $a_0, a_1, \ldots, a_{n-1}$ can be specified as initial conditions and a_k can be found by differentiating Equation 20 successively with respect to t and then substituting t = 0, which yields

$$u_{n+1} = \frac{\partial f_n}{\partial u_0} u_1 + \ldots + \frac{\partial f_n}{\partial u_{n-1}} u_n + \frac{\partial f_n}{\partial t} = f_{n+1} (u_0, u_1, \ldots, u_{n-1}, u_n, t) \tag{24}$$

and for t = 0

$$\begin{aligned} a_n &= f_n (a_0, a_1, \ldots, a_{n-1}, 0) \\ a_{n+1} &= f_{n+1} (a_0, a_1, \ldots, a_n, 0) \end{aligned} \tag{25}$$

Equation 25 should then be solved for each a_k.

To see the equivalence, assume that Equation 1 is partitioned at the highest derivative, as already shown by Equation 16, and let

$$\frac{dx^m}{dt^m} = \sum_{k=0}^{\infty} C_k t^k \tag{26}$$

From Equations 26 and 22 the following relation is obtained:

$$C_k = \frac{a_{a+k}}{k!} \quad (k \geq 0) \tag{27}$$

However, the author feels that the criticisms of Bohn and Cosgriff are not totally warranted. A review of Wolf's published work would reveal that the method of Frobenius is a special case of the partition theory. The Frobenius method only gives rise to a power series solution, but the partition method gives rise to a general class of solution forms. The nature of these solutions depends on the point of partitioning and the form of expansion of the auxiliary forcing function which, together, provide flexible control of the nature of solution forms. Reference 43 contains examples which illustrate this. They show that although the solution exists and is unique, the form of the solution is not unique but is a function of the point of partition. Thus, by using the partition technique, the solution can be obtained as a power series, Dirichlet series, trigonometric series, or orthogonal series, whereas the Frobenius method permits only a power series solution. Therefore, as a result of the partition theory, the following expansion of A(t), for example, can be justified:

$$A(t) = \sum_{n=0}^{\infty} C_n e^{-\lambda_n t} \tag{28}$$

and if

$$\lambda_n = an \tag{29}$$

the system Equation 1 now has the following solution:

$$x(t) = \sum_{n=0}^{\infty} C_n P_n(t) \tag{30}$$

where $P_n(t)$ are the exponential moments of the folded impulsive response of the linear part given by

$$P_n(t) = \int_0^t e^{-nat} y(t - \tau)\, dt \tag{31}$$

It is evident from expanding e^{-nat} into a power series that a relation exists between the exponential and the impulsive moments of $y(t - \tau)$, $Q_n(t)$. By choosing λ_n as in Equation 29 we have the special case of the Dirichlet series known as the power Dirichlet series. This terminology is adopted since the solution is given in a power series of Z, when $z = e^{-at}$ and λ_n is given by Equation 29. As an example, consider a system described by [47]:

$$\frac{dx}{dt} + \delta x + \alpha x^2 = \beta \tag{32}$$

where δ, α, β are constants. The initial conditions are given as x(0) = 0. Equation 32 may be partitioned in two ways: one way would be at the highest derivative and the other way would include the x itself in addition to the derivative. Consider using the highest derivative first:

$$\frac{dx}{dt} = \beta - \delta x - \alpha x^2 \tag{33}$$

Thus

$$\frac{dx}{dt} = \sum_{n=0}^{\infty} C_n e^{-nat} \tag{34}$$

The auxiliary function now becomes

$$\sum_{n=0}^{\infty} C_n e^{-ant} = \beta - \delta x - \alpha x^2 \tag{35}$$

Integrating once, Equation 34 becomes

$$x = \left(\sum_{n=1}^{\infty} \frac{C_n}{-na} e^{-nat} + k \right) \tag{36}$$

where k = constant of integration. Substituting Equation 36 into Equation 35 results in the following:

$$\sum_{n=0}^{\infty} C_n e^{-nat} = \beta + \delta \sum_{n=1}^{\infty} \frac{C_n}{na} e^{-nat} - \delta k - \alpha \left(\sum_{n=0}^{\infty} \frac{C_n}{-na} e^{-nat} + k \right)^2 \tag{37}$$

Let

$$z = e^{-at} \tag{38}$$

so that

$$\sum_{n=0}^{\infty} C_n Z^n = \beta + \delta \sum_{n=1}^{\infty} \frac{C_n}{na} Z^n - \delta k - \alpha \sum_{n=1}^{\infty} \left(\frac{C_n}{na} Z \right)^2$$

$$+ 2\alpha k \sum_{n=1}^{\infty} \frac{C_n}{na} Z - \alpha k^2 \tag{39}$$

Upon integrating, we notice that the lower bound becomes unity instead of zero. This is necessary since $C_0 = 0$. If $C_0 \neq 0$, then the sum would become infinite at n = 0. A necessary requirement for all the coefficients is that they be finite. Therefore, the recursion formula is: for n > 0

$$C_n = (\beta - \delta k - \alpha k^2) C_n^{(0)} + \frac{\delta C_n}{na} - \alpha C_{n,0}^{(2)} + 2\alpha k \frac{C_n}{na} \tag{40}$$

where

$$C_{n,0}^{(2)} = \sum_{k=0}^{\infty} \frac{C_k C_{n-k}}{\alpha^2 k(n - k)}$$

For n = 0, we determine k as follows

$$\alpha k^2 + \delta k - \beta = 0 \tag{41}$$

or

$$k = -\frac{\delta \pm \sqrt{\delta^2 + 4\alpha\beta}}{2\alpha} \tag{42}$$

For n = 1,

$$C_1 = \frac{\delta C_1}{a} - \alpha C_{1,0}^{(2)} + 2\alpha k \frac{C_1}{a}$$

or

$$C_1 \left(\frac{\delta}{a} + \frac{2\alpha k}{a}\right) = 0 \tag{43}$$

The second factor is not zero; hence

$$C_1 = 0 \tag{44}$$

For n = 2, it is possible to determine the characteristic exponent a:

$$C_2 = \frac{\delta C_2}{2a} - \frac{\alpha C_1^2}{a^2} + \frac{2\alpha k C_2}{2a} \tag{45}$$

$$C_2 \left(1 - \frac{\delta}{2a} + \frac{\alpha C_1^2}{a^2} - \frac{2\alpha k}{2k}\right) = 0 \tag{46}$$

Either or both factors will be zero. We shall assume that the second factor is zero to determine a. By taking $C_1 = 0$ from Equation 46 and substituting the value of k given by Equation 41, we get the result:

$$a = \pm 1/2 \sqrt{\delta^2 + 4\alpha\beta} \tag{47}$$

This result is known to be correct since it can be compared with the solution obtained by separating the variables (see Appendix A). The characteristic exponent here has two values. Both must be used in Equation 36. This gives rise to two Dirichlet series. If the characteristic exponent had k values, then k series would be needed in addition to the constant of integration which may arise. This procedure is similar to the one used in linear differential equations. The remainder of the solution follows a procedure similar to that described elsewhere [13].

If the second partition scheme is used, a different form of x(t) is obtained; this is easily checked. Also, the equation of the auxiliary forcing term for A(t) in the forms

$$A(t) = \sum_{n=0}^{\infty} C_n t^n e^{-at} \tag{48}$$

$$A(t) = \sum_{n=0}^{\infty} C_n t^n e^{-nat} \tag{49}$$

$$A(t) = \sum_{n=0}^{\infty} C_n t^n \tag{50}$$

eventually follows the same procedure and yields different and interesting cases for investigations. These cases enable the engineer to gain further insight into the nonlinear system under consideration. Suppose, e.g., the form of Equation 50 is chosen and the following results are obtained for the same system described by Equation 32. The moments are

$$Q_n(t) = \frac{t^{n+1}}{n+1} \tag{51}$$

Thus

$$x(t) = \sum_{n=0}^{\infty} \frac{C_n}{n+1} t^{n+1} \tag{52}$$

and it is possible to obtain the recurrence relations

$$C_n = \beta C_n^{(0)} - \delta \frac{C_{n-1}}{n} - \alpha C_{n-2}^{(2)} \tag{53}$$

Hence the first few coefficients are

$$
\begin{aligned}
C_0 &= \beta \\
C_1 &= -\delta\beta \\
C_2 &= \frac{\delta^2\beta}{2} - \alpha\beta^2 \\
C_3 &= \frac{\delta^3\beta}{1} + \frac{4\delta\alpha\beta^2}{3}
\end{aligned}
\tag{54}
$$

Professor S. S. L. Chang [42] (New York University) suggested that the partition should be made so that N, the nonlinear term, is the smallest possible. Professor Chang offered the following illustration:

$$0.01\,\ddot{x} + \sinh \dot{x} + x = 0 \tag{55}$$

For this equation, it is better to use

$$
\begin{aligned}
Z(D) &= 0.01\,D^2 + D + 1 \\
N &= (\sinh \dot{x} - \dot{x})
\end{aligned}
\tag{56}
$$

than

$$
\begin{aligned}
Z(D) &= 0.01\,D^2 + 1 \\
N &= \sinh \dot{x}
\end{aligned}
\tag{57}
$$

It is important to mention here that the types of nonlinearities which can be solved by use of the partition theory can be divided into two classes:

(1) Algebraic Nonlinearities: These are exemplified by a polynomial in x and its admissible derivatives, and they are described by:

$$N(x) = \sum_{k=0}^{N} a_k x^k \tag{58}$$

$$N(x) = \sum_{m=0}^{M} \sum_{k=0}^{N} a_{mk} \{x^{(k)}\}^m \tag{59}$$

$$N(x) = \sum_{m=0}^{M} \sum_{k=0}^{N} a_m a_k x^{(m)} x^{(k)} \tag{60}$$

where x = x(t) and the superscript (k) indicates the k-th derivative of x with respect to time.

(2) Transcendental Nonlinearities: These are restricted to functions of x(t) which have an inverse whose derivative is a rational function, so that the system equation can be easily transformed to equations with algebraic nonlinearities. Typical examples are

$$Z(D)x(t) + be^{x(t)} = g(t) \tag{61}$$

$$Z(D)x(t) + C \tan h \; x(t) = g(t) \tag{62}$$

$$J \ddot{\theta} + r \dot{\theta} + \alpha \sin \theta = T \tag{63}$$

where b, c, J, r, α, and T are constants.

Equation 61 occurs in problems dealing with physical electronics, Equation 62 in physical systems containing saturating types of nonlinearities, and Equation 63 in synchronous machinery.

To show how these types of equations can be transformed into equations containing algebraic nonlinearities, let us take Equation 61 as an example. If we assume

$$W = e^{x} \tag{64}$$

then

$$x = \log W \tag{65}$$

Substituting Equations 65 and 64 into Equation 61 gives

$$Z(D) \log (W) + bW = g(t) \tag{66}$$

Differentiating Equation 66 with respect to (t) gives

$$Z(D) \frac{\dot{W}}{W} + b\dot{W} = \dot{g}(t) \tag{67}$$

if we assume that

$$Z(D) = \frac{1}{Y(D)} \tag{68}$$

Substituting Equation 68 into Equation 67 gives

$$\dot{W} - W[Y(D)\dot{g}(t)] + bW[Y(d)\dot{W}] = 0 \tag{69}$$

Certainly, Equation 69 cannot be solved for all forms of Z(D) by the method presented, because the term $Y(D)\dot{W}$ which appears in the equation is difficult to handle. Moreover, Equation 69, as given, is an integral differential equation. For the special case in which Z(D) is the first-order differential operator, Equation 69 reduces to the following form:

$$\dot{W} - \frac{1}{D}[\dot{g}(t)] \, W + bW \frac{1}{D}(\dot{W}) = 0 \tag{70}$$

When the operator 1/D is replaced by an integration operation, Equation 70 yields

$$\dot{W} - g(t)W + bW^2 = 0 \tag{71}$$

Equation 71 can be solved by partitioning techniques. When Z(D) is a linear combination of x and one or more of its derivatives, the problem is more complicated but can be explained by the case in which

$$Z(D)x \equiv \frac{dx}{dt} + x \tag{72}$$

Substituting for Z(D) from Equation 72 into Equation 67 gives

$$(D + 1)\frac{\dot{W}}{W} + b\dot{W} = \dot{g}(t) \tag{73}$$

or

$$\frac{d}{dt}\left(\frac{\dot{W}}{W}\right) + \frac{\dot{W}}{W} + b\dot{W} = \dot{g}(t) \tag{74}$$

$$\frac{W\ddot{W} - \dot{W}^2}{W^2} + \frac{\dot{W}}{W} + b\dot{W} = \dot{g}(t) \tag{75}$$

Equation 75 can be put in the form:

$$\ddot{W} + \dot{W} - \dot{g}(t)W + \frac{bW^2\dot{W} - \dot{W}^2}{W} = 0 \tag{76}$$

Equation 76 is of the form:

$$Z(D)W + N(W, \dot{W}) = 0 \tag{77}$$

where

$$Z(D)W = [D^2 + D - \dot{g}(t)]W$$

and

$$N(W, \dot{W}) = bW^2\dot{W} - \dot{W}^2/W \tag{78}$$

Further investigations of $N(W, \dot{W})$ show that it satisfies the Lipschitz condition.

A review of A. A. Wolf's published work, by Professor L. F. Kazda and the author [43], revealed that certain questions have been left unanswered, which the novice, when trying to apply the method presented, would naturally like to have summarized. These questions are:

(1) What insight can be gained about the stability of a system for a general class of forcing functions by using the partitioning method of analysis?

(2) Under what conditions would partitioning at points other than the highest derivative be desirable?

(3) How does the partitioning method help one to understand the behavior of the amplitude and frequency of oscillations with time? Consider, e.g., the following nonlinear differential equation:

$$\ddot{x} + \omega_0 x + x^3 = 0 \tag{79}$$

Reference 43 gives answers to the above questions.

In the author's opinion the partitioning technique has not been used in the field of nonlinear synthesis as much as in the field of nonlinear analysis. However, Professor Y. H. Ku (Moore School of Electrical Engineering, University of Pennsylvania) has pointed out that it is precisely in the field of synthesis rather than the field of analysis that the new method is most useful. Dr. A. A. Wolf [43] has suggested a synthesis procedure as follows.

Let Y(s) be the transfer function of the linear part and let $\{s_n\}$ be a set of singularities of the system response in the complex plane generally supposed to be finite in number. Then, given a specified forcing function g(t), what is the nature of the nonlinear part N that enables it to satisfy these conditions?

The synthesis procedure would be as follows:

(1) From the singularities and initial conditions, determine C_n in closed form as shown in Reference 2.

(2) Use the C_n coefficients to determine the response of the system by using the moment theorem.

(3) Then determine the nonlinear part of the system by using the canonical Equation 1.

(4) Finally, by considering Figure 6, the system is obtained easily.

As an example [43], let the linear part of a system be given as:

$$Y(s) = \frac{1}{s + 2} \tag{80}$$

If the system is forced by

$$g(t) = e^{-2t} \tag{81}$$

find the coefficients of the nonlinear terms so that the response has a pole at s = -1 and the nonlinear part has no derivatives and does not exceed the second degree. The solution can be effected by using the recurrence equations of References 45, 46, and 47, or it may be solved directly. In the latter case, the solution is almost trivial. From

$$Z(s) = \frac{1}{Y(s)} = S + 2 \tag{82}$$

the required system equation form is

$$dx/dt = 2x + N(x) = g(t) \tag{83}$$

$$N(x) = g(t) - \dot{x} - 2x \tag{84}$$

Note that

$$N(x) = \sum_{k=0} a_k x^k \tag{85}$$

so that

$$\sum_{k=0}^{2} a_k x^k = g(t) - \dot{x} - 2x \tag{86}$$

From the problem, it is given that

$$X(s) = \frac{1}{s + 1} \tag{87}$$

from which

$$x(t) = e^{-t} \tag{88}$$

so that

$$a_0 + a_1 d^{-t} + a_2 e^{-2t} = e^{-2t} - e^{-t} \tag{89}$$

From the identity of Equation 89 it is clear that

$$a_0 = 0 \tag{90}$$

$$a_1 = -1 \tag{91}$$

$$a_2 = 1 \tag{92}$$

Thus the nonlinear component satisfying the problem is

$$N(x) = -x + x^2 \tag{93}$$

1.2. SUMMARY AND RESEARCH OBJECTIVES

A short review of a generalized mathematical theory for analyzing a certain class of nonlinear systems has been presented. It was stated that under a broad set of conditions imposed on the linear, nonlinear, and forcing functions, a class of systems exists for which it is possible to develop exact and unique mathematical solutions. Although linear differential equations generally have unique solutions, examples show that nonlinear differential equations generally do not have such properties. The techniques of partitioning for systematic solution of actual physical problems in connection with feedback systems also were reviewed. Examples were shown to illustrate that a power series solution of differential equations is just a special case of the partitioning technique.

It is interesting to note that a complete synthesis theory can be developed by using the system shown in Figure 6 as a basis. In addition, it is possible to show that by appropriate partitioning almost any desired configuration can be obtained in both single and multiloop feedback systems, a fact of paramount importance in engineering today. For these reasons, the author felt that it would be of interest to control engineers to extend Wolf's work to include this new class of nonlinear systems. The objectives of this extension are

(1) To develop, by using the partitioning technique, a theory for the analysis of a certain class of systems, the behavior of which can be described by a single-degree-of-freedom nonlinear differential equation, with time-varying parameters.

(2) To apply the partitioning technique and the state-variable approach in further investigation of the system's behavior and stability.

(3) To extend items (1) and (2) above to include systems with more than one degree of freedom.

(4) To analyze the system numerically by using a digital computer for systematic approximations of previous theoretical results. The Runge-Kutta methods of analysis are to be adopted for the purpose of comparison.

2
A THEORY FOR THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR TIME-VARYING PHYSICAL SYSTEMS

The partitioning technique introduced in Section 1 is used here to analyze a certain class of nonlinear physical systems, whose dynamic behavior can be represented by a nonlinear differential equation having some time-varying terms and a forcing function. Most of the material in this section describes work done under the guidance of Professor L. F. Kazda while the author was at The University of Michigan [2]. It will be shown that by placing certain restrictions on the system equations a unique solution which belongs to L_2 space can be obtained.[5] This result is useful because the analysis presented gives a means of controlling the amount of energy associated with the system response; but, it is also limiting because the shape of an L_2 function may not be desirable in all applications. (This disadvantage will be discussed in Section 3.) The exact analytic solution is given as the limit of a sequence of Picard iterates $\{x_n\}$ each of which is also an L_2 function. Uniform convergence of these iterates is shown to be guaranteed by the above restrictions. The well-known mean square error criterion is then applied to find the number of iterates necessary to approximate the solution of this class of nonlinear differential equations. An upper-bound function $\beta(t)$, which also belongs to L_2 space, is calculated. Variations of the solution with respect to initial conditions and system parameters are investigated. Finally, some applications to specific nonlinear problems will be displayed to illustrate the method presented.

2.1. SYSTEM DESCRIPTION AND DEFINITIONS

Consider a certain class of physical systems that can be described by the general, nonlinear, time-varying differential equation of order n, namely:

$$G(x, \dot{x}, \ddot{x}, \ldots, x^{(n)}t) = 0 \tag{94}$$

where x(t) and its first (n - 1) derivatives are given at t = 0. Furthermore, assume that Equation 94 can be put in the form

$$\sum_{m=0}^{n} \left[a_m(t)D^m\right] x(t) + N(x, \dot{x}, \ldots, t) = g(t) \tag{95}$$

where

$[D]$ = d/dt = the differential operator

$a_m(t)$, s = time-varying parameters

$g(t)$ = the forcing function[6]

$N(x, \dot{x}, \ldots, t)$ = the nonlinear term

t = the independent variable

Equation 95 can be written in the following form:

$$L(D, t)x(t) + N(x) = g(t) \tag{96}$$

[5] For an explanation of L_2 spaces, see References 4, 30, 38, and 40. Also, Appendix B includes information on some of the properties of L_2 spaces which are considered in this report.

[6] See Appendix C for situations in which there are forcing functions of the form K(D, t)g(t).

where L(D, t) is the linear operator operating on the system output x(t) and is given by

$$L(D, t)x(t) = \sum_{m=0}^{n} \left[a_m(t)D^m\right] x(t) \tag{97}$$

The following definitions [17, 48, 49] can now be introduced:

(1) Let W(t, u) represent the impulsive response of the linear part of Equation 96 which is defined by Equation 98:[7]

$$L(D, t)\, W(t, u) = \delta(t - u) \tag{98}$$

where $\delta(t - u)$ is the unit impulse applied at t = u.

(2) Let $x_0(t)$ be the response of the linear part with g(t) as the input forcing function and with the same initial conditions as in Equation 94. If zero initial conditions are assumed, $x_0(t)$ is defined as follows:

$$x_0(t) = \int_0^t W(t, u)\, g(u)\, du \tag{99}$$

This means that Equation 99 represents only the particular solution of the linear equation $L(D, t)x(t) = g(t)$. In general, the analysis is true when $x_0(t)$ is the complete solution which contains the effect of initial conditions of the original nonlinear system and when the system restrictions given in Section 2.2 are satisfied.

(3) Let

$$F[t, u, x(t)] = W(t, u)\, N(x, \dot{x}, \ldots, x^{(n-1)}, t) \tag{100}$$

be a nonlinear function obtained by combining the actual system nonlinearity N and the weighting function W(t, u); hence, it can be called a weighted nonlinear function. In general, F depends on $x, \dot{x}, \ldots,$ and $x^{(n-1)}$.

2.2. MATHEMATICAL RESTRICTIONS AND THE SYSTEM ANALYSIS

In order to present a theory that applies to a general class of problems, a number of mathematical restrictions are now placed on the system parameters and on x(t) and F(t, u, x), which were defined in Section 2.1. For simplicity of analysis, the weighted nonlinear function F is assumed to be a function of t, u, and x and is not dependent on $\dot{x}, \ddot{x}, \ldots$. For example, F can assume the following forms:[8]

---

[7] See also Appendix C.

[8] The extension to more complex nonlinearities is explained in Appendix F.

$$F(t, u, x) \triangleq W(t, u) \sum_{k=1}^{\ell} a_k x^k$$

$$F(t, u, x) \triangleq W(t, u) [\exp (x)] \tag{101}$$

The restrictions required are:

(1) $x_0(t)$ is an L_2 function[9] in the interval of interest [0, T]

$$0 < T \leq +\infty \tag{102}$$

(2) For a given region D, [$|x| \leq d(t)$, $0 \leq u \leq t \leq T$], the function F(t, u, x) satisfies the following conditions:

(a) F(t, u, x) satisfies the following conditions; namely, if the two triples (t, u, z_1) and (t, u, z_2) are in the domain D, then

$$|F(t, u, z_1) - F(t, u, z_2)| \leq K(t, u)|z_1 - z_2| \tag{103}$$

where K(t, u) is an L_2 function in the given domain, that is,

$$\int_0^t K^2(t, u)\, du \leq \alpha^2(t) \quad 0 \leq t \leq T$$

and (104)

$$\int_0^T \alpha^2(t)\, dt \leq A^2 = \text{constant}$$

(b)

$$\int_0^t |F[t, u, x_0(u)]|\, du \leq n(t) \tag{105}$$

where n(t) belongs to $L_2[0, T]$, that is,

$$\int_0^T n^2(t)\, dt \leq C^2 = \text{constant} \tag{106}$$

and A^2 and C^2 are two finite positive numbers.

(c) It is required that:

$$|x_0(t)| \leq d(t)$$

$$q(t) \leq d(t) \tag{107}$$

[9]The definition of an L_2 function on the real line and on the plane is given in Appendix B. Also, restriction (1) can be written as: $x_0(t)$ belongs to $L_2[0, T]$.

where

$$x_1(t) \triangleq x_0(t) - \int_0^t F(t, u, x_0(u))\, du$$

and (108)

$$q(t) \triangleq |x_1(t)| + C\,\alpha(t) \sum_{k=0}^{\infty} \frac{A^k}{\sqrt{k!}}$$

where A, C, and $\alpha(t)$ are as defined before.

(3) The L_2 functions $x_0(t)$, n(t), and $\alpha(t)$; the time-varying parameters $a_i(t)$; and the enforced bound d(t) are assumed continuous and bounded over [0, T]. (109)

It should be noted that the function d(t), appearing in the definition of the domain D, may or may not be an L_2 function. This depends on the particular problem. Its use as a function of t gives flexibility to an analysis of problems in which the interval of interest is infinite. Unfortunately, it is not helpful for all problems, as illustrated by the following special examples:

Let

$$F(t, u, x) = W(t, u)\, N(x) = e^{-\lambda(t-u)} x \tag{110}$$

Then

$$|F(t, u, z_1) - F(t, u, z_2)| = e^{-\lambda(t-u)} |z_1 - z_2| \le K(t, u) |z_1 - z_2| \tag{111}$$

Therefore,

$$K(t, u) \ge e^{-\lambda(t-u)} \tag{112}$$

Thus, in this particular case, K(t, u) is independent of the choice of d(t) and does not belong to L_2 space for infinite intervals, but only for finite intervals. Consequently, for problems involving infinite intervals, the linear terms of x should be included in the linear operator L(D, t)x, and not in N(x).

Again, assuming that

$$N(x) = \sum_{n=2}^{\infty} a_n x^n \tag{113}$$

then

$$F(t, u, z_1) - F(t, u, z_2) = W(t, u)(z_1 - z_2) \sum_{n=2}^{\infty} a_n \left[z_1^{n-1} + z_2 z_1^{n-2} + \ldots + z_2^{n-2} z_1 z_2^{n-1}\right] \tag{114}$$

Letting $|z| \leq d(t) > 0$ gives

$$|F(t, u, z_1) - F(t, u, z_2)| = |W(t, u)| \left| \sum_{n=2}^{\infty} a_n \left[z_1^{n-1} + z_2 z_1^{n-2} + \ldots + z_2^{n-2} z_1 + z_2^{n-1}\right]\right| \cdot |z_1 - z_2| \tag{115}$$

Defining:

$$Q(t) = \max_{\substack{z_1 \leq d(t) \\ z_2 \leq d(t)}} \sum_{n=2}^{\infty} a_n \left| z_1^{n-1} + z_2 z_1^{n-2} + \ldots + z_2^{n-2} z_1 + z_2^{n-1} \right| \tag{116}$$

gives

$$|F(t, u, z_1) - F(t, u, z_2)| \leq W(t, u)Q(u)|z_1 - z_2| = K(t, u)|z_1 - z_2| \tag{117}$$

Hence,

$$K(t, u) = W(t, u)Q(u) \tag{118}$$

Therefore, for problems with infinite intervals, d(t) and hence Q(t) should be chosen so that

$$\left[\int_0^t K^2(t, u)\,du\right]^{1/2} = \left[\int_0^t W^2(t, u)\,Q^2(u)\,du\right]^{1/2} = \alpha(t) \in L_2[0, \infty] \tag{119}$$

Now, applying the partitioning technique to the system described in Equation 96 yields

$$L(D, t)\,x(t) = g(t) - N[x(t)] = f(t) \tag{120}$$

Equation 120 can be considered as a linear differential equation with f(t) as an auxiliary forcing function. This function is not unique since it depends on how Equation 96 is partitioned. For a particular choice of partition, N(x) may include some of the linear terms of Equation 96.[10] Equation 120 is known as the linear partitioned system.

If zero initial conditions are assumed, x(t) can be obtained from

$$x(t) = \int_0^t W(t, u)\,f(u)\,du \tag{121}$$

[10] See the rules of partitioning in Section 1, or see Reference 47.

or, equivalently, from

$$x(t) = \int_0^t W(t, u)\,\{g(u) - N[x(u)]\}\,du$$

$$x(t) = \int_0^t W(t, u)\,g(u)\,du - \int_0^t W(t, u)\,N[x(u)]\,du \tag{122}$$

$$x(t) = x_0(t) - \int_0^t F[t, u, x(u)]\,du$$

Equations 96 and 122 are equivalent, and $x_0(t)$ and F are as previously defined. Also, Equation 122 still holds for problems with nonzero initial conditions when $x_0(t)$ is the complete solution of the linear equation $L(D, t)x_0(t) = g(t)$, which contains the effect of these initial conditions. Equation 122 is a nonlinear integral equation of the Volterra type [26, 38, 39], and it can be shown that for the class of systems represented by Equation 96 and the given system restrictions, theorems A, B, C, and D (which follow) are true. In general, the proof of theorems A and C follows the method of Vito Volterra for dealing with nonlinear integral equations [38]. Since the proofs are important for an understanding of the next sections, they are given in detail.

2.3. THEOREM A

Let a set of functions $x_0, x_1, \ldots, x_n$, each of which is a function of time, be defined according to the following recurrence relation:

$$x_{n+1}(t) = x_0(t) - \int_0^t F[t, u, x_n(u)]\,du \tag{123}$$

where $n = 0, 1, 2, \ldots$, and $x_0(t)$ is defined as in Equation 99. Then, under the restrictions placed on the system (Equations 102 through 109), the sequence $\{x_n\}$ belongs to an L_2 space and is in the domain $D \equiv [\,|x| \leq d(t), 0 \leq u \leq t \leq T\,]$.

<u>Proof</u>

It can be shown that when t lies in the interval [0, T], the sequence $\{x_n\}$ stays inside D.

Assume (as an induction hypothesis) that $x_1(t), x_2(t), \ldots, x_n(t)$ are inside D; then Equation 123 shows that $x_{n+1}(t)$ is defined.

Replacing n + 1 by n in Equation 123 gives

$$x_n(t) = x_0(t) - \int_0^t F[t, u, x_{n-1}(u)]\,du \tag{124}$$

Subtracting corresponding sides of Equations 123 and 124 and squaring gives

$$[x_{n+1}(t) - x_n(t)]^2 = \left[\int_0^t F[t, u, x_n(u)]\, du - \int_0^t F[t, u, x_{n-1}(u)]\, du\right]^2$$

$$= \left[\int_0^t |F[t, u, x_n(u)] - F[t, u, x_{n-1}(u)]|\, du\right]^2 \tag{125}$$

Making use of the restrictions gives

$$[x_{n+1}(t) - x_n(t)]^2 \leq \left[\int_0^t K(t, u)\,|x_n(u) - x_{n-1}(u)|\, du\right]^2 \tag{126}$$

Applying Schwarz's inequality to Equation 126 gives

$$[x_{n+1}(t) - x_n(t)]^2 \leq \int_0^t K^2(t, u)\, du \int_0^t [x_n(u) - x_{n-1}(u)]^2\, du$$

$$\leq \alpha^2(t) \int_0^t [x_n(u) - x_{n-1}(u)]^2\, du \tag{127}$$

for n = 1, 2, ... For n = 1, Equation 127 gives

$$[x_2(t) - x_1(t)]^2 \leq \alpha^2(t) \int_0^t (x_1 - x_0)^2\, du \tag{128}$$

The integral on the right-hand side of Equation 128 can be obtained from Equation 123 by putting $n = 0$, as follows:

$$x_1(t) = x_0(t) - \int_0^t F[t, u, x_0(u)]\, du \tag{129}$$

Equation 129 gives

$$\int_0^T [x_1(t) - x_0(t)]^2\, dt \leq \int_0^T n^2(t)\, dt \leq C^2 \tag{130}$$

Thus,

$$[x_2(t) - x_1(t)]^2 \leq C^2\, \alpha^2(t) \tag{131}$$

and

$$\int_0^T (x_2 - x_1)^2 \leq C^2 A^2 \tag{132}$$

Appendix D shows that

$$\int_0^T [x_{n+1}(t) - x_n(t)]^2 dt \leq \frac{C^2 A^{2n}}{n!} \tag{133}$$

Equation 133 gives

$$\int_0^T [x_n(t) - x_{n-1}(t)]^2 dt \leq C^2 \frac{A^{2(n-1)}}{(n - 1)!} \tag{134}$$

Substituting Equation 134 into Equation 127 gives

$$[x_{n+1}(t) - x_n(t)]^2 \leq C^2 \alpha^2(t) \frac{A^{2(n-1)}}{(n - 1)!} \tag{135}$$

for n = 1, 2, . . .

Now,

$$|x_{n+1}(t)| = |x_1(t) + [x_2(t) - x_1(t)] + [x_3(t) - x_2(t)] + \ldots + \ldots + [x_{n+1}(t) - x_n(t)]|$$

$$\leq |x_1(t)| + |x_2(t) - x_1(t)| + \ldots + |x_{n+1}(t) - x_n(t)| \tag{136}$$

Substituting from Equation 135 into Equation 136 gives

$$|x_{n+1}(t)| \leq |x_1(t)| + C\,\alpha(t) + C\,\alpha(t)\frac{A}{\sqrt{1!}} + C\,\alpha(t)\frac{A^2}{\sqrt{2!}} + \ldots + \ldots + C\,\alpha(t)\frac{A^{n-1}}{\sqrt{(n - 1)!}}$$

$$= |x_1(t)| + C\,\alpha(t)\sum_{k=0}^{\infty}\frac{A^k}{\sqrt{k!}} = q(t) \tag{137}$$

But, condition 108 shows that $q(t) \leq d(t)$; thus,

$$|x_{n+1}(t)| \leq d(t) \quad 0 \leq t \leq T \tag{138}$$

Since $|x_0(t)$ and $x_1(t)$ are in D, the sequence $\{x_n(t)\}$ given by Equation 123 is meaningful and is also in D.

Note that $x_0(t)$ is assumed to belong to $L_2[0, T]$ and Equation 130 shows that $x_1(t)$ also belongs to $L_2[0, T]$; hence Equation 135 shows that the sequence $\{x_n(t)\}$ is in $L_2[0, T]$ for n = 1, 2, 3, . . .

2.4. THEOREM B

If conditions 102 through 109 are satisfied, then the sequence of functions $x_0, x_1, x_2, \ldots,$ x_n of Theorem A converges in the quadratic mean to a function $\varphi(t)$. It also converges uniformly in the ordinary sense to a function $\psi(t)$. The two functions $\varphi(t)$ and $\psi(t)$ are identically equal to a function x(t) for all t in the region $0 \leq t \leq T$, where T is the period of interest.

Proof

For convergence in the mean, we have to prove first that, given two numbers m and n so that m > n,

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_0^T [x_m(u) - x_n(u)]^2 du = 0 \tag{139}$$

Now, since

$$(x_m - x_n) = (x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \ldots + (x_{n+1} - x_n) \tag{140}$$

therefore

$$|x_m - x_n| \leq \sum_{k=m-1}^{k=n} |x_{k+1} - x_k| \tag{141}$$

Substituting Equation 134 into Equation 141 gives

$$|x_m - x_n| \leq \sum_{k=m-1}^{k=n} C\,\alpha(t) \frac{A^{(k-1)}}{\sqrt{(k-1)!}} \tag{142}$$

Substituting Equation 142 into the left-hand side of Equation 139 gives

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_0^T [x_m(u) - x_n(u)]^2 du = \lim_{\substack{n\to\infty \\ m\to\infty}} \int_0^T C^2\alpha^2 \left[\sum_{k=m-1}^{k=n} \frac{A^{K-1}}{\sqrt{(k-1)!}}\right]^2 du$$

$$= \lim_{\substack{n\to\infty \\ m\to\infty}} C^2A^2 \left[\sum_{k=m-1}^{n} \frac{A^{(k-1)}}{\sqrt{(k-1)!}}\right]^2 = 0 \tag{143}$$

Therefore, Equation 139 is satisfied.

Since x_m is an L_2 function, x_n is also an L_2 function; hence, by using the first theorem given in Appendix B one can conclude that there exists a function $\varphi(t)$ of L_2 space to which the sequence $\{x_n\}$ converges in the mean, and this function $\varphi(t)$ satisfies the following equation:

$$\lim_{n\to\infty} \int_0^T [x_n(u) - \varphi(u)]^2 du = 0 \tag{144}$$

or, in shortened notation,

$$\text{Lim } x_n(u) = \varphi(u) \tag{145}$$

To prove convergence in the ordinary sense, one can form the series:

$$x_1 + (x_2 - x_1) + (x_3 - x_2) + \ldots \tag{146}$$

In this series,

$$(x_n - x_{n-1}) \leq C\, \alpha(t) \frac{A^{n-2}}{\sqrt{(n-2)!}} \tag{147}$$

Since $\alpha(t)$ and $x_1(t)$ are assumed finite, one can easily conclude the required uniform convergence by the M-test. But the n-th partial sum of series 146 is given by

$$x_n = x_1 + (x_2 - x_1) + \ldots + (x_n - x_{n-1}) \tag{148}$$

Therefore as $n \to \infty$, x_n has a limit.

Let

$$\lim_{n\to\infty} x_n(t) = \psi(t) \tag{149}$$

Note that uniform convergence of the sequence x_n can be easily concluded from Equation 142, since this equation shows that the sequence x_n is a Cauchy sequence in the ordinary sense.

Now, by applying H. Weyl's lemma (stated in Appendix B), we can conclude that

$$\psi(t) = \varphi(t) \tag{150}$$

almost everywhere. Let

$$\psi(t) = \varphi(t) = x(t),\ 0 \leq t \leq T \tag{151}$$

Equation 150 holds everywhere because the functions are continuous. The existence, uniform continuity, and boundedness of the first n derivatives of x(t) are discussed in theorem E of Section 3.

2.5. THEOREM C

Under the conditions placed on the linear, nonlinear, and forcing functions of the system given by Equation 96, the function x(t) as defined in Equation 151 is a solution of the differential Equation 96, and is the unique solution for a given initial state.

Proof

To prove that x(t) is a solution of the differential Equation 96, one can easily form the following equation:

$$[x_n - \varphi(t)]^2 = \left\{\varphi(t) - x_0(t) + \int_0^t F[t, u, x_{n-1}(u)]du\right\}^2 \tag{152}$$

Integrating both sides of Equation 152 in the interval (0, T) and using Equation 144 gives

$$\lim_{n\to\infty} \int_0^T [x_n - \varphi(t)]^2 dt = \lim_{n\to\infty} \int_0^T \left\{[\varphi(t) - x_0(t)] + \int_0^t F[t, u, x_{n-1}(u)]du\right\}^2 = 0 \tag{153}$$

Since

$$\underset{n\to\infty}{\text{L.i.m.}}\ x_{n-1}(t) = \underset{n\to\infty}{\text{L.i.m.}}\ x_n(t) = \varphi(t) \tag{145}$$

therefore,

$$\varphi(t) - x_0(t) + \int_0^t F[t, u, \varphi(u)]du = 0 \tag{154}$$

hence, $\varphi(t)$ satisfies the integral Equation 125. Consequently, x(t) is a solution of the differential Equation 96. To prove uniqueness, assume $\varphi(t)^*$ to be another solution for Equation 96. Therefore,

$$\varphi(t) = x_0(t) + \int_0^t F[t, u, \varphi(u)]du \tag{155}$$

$$\varphi^*(t) = x_0(t) + \int_0^t F[t, u, \varphi^*(u)]du \tag{156}$$

Subtracting Equations 156 and 155 and squaring gives

$$[\varphi^*(t) - \varphi(t)]^2 = \left(\int_0^t \{F[t, u, \varphi(t)] - F[t, u, \varphi^*(u)\}du\right)^2 \tag{157}$$

Applying the restrictions on the system gives

$$[\varphi^*(t) - \varphi(t)]^2 \le \int_0^t K^2(t, u)\, du \int_0^t [\varphi^*(u)]^2 du \le \alpha^2(t) \int_0^T [\varphi^*(u) - \varphi(u)]^2 du \tag{158}$$

Let

$$\int_0^T [\varphi^*(t) - \varphi(t)]^2 dt = S^2 \tag{159}$$

Therefore,

$$[\varphi^*(t) - \varphi(t)] \leq S^2 \alpha^2(t) \tag{160}$$

Substituting Equation 160 into Equation 158 and repeating n times[11] gives

$$\int_0^T [\varphi^*(t) - \varphi(t)]^2 dt \leq S^2 \frac{A^{2n(*)}}{n!} \tag{161}$$

Letting $n \to \infty$ gives

$$\int_0^T [\varphi^*(t) - \varphi(t)]^2 dt = 0 \tag{162}$$

Hence $\varphi = \varphi^*$, and the solution x(t) of the system equation is unique.

2.6. THE APPROXIMATION OF THE SOLUTION IN THE MEAN SQUARE SENSE

According to the preceding theorems, the solution x(t) of the system Equation 96 was found to be the limit of a sequence of iterations $\{x_n\}$ in the L_2 space as the value of n tends to infinity. It may be reasonable to ask whether there exists a number H such that for all $n \geq H$, the error $\varepsilon_n(t)$ remains within a specified value. It is apparent from the mathematical definition that the number H exists. A reasonable error criterion for the calculation of H is the mean square error criterion. Thus, one can form the following equation:

$$\int_0^T \varepsilon_n^2(u)\, d(u) \leq q^2, \text{ for } n \geq H \tag{163}$$

where

$$\varepsilon_n^2 = (x - x_n)^2 \tag{164}$$

x_n is an approximation for the actual solution x(t) of the system under consideration, and q^2 is a prescribed positive error.

It should be noted that the analysis also leads to an estimate of ε_n itself. The author uses the mean square error in the analysis, because this leads to a result which is simple to use in the applications.

---

[11]After the first substitution, $[\varphi^* - \varphi]^2 \leq S^2\alpha^2(t) \int_0^t \alpha^2(z)\,dz$. By repeated substitution, $[\varphi^*(t) - \varphi(t)]^2 \leq S^2\alpha^2(t) \int_0^t \alpha^2(Y)\,dY \int_0^Y \alpha^2(z)\,dz \ldots$. Integrating both sides of this equation gives Equation 161.

Now, by making use of the following equation:

$$|x_m - x_n| \leq \sum_{k=m-1}^{k=n} \frac{C\alpha(t)A^{(k-1)}}{\sqrt{(k-1)!}} \tag{142}$$

holding n fixed, and letting $m \to \infty$, one gets

$$\varepsilon_n = |x - x_n| \leq \sum_{k=n}^{\infty} \left| \frac{C\alpha(t)A^{k-1}}{\sqrt{(k-1)!}} \right| \tag{165}$$

Therefore,

$$\varepsilon_n \leq C\alpha(t)\left(\sum_{k=n-1}^{\infty} \frac{A^k}{\sqrt{k!}} \right) \tag{166}$$

$$\varepsilon_n^2 < C^2\alpha^2\left(\sum_{k=n-1}^{\infty} \frac{A^k}{\sqrt{k!}} \right)^2 \tag{167}$$

Substituting Equation 167 into Equation 163 and making use of Equation 104 gives:

$$\int_0^T \varepsilon_n^2(u)\,du \leq C^2A^2\left(\sum_{k=n-1}^{\infty} \frac{A^k}{\sqrt{k!}} \right)^2 = q^2 \tag{168}$$

However, Equation 168 is very difficult to solve for n if constants C^2, A^2, and q^2 are specified. But, a simplification for this situation can be obtained by making use of the following inequality [8, p. 13]:

$$\sum_{k=n-1}^{\infty} \frac{A^k}{\sqrt{k!}} < \frac{A^{n-1}}{\sqrt{(n-1)!}} \sum_{k=0}^{\infty} \frac{A^k}{\sqrt{k!}} \tag{169}$$

Substituting inequality 169 into Equation 168 gives

$$q^2 = \frac{C^2A^{2n}}{(n-1)!}\left(\sum_{k=0}^{\infty} \frac{A^k}{\sqrt{k!}} \right)^2 \tag{170}$$

Equation 170 gives an upper bound for the error when the actual solution x(t) is approximated by the function x_n, in the sense of the mean square. The infinite series of Equation 170 can be shown to converge for all finite values of A. In addition, as the number of iterates n tends to infinity, the square of the error q^2 tends to zero. This condition is explained by theorem C since the actual accurate solution can then be obtained. In general, the resulting square error q^2 depends on the number of iterates n and on the constants A^2 and C^2, already defined by Equations 104 and 106 respectively. Therefore, for a given mean square error q^2, the number of iterates necessary to approximate the solution is dependent upon A^2 and C^2 and, hence, upon how the system Equation 96 is partitioned. When Picard iterates are used to yield the approximate solution of the system, it is desirable to partition the equation so that fewer iterates are needed for a given mean square error q^2. This reduces the labor required for the solution of the system equation. Unfortunately, it is very difficult to formulate a rule for choosing the partitioning so as to obtain the minimum number of iterates. However, for a given n-th order equation, there are an infinite number of choices for partitioning the equation.[12] At each choice, the number of iterates (n) necessary to yield a mean square error q^2 is obtained by using Equation 170 after calculating A^2 and C^2. In order to compare two partitionings, let us assume that for the same q^2 we have C_1, A_1, n_1 and C_2, A_2, n_2 as the C, A, n numbers for the first and second partitionings, respectively. Applying Equation 170 yields

$$C_1^2 \frac{A_1^{2n_1}}{(n_1 - 1)!} \left(\sum_{k=0}^{\infty} \frac{A_1^k}{\sqrt{k!}} \right)^2 = C_2^2 \frac{A_2^{2n_2}}{(n_2 - 1)!} \left(\sum_{k=0}^{\infty} \frac{A_2^k}{\sqrt{k!}} \right)^2 \tag{171}$$

or

$$\frac{(n_2 - 1)!}{(n_1 - 1)!} \frac{A_1^{2n_1}}{A_2^{2n_2}} = \frac{p_2}{p_1} \tag{172}$$

where

$$p_1 = C_1^2 \left(\sum_{k=0}^{\infty} \frac{A_2^k}{\sqrt{k!}} \right)^2$$

and (173)

$$p_2 = C_2^2 \left(\sum_{k=0}^{\infty} \frac{A_2^k}{\sqrt{k!}} \right)^2$$

[12] See the rules of partitioning, Section 1.

An examination of Equation 172 shows that in order to have $n_2 > n_1$, $A_2^{\,2}$ should be greater than $A_1^{\,2}$ and $C_2^{\,2}$ should be greater than $C_1^{\,2}$.

When A is less then unity, the infinite series of Equation 170 will be quickly convergent. For a given A^2, C^2, and q^2, the value of n that satisfies Equation 170 is best obtained by the following graphical procedure. Consideration of Equation 170 gives

$$\frac{A^{2n}}{(n-1)!} = \frac{q^2}{C^2\left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)^2} \tag{174}$$

or

$$\frac{A^{2n}}{(n-1)!} = Q^2 \tag{175}$$

where

$$Q^2 = \frac{q^2}{C^2\left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)^2} \tag{176}$$

The left-hand side of Equation 174 can be drawn for different values of n by using a given value of A. A straight line drawn parallel to the n axis from a distance Q^2 intersects the last drawn curve at the required point. As an illustration, the left-hand side of Equation 174 is drawn for the cases in which A = 1 and A = 0.364 < 1 in Figure 7. The case in which A = 2 > 1 is also plotted in Figure 7 but, for convenience, not to the same scale. The Figure shows that if the values of Q^2, as defined by Equation 176, are chosen so that $Q^2 < A^2 = \left.\frac{A^{2n}}{(n-1)!}\right|_{n=1}$, then there will be only one value of n corresponding to a given mean square value of error. This follows because a straight line drawn parallel to the n axis at a distance Q^2 from it will intersect the curve $\frac{A^2}{(n-1)!}$ at one and only one point. If Q^2 is chosen so that $Q^2 > A^2 = \left.\frac{A^{2n}}{(n-1)!}\right|_{n=1}$, then the magnitude of A^2 will determine which of the following situations will occur:

(1) If $A^2 < 1$, then no solution will give a positive value of n because the curve $\frac{A^{2n}}{(n-1)!}$ is monotonic decreasing, and has a maximum value which is equal to A^2 and occurs at n = 1. Thus a straight line drawn parallel to the n axis with $Q^2 > A^2$ will not intersect the curve $\frac{A^{2n}}{(n-1)!}$ for any positive value of n.

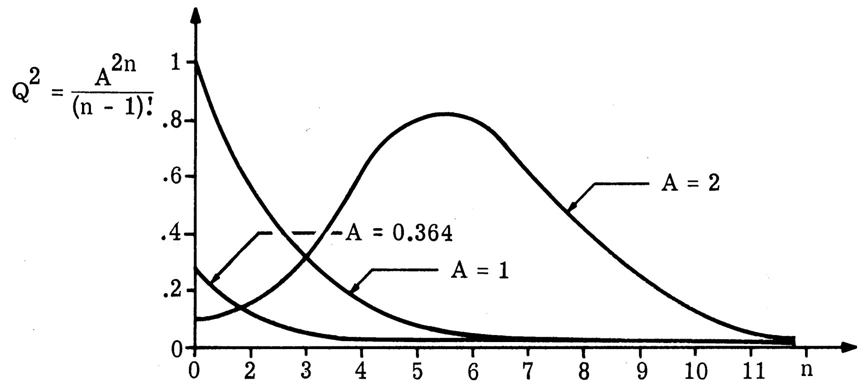

FIGURE 7. A PLOT OF $\frac{A^{2n}}{(n-1)!}$ VERSUS n

(2) If $A^2 > 1$, then the corresponding curve $\frac{A^{2n}}{(n-1)!}$ increases as n increases, until it attains its maximum value at a particular value of n. With this behavior of the function $\frac{A^{2n}}{(n-1)!}$, one expects the following two situations to happen:

(a) If $Q^2 > \max \frac{A^{2n}}{(n-1)!}$, then no solution will give a particular value of n, because there will be no intersection of the curve $\frac{A^{2n}}{(n-1)!}$ with the straight line Q^2 drawn parallel to the n axis.

(b) If Q^2 lies between the value of the function $\frac{A^{2n}}{(n-1)!}$ at n = 1 and its maximum value, that is $A^2 < Q^2 < \max_n \frac{A^{2n}}{(n-1)!}$, then there will be two values of n for a given value of Q^2, that is, for a given mean square error. This is not surprising because it indicates that among the sequence $\{x_n\}$, there are two elements x_{n_1} and x_{n_2} $(n_1 \leq n_2)$ which will approximate the solution x(t) for the same mean square error. For the purpose of simplicity, one should select x_{n_1} which has the smaller index n_1.

2.7. CALCULATION OF AN UPPER-BOUND FUNCTION $\beta(t)$ FOR THE SYSTEM RESPONSE x(t)

To calculate an upper bound $\beta(t)$ for the system response x(t), one should note that the sequence $\{x_n(t)\}$ is uniformly bounded by q(t). This is given by the following equation:

$$|x_{n+1}(t)| \leq q(t) = |x_1(t)| + C\,\alpha(t) \sum_{0}^{\infty} \frac{A^k}{\sqrt{k!}} \tag{137}$$

for n = 1, 2, . . . , ∞. But theorem B shows that

$$\lim_{n\to\infty} x_n(t) = x(t) \tag{177}$$

uniformly. Therefore,

$$|x(t)| \leq q(t) \tag{178}$$

Also, since

$$x_1(t) = x_0(t) - \int_0^t F[t, u, x_0(u)]du \tag{179}$$

it follows that

$$|x_1(t)| \leq |x_0(t)| + |n(t)| \tag{180}$$

Now, define a function $\beta(t)$ so that

$$\beta(t) = |x_0(t)| + |n(t)| + m\alpha(t) \tag{181}$$

where

$$m = c \sum_0^{\infty} \frac{A^k}{\sqrt{k!}} \tag{182}$$

and compare Equations 137, 180, and 181 which give

$$q(t) \leq \beta(t) \tag{183}$$

The upper-bound function $\beta(t)$ should obviously stay inside the domain bounded by the enforced bound d(t).[13]

$$0 \leq |x(t)| \leq q(t) \leq \beta(t) \leq d(t) \tag{184}$$

From Equation 184 it is obvious that both q(t) and $\beta(t)$ work as upper bounds for the system response x(t) inside the enforced bound d(t). Although q(t) is a lower bound than $\beta(t)$, and is a better bound from this point of view, the latter has the advantage that is may easily be obtained in calculations, as shown in Equation 181.

2.8. VARIATION OF THE SOLUTION WITH RESPECT TO THE INITIAL CONDITIONS AND THE SYSTEM PARAMETERS (THEOREM D)

Let the initial conditions $x^{(i)}(0)$ change to $x^{(i)}(0) + \Delta x^{(i)}(0)$, (i = 0, 1, . . . , n - 1), then $x_0(t)$, which was defined by Equation 99 and which in addition contains the effect of the initial

[13]See Figure 12.

conditions, will change to $[x_0(t) + w(t)]$. Instead of the solution x(t), there will arise a new solution X(t), as given by

$$X(t) = [x_0(t) + w(t)] - \int_0^t F[t, u, X(u)]du \tag{185}$$

If we require that $\Delta x^{(i)}(0) = \bar{\xi}$ so that $|\xi| \leq \delta$, and that this change in the initial conditions will keep $[w(t)] \in L_2$ in the interval of interest,[14] then the existence and uniqueness of the system Equation 185 follows exactly as before. This particular change of initial conditions of the original nonlinear system requires an investigation of the behavior of the norm of the difference of the two solutions X(t) and x(t), symbolized as $||X(t) - x(t)||$.[15] This investigation is carried on in the following theorem.

Theorem D

Under the conditions (given by Equations 102 through 109 placed on the linear, nonlinear, and forcing function terms of the system described in Equation 96, the system solution is uniformly continuous with respect to the initial conditions $x^{(i)}(0)$. If the system contains a parameter λ, then the solution also is uniformly continuous with respect to λ. Therefore, $x(t, x^{(i)}(0), \lambda)$ is uniformly continuous with respect to $x^{(i)}(0)$ and λ.

Proof

Subtracting Equations 185 and 124 gives

$$[X(t) - x(t)] = w(t) - \left\{\int_0^t (F[t, u, X(u)] 1 F[t, u, x(u)] du\right\} \tag{186}$$

Applying the Lipschitz condition to Equation 186 gives

$$[X(t) - x(t)] = [w(t)] - \int_0^t \{Kt(u) [X(u) - x(u)]\}du \tag{187}$$

Squaring both sides and using the Schwarz inequality gives

$$[X(t) - x(t)]^2 \leq 2\left\{w(t)^2 + \alpha^2(t) \int_0^t [X(u) - x(u)^2]du\right\} \tag{188}$$

---

[14] Since $x_0(t) \in L_2[0, T]$, and $w(t) \in L_2[0, T]$; this implies that $[x_0(t) + w(t)] \in L_2[0, T]$.

[15] $||X(t) - x(t)||^2 = \int_0^t (X - x)^2 dt$

Letting $S^2 = \int_0^t [X(u) - x(u)]^2 du$, and substituting into Equation 188 gives

$$[X(t) - x(t)]^2 \leq 2[w(t)^2 + S^2\alpha^2(t)] \tag{189}$$

Substituting Equation 189 into Equation 188 and repeating the first substitution gives

$$[X(t) - x(t)]^2 \leq 2\left[w(t)^2 + 2\varepsilon^2\alpha^2(t)\left(1 + 2\frac{(A^2)}{1!} + \frac{2^2(A^2)^2}{2!} + \frac{2^3(A^2)^3}{3!} + \ldots + \frac{2^n(A^2)^n}{n!}\right)\right] + 2\alpha(t)^2 S^2 \frac{2^n(A^2)^n}{n!} \tag{190}$$

where

$$\varepsilon^2 = \int_0^t w^2(t)\,dt < \infty$$

The integral appearing in Equation 191 is finite since $w(t) \in L_2$ by definition, ε^2 can ar be made small by a proper choice of $\Delta x^{(i)}(0)$. As n tends to infinity the last term in E 191 drops out and we are left with

$$[X(t) - x(t)]^2 \leq 2\left[w^2(t) + 2\varepsilon^2\alpha^2 \sum_{n=0}^{\infty} \frac{(2A^2)^n}{n!}\right]$$

or

$$[X(t) - x(t)]^2 \leq 2\left[w^2(t) + 2\varepsilon^2\alpha^2 e^{2A^2}\right] \tag{192}$$

Integrating both sides of Equation 192 in the interval $0 \leq t \leq T$ gives

$$||X(t) - x(t)||^2 \leq 2\varepsilon^2 \left[1 + (2A^2)\, e^{(2A^2)}\right] \tag{193}$$

or

$$||X(t) - x(t)||^2 \leq \varepsilon_1^{\,2} \tag{194}$$

where

$$\varepsilon_1^{\,2} = 2\varepsilon^2\left[1 + (2A^2)\, e^{(2A^2)}\right] \tag{195}$$

Equation 194 shows that the two solutions x(t) and X(t) may be made to differ by a small quantity, as we please. A proof proceeding on similar lines as the above shows that if the system equation involves a parameter λ, then the solution x(t) is continuous with respect to λ. Hence the theorem is true.

2.9. THREE EXAMPLE CALCULATIONS

Example One

Let us consider a system whose behavior is governed by the following nonlinear differential equation:

$$\frac{dx}{dt} + x + x^2 = e^{-2t} \quad (t = 0, x = -1) \tag{196}$$

This can represent either the electric circuit of Figure 8 or the nonlinear feedback systems of Figure 9. Now, partitioning Equation 196 at the second term gives

$$\frac{dx}{dt} + x = e^{-2t} - x^2 = f(t) \tag{197}$$

The impulse response of the left-hand side of Equation 197 is obtained by solving the following differential equation:

$$\frac{dx}{dt} + x = \delta(t - u) \tag{198}$$

This solution is

$$W(t, u) = W(t, u) = +e^{-(t-u)} \tag{199}$$

The term x_0, as defined before, is the solution of the linear equation:

$$\frac{dx}{dt} + x = e^{-2t} \quad (t = 0, x = -1) \tag{200}$$

Therefore,

$$x_0(t) = -e^{-2t}$$

Now, since

$$F[t, u, x(t)] = e^{-(t-u)}x^2 \tag{201}$$

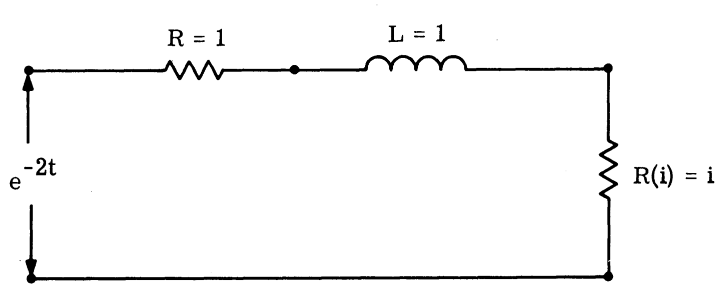

FIGURE 8. AN RL CIRCUIT WITH A NONLINEAR RESISTOR

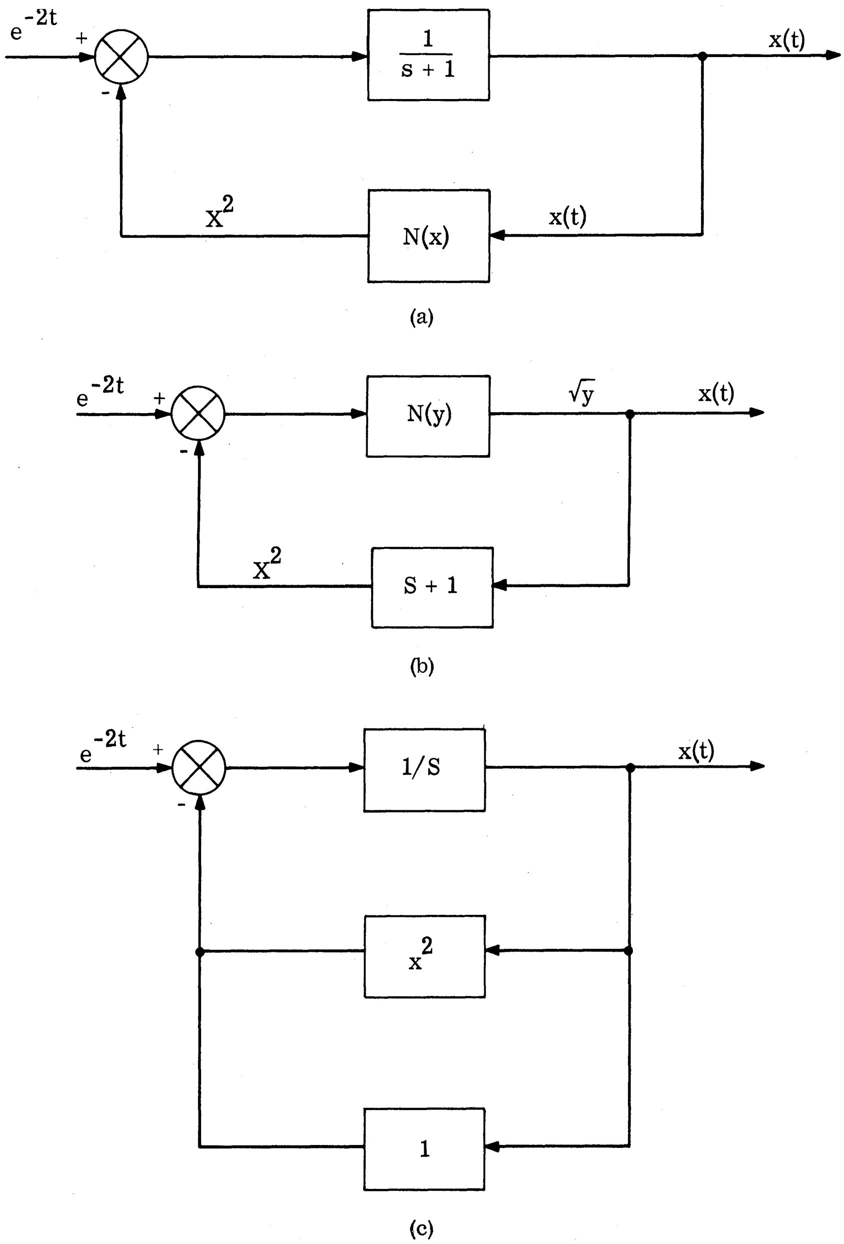

FIGURE 9. EQUIVALENT REPRESENTATIONS OF THE SYSTEM GOVERNED BY EQUATION 185

then

$$F[t, u, x_0(t)] = e^{-(t-u)}x_0^2 = e^{-(t-u)}e^{-4t} \qquad (202)$$

Applying condition 105 of the restrictions to Equation 202 gives n(t)

$$n(t) = \int_0^t F[t, u, x_0(u)]du = \int_0^t e^{-(t-u)}e^{-4u} = 1/3\ (e^{-t} - e^{-4t}) \qquad (203)$$

Also F(t, u, x) satisfies the Lipschitz condition in the domain $D = [x \leq 1, (0 \leq u \leq t \leq T]$, since

$$F(t, u, x_1) - F(t, u, x_2) = e^{-(t-u)}\left[x_1^2 - x_2^2\right] = e^{-(t-u)}|x_1 - x_2| \cdot |x_1 + x_2|$$

$$\leq 2x_{max}\ e^{-(t-u)}|x_1 - x_2| \qquad (204)$$

Hence

$$|F(t, u, x_1) - F(t, u, x_2)| \leq 2e^{-(t-u)}|x_1 - x_2| \qquad (205)$$

Thus: $K(t, u) = 2e^{-(t-u)}$. It can easily be shown that $x_0(t)$, n(t), and K(t, u) are L_2 functions, in the interval $0 \leq t \leq T$, where T is any finite period of interest. Therefore, we can conclude that there exists a unique solution that belongs to an L_2 space in the same interval. Moreover, an upper-bound function can be obtained by using Equation 181 and 182 as follows. Let

$$\beta(t) = |x_0(t)| + |n(t)| + m\alpha(t) \qquad (181)$$

and

$$m = c \sum_0^{\infty} \frac{A^k}{\sqrt{k!}} \qquad (182)$$

Then to calculate C, use Equation 106. This gives

$$C = 1/3 \left[1 - 1/2\left(\frac{e^{-2t}}{2} + \frac{e^{-8T}}{2} - 2/5\ e^{-5T}\right)\right]^{1/2} \qquad (206)$$

Also, applying Equations 103 and 104 gives

$$\alpha(t) = 2(1 - e^{-2t})^{1/2} \qquad (207)$$

and

$$A = 2\left(-1 + T + \frac{e^{-2T}}{2}\right)^{1/2} \qquad (208)$$

and T is finite. Then choosing T = 1 gives A = 0.364, C = 0.3, and m = .143. Now, by substituting into Equation 181, we can conclude that the absolute value of the system solution x(t) will always remain below an L_2 function $\beta(t)$ given by

$$x(t) \leq \beta(t) = e^{-2t} + 1/2\,(e^{-t}e^{-4t}) + .2(1 - e^{-2t})^{1/2} \qquad (209)$$

It would be interesting to find the exact solution for the system given by Equation 196. It should be noted that the restrictions imposed by A. A. Wolf (stated in Section 1) fit closely here, so that one can apply Wolf's technique as follows: Partitioning the differential Equation 196 at the highest derivative gives

$$\frac{dx}{dt} = e^{-2t} - x - x^2 = f(t) \qquad (210)$$

The impulse response W(t) of the left-hand side of Equation 210 is h(t), a unit step function.[16] The solution x(t) can be written in the form of a power series as follows:

$$x(t) = K + \sum_{n=0}^{\infty} C_n Q_n(t) \qquad (211)$$

where the moment $Q_n(t)$ is given by

$$Q_n(t) = \int_0^t u^n\, W(t - u)\, du \qquad (212)$$

$$Q_n(t) = \frac{t^{n+1}}{n + 1} \qquad (213)$$

Therefore,

$$x(t) = K + \sum_{n=0}^{\infty} \frac{C_n}{n + 1} t^{n+1} \qquad (214)$$

But when t = 0 and x = -1, the result is K = -1. Substituting Equation 214 with K = -1 into Equation 210 gives

$$\sum_{n=0}^{\infty} C_n t^n = e^{-2t} + 1 \sum_{n=0}^{\infty} \frac{C_n}{n + 1} - \left(-1 + \sum_{n=0}^{\infty} \frac{C_n}{n + 1} t^{n+1}\right)^2 \qquad (215)$$

[16]The definition of the unit step function is

$$h(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$

Now, recall that

$$e^{-2t} = \sum_{n=0}^{\infty} \frac{(-2t)^n}{n!} \tag{216}$$

Substituting Equation 216 into Equation 215 yields

$$\sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} \frac{(-2t)^n}{n!} + \sum_{n=0}^{\infty} \frac{C_n}{n+1} t^{n+1} - \sum_{n=0}^{\infty} C_n^{(2)} t^{n+2} \tag{217}$$

from which it is possible to obtain the recurrence relation:

$$C_n = \frac{(-2)^n}{n!} + \frac{C_{n-1}}{n} - C_{(n-2)}^{(2)} \tag{218}$$

where $C_{(n-2)}^{(2)}$, explained in Reference 47, means

$$C_{(n-2)}^{(2)} = \sum_{r=0}^{n-2} \frac{C_r}{r+1} \frac{C_{n-r-2}}{n-r-1} \tag{219}$$

The first few coefficients in this case are: $C_0 = 1$, $C_1 = 1$, $C_2 = 1/2$, and $C_3 = -1/6$. The following solution is obtained when these coefficients are substituted into Equation 214:

$$x(t) = -1 + t - \frac{t^2}{2} + \frac{t^3}{3!} - \ldots = -e^{-t} \tag{220}$$

A comparison of the actual solution x(t), given in Equation 220, in found in Table I. For comparison, x(t) and $\beta(t)$ are also plotted in Figure 10.

Example Two

As a second example, consider the nonlinear time-varying feedback control system shown in Figure 11. This can be described by the following equation:

$$\frac{dx}{dt} + a(t)\,x + 0.2X^2 = g(t) \tag{221}$$

where $a(t)$ = time-varying parameter $\dfrac{2t}{1+t^2}$

$g(t)$ = forcing function = $\dfrac{0.2}{(1+t^2)^2}$ (222)

$x(0+) = 1$

TABLE I. SOME NUMERICAL VALUES OF |x(t)| AND β(t) FOR THE SYSTEM GOVERNED BY EQUATION 196

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
|x(t)|	1	0.91	0.82	0.75	0.685	0.62	0.563	0.512	0.465	0.62	0.37
β(t)	1	0.98	0.915	0.845	0.772	0.674	0.614	0.569	0.52	0.478	0.437

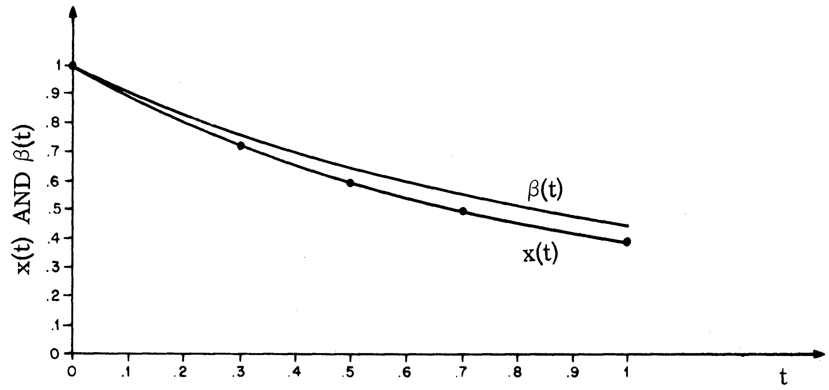

FIGURE 10. A PLOT OF THE SOLUTION x(t) AND THE UPPER-BOUND FUNCTION β(t) FOR THE SYSTEM GOVERNED BY EQUATION 196

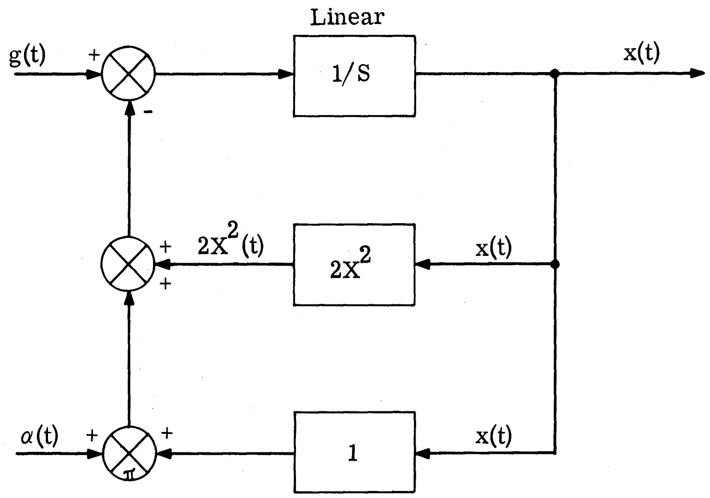

FIGURE 11. REPRESENTATION OF THE SYSTEM GOVERNED BY EQUATION 221

Partitioning Equation 221 at the second term gives

$$\frac{dx}{dt} + \frac{2t}{1 + t^2} x(t) = 0.2 \left[\frac{1}{(1 + t^2)^2} - x^2 \right] \tag{223}$$

The next step is to determine the impulse response W(t, u) of the left-hand side of Equation 223. This can be obtained by solving the following differential equation:

$$\frac{dW(t, u)}{dt} + \frac{2t}{1 + t^2} W(t, u) = \delta(t - u) \tag{224}$$

The solution of Equation 224 is

$$\begin{aligned} W(t, u) &= \frac{1 + u^2}{1 + t^2}, \quad u < t \\ &= 0, \qquad u > t \end{aligned} \tag{225}$$

The term $x_0(t)$, as defined before, is the solution of the following linear differential equation:

$$\frac{dx}{dt} + \frac{2t}{1 + t^2} x(t) = \frac{0.2}{(1 + t^2)^2} \tag{226}$$

and is subject to the boundary condition x(t) = 1.

Solving for $x_0(t)$ gives

$$x_0(t) = \frac{1 + 0.2 \tan^{-1} t}{1 + t^2} \tag{227}$$

Now, since

$$F[t, u, x(u)] = 0.2 \frac{1 + u^2}{1 + t^2} x^2(u) \tag{228}$$

Then

$$f[t, u, x_0(u)] = \frac{0.2[1 + 0.2 \tan^{-1}(u)]^2}{(1 + t^2)(1 + u^2)} \tag{229}$$

Applying condition 105 of the mathematical restrictions given in Section 2.2 gives

$$n(t) = \frac{0.2 \tan^{-1}(t)}{1 + t^2} \left[1 + 0.2 \tan^{-1}(t) + \frac{.04}{3} (\tan^{-1} t)^2 \right] \tag{230}$$

By choosing a domain D = [|x| < 1, 0 ≤ u ≤ t ≤ T] and proceeding as in the last example, it can be shown that:

$$K(t, u) = \frac{0.4(1 + u^2)}{(1 + t^2)} \tag{231}$$

Recognition of $x_0(t)$, n(t), and K(t, u) as L_2 functions in the interval 0 ≤ t ≤ T leads to the conclusion that there exists a unique solution which belongs to the L_2 space in the same interval. The solution x(t) will always remain below an L_2 function $\beta(t)$, which can be calculated as follows:

$$C^2 = \int_0^T \frac{.04 \tan^{-1}(t)}{(1 + t^2)^2} \left[1 + 0.2(\tan^{-1} t) + \frac{.04}{3} (\tan^{-1} t)^2\right]^2 dt \tag{232}$$

$$\alpha^2(t) = 0.16 \frac{t + 2/3\, t^3 + t^5/5}{(1 + t^2)^2} \tag{233}$$

$$A^2 = 0.16 \int_0^T \frac{t + 2/3\, t^3 + t^5/5}{(1 + t^2)^2} dt \tag{234}$$

where T in finite. The choice of T = 1 gives A = 0.111, C = 0.093, and m = 0.012. Substituting $x_0(t)$, n(t), m, and $\alpha(t)$ in Equation 181 gives the upper-bound function $\beta(t)$:

$$\beta(t) = \frac{1}{1 + t^2} \left\{1 + 0.4 \tan^{-1}(t) + 0.4 [\tan^{-1}(t)]^2 + \frac{0.008}{3} [\tan^{-1}(t)]^3 + \frac{0.00192}{1 + t^2} \frac{(t + 2\, t^3 + t^5/5)}{3}\right\} \tag{235}$$

Therefore,

$$\beta(t) \geq \frac{1}{1 + t^2} \tag{236}$$

Inspection of Equation 221 reveals that $x(t) = \frac{1}{1 + t^2}$ is the exact solution. Hence, $x(t) \leq \beta(t)$ all t ($0 \leq t \leq 1$), as it should be.

Example Three

Remarks: In examples one and two, the interval of interest was chosen to be finite, and there was no difficulty encountered in apply the technique presented in this section. The following example, however, is intended to illustrate to the readers how this technique can be applied to problems in the infinite interval of time. In the given domain D, one should ascertain that the functions $x_0(t)$, n(t), and K(t, u), as previously defined, belong to L_2 space. Moreover, inequalities 108 should be satisfied; otherwise, the technique fails to work. It is obvious that example one works only for the case in which T, as defined before, is finite; it does not work if T is infinite. This is because as $T \to \infty$, A tends to ∞, as is clearly shown in Equation 208. The problem presented in example one is again presented, but modifications have been made to permit work in the infinite interval of time.

The example can now be given. Consider the system whose behavior is governed by the following nonlinear differential equation:

$$\frac{dx}{dt} + x + ax^2 = e^{-2t} \quad (t = 0, x = -1) \tag{237}$$

where a is a constant parameter. The parameter a must be chosen in such a way that the system response remains in a given domain D, as given by the following identity:

$$D \equiv [|x| \leq d(t) = 2e^{-0.5t}, \ 0 \leq u \leq t \leq \infty] \tag{238}$$

Partitioning Equation 237 at the second term gives

$$\frac{dx}{dt} + x = e^{-2t} - ax^2 \tag{239}$$

The impulse response of the left-hand side of the above equation is given by

$$W(t, u) = W(t - u) = e^{-(t-u)} \tag{240}$$

The term $x_0(t)$, as defined before, is found in example one to be[17]

$$x_0(t) = -e^{-2t} \tag{241}$$

Since

$$F[t, u, x(t)] = a\, e^{-(t-u)} x^2 \tag{242}$$

n(t) can be found by applying hypothesis 105 of the system restrictions:

$$n(t) = \frac{a}{3}(e^{-t} - e^{-4t}) \tag{243}$$

By following the same procedures used to obtain Equation 204 of example one, the weighted nonlinear function F(t, u, x) can be shown to satisfy the following condition:

$$|F(t, u, x_1) - F(t, u, x_2)| \leq 2a\, x_{max}\, e^{-(t-u)} |x_1 - x_2| \leq 2a\, d(t)\, e^{-(t-u)} |x_1 - x_2| \tag{244}$$

[17] If $|x_0(t)|$ is not inside d(t), the analysis fails.

Thus, K(t, u) can be given by the following equation:

$$K(t, u) = 4ae^{-(t-u)} e^{-0.5t} = 4ae^{-t}e^{+0.5t} \quad (245)$$

and K(t, u) can easily be shown to belong to L_2 space in the given domain because

$$\alpha^2(t) = \int_0^t k^2(t, u)\, du = 16\, a^2 \left[e^{-t} - e^{-2t}\right] \quad (246)$$

and

$$A^2 = \int_0^\infty \alpha^2(t)\, dt = 8a^2 \quad (247)$$

Also, $x_0(t)$ and n(t) as defined by Equations 241 and 243 can easily be shown to be $L_2(0, \infty)$.

Now, the parameter a must be chosen in such a way that the upper-bound function $\beta(t, a)$, as defined by Equation 181, remains below d(t) for all t, $[0 \le t \le \infty]$.

Substituting for $x_0(t)$, n(t), and $\alpha(t)$ in Equation 181 gives

$$|x(t, a)| \le \beta(t, a) = |e^{-2t}| + \frac{a}{3}(e^{-t} - e^{-4t}) + m(e^{-t} - e^{-2t})^{1/2} \quad (248)$$

where

$$m = c \sum_{k=1}^{\infty} \frac{A^k}{\sqrt{k!}} \quad (249)$$

and c is given by

$$c^2 = \int_0^\infty n^2(t)\, dt = \frac{a^2}{40} \quad (250)$$

The parameter a, as it appears in Equation 248, controls both the absolute value of x(t) and the upper-bound function $\beta(t)$ in such a way that inequalities 108 must be satisfied. This, of course, permits flexibility in the application of the given technique. If the parameter a were not present, the inequalities 108 would not be satisfied and the technique would fail to work.[18]

[18]Where this is true, $\beta(t)$ may assume the shape of the curve $\beta^*(t)$, shown in Figure 12.

An investigation of Equation 248 reveals that a value of 0.25 assigned to the parameter a satisfies inequalities 108 and gives the following results:

$$\left.\begin{aligned} \alpha(t) &= (e^{-t} - e^{-2t})^{1/2} \\ n(t) &= \frac{1}{12}(e^{-t} - e^{-4t}) \\ c &= 0.039 \\ A &= 0.707 \\ m &= 0.048 \end{aligned}\right\} \qquad (251)$$

Substituting from Equation 251 into Equation 248 gives

$$|x(t)| \leq \beta(t) = |e^{-2t}| + 0.083\,(e^{-t} - e^{-4t}) + 0.048\,(e^{-t} - e^{-2t})^{1/2} \qquad (252)$$

The upper-bound function $\beta(t)$ and the function d(t) are plotted in Figure 12 for the purpose of comparison. Some numerical values of these two functions can also be found in Table II.

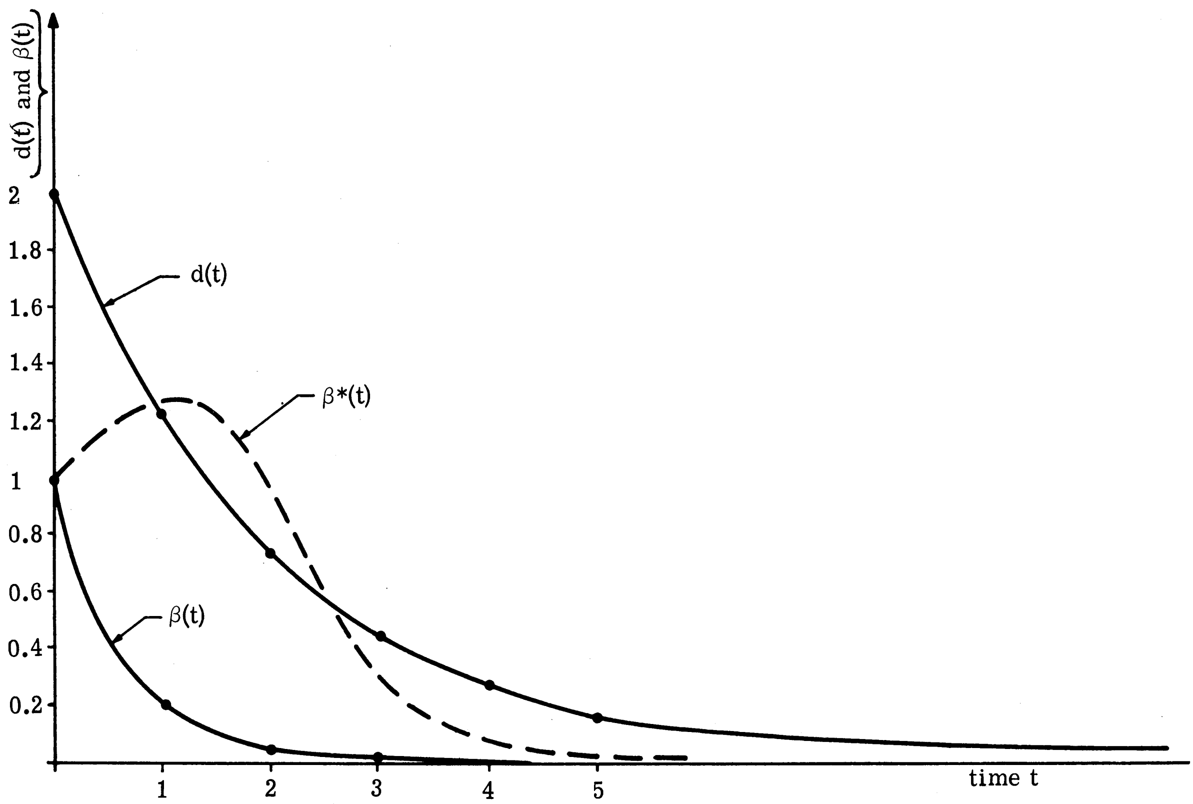

FIGURE 12. A PLOT OF d(t) AND β(t) FOR THE SYSTEM GOVERNED BY EQUATION 237

TABLE II. SOME NUMERICAL VALUES OF d(t) AND β(t) FOR THE SYSTEM DESCRIBED IN EQUATION 237

t	0	1	2	3	4	5	∞
d(t)	2	1.216	0.7220	0.4464	0.2706	0.1638	0
β(t)	1	0.1989	0.0447	0.0171	0.00832	0.00451	0

To obtain the approximate solution of the new system equation, namely:

$$\frac{dx}{dt} + x + \frac{x^2}{4} = e^{-2t} \tag{253}$$

one has to substitute for C and A in the error Equation 170

$$\delta_n^{\,2} = \frac{C^2 A^{2n}}{(n-1)!} \sum_{k=0}^{\infty} \frac{A^k}{\sqrt{k!}} \tag{170}$$

Then, continuing the above substitution in Equation 170 gives

$$\delta_n^{\,2} = \frac{(0.038)^2\,(2.5522)^2\,(0.5)^2}{(n-1)!} \tag{254}$$

For n = 1, Equation 254 gives

$$\delta_1^{\,2} = 0.0039 \tag{255}$$

The first-order approximate solution $x_1(t)$ for the above value of mean square error $\delta_1^{\,2}$ can be given by

$$x_1(t) = x_0(t) - n(t) \tag{256}$$

Substituting from Equations 241 and 243 into Equation 256 gives

$$x_1(t) = -[e^{-2t} - \frac{1}{12}(e^{-t} - e^{-4t})] \tag{257}$$

Inspection of Equations 257 and 252 reveals that the modulus of the approximate solution $x_1(t)$ remains below the upper-bound function β(t) for all t, as it should be.

2.10. CONCLUSIONS AND REMARKS

The partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation. It was found that when certain restrictions are placed on the linear, nonlinear, and forcing function terms, this equation has a unique solution which is the limit of a sequence of iterates $\{x_n\}$, each of which, together with the system output x(t), belongs to an L_2 space. The solution was shown to remain below some upper-bound function $\beta(t)$ in the L_2 space for every t in the period of interest $0 \leq t \leq T$. The upper-bound function $\beta(t)$ defined by Equation 181 can be easily calculated since it is a linear combination of functions defined by the class of system under consideration, and it may be changed by changing the system parameters. This method should help control engineers gain insight into the stability and boundedness of system responses, and hence, is a first step toward solving the problem of synthesis.

Equation 170, derived in this section, gives an upper bound for the error when the actual solution x(t) is approximated by the n-th iterate x_n in the sense of the mean square. The number of iterations n required for a given approximation was shown to depend on how the system equation is partitioned and, therefore, one can compare, with the help of Equation 170, different partitions in order to have n as small as possible. It should be pointed out that since the L_2 space is a complete space, the solution x(t) is guaranteed to be in the same L_2 space of the sequence of iterates $\{x_n\}$, a fact which is not true for the case studied by A. A. Wolf in Reference 47. This is because the limiting process of a sequence of functions of exponential type may give rise to functions of higher order but not of the same exponential type. The following example, introduced by A. A. Wolf [47, p. 22], illustrates this:

$$\dot{x} + x^2 = e^{-t} \tag{258}$$

$$x_0(t) = 1 - e^{-t} \tag{259}$$

$$x_{n+1}(t) = (1 - e^{-t}) - \int_0^t x_n^2(u)\,du \tag{260}$$

then, as $n \to \infty$, x_n approaches infinity. The resulting limit is clearly of higher order and not of exponential type.

In a given domain D, the modified Lipschitz condition defined by Equation 102 is sufficient, but not necessary, for the existence of Picard iterates $\{x_n\}$ and the uniqueness of the solution x(t). The Lipschitz condition implies continuity of the weighted nonlinear function F[t, u, x(t)] with respect to x but not to t or u. The physical significance of the condition is that when t and u are fixed, the resulting function F(x) has bounded difference quotients; the bound is given by the L_2 function K(t, u) defined in this section [29, p. 38]. If the domain D is convex (that is, D contains the line segments connecting any two points in D), then an application of the mean-value theorem of differential calculus will show that the existence and boundedness of the partial derivatives of F(t, u, x), with respect to (x) in the given domain, are sufficient for F(t, u, x) to satisfy the Lipschitz condition [8, p. 54]. This means that the rate of change of the weighted nonlinear function should not be rapid. Nonlinearities with sharp breaks would not satisfy the Lipschitz condition and should be avoided, but since they are not too likely to occur in nature, the given theory can be applied to a broad class of physical systems [47, p. 191].

Although uniqueness does not always imply the convergence of successive approximations, the restrictions placed on the theorems given in this chapter are sufficient to assure convergence [8]. (Examples of convergence are given in Appendix E.)

The conversion of the system differential Equation 96 into the equivalent integral Equation 122 proves an interesting relation between circuit configuration and feedback-system configuration. Equation 96 is composed of both a linear and nonlinear set of terms and a forcing function; therefore, it can represent the electric circuit of Figure 13. The integral Equation 122 can represent the feedback system of Figure 14, which has the same output as the current flowing in the circuit of Figure 13. This establishes a relation between circuits and feedback systems.

The class of forcing functions g(t) allowed in this work is defined completely by condition 102. This condition states that the forcing function should be such that when applied to a linear system consisting of the linear terms L(D, t) (defined by Equation 97 and shown in Figure 15 the energy associated with the response x_0 of the linear system should be finite.[19] This condition includes many physical systems of interest. By an application of the Schwarz inequality to Equation 99, it can be shown that

$$||x_0|| \leq ||g|| \; ||W t, u|| \tag{261}$$

i.e., condition 102 can be satisfied by choosing functions g(t) and W(t, u) that have bounded norms.

Finally, it is worthwhile to note that the method of analysis presented in this chapter is best suited for handling nonlinear systems with parameters, because it gives control engineers an easy and flexible means for choosing parameters that will satisfy certain requirements in operation during a specified time. This is clearly shown in example three and will be elaborated upon in the following section.

[19] See Reference 25, page 123, for the following definition: "For an arbitrary real valued function of time x(t), we call the quantity $x(t)^2$ the instantaneous power associated with x(t). The total energy associated with this function is the integral $\int_{\infty}^{\infty} x(t)^2\, dt$ when this integral exists."

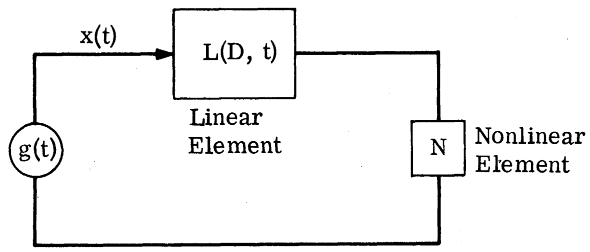

FIGURE 13. ELECTRIC CIRCUIT CONFIGURATION OF EQUATION 96

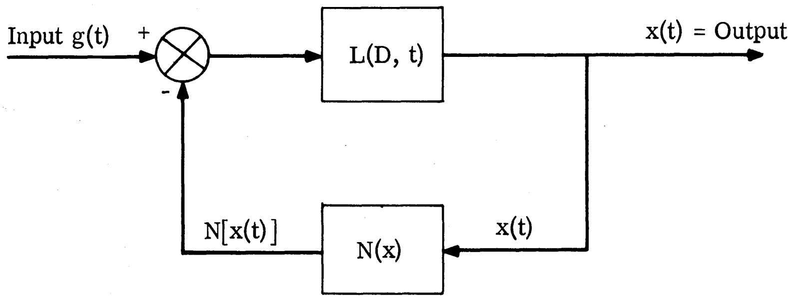

FIGURE 14. A FEEDBACK SYSTEM REPRESENTATION OF THE NONLINEAR INTEGRAL EQUATION 122

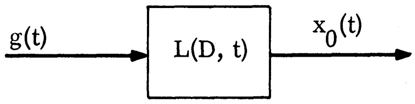

FIGURE 15. A LINEAR SYSTEM REPRESENTATION OF EQUATION 97

3
STABILITY ANALYSIS

This section discusses the use of the partitioning technique and the state-variable technique to study a class of control systems whose behavior can be represented by Equation 96. Placing suitable restrictions on the system equation revealed that the state variables $(v_1, v_2, \ldots, v_n)$, which represent the state of the system in the phase space, belong to an L_2 space. These restrictions include those presented in Section 2 as sufficient, but not necessary, for the existence and uniqueness of the system's response. Two definitions of the norm of the state of the system were used to study asymptotic stability in the Lyapunov sense. This method is useful for an analysis of the system trajectory at any time t in the period of interest. An expression for the upper-bound state is calculated, and an example is included to illustrate the method presented.

3.1. SYSTEM RESTRICTIONS

The notations, definitions, and system description introduced in Section 2 are followed in the present analysis. The system must satisfy a number of mathematical restrictions which define the class of admissible systems under consideration. These restrictions are:

(1) $x_0^{(m)}(t)$ belong to an L_2 space over an interval $[0, T]$, and are assumed bounded.

$m = 0, 1, \ldots, (n - 1)$, where n is the order of the highest derivative of the linear part of the system equation.

(m) = the m-th order derivative with respect to the independent variable t. (262)

(2) In a given region D, $[\,|x| \leq d(t), 0 \leq u \leq t \leq T]$, $F^{(m)}(t, u, x)$[20] satisfies the following two conditions:

(a) $F^{(m)}[t, u, x(u)]$ satisfies the modified Lipschitz condition; that is, if the two triples (t, u, z_1) and (t, u, z_2) are in D, then

$$|F^{(m)}(t, u, z_1) - F^{(m)}(t, u, z_2)| \leq K_m(t, u,) |z_1 - z_2| \tag{263}$$

where $[K_m(t, u)]$ are $L_2[0, T]$, that is

$$\begin{aligned} &\int_0^t K_m^2(t, u)\, du \leq \alpha_m^2(t) \\ &\int_0^T \alpha_m^2(t)\, dt \leq A_m^2 \end{aligned} \tag{264}$$

[20] $F^{(m)}(t, u, x) \triangleq \dfrac{\partial^m F(t, u, x)}{\partial t^m}$ and, as before, F is a function of x only, and not $\dot{x}$, etc.

(b) $$\int_0^t F^{(m)}[t, u, x_0(u)]\, du \leq n_m(t)$$

where $n_m(t)$ are bounded and belong to $L_2(0, T)$, that is,

$$\int_0^T n_m^2(t) \leq c_m^2 \tag{266}$$

where (m) and m are as defined above

A_m^2 and C_m^2 are finite positive numbers

T is the period of interest $(0 \leq T \leq +\infty)$

The functions $\alpha_m^2(t)$, $n_m(t)$ are continuous and bounded over $[0, T]$

(3) Condition 108 is still required; that is,

$$\begin{aligned} q(t) &\leq d(t) \\ |x_0(t)| &\leq d(t) \end{aligned} \tag{267}$$

where $x_0(t)$ and q(t) are as defined by Equations 99 and 107.

(4) The time-varying parameters $a_i(t)$ and $|N(x, t) - g(t)|$ are bounded over $[0, T]$. (268)

It should be noted that when m = 0, restrictions 262 through 268 include the set of restrictions defined in Section 2.2. These restrictions, that is, those in which m = 0, were shown to be sufficient for the existence and uniqueness of the solution x(t), which is the limit of a sequence of iterates $\{x_n(t)\}$ belonging to the L_2 space, and given by the following equation:

$$x_{n+1} = x_0(t) - \int_0^t F[t, u, x_n(u)]\, du \qquad (n = 0, 1, 2, \ldots) \tag{269}$$

3.2. THE STATE-VARIABLE REPRESENTATION

Consider the system described by Equation 96 under the restrictions placed on it by Equations 262 through 265 and let:

$$\begin{aligned} x &= v_1 \\ \frac{dx}{dt} &= v_2 \\ \frac{dx^{n-1}}{dt^{n-1}} &= v_n \end{aligned} \tag{270}$$

These equations become

$$
\begin{aligned}
\frac{dv_1}{dt} &= v_2 && = f_1(t, v_1, v_2, \ldots, v_n) \\
\frac{dv_2}{dt} &= v_3 && = f_2(t, v_1, v_2, \ldots, v_n) \\
&\vdots \\
\frac{dv_n}{dt} &= a_n(t)\, v_1 - a_{n-1}(t)\, v_2 \ldots + N_1 && = f_n(t, v_1, v_2, \ldots, v_n)
\end{aligned}
\tag{271}
$$

where

$$N_1 = N_1(x, t) = g - N \tag{272}$$

Equation 96 can be put into the following matrix equation form:

$$
\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \\ \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \\ 0 & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & \cdot \\ -a_n(t) & -a_{n-1}(t) & -a_0(t) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ \\ \cdot \\ N_1 \end{bmatrix}
\tag{273}
$$

Again, Equation 273 can be represented by the following two equations:

$$\frac{d}{dt}[\bar{v}] = [A(t)]\,[\bar{v}] + [N_1] \tag{274}$$

$$\frac{d}{dt}[\bar{v}] = \bar{f}(t, \bar{v}) \tag{275}$$

where $\bar{v}$ = a vector comprised of the state variable $(v_1, v_2, \ldots, v_n)$; it can be used to represent the state of the system in the phase space in which each succeeding axis represents the rate of change of the quantity measured along the one preceding it

$\bar{f}$ = a column vector made up of the functions $(f_1, f_2, \ldots, f_n)$, as defined by Equation 271

$[A(t)]$ = an n by n matrix, as defined by Equation 273

3.3. THE PROPERTIES OF THE STATE VARIABLES ($v_1, v_2, \ldots, v_n$) OF THE CLASS OF SYSTEMS UNDER CONSIDERATION (THEOREM E)

Theorem E summarizes some of the properties of the state variables which are of interest in this study.

Theorem E

Under the restrictions (given by Equations 262 through 268) placed on the linear, nonlinear, and forcing function terms, the state variables $v_i(t)$ belong to $L_2[0, T]$, and are bounded by corresponding arbitrary $L_2[0, T]$ functions $\beta_i(t)$. Since $|N_1(t, v_1)|$ and $|a_i(t)|$, which were defined before, are assumed bounded over the interval $[0, T]$, then the state variables $v_i(t)$ are uniformly continuous[21] over the same interval $[0, T]$. In particular, if $T = +\infty$, then it follows that $\lim_{t\to\infty} v_i(t) = 0$ $(i = 1, 2, \ldots, n)$.

Proof

The following two lemmas are needed to prove the theorem:

Lemma One[22]

If a function f(t) has a bounded derivative on any interval (finite or infinite), then f(t) is uniformly continuous over the same interval.

Lemma Two [50, p. 86][23]

If a function f(t) is uniformly continuous over the interval $[0, \infty]$ and if $\int_0^\infty f^2(t)\, dt$ is finite, then it follows that $\lim_{t\to\infty} f(t) = 0$.

Recall from Equation 122 that the state variable $v_1(t) = x(t)$ can be given by

$$x(t) = v_1(t) = x_0(t) - \int_0^t F[t, u, v_1(u)]du \tag{276}$$

and that

$$\frac{d}{dt}\int_0^t g(t, u)\, du = g(t, t) + \int_0^t \frac{d}{dt} g(t, u)\, dt \tag{277}$$

[21] See Appendix H for the definitions of uniform continuity.

[22] Lemma One can be easily obtained by an application of the law of the mean: $|f(t_2) - f(t_1)| = f(t^*)\,|t_2 - t_1| \le K|t_2 - t_1|$, $t_1 \le t^* \le t_2$

[23] See also Appendix H.

Upon differentiating Equation 276 and using Equation 277, the following equation is obtained:

$$\frac{dx}{dt} = \frac{dv_1(t)}{dt} = \frac{d}{dt} x_0(t) + F[t, tv_1(t)] - \int_0^t \frac{d}{dt} F[t, u, v_1(u)]du \tag{278}$$

However, since W(t, t) = 0, it follows that

$$F[t, tv_1(t)] = W(t, t) N[v_1(t)] = 0 \tag{279}$$

Equation 278 can be put in the form:

$$x(t)^{(1)} = v_2 = v_1^{(1)} = x_0^{(1)}(t) - \int_0^t F^{(1)}[t, u, v_1(u)]du \tag{280}$$

where, as defined before, the superscript (1) denotes the differential operator d/dt, and

$$F^{(1)}[t, u, v_1(u)] = W^{(1)}(t, u) N_1[v_1(u)] \tag{281}$$

From the properties of the impulse response,[24] it follows that

$$\begin{aligned} W(t, t) = W^{(1)}(t, t) = \ldots = W^{(n-2)}(t, t) = 0 \\ W^{(n-1)}(t, t) = \frac{1}{a_n(t)} \neq 0 \end{aligned} \tag{282}$$

Using Equation 282 and differentiating Equation 276 k times yields

$$x^{(k)} = v_{k+1} = x_0^{(k)}(t) - \int_0^t F^{(k)}[t, u, v_1(u)]du \tag{283}$$

Equation 283 gives all the state variables, $v_1, v_2, \ldots, v_n$. When k = 0, it yields the state variable v_1, which was shown by theorems A, B, and C to exist, to be unique, and to belong to an L_2 space.

Now, to prove that all the v_i, (i = 1, 2, . . . , n) in general belong to an L_2 space, it is necessary to define a sequence $x_m^{(k)}$ according to the following recurrence relation:

$$x_{m+1}^{(k)} = x_0^{(k)}(t) - \int_0^t F^{(k)}[t, u, x_m(u)]du \tag{284}$$

where k = 0, 1, 2, . . . , n - 1
m = 0, 1, 2, . . .

[24] See Appendix C.

Equation 284 can be used as follows to obtain $x_m^{(k)}$:

$$x_m^{(k)} = x_0^{(k)}(t) - \int_0^t F^{(k)}[t, u, x_{m-1}(u)]du \tag{285}$$

Subtracting Equation 285 from Equation 284 and squaring gives

$$\left(x_{m+1}^{(k)} - x_m^{(k)}\right)^2 = \left(\int_0^t \left\{F^{(k)}[t, u, x_m(u)] - F^{(k)}[t, u, x_{m-1}(u)]\right\} du\right)^2$$

$$\leq \left[\int_0^t K_k(t, u) \,|x_m - x_{m-1}|\right]^2$$

$$\leq \left\{\int_0^t K_k(t, u)^2 du \int_0^t [x_m(u) - x_{m-1}(u)]^2 du\right\} \tag{286}$$

Equation 264 can be used to put Equation 286 into the following form:[25]

$$\left(x_{m+1}^{(k)} - x_m^{(k)}\right)^2 \leq \alpha_k^2(t) \int_0^T [x_m(u) - x_{m-1}(u)]^2 du \tag{287}$$

But Equation 134 gives

$$\int_0^T [x_m(u) - x_{m-1}(u)]^2 du \leq \frac{C_0^2 A_0^{m-2}}{(m-2)!} \tag{288}$$

Therefore, substituting Equation 288 into Equation 287 gives

$$\left[x_{m+1}^{(k)} - x_m^{(k)}\right]^2 \leq \frac{C_0^2 A_0^{m-2}}{(m-2)!} \alpha_k^2(t) \tag{289}$$

where m = 2, 3, . . .

k = 0, 1, . . . , n - 1, n = the order of the system

Equation 289 shows that all $\left\{x_m^{(k)}\right\}$ are L_2 functions. It can easily be proved that $\lim_{m\to\infty} x_m^{(k)}$ exists and is given by[26]

$$\lim_{m\to\infty} x_m^{(k)} = x^{(k)} = v_{k+1} \tag{290}$$

where k = 0, 1, . . . , n - 1, n = order of the system.

[25] $\int_0^t [x_m(u) - x_{m-1}(u)]^2 du \leq \int_0^t [x_m(u) - x_{m-1}(u)]^2 du, \; t \leq T$

[26] See Appendix G.

Therefore, it can be concluded that all the state variables v_i (i = 1, 2, . . . , n) belong to an L_2 space in the assumed interval of interest T $(0 \leq t \leq T)$.

Using Equation 528 of Appendix G, an upper-bound function for each of the state variables $v_i(t)$ can be calculated so that

$$v_i(t) \leq \beta_i(t) \text{ for } (i = 1, 2, \ldots, n)\ (0 \leq t \leq T) \tag{291}$$

The functions $\beta_i(t)$ satisfying inequality 291 are given by

$$\beta_i(t) = x_0^{(i-1)}(t) + |n_{i-1}(t)| + m\,|\alpha_{i-1}(t)| \tag{292}$$

where $x_0^{(i-1)}(t)$, $n_{i-1}(t)$, and $\alpha_{i-1}(t)$ are functions defined by the restrictions placed on the system for i = 1, 2, . . . , n, and m is a positive constant given by

$$m = C_0\left[1 + A_0 \sum_{k=0}^{\infty} \frac{A_0^{\ k}}{\sqrt{k!}}\right] \tag{293}$$

It is clear from Equation 292 that the functions $\beta_i(t)$ lie in the L_2 space, since the right-hand side of Equation 291 is a linear combination of L_2 functions. Also, the upper-bound functions can be changed as desired since they depend only on known functions defined by the system restrictions.

If the period of interest T is chosen to be infinite, then from the properties of the L_2 spaces, we can conclude that

$$\int_0^{\infty} v_i^{\ 2}\, dt < \infty \ (i = 1, 2, \ldots, n) \tag{294}$$

It is not necessary for a continuous function which is square integrable on the interval $[0, +\infty)$ to have a zero limit at $t = +\infty$ since in general the limit may not exist.[27] However, by considering the matrix Equation 273, it can easily be shown (since the time-varying parameters $a_n(t)$ and the nonlinear function $N_1(t)$ are assumed bounded) that each of the state variables $v_i(t)$ has a bounded derivative in the interval of interest (assumed infinite). Thus Lemma One shows that $v_i(t)$ for i = 1, 2, . . . , n are uniformly continuous over the same interval. Now, by considering Equation 294 and by applying Lemma Two, it can be concluded that

$$\lim_{t \to \infty} v_i(t) = 0 \text{ for } i = 1, 2, \ldots, n \tag{295}$$

[27] See Appendix H.

3.4. STABILITY CONSIDERATIONS

Consider the system described by the vector Equation 275. An equilibrium position $\bar{v}_e$ exists such that all the state derivatives dv_i/dt for i = 1, 2, . . . , n are simultaneously zero. It can easily be seen from Equations 271 that, for the system under consideration, the origin of the phase space $\bar{v} = (v_1, v_2, \ldots, v_n) = 0$ is a position of equilibrium if the condition

$$N_1(\bar{v}, t) = 0 \text{ at } \bar{v}(t) = 0 \tag{296}$$

is satisfied. Thus, under condition 296 the solution $\bar{v} = \bar{0}$ is an equilibrium position of Equation 275 and its stability is in question. With the passage of time, the state vector $\bar{v}$ whose components are $v_1, v_2, \ldots, v_n$ traces a curve in the phase space known as the system trajectory which describes all possible states of the system. Figure 16 is a possible trajectory in the n-dimensional space. It may be thought of as the projection of a solution of the equation, plotted in a space containing the n-space axis and time axis, into the state space. This is shown in Figure 17.

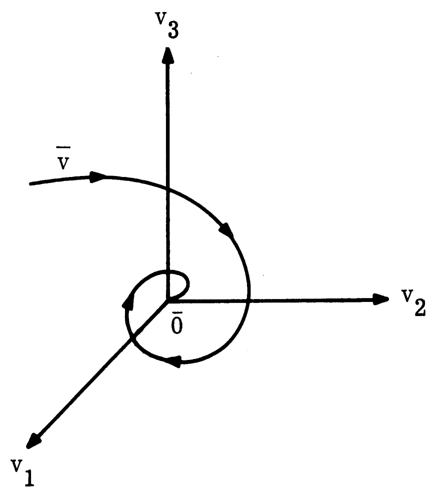

FIGURE 16. SYSTEM TRAJECTORY

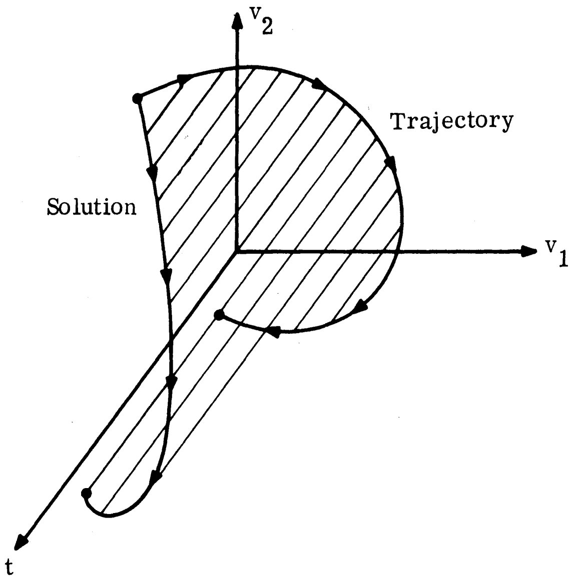

FIGURE 17. PROJECTION OF A SOLUTION CURVE [12]

A simple measure of the departure of one possible state $\bar{v}$ from the equilibrium position $\bar{v}$ = 0 can be obtained from the norm of $\bar{v}$ denoted by $||\bar{v}||$. The norm is defined as a function which assigns to every vector $\bar{v}$ in a linear space a real number denoted by $||\bar{v}||$ so that [1, p. 406]:

$$\begin{aligned} &||\bar{v}|| \text{ is defined for every } \bar{v} \text{ in the space} \\ &||\bar{v}|| = 0 \text{ if and only if } \bar{v} = 0 \\ &||\bar{v}|| > \bar{0} \text{ for all } \bar{v} > \bar{0} \\ &||u + \bar{v}|| \leq ||\bar{v}|| + ||\bar{u}|| \text{ for all } \bar{u} \text{ and } \bar{v} \\ &||\alpha\bar{v}|| = |\alpha| \cdot ||\bar{v}|| \text{ for all } \bar{v}, \text{ and } \alpha \text{ is a real constant} \end{aligned} \tag{297}$$

Some commonly used norms are [8, p. 17]:

$$||\bar{v}|| = \left[\sum_{i=1}^{n} \left(v_i^{\,2}\right)\right]^{1/2} \tag{298}$$

$$||v|| = \max_i \{v_i\} \tag{299}$$

$$||\bar{v}|| = \sum_{i=1}^{n} |v_i| \tag{300}$$

$$\|\bar{v}\| = \left[\int \cdots_R \int |v(s_1, \ldots, s_\nu|^2 \, ds_1, \ldots, ds_\nu\right]^{1/2} \tag{301}$$

where R is a fixed region of the $s_1, \ldots, s_\nu$ space, and $|\bar{v}|$ denotes the modulus of the vector $\bar{v}$ in the usual way. It can be shown that the norms, as defined by Equations 298 through 301, satisfy the conditions for the norm as stated under definition 297. Thus, it can be said that for the system under consideration that an output state $\bar{u}$ is greater than an output state $\bar{v}$ if and only if $\|\bar{u}\| > \|\bar{v}\|$.

If a system is perturbed slightly from its equilibrium state at the origin, it is desirable in many applications to have all subsequent motions remain in a correspondingly small neighborhood about the origin. This is shown diagramatically in Figure 18 and means that the system under consideration is stable. However, in order to investigate stability it is necessary to define precisely the concept of stability to be applied.

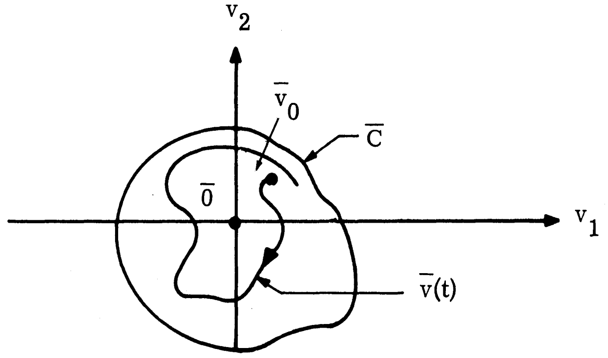

FIGURE 18. GEOMETRIC DEFINITION OF STABILITY. $\bar{C}$ = upper-bound state, $\bar{v}$ = system trajectory, $\bar{v}_0$ = initial state, $\bar{0}$ = equilibrium state.

Definition of Stability [7, 12, 14]

The equilibrium position $\bar{v} = 0$ is called stable in the sense of Lyapunov if, for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ so that $\|\bar{v}(t)\| \leq \epsilon$ whenever $|\bar{v}(0)| \leq \delta$ for all $t > 0$. In other words, the equilibrium position $\bar{v} = \bar{0}$ is stable if the initial magnitude of the norm of the state vector is small enough so that $\|\bar{v}\|$ can be made to remain below an arbitrary upper bound. If $\bar{v}(t)$ approaches zero as t approaches infinity, that is, if

$$\lim_{t\to\infty} v_i(t) = 0 \quad (i = 1, 2, \ldots, n)$$

the equilibrium position is called asymptotically stable. If δ is independent of the choice of the initial time, then it follows that the system is uniformly asymptotically stable.

It is necessary to confine the application of this definition of stability to those cases which satisfy the definitions of the norm $||\bar{v}||$ given by Equations 297. However, for the purposes of this report, there are two reasons which make it desirable to study the behavior of the system under the two definitions of the norm given by Equations 298 and 301:

(1) The Euclidean norm defined by Equation 298 gives the magnitude or length of the state vector $\bar{v}(t)$ during the period of interest.

(2) The norm defined by Equation 301 gives a measure of the mean square value of the state vector $\bar{v}(t)$ during the period of interest.

Hence, the two definitions of the norm $||\bar{v}||$ adopted in this study govern the transient behavior of the system under consideration, at any particular time t and also on the average during a given interval of time, a situation which is of interest to many system engineers.

Now, if the functions $x_0^{(m)}(t)$, for any initial state v_0, satisfy conditions 262 of the restrictions placed on the system, i.e., if $x_0^{(m)}(t)$ belongs to an L_2 space, and if conditions 263 through 268 are true, then it can be concluded that

(1) The system under consideration possesses a unique solution existing in a domain to the right of the initial state.

(2) The system trajectory $\bar{v}(t)$ approaches $\bar{0}$ as the time t approaches infinity.

(3) The norm $||\bar{v}||$, as defined by Equation 298, is bounded by an arbitrary upper norm $||\bar{\beta}(t)||$ given by

$$||\bar{\beta}(t)|| = \left[\sum_{i=1}^{n} \left\{\beta_i^2(t)\right\}\right]^{1/2} \tag{302}$$

where $\beta_i(t)$ is as previously defined by Equation 292.

(4) The norm $||\bar{v}||$ as defined by Equation 303 (a special case[28] of Equation 301), namely

$$||\bar{v}|| = \left[\int_0^\infty |\bar{v}(t)|^2\, dt\right]^{1/2} \tag{303}$$

exists and is less than an arbitrary positive number N, given by the following equation:

$$N^2 = \int_0^\infty |\bar{\beta}(t)|^2\, dt \tag{304}$$

[28] For the special case, $\nu = 1$, $s = t$, and R = the interval $0 \leq t \leq \infty$.

where

$$|\bar{\beta}(t)| = \left[\sum_{i=1}^{n} \beta_i^{\,2}(t)\right]^{1/2} \tag{305}$$

The existence of the two integrals given by Equations 303 and 304 can easily be seen because both of the vector functions $\bar{v}(t) = [v_1(t), \ldots, v_n(t)]$ and $\bar{\beta}(t) = [\beta_1(t), \ldots, \beta_n(t)]$ are L_2 functions, as previously proved.

(5) The system trajectory $\bar{v}$ is uniformly continuous with respect to the initial conditions.[29]

Consequently, the system under consideration is asymptotically stable according to the definition of stability given previously. For autonomous systems which are free and stationary, asymptotic stability implies uniform asymptotic stability. This is not true for nonstationary systems which are, in general, the systems discussed in this report. Unfortunately, in nonstationary systems, asymptotic stability alone does not guarantee that bounded inputs will give rise to bounded outputs; uniform asymptotic stability is needed. This is indicated by the following example suggested by Kalman and Betram [14]: Consider the nonstationary system

$$\frac{dx}{dt} + \frac{x}{t} = g(t) \qquad 0 > t_0 \geq t \tag{306}$$

(For values of t_0 less than 0, this equation has finite escape time and does not define a dynamic system.) The impulse response of this system is given by

$$W(t, t_0) = \frac{t_0}{t} \qquad t \geq t_0 > 0 \tag{307}$$

Evidently, the system is asymptotically stable. However, the unit step response of this system, which corresponds to a zero state at some time $t > 0$, is given by

$$x(t) = \int_0^t W(t, \tau)\, d\tau = (t - t_0)^2/2t \tag{308}$$

which tends to ∞ with t!

[29]This statement is deduced from arguments presented in Section 2.8.

3.5. AN EXAMPLE CALCULATION: THE ANALYSIS OF A SECOND-ORDER NONLINEAR CONTROL SYSTEM

Consider the system equations:

$$\begin{aligned} \frac{dv_1}{dt} &= 0 + K_{12}v_2 &&= f_1(v_1, v_2, t) \\ \frac{dv_2}{dt} &= K_{21}v_1 + K_{22}v_2 + K_{2N}(t)N(v_1, v_2) &&= f_2(t_1, v_2, t) \end{aligned} \tag{309}$$

with initial conditions at t = 0, $v_1 = v_2 = 0$

where $K_{2N}(t)$ = a time-varying parameter

K_{12}, K_{22}, K_{21} = constant parameters

v_1 and v_2 = the state variables that represent the state of the system in the phase plane at any instant of time

$N(v_1, v_2)$ = a nonlinear term

Combining Equations 309 by eliminating v_2 gives

$$\frac{1}{K_{12}}\frac{d^2v_1}{dt} = K_{21}v_1 + K_{22}\left(\frac{1}{K_{12}}\frac{dv_1}{dt}\right) + K_{2N}(t)\,N\left(v_1, \frac{1}{K_{12}}\frac{dv_1}{dt}\right) \tag{310}$$

Equation 310 is equivalent to the following equation:

$$\ddot{v}_1 = K_{21}K_{12}v_1 + K_{22}\dot{v}_1 + K_{12}K_{2N}(t)\,N\left(v_1, \frac{\dot{v}_1}{K_{12}}\right) \tag{311}$$

where $\dot{v}_1 = \frac{dv_1}{dt}$ and $\ddot{v}_1 = \frac{d^2v_1}{dt^2}$

Consider the nonlinear control system shown in Figure 19. The system shown in Figure 19 can be described as follows:

$$J\ddot{c}(t) + (B + K_aK_b)\dot{c}(t) + K_aK_cc(t) = K_aK_{2N}(t)N(v_1) \tag{312}$$

Recall that

$$\begin{aligned} v_1 &= r(t) - c(t) \\ \dot{v}_1 &= \dot{r}(t) - \dot{c}(t) \\ \ddot{v}_1 &= \ddot{r}(t) - \ddot{c}(t) \end{aligned} \tag{313}$$

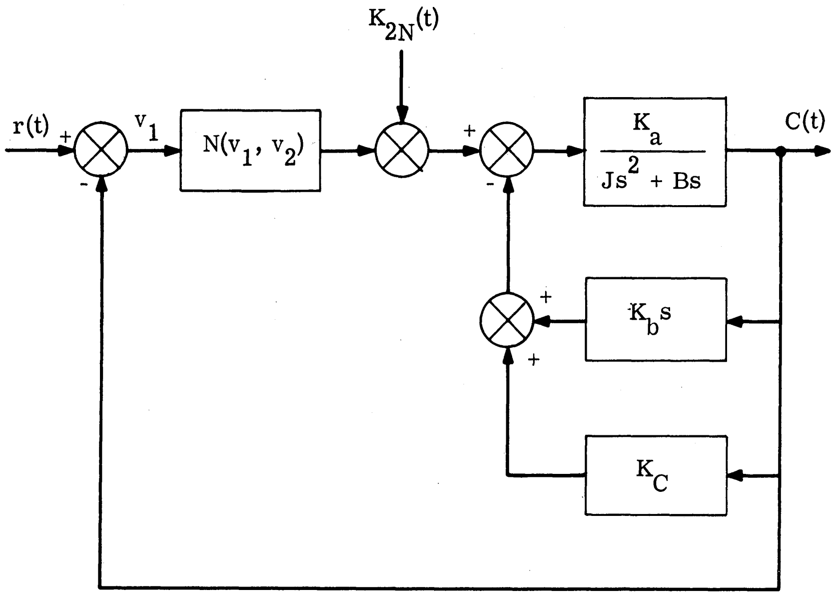

FIGURE 19. A SECOND-ORDER NONLINEAR CONTROL SYSTEM

Equation 312 gives

$$J\ddot{v}_1(t) + (B + K_aK_b)\dot{v}_1(t) + K_aK_cv_1(t) + K_aK_{2N}(t)N(v_1) = J\ddot{r}(t) + (B + K_aK_b)\dot{r} + K_aK_cr \quad (314)$$

To study the homogeneous differential equation of the system Equation 314, let r(t) = 0. This gives

$$\ddot{v}_1(t) + \left(\frac{B + K_aK_b}{J}\right)\dot{v}_1(t) + \left(\frac{K_aK_c}{J}\right)v_1(t) + \left(\frac{K_a}{J}\right)K_{2N}(t)N(v_1) = 0 \quad (315)$$

The homogeneous Equation 315 of the second-order control system shown in Figure 19 is identical to Equations 309 or their equivalent Equation 311 if

$$\frac{B + K_aK_b}{J} = -K_{22}$$

$$\frac{K_aK_c}{J} = -K_{21}K_{12} \quad (316)$$

$$\frac{K_a}{J} = -K_{12}$$

Equations 316 relate the parameters K_{12}, K_{21}, and K_{22} of Equations 309 to the parameters B, K_a, K_b, K_c, and J of the control system. This means that changing the set of parameters in Equations 309 is equivalent to changing the control system parameters; the converse is also true. From Equations 309 it can be seen that if

$$N(v_1, v_2) = 0 \tag{317}$$

at $v_1 = v_2 = 0$, then the origin of the phase space will be an equilibrium position at which dv_1/dt and dv_2/dt will be simultaneously zero for all t.

Let the functions of interest for $N(v_1, v_2)$ and $K_{2N}(t)$, be

$$\begin{aligned} N(v_1, v_2) &= v_1^{\,2} \\ K_{2N}(t) &= e^{+2\alpha t} \end{aligned} \tag{318}$$

where α is any arbitrary constant, yet to be determined. Now, the problem at hand is: Given a nonlinear function $N(v_1, v_2)$ and the specific form of $K_{2N}(t)$ for the control system shown in Figure 19, choose the parameters K_{12}, K_{21}, K_{22} and α so that (1) The actuating error signal v_1 and the rate of change of signal $\dot{v}_1$ belong to an L_2 space in the interval $0 \le t \le \infty$. (2) Given two specific numbers H_1 and H_2, the system satisfies the following requirements:

$$\begin{aligned} \int_0^\infty v_1^{\,2}(t)\,dt &\le H_1^{\,2} \\ \int_0^\infty \dot{v}_1^{\,2}\,dt &\le H_2^{\,2} \end{aligned} \tag{319}$$

This means that the unknown parameters are required to keep the integrated square error $[v_1(t)]$ and the integrated square rate of change of the error $[\dot{v}_1(t)]$ below certain specified levels. Equations 319 also require that the state variables v_1 and $v_2 = \dot{v}_1$ should finally reach the equilibrium position at the origin of the phase plane, shown in Figure 20.

To solve the problem the required parameters should be chosen so that the system under consideration satisfies the restrictions already studied. Now, $x_0(t)$ is the solution of the linear equation:

$$\ddot{v}_1(t) + \frac{B + K_a K_b}{J}\dot{v}_1(t) + \frac{K_a K_c}{J}, v_1(t) = 0 \tag{320}$$

with the initial conditions that $t = 0$, $x_0 = 1$, and $\dot{x}_0 = 0$.

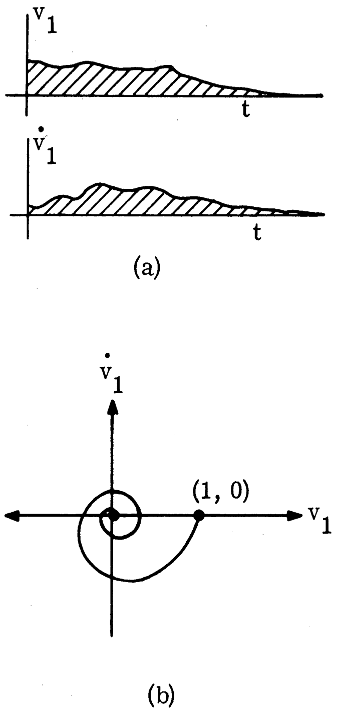

FIGURE 20. REQUIRED SYSTEM CHARACTERISTICS. (a) Time domain. (b) Phase plane domain.

Therefore $x_0(t)$ and $x_0^{(1)}(t)$ take the form

$$x_0(t) = \frac{1}{m_1 - m_2}\left(-m_2 e^{m_1 t} + m_1 e^{m_2 t}\right), \; x_0^{(1)}(t) = \frac{1}{m_1 - m_2}\left(-m_1 m_2 e^{m_1 t} + m_1 m_2 e^{m_2 t}\right) \qquad (321)$$

where m_1 and m_2 are the roots of the characteristic Equation 320 as follows:

$$m_i = -\frac{B + K_a K_b}{2J} \pm \sqrt{\frac{(B + K_a K_b)^2}{4J^2} - \frac{K_a K_c}{J}} \qquad (i = 1 \text{ and } 2) \qquad (322)$$

Also, W(t, u) is the solution of the following equation:

$$\ddot{v}_1(t) + \frac{B + K_a K_b}{J}\dot{v}_1(t) + \frac{K_a K_c}{J} v_1(t) = \delta(t, u) \qquad (323)$$

with the initial conditions that at t = u, W(t, t) = 0, $W^{(1)}(t, t) = 1$ where $\delta(t, u)$ is the unit impulse applied at t - u. Therefore,

$$W(t, u) = W(t - u) = \frac{1}{m_1 - m_2}\left[e^{m_1(t-u)} - e^{m_2(t-u)}\right] h(t - u) \tag{324}$$

where m_1, m_2 are as defined before, and h(t) is the unit step function. The weighted nonlinear function F[t, u, x(t)] can now be given:

$$F(t, u, v_1) = \frac{m_3}{m_1 - m_2}\left[e^{m_1(t-u)} - e^{m_2(t-u)}\right] e^{+2\alpha u} v_1^{\,2} \tag{325}$$

where

$$m_3 = K_a/J \tag{326}$$

Also,

$$F^{(1)}(t, u, v_1) = \frac{m_3}{m_1 - m_2}\left[m_1\, e^{m_1(t-u)} - m_2\, e^{m_2(t-u)}\right] e^{+2\alpha u} v_1^{\,2} \tag{327}$$

Both $F(t, u, v_1)$ and $F^{(1)}(t, u, v_1)$ satisfy the Lipschitz condition in the domain $D = [|x| \leq 1\,(0 \leq u \leq t \leq \infty)]$, since

$$|F(t, u, x_1) - F(t, u, x_2)| \leq \frac{2m_3}{m_1 - m_2}\left[e^{m_1 t} - e^{(m_1-2\alpha)u} - e^{m_2 t} e^{-(m_2-2\alpha)u}\right] |x_1 - x_2| \tag{328}$$

$$F^{(1)}(t, u, x_1) - F^{(1)}(t, u, x_2)| \leq \frac{2m_3}{m_1 - m_2}\left[m_1\, e^{m_1(t)} e^{-(m-2\alpha)u} - m_2\, e^{m_2(t)} e^{-(m_2-2\alpha)u}\right] |x_1 - x_2| \tag{329}$$

Therefore, from Equations 328 and 329, we have

$$K_0(t, u) = \frac{2m_3}{m_1 - m_2}\left[e^{m_1 t} e^{-(m+2\alpha)u} - e^{m_2 t} e^{-(m_2-2\alpha)u}\right] \tag{330}$$

$$K_1(t, u) = \frac{2m_3}{m_1 - m_2}\left[m_1\, e^{m_1 t} e^{-(m_1-2\alpha)u} - m_2\, e^{m_2 t} e^{-(m_2-2\alpha)u}\right] \tag{331}$$

Also, applying Equations 265 of the restrictions gives

$$n_0(t) = \int_0^t \frac{m_3}{(m_1 - m_2)^3}\left[e^{m_1(t-u)} - e^{m_2(t-u)}\right]\left[-m_2 e^{m_1 u} + m_1 e^{m_2 u}\right]^2 e^{+2\alpha u}\,du$$

$$= C_{01} e^{2(m_1+\alpha)t} + C_{02} e^{2(m_2+\alpha)t} + C_{03} e^{(m_1+m_2+2\alpha)t} + C_{04} e^{m_1 t} + C_{05} e^{m_1 t} \qquad (332)$$

and

$$n_1(t) = \int_0^t \frac{m_3}{(m_1 - m_2)^3}\left[m_1 e^{m_1(t-u)} - m_2 e^{m_2(t-u)}\right]\left[m_2 e^{m_1 u} + m_1 e^{m_2 u}\right]^2 e^{+2\alpha u}\,du$$

$$= C_{11} e^{2(m_1+\alpha)t} + C_{12} e^{2(m_2+\alpha)t} + C_{13} e^{(m_1+m_2+\alpha)t} + C_{14} e^{m_1 t} + C_{15} e^{m_2 t} \qquad (333)$$

where

$$C_{01} = \frac{m_3}{(m_1 - m_2)^2}\left\{\frac{m_2^2}{m_1 + 2\alpha} - \frac{m_2^2}{2m_2 + 2\alpha - m_2}\right\}$$

$$C_{02} = \frac{m_3}{(m_1 - m_2)^3}\left\{\frac{m_1^2}{2m_2 + 2\alpha - m_2} - \frac{m_1^2}{m_2 + 2\alpha}\right\}$$

$$C_{03} = \frac{m_3}{(m_1 - m_2)^3}\left\{\frac{-2m_1 m_2}{m_2 + 2\alpha} + \frac{2m_1 m_2}{m_1 + 2\alpha}\right\}$$

$$C_{04} = \frac{m_3}{(m_1 - m_2)^3}\left\{\frac{-m_2^2}{m_1 + 2\alpha} + \frac{2m_1 m_2}{m_2 + 2\alpha}\,\frac{m_1^2}{2m_2 + 2\alpha - m_1}\right\}$$

$$C_{05} = \frac{m_3}{(m_1 - m_2)}\left\{\frac{m_2^2}{2m_1 + 2\alpha - m_2} - \frac{2m_1 m_2}{m_1 + 2\alpha} + \frac{m_1^2}{m_2 + 2\alpha}\right\}$$

$$C_{11} = \frac{m_3}{(m_1 - m_2)^3}\left\{\frac{m_2^2 m_1}{m_1 + 2\alpha} - \frac{m_2^3}{2m_1 + 2\alpha - m_2}\right\} \qquad (334)$$

(Equation 334 continued)

$$C_{12} = \frac{m_3}{(m_1 - m_2)^3}\left\{\frac{m_1^3}{2m_2 + 2\alpha - m_1} - \frac{m_1^2 m_2}{m_2 + 2\alpha}\right\}$$

$$C_{13} = \frac{m_3}{(m_1 - m_2)^3}\left\{\frac{-2m_1^2 m_2}{m_2 + 2\alpha} + \frac{2m_1 m_2^2}{(m_1 + 2\alpha)}\right\}$$

$$C_{14} = m_1 C_{04}$$

$$C_{15} = m_2 C_{05}$$

Applying condition 264 of the restrictions gives

$$\alpha_0^2(t) = \int_0^t K_0^2(t, u)\, du = d_{01}\, e^{(m_1+m_2)t} + d_{02}\, e^{2m_1 t} + d_{03}\, e^{2m_2 t} + d_{04}\, e^{+4\alpha t} \tag{335}$$

and

$$\alpha_1^2 = \int_0^t K_1^2(t, u)\, du = d_{11}\, e^{(m_1+m_2)t} + d_{12}\, e^{2m_1 t} + d_{13}\, e^{2m_2 t} + d_{14}\, e^{+4\alpha t} \tag{336}$$

where

$$d_{01} = \frac{4m_3^2}{(m_1 - m_2)^2}\, \frac{2}{m_1 + m_2 - 4\alpha}$$

$$d_{02} = \frac{4m_3^2}{(m_1 - m_2)^2}\, \frac{1}{2(m_1 - 2\alpha)}$$

$$d_{03} = \frac{4m_3^2}{(m_1 - m_2)^2}\, \frac{1}{2(m_2 - 2\alpha)}$$

$$d_{04} = \frac{4m_3^2}{(m_1 - m_2)^2}\, \frac{1}{2(m_1 - 2\alpha)} + \frac{1}{2(m_2 - 2\alpha)} - \frac{2}{(m_1 + m_2 - 4\alpha)} \tag{337}$$

$$d_{11} = m_1 m_2 d_{01}$$

$$d_{12} = m_1^2 d_{02}$$

$$d_{13} = m_2^2 d_{03}$$

$$d_{14} = \frac{-4m_3^2}{(m_1 - m_2)^2} \frac{m_1^2}{2(m_1 - 2\alpha)} + \frac{m_2^2}{2(m_2 - 2\alpha)} - \frac{2m_1 m_2}{m_1 + m_2 - 2\alpha}$$

An examination of the functions $x_0^{(i)}(t)$, $n_i(t)$, and $\alpha_i(t)$ for i = 0 and 1 reveals that these functions belong to an L_2 space (in $0 \leq t \leq \infty$), if the values of m_1, m_2, and α (hence, the system parameters) are properly chosen. These functions are also uniformly continuous and, if they belong to L_2 space in the infinite interval, they will have zero limits at $t = +\infty$.

In order to evaluate the two integrals in Equations 319, one has to make use of Equations 292 and 293 to evaluate A_0 and C_0, as follows:[30]

$$A_0 = \left[\int_0^\infty \alpha_0^2(t)\,dt\right]^{1/2} = \left[\frac{-d_{01}}{m_1 + m_2} - \frac{d_{02}}{2m_1} - \frac{d_{03}}{2m_2} - \frac{d_{04}}{4\alpha}\right]^{1/2} \tag{338}$$

$$C_0 = \left[\int_0^\infty n_0^2(t)\,dt\right]^{1/2} = 4\left[\frac{-C_{01}^2}{4(m_1 + \alpha)} - \frac{C_{02}^2}{4(m_2 + \alpha)} - \frac{C_{03}^2}{2(m_1 + m_2 + 2\alpha)} - \frac{C_{04}^2}{m_1} - \frac{C_{05}^2}{m_2}\right]^{1/2} \tag{339}$$

Further, if the two integrated square error values H_1^2 and H_2^2 given by Equation 319 are specified as $H_1^2 = 12$ and $H_2^2 = 32$, then it can be shown that the following set of parameters satisfies the requirements:

$$J = K_a = K_b = 1$$

$$B = K_c = 2 \tag{340}$$

$$\alpha = 1.5$$

The parameters of Equation 340 correspond to $m_1 = -1$, $m_2 = -2$, $m_3 = 1$, and $\alpha = -1.5$.

In terms of Equation 315, this also corresponds to

$$K_{12} = +1$$

$$K_{22} = -3 \tag{341}$$

$$K_{21} = +2$$

[30] In evaluating these integrals, it was assumed that m_1, m_2, and α are negative numbers. Use was made of the fact that $(a + b + c + d + e)^2 \leq 4(a^2 + b^2 + c^2 + 2d^2 + 2e^2)$.

By using the set of parameters derived in the previous analysis, the Runge-Kutta method, and a digital computer, the system was analyzed numerically. The values of $v_1(t)$ and $v_2(t)$ for different time intervals are shown in Table III and are plotted in Figure 21. The phase plane behavior of the system is shown in Figure 22. Notice that this type of analysis in which the interval of interest is infinite is not suitable for digital computation from a practical point of

TABLE III. SOME NUMERICAL VALUES OF $v_1(t)$, $v_2(t)$ FOR THE SECOND-ORDER NONLINEAR CONTROL SYSTEM

t	0.1	0.2	0.3	0.4	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$v_1(t)$	0.99	0.954	0.908	0.857	0.802	0.538	0.344	0.215	0.134	0.081	0.049	0.027	0.017	0.010
$v_2(t)$	-0.245	-0.401	-0.493	-0.541	-0.556	-0.464	-0.318	-0.205	-0.129	-0.080	-0.049	-0.027	-0.016	-0.010

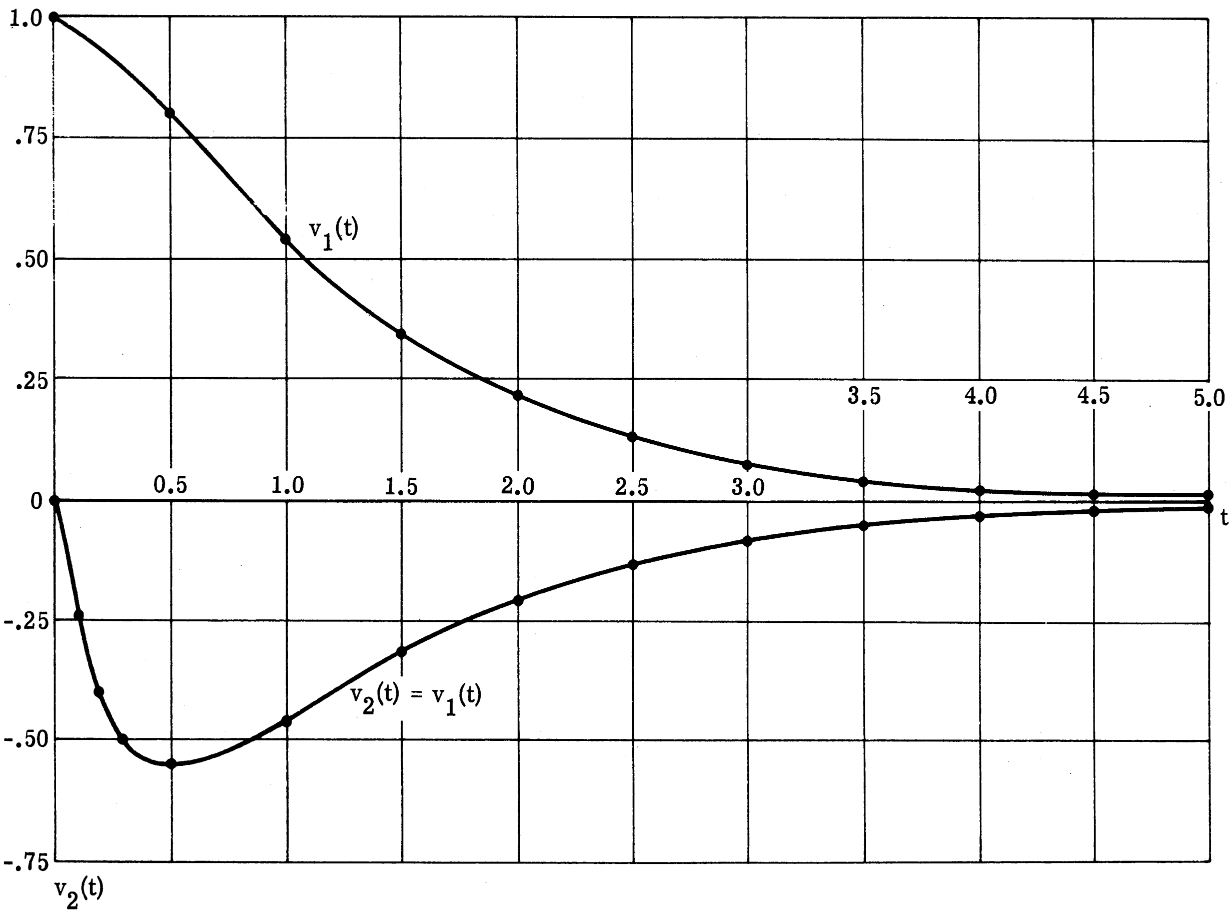

FIGURE 21. THE SYSTEM TRANSIENT RESPONSE

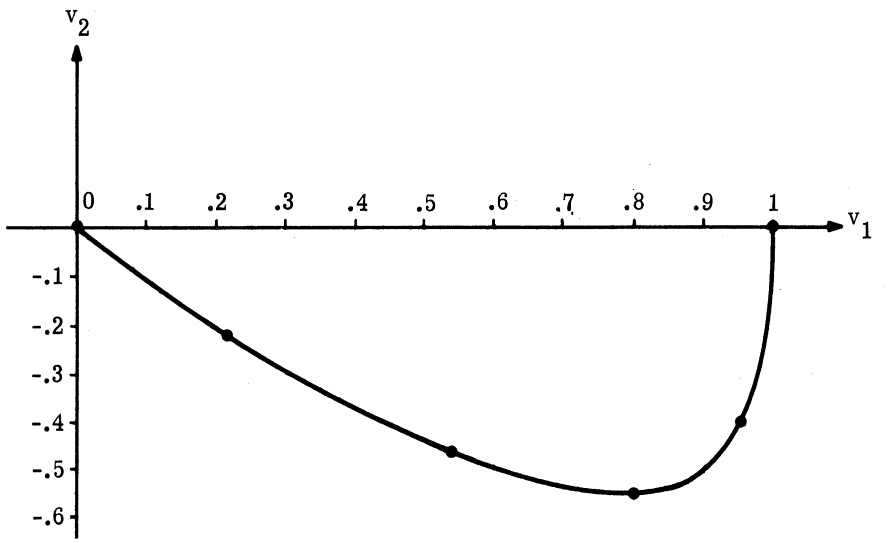

FIGURE 22. PHASE PLANE TRAJECTORY OF THE SECOND-ORDER NON-LINEAR CONTROL SYSTEM

view. However, in the example chosen the values of $v_1(t)$ and $v_2(t)$ happened to decay very quickly with time so that beyond a certain time interval the response functions $v_1(t)$ and $v_2(t)$ were considered approximately equal to zero.

3.6. CONCLUSIONS AND REMARKS

The partitioning technique and the state variable approach have been used to analyze and study a system whose dynamic behavior can be represented by a nonlinear differential equation containing some linear terms, some nonlinear terms, and a forcing function term. Section 2 proved that this system possesses a unique solution which belongs to an L_2 space, and Section 3 proved that under suitable restrictions all the state variables $v_1, v_2, \ldots, v_n$ belong to an L_2 space. By studying the definitions of asymptotic stability in the sense of Lyapunov, the system, as restricted, was found to satisfy the definition. Hence, the restrictions placed on the system are sufficient, not only to prove uniqueness of a solution which belongs to L_2 space, but also to produce asymptotically stable systems in the sense of Lyapunov. Further, by selecting two different definitions of the norm, it was possible to study the behavior of the system trajectory at any time t and on the average during the interval of interest. Studying the average behavior of the system trajectory is not usually desirable because in many applications, although the average value of the system trajectory cannot exceed a certain specified value, the actual system response may have an undesirable shape as shown in Figure 23. Therefore, an expression is obtained for an upper-bound state vector within which the system always remains during operation. This upper-bound state vector is found to depend only on the system restrictions; hence, it can be changed to suit a specific application. An example was given to illustrate the method presented. The author believes that the analysis presented herein should help system engineers to analyze or synthesize systems containing nonlinear characteristics.

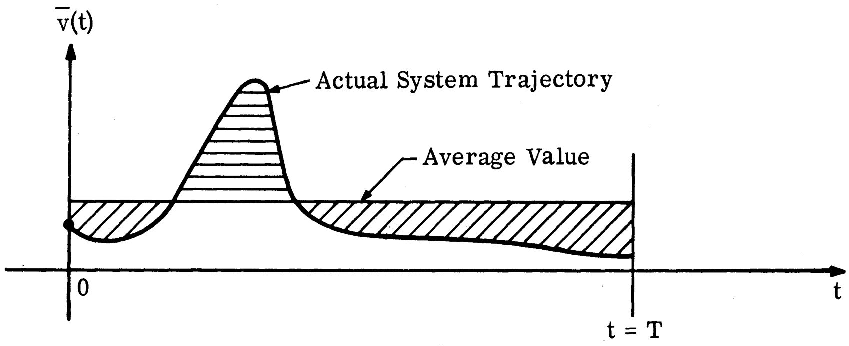

FIGURE 23. ACTUAL AND AVERAGE SYSTEM BEHAVIOR

4
EXTENSION OF THE ANALYSIS TO MULTILOOP SYSTEMS

This section extends the analysis to multiloop nonlinear systems whose dynamic performance can be represented by n simultaneous nonlinear differential equations with time-varying coefficients. The procedures are similar to those used in the analysis presented in Section 3. The new admissible class of systems is defined by placing suitable restrictions on the linear, nonlinear, and forcing function terms. These restrictions are found to be sufficient, but not necessary, to produce unique and stable multiple outputs all of which belong to an L_2 space in the interval of interest. The multiple outputs are given as the limit in the mean and, in the ordinary sense, of a sequence of functions in the L_2 space.

4.1. INTRODUCTION

Thus far we have analyzed a class of systems whose behavior is described by a single equation (96). That equation can be considered the canonical form for nonlinear systems since any such system can be described by an equation of that type. However, there are situations in which:

(1) The system equation will not reduce easily to the canonic form.

(2) A system contains coupling between different subsystems.

(3) A system has more than one input (a multiple-input system). The input may be desired or undesired; for example, noise within the system would be undesired. The noise should be a known function of time.

(4) A system has more than one desirable output (a multiple-output system).

4.2. SYSTEM DESCRIPTION AND DEFINITIONS

It is of interest to control engineers to extend the analysis to include the situations cited. Therefore, it is necessary to consider the general case of n simultaneous nonlinear differential equations of the following form:

$$\begin{aligned} &L_{11}(D, t)x_1(t) + N_{11}[x_1(t)] + L_{12}(D, t)x_2(t) + \ldots + L_{1n}(D, t)x_n(t) + N_{1n}[x(t)] = g_1(t) \\ &L_{21}(D, t)x_1(t) + N_{21}[x_1(t)] + L_{22}(D, t)x_2(t) + \ldots + L_{2n}(D, t)x_n(t) + N_{2n}[x(t)] = g_2(t) \\ &\vdots \\ &L_{n1}(D, t)x_1(t) + N_{n2}[x_1(t)] + L_{n2}(D, t)x_2(t) + \ldots + L_{nn}(D, t)x_n(t) + N_{nn}[x(t)] = g_n(t) \end{aligned} \quad (342)$$

where $L_{ik}(D, t)$ = the linear operators with time-varying coefficients:

$i = 1, 2, \ldots, n$

$k = 1, 2, \ldots, n$

N_{ik} = the nonlinear operators with

$i = 1, 2, \ldots, n$

$k = 1, 2, \ldots, n$

$x_1(t), x_2(t), \ldots, x_n(t)$ = the system output

$g_1(t), g_2(t), \ldots, g_n(t)$ = the system input

Equation 342 can be put in the matrix form:

$$\begin{bmatrix} L_{11}(D, t) & L_{12}(D, t) & \ldots & L_{1n}(D, t) \\ L_{21}(D, t) & L_{22}(D, t) & \ldots & \\ \vdots & & & \\ L_{n1}(D, t) & \cdot & \ldots & L_{nn}(D, t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} N_{11} & N_{21} & \ldots & N_{1n} \\ N_{21} & & & \\ \vdots & & & \\ N_{n1} & & & N_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \quad (343)$$

Further, Equations 343 can be put in the following compact form:

$$[L(D, t)]x] + [N]x] = g] \tag{344}$$

where, as defined in Equations 343, [L(D, t)] is a square matrix of the linear operators, [N] is a square matrix containing the nonlinear operators, and x] and g] are two column matrices for the output variables $\{x_i\}$ and the deterministic input functions g_i; the independent variable is the time t. If there is only one input g(t) and one output x(t), Equation 344 reduces to the special and simpler case analyzed in Sections 2 and 3. Here, however, we are interested in studying the behavior of the complex system — represented by the matrix Equation 344 — when some specified constraints are placed on each of the elements $\{L_{ik}\}$, $\{F_{ik}\}$, and $\{g_i\}$ for $\begin{matrix} i = 1, 2, \ldots, n \\ k = 1, 2, \ldots, n \end{matrix}$ in a systematic manner. The author believes that this will help solve problems of synthesis of complicated systems when deterministic inputs are assumed.

To simplify the analysis, the case of two simultaneous differential equation will be considered. (The generalization of the results to systems of n simultaneous equations should follow easily from these arguments.)

$$\begin{aligned} L_{11}(D, t)x_1(t) + N_{11}(x_1) + L_{12}(D, t)x_2 + N_{12}(x_2) &= g_1(t) \\ L_{21}(D, t)x_1(t) + N_{21}(x_1) + L_{22}(D, t)x_2 + N_{22}(x_2) &= g_2(t) \end{aligned} \tag{345}$$

The following eight definitions use the notations presented in Section 2.1:

(1) $W_{11}(t, u) \triangleq$ the impulse response of the linear term $L_{11}(D, t)$

(2) $W_{22}(t, u) \triangleq$ the impulse response of the linear term $L_{22}(D, t)$

(3) $F_{11}(t, u, x_1) \triangleq W_{11}(t, u)N_{11}(x_1)$, and $N_{11}(x_1)$ is a function of x_1 only

(4) $F_{12}(t, u, x_2) \triangleq W_{11}(t, u)[L_{12}(D, t)x_2 + N_{12}(x_2)]$, and $N_{12}(x_2)$ is a function of x_2 only

(5) $F_{22}(t, u, x_2) \triangleq W_{22}(t, u)N_{22}(x_2)$, and $N_{22}(x_2)$ is a function of x_2 only (346)

(6) $F_{21}(t, u, x_1) \triangleq W_{22}(t, u)[L_{21}(D, t)x_1 + N_{21}(x_1)]$, and $N_{21}(x_1)$ is a function of x_1 only

(7) x_{10} is the solution of the linear equation $L_{11}(D, t)x_{10} = g_1(t)$ having the same initial conditions as $x_1(t)$; that is, $x_{10}(t) = \int_0^t W_{11}(t, u)g_1(u)\,du$ + terms due to initial conditions

(8) x_{20} is the solution of the linear equation $L_{22}(D, t)x_{10} = g_1(t)$ having the same initial conditions as $x_2(t)$; that is, $x_{20}(t) = \int_0^t W_{22}(t, u)g_2(u)du$ + terms due to initial conditions

By applying the partitioning technique to Equations 345 in such a way that only the linear operators L_{11} and L_{22} remain to the left-hand side of these equations, and by then using the above definitions, Equations 345 can be transformed into the following equations in integral form:

$$x_1(t) = x_{10}(t) - \int_0^t F_{11}[t, u, x_1(u)]\,du - \int_0^t F_{12}[t, u, x_2(u)]\,du$$
$$x_2(t) = x_{20}(t) - \int_0^t F_{21}[t, u, x_1(u)]\,du - \int_0^t F_{22}[t, u, x_2(u)]\,du \quad (347)$$

The relation between circuit configurations and feedback system configurations for the single-degree-of-freedom case has been investigated. Similarly, the n simultaneous nonlinear differential equations can be interpreted as representative of an n-terminal pair (n-port) network, and the equivalent integral equations as representative of a multiloop nonlinear system. To illustrate the latter, consider the system which as two inputs, $g_1(t)$ and $g_2(t)$, and two outputs, $x_1(t)$ and $x_2(t)$. The differential equations which govern these functions are given by Equations 345 and can be shown to represent the electric circuit of Figure 24. The circuit shown in Figure 24 has the following equivalences:

(1) $g_1(t)$ and $g_2(t)$ represent input voltages at terminals (1) and (2)

(2) $x_1(t)$ and $x_2(t)$ represent electric current at terminals (1) and (2)

(3) $[L_{11}(D, t)x_1 + N_{11}(x_1)]$ represents the voltage drop on the linear and nonlinear elements of loop (1) due to the flow of current $x_1(t)$

(4) $[L_{22}(D, t)x_2 + N_{22}(x_1)]$ represents the voltage drop on the linear and nonlinear elements of loop (2) due to the flow of current $x_2(t)$

(5) $[L_{12}(D, t)x_2 + N_{12}(x_2)]$ represents, in a general sense, a voltage source in loop (1) that depends only on the current x_2

(6) $L_{21}(D, t)x_1 + N_{21}(x_1)$ represents, in the same way, a voltage source in loop (2) which depends only on the current x_1

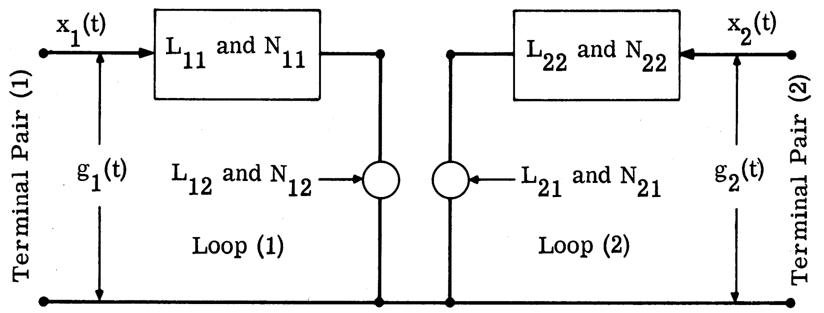

FIGURE 24. A TWO-TERMINAL PAIR, ELECTRIC CIRCUIT REPRESENTATION OF THE DIFFERENTIAL EQUATION 345

The integral Equations 347 can be shown to represent the multiloop nonlinear feedback systems of Figure 25 or the equivalent conjugate system of Figure 26. The difference between Figures 25 and 26 arises from the difference between the elements in the two minor loops. In Figure 26, the linear time-varying elements of the minor loops, described by $W_{11}(t, u)$ and $W_{22}(t, u)$ in Figure 25, are replaced by the inverse of the nonlinear elements N_{11} and N_{22}, and are given the names N_{11}^{-1} and N_{22}^{-1}, respectively. Similarly, the linear operators L_{11} and

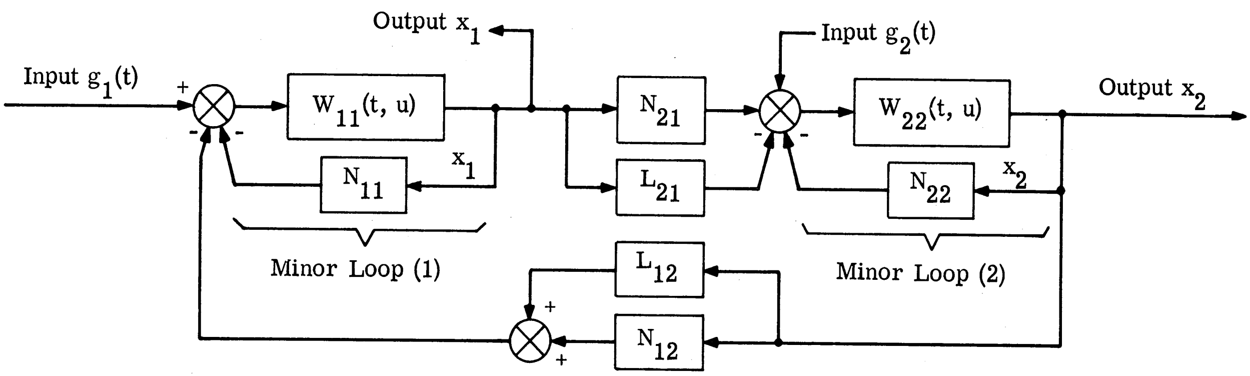

FIGURE 25. A MULTILOOP NONLINEAR FEEDBACK SYSTEM

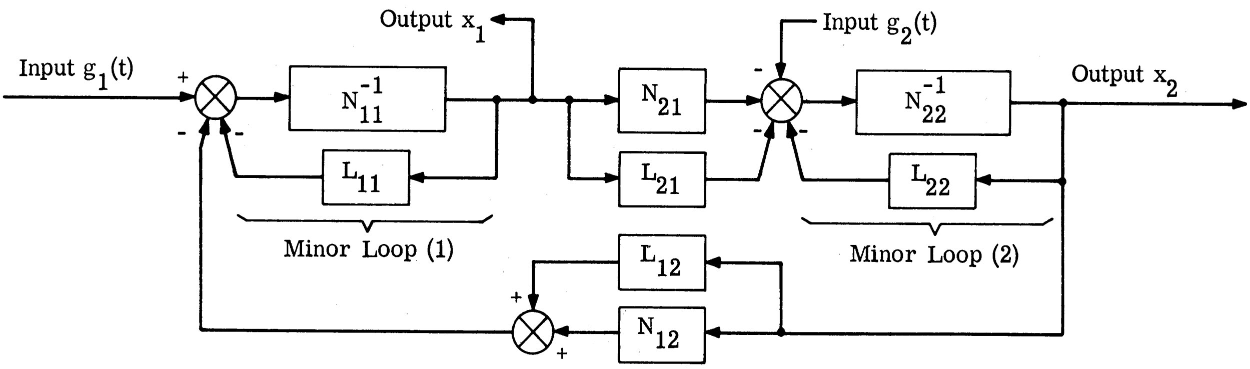

FIGURE 26. THE CONJUGATE MULTILOOP NONLINEAR FEEDBACK SYSTEM

L_{22} in Figure 26 replace the nonlinear elements N_{11} and N_{22} of Figure 25. When the linear elements are time invariant, the following relations are true:

$$W_{11}(t) = L^{-1}\left\{\frac{1}{L_{11}(s)}\right\} = L^{-1}[Y_{11}(s)] = \frac{1}{2\pi J}\int_{\sigma-J\infty}^{\sigma+J\infty} Y_{11}(s)\, e^{+st}\, ds \tag{348}$$

$$W_{22}(t) = L\left\{\frac{1}{L_{22}(s)}\right\} = L^{-1}[Y_{22}(s)] = \frac{1}{2\pi J}\int_{\sigma-J\infty}^{\sigma+J\infty} Y_{22}(s)\, e^{st}\, ds \tag{349}$$

where L^{-1} is the inverse Laplace transform operator and $Y_{11}(s)$ and $Y_{22}(s)$ are the linear system transfer functions.

Equations similar to 348 and 349 can be derived for time-varying linear elements in terms of the system function [48, 49]. The system function of a time-varying system is defined as the function Y(s, t) so that Y(s, t) e^{st} represents the steady-state response[31] of the system to the input g(t) = e^{st}. References 48 and 49 show that Y(s, t) and W(t, u) are related to each other by the relations

$$Y(s, t) = \int_{-\infty}^{\infty} W(t, u)\, e^{-s(t-u)}\, du \tag{350}$$

and

$$W(t, u) = \frac{1}{2\pi J}\int_{\sigma-J\infty}^{\sigma+J\infty} Y(s, t)\, e^{s(t-u)}\, ds \tag{351}$$

The salient characteristic of Y(s, t) is that the response x(t) to any input $g_1(t)$ may be expressed in terms of y(s, t) and g(t) through the relation

$$x(t) = L^{-1}[Y(s, t)G(s)] \tag{352}$$

where G(s) is the Laplace transform of g(t) and, as before, L^{-1} represents the inverse Laplace transformation. In evaluating the inverse Laplace transform of Y(s, t)G(s), t should be treated as if it were a constant parameter.

[31] In Reference 49, L. A. Zadeh demonstrates that Y(Jω, t) is the steady-state solution of the differential equation: L(D + Jω; t) Y(Jω, t) = 1.

4.3. THE MATHEMATICAL RESTRICTIONS AND THE SYSTEM ANALYSIS (THEOREMS F, G, AND H)

In this section, the new admissible class of systems is defined by placing suitable restrictions on the linear, nonlinear, and forcing function terms of the system equations. The physical meanings and the limitations in applications given in Section 2.2 apply here. In particular, the remarks about problems involving the infinite interval of time are assumed to be understood by the reader. To simplify the analysis, the weighted nonlinear functions F_{11} and F_{21} are assumed to be functions of x_1 alone, and F_{22} and F_{12} are assumed to be functions of x_2 alone. However, this is not a serious restriction because the systems analyzed here are the same as those given in Figures 25 and 26. The only exception is that the linear branches L_{21} and L_{12} are omitted. Now, the following three mathematical restrictions are required:

(1) $$x_{i0}(t) \in L_2[0, t] \qquad 0 < T \leq +\infty \tag{353}$$

(2) For a given region D; $[|x_1| \leq d_1(t), |x_2| \leq d_2(t), 0 \leq u \leq t \leq T]$, the weighted nonlinear functions $F_{ik}(t, u, x_k)$ are assumed to satisfy these conditions:

(a) If the two triplets (t, u, z_1) and (t, u, z_2) are in D, then

$$|F_{ik}(t, u, z_1) - F_{ik}(t, u, z_2)| \leq K_{ik}(t, u)|z_1 - z_2| \tag{354}$$

(b) $$\int_0^t F_{ik}[t, u, z_{k0}(u)]\, du \leq n_{ik}(t) \tag{355}$$

where $K_{ik} \in L_2(0, T)$, that is,

$$\int_0^t K_{ik}^2(t, u)\, du \leq \alpha_{ik}^2(t) \leq \alpha^2(t)$$

and (356)

$$\int_0^T \alpha_{ik}^2(t)\, dt \leq \int_0^T \alpha^2(t)\, d(t) \leq A^2 < \infty$$

$n_{ik} \leq n(t) \in L_2[0, T]$, that is

$$\int_0^T n_{ik}^2(t)\, dt \leq C_{ik}^2 \leq \int_0^T n^2(t)\, dt \leq C^2 < \infty \tag{357}$$

and i = 1 or 2

(c) It is required that:

$$|x_{i0}(t)| \leq d_i(t)$$

(358)

$$|q_i(t)| \leq d_i(t)$$

where i = 1 or 2

and where

$$x_{i1}(t) = x_{i0}(t) - \int_0^t F_{i1}[t, u, x_{10}(u)]\,du - \int_0^t F_{i2}[t, u, x_{20}(u)]\,du$$

$$q_i(t) \triangleq |x_{i1}(t)| + C\,|\alpha(t)| \sum_{m=0}^{\infty} \frac{2^{(m+2)/2} A^m}{\sqrt{m!}} \tag{359}$$

(3) The L_2 functions $x_{i0}(t)$, $n_{ik}(t)$, and $\alpha(t)$; the time-varying parameters appearing in the system's equations; and the enforced bounds $d_i(t)$ are continuous and bounded over [0, T]. (360)

The method of successive approximations will now be used to prove theorems which will describe some of the properties of the system under consideration. The following inequality will be used in the proof of the theorems:

$$(x + y)^2 \leq 2(x^2 + y^2) \tag{361}$$

Theorem F

Let a set of functions $\{x_{10}, x_{11}, \ldots, x_{1n}\}$ and $\{x_{20}, x_{21}, \ldots, x_{2n}\}$ be defined by the following recurrence relations:

$$x_{1n} \triangleq x_{10} - \int_0^t F_{11}[t, u, x_1(u)_{n-1}]\,du - \int_0^t F_{12}[t, u, x_2(u)_{n-1}]\,du$$

$$x_{2n} \triangleq x_{20} - \int_0^t F_{21}[t, u, x_1(u)_{n-1}]\,du - \int_0^t F_{22}[t, u, x_2(u)_{n-1}]\,du \tag{362a}$$

where n = 0, 1, 2, ...

Then, as specified by the restrictions placed on the system (given by Equations 353 through 360), this set of functions belongs to L_2 space and is inside the domain D.

Theorem G

Under the restrictions specified by Equations 353 through 355, both sequences of iterations $\{x_{1n}\}$ and $\{x_2 n\}$ (defined by Equations 356) converge in the mean to the functions $\varphi_1(t)$ and $\varphi_2(t)$, respectively. They also converge in the ordinary sense to the functions $\psi_1(t)$ and $\psi_2(t)$, respectively. Hence,

$$\psi_1(t) = \varphi_1(t) \triangleq x_1(t) \qquad \text{and} \qquad \psi_2(t) = \varphi_2(t) \triangleq x_2(t) \tag{362b}$$

Theorem H

Under the restrictions specified by Equations 353 through 355, the functions $x_1(t)$ and $x_2(t)$ (defined by Equation 357) are the solutions of the system of simultaneous differential Equations 345 and are the unique solutions for a given initial state.

Proof of Theorem F

Let n = 1 in Equation 356. This gives

$$x_{11}(t) = x_{10}(t) - \left\{\int_0^t F_{11}[t, u, x_{10}(u)]\,du + \int_0^t F_{12}[t, u, x_{20}(u)]\,du\right\}$$

$$x_{21}(t) = x_{21}(t) - \left\{\int_0^t F_{21}[t, u, x_{20}(u)]\,du + \int_0^t F_{22}[t, u, x_{20}(u)]\,du\right\} \qquad (362c)$$

The conditions 358 of the restrictions show that the elements $x_{10}(t)$, $x_{11}(t)$, $x_{20}(t)$, and $x_{21}(t)$ are inside D. However, it must be shown by induction that the two sequences $\{x_{1n}\}$ and $\{x_{2n}\}$ are also inside D. Assume that the two elements $x_{1n}(t)$ and $x_{2n}(t)$ are in D; then $x_{1n+1}(t)$ and $x_{2n+1}(t)$ are defined and are given by

$$x_{1n+1}(t) = x_{10}(t) - \int_0^t F_{11}[t, u, x_{1n}(u)]\,du - \int_0^t F_{12}[t, u, x_{2n}(u)]\,du$$

$$x_{2n+1}(t) = x_{20}(t) - \int_0^t F_{21}[t, u, x_{1n}(u)]\,du - \int_0^t F_{22}[t, u, x_{2n}(u)]\,du \qquad (363)$$

Finally, it must be shown that $x_{1n+1}(t)$ and $x_{2n+1}(t)$ are in D. Equations 362c give

$$[x_{11}(t) - x_{10}(t)]^2 = \left\{\int_0^t F_{11}[t, u, x_{10}(u)]\,du + \int_0^t F_{12}[t, u, x_{20}(u)]\,du\right\}^2$$

$$[x_{21}(t) - x_{21}(t)]^2 = \left\{\int_0^t F_{21}[t, u, x_{10}(u)]\,du + \int_0^t F_{22}[t, u, x_{20}(u)]\,du\right\}^2 \qquad (364)$$

Making use of the inequalities 361 and the system restrictions 356 and 357 gives

$$[x_{11}(t) - x_{10}(t)]^2 \le 4n^2(t)$$

$$[x_{21}(t) - x_{20}(t)]^2 \le 4n^2(t) \qquad (365)$$

Equations 365 give

$$\int_0^T [x_{11}(t) - x_{10}(t)]^2 dt \triangleq ||x_{11}(t) - x_{10}(t)||^2 \le 4C^2$$
$$\int_0^T [x_{21}(t) - x_{21}(t)]^2 dt \triangleq ||x_{21}(t) - x_{20}(t)||^2 \le 4C^2 \tag{366}$$

Subtracting $x_{1n}(t)$ from $x_{1n+1}(t)$, as given in Equations 362a and 363, respectively, and then squaring, gives

$$\begin{aligned}[x_{1n+1}(t) - x_{1n}(t)]^2 &= \left(\int_0^t \left\{F_{11}[t, u, x_{1n}(u)] - F_{11}[t, u, x_{1n-1}(u)]\right\} du \right. \\ &\quad \left. + \int_0^t \left\{F_{12}[t, u, x_{2n}(u)] - F_{12}[t, u, x_{2n-1}(u)] du\right\}\right)^2 \\ &\le 2\left(\int_0^t \left\{F_{11}[t, u, x_{1n}(u)] - F_{11}[t, u, x_{1n-1}(u)]\right\} du\right)^2 \\ &\quad + 2\left(\int_0^t \left\{F_{12}[t, u, x_{2n}(u)] - F_{12}[t, u, x_{2n-1}(u)]\right\} du\right)^2 \end{aligned} \tag{367}$$

Making use of Equations 355 and 356 of the restrictions, Equations 367 give

$$[x_{1n+1}(t) - x_{1n}(t)]^2 \le 2\alpha^2(t)\int_0^t [x_{1n}(u) - x_{1n-1}(u)]^2 du + 2\alpha^2(t)\int_0^t [x_{2n}(u) - x_{2n-1}(u)]^2 du \tag{368a}$$

By a similar argument, it can be shown that

$$[x_{2n+1}(t) - x_{2n}(t)]^2 \le 2\alpha^2(t)\int_0^t [x_{2n}(u) - x_{2n-1}(u)]^2 du + 2\alpha^2(t)\int_0^t [x_{2n}(u) - x_{2n-1}(u)]^2 du \tag{368b}$$

Letting n = 1 in the two equations above and using Equation 366 gives

$$[x_{12}(t) - x_{11}(t)]^2 \le (2^2)^2 C^2\alpha^2(t)$$
$$[x_{22}(t) - x_{21}(t)]^2 \le (2^2)^2 C^2\alpha^2(t) \tag{369}$$

Again, by following the same argument, it can be shown that

$$[x_{13}(t) - x_{11}(t)]^2 \le (2^2)^3 C^2 \alpha^2(t) \int_0^t \alpha^2(u)\,du \le \frac{(2^2)^3 C^2 \alpha^2 A^2}{1!}$$

$$[x_{23}(t) - x_{22}(t)]^2 \le (2^2)^3 C^2 \alpha^2(t) \int_0^t \alpha^2(u)\,du \le \frac{(2^2)^3 C^2 \alpha^2 A^2}{1!} \quad (370)$$

Also

$$[x_{14}(t) - x_{13}(t)]^2 \le \frac{(2^2)^4 C^2 \alpha^2(t) A^2}{2!}$$

$$[x_{24}(t) - x_{23}(t)]^2 \le \frac{(2^2)^4 C^2 \alpha^2(t) A^2}{2!} \quad (371)$$

In general, it follows that

$$[x_{1n+2}(t) - x_{1n+1}(t)]^2 \le \frac{2^{2(n+2)} A^{2n} C^2 \alpha^2(t)}{n!}$$

$$[x_{2n+2}(t) - x_{2n+1}(t)]^2 \le \frac{2^{2(n+2)} A^{2n} C^2 \alpha^2(t)}{n!} \quad (372)$$

where $n = 0, 1, 2, \ldots$

Now

$$|x_{1n+1}(t)| \le |x_1(t)| + |x_{12}(t) - x_{11}(t)| + \ldots + |x_{1n+1}(t) - x_{1n}(t)|$$

and (373)

$$|x_{2n+1}(t)| \le |x_{21}(t)| + |x_{22}(t) - x_{21}(t)| + \ldots + |x_{2n+1}(t) - x_{2n}(t)|$$

Substituting from Equation 372 into Equation 373 gives

$$|x_{1n+1}(t)| \le |x_{11}(t)| + C\alpha(t) \sum_{m=0}^{\infty} \frac{2^{(m+2)} A^m}{\sqrt{m!}} \le d_1(t)$$

$$|x_{2n+1}(t)| \le |x_{21}(t)| + C\alpha(t) \sum_{m=0}^{\infty} \frac{2^{(m+2)} A^m}{\sqrt{m!}} \le d_2(t) \quad (374)$$

Thus, $x_{1n+1}(t)$ and $x_{2n+1}(t)$ are in D, and it follows by induction that the two sequences $\{x_{1n}(t)\}$ and $\{x_{2n}(t)\}$ are in D for all n. Since $x_{10}(t)$ and $x_{20}(t)$ are assumed to be $L_2[0, T]$, Equations 364 show that $x_{11}(t)$ and $x_{21}(t)$ are $L_2[0, T]$. In general, Equation 372 shows that $\{x_{1n}(t)\}$ and $\{x_2n(t)\}$ are $L_2[0, T]$; therefore, the proof of the theorem is complete.

Proof of Theorem G

By the reasoning used to prove Theorem B of Section 2, $\{x_{1n}\}$ and $\{x_{2n}\}$ can be proved to be two sets of Cauchy sequences or fundamental sequences[32]. Therefore, from the properties of the L_2 space, it can be concluded that two functions $\varphi_1(t)$ and $\varphi_2(t)$ exist in the same L_2 space in which $\{x_{1n}\}$ and $\{x_{2n}\}$ converge in the mean. In other words,

$$\lim_{n \to \infty} \int_0^T [x_{1n}(t) - \varphi_1(t)]^2 \, dt = 0$$

$$\lim_{n \to \infty} \int_0^T [x_{2n}(t) - \varphi_2(t)]^2 \, dt = 0 \tag{375}$$

Equations 375 show that for every $\varepsilon > 0$, there exists a natural number N such that

$$||x_{1n}(t) - \varphi_1(t)|| \leq \varepsilon$$

$$||x_{2n}(t) - \varphi_2(t)|| \leq \varepsilon \tag{376}$$

for all $n > N$. Now, to prove convergence in the ordinary sense, one has to show that the infinite series

$$x_{11}(t) + [x_{12}(t) - x_{11}(t)] + [x_{13}(t) - x_{12}(t)] + \ldots$$

$$x_{21}(t) + [x_{22}(t) - x_{21}(t)] + [x_{23}(t) - x_{22}(t)] + \ldots \tag{377}$$

are uniformly convergent. Equations 372 show that

$$[x_{1n+1}(t) - x_{1n}(t)] \leq \frac{C\alpha(t)\, 2^{n+1} A^{n-1}}{(n-1)!}$$

$$[x_{2n+1}(t) - x_{2n}(t)] \leq \frac{C\alpha(t)\, 2^{n+1} A^{n-1}}{(n-1)!} \tag{378}$$

[32] Definition (from Reference 30, p. 169): "A sequence $\{f_n\}$ of points of the space L_2 is said to be a Cauchy sequence or a fundamental sequence, if to every $\varepsilon > 0$ there corresponds a natural number N such that $||f_n - f_m|| < \varepsilon$ for all $m, n \geq N$."

where $\alpha(t)$, C, and A are finite. One can easily conclude the uniform convergence by using Equations 378 and applying the M test on series 377. From evidence that the n-th partial sum of the infinite series 377 is x_{1n} and x_{2n}, it can be concluded that the two sequences $\{x_{1n}(t)\}$ and $\{x_{2n}(t)\}$ are convergent.

Let

$$\lim_{n\to\infty} x_{1n}(t) = \psi_1(t) \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n}(t) = \psi_2(t) \tag{379}$$

From the use of H. Weyl's lemma (Appendix B), it can be concluded that

$$\begin{aligned} \psi_1(t) &= \varphi_1(t) \\ \psi_2(t) &= \varphi_2(t) \end{aligned} \tag{380}$$

Proof of Theorem H

One can easily form the following equations to prove that the limiting functions $\varphi_1(t)$ and $\varphi_2(t)$ are the solutions of the two simultaneous differential Equations 345:

$$\begin{aligned} [x_{1n}(t) - \varphi_1(t)]^2 &= \left\{\varphi_1(t) - x_{10} + \int_0^t F_{11}[t, u, x_{1n-1}(u)]\,du + \int_0^t F_{12}[t, u, x_{2n-1}(u)]\,du\right\}^2 \\ [x_{2n}(t) - \varphi_2(t)]^2 &= \left\{\varphi_2(t) - x_{20} + \int_0^t F_{21}[t, u, x_{1n-1}(u)]\,du + \int_0^t F_{22}[t, u, x_{2n-1}(u)]\,du\right\}^2 \end{aligned} \tag{381}$$

Then, one can obtain the following equations by integrating both sides of Equation 381 between the limits 0 and T, taking the limits as $n \to \infty$, and making use of Equations 375:

$$\begin{aligned} \varphi_1(t) - x_{10} + \lim_{n\to\infty}\int_0^t F_{11}[t, u, x_{1n-1}(u)]\,du + \lim_{n\to\infty}\int_0^t F_{12}[t, u, x_{2n-1}(u)]\,du = 0 \\ \varphi_2(t) - x_{20} + \lim_{n\to\infty}\int_0^t F_{21}[t, u, x_{1n-1}(u)]\,du + \lim_{n\to\infty}\int_0^t F_{22}[t, u, x_{2n-1}(u)]\,du = 0 \end{aligned} \tag{382}$$

Therefore, $\varphi_1(t)$ and $\varphi_2(t)$ satisfy the integral Equations 347 and are the solutions of the differential equations.

Uniqueness of the solution is assured[33] since the sequences $\{x_{1n}\}$ and $\{x_{2n}\} \in L_2$; however, the proof can be easily demonstrated by considering the system's equations themselves.

[33] See, for example, Reference 30, page 169, theorem (2). If $x_{1n} \to \varphi_1$ and $x_{1n} \to \varphi_1^*$, then $||\varphi_1 - \varphi_1^*|| \le ||\varphi_1 - x_{1n}|| + ||x_{1n} - \varphi_1^*||$. Since the right-hand side has a zero limit, it follows that $||\varphi_1 - \varphi_1^*|| = 0$ and $\varphi_1 = \varphi_1^*$. The same is true for x_{2n}.

Assume two other solutions φ_1^* and φ_2^* and form the following equation:

$$(\varphi_1 - \varphi_1^*)^2 = \left\{\int_0^t \left[F_{11}(t, u, \varphi_1) - F_{11}(t, u, \varphi_1^*)\right] du - \int_0^t \left[F_{11}(t, u, \varphi_2) - F_{12}(t, u, \varphi_2^*)\right] du\right\}^2$$

$$(\varphi_2 - \varphi_2^*)^2 = \left\{\int_0^t \left[F_{21}(t, u, \varphi_1) - F_{21}(t, u, \varphi_1^*)\right] du - \int_0^t \left[F_{22}(t, u, \varphi_2) - F_{22}(t, u, \varphi_2^*)\right] du\right\}^2 \quad (383)$$

Then apply the Lipschitz condition; this gives

$$(\varphi_1^* - \varphi_1)^2 \le 2\alpha^2(t)\int_0^t \left[(\varphi_1 - \varphi_1^*)^2 + (\varphi_2 - \varphi_2^*)^2\right] du$$

$$(\varphi_2^* - \varphi_2)^2 \le 2\alpha^2(t)\int_0^t \left[(\varphi_1 - \varphi_1^*)^2 + (\varphi_2 - \varphi_2^*)^2\right] du \quad (384)$$

Adding Equations 384 together gives

$$\left[(\varphi_1^* - \varphi_1)^2 + (\varphi_2^* - \varphi_2)\right] \le 4\alpha^2(t)\int_0^t \left[\varphi_1^* - \varphi_1)^2 + (\varphi_2^* - \varphi_2)^2\right] du \quad (385)$$

Letting $\int_0^T \left[(\varphi_1^* - \varphi_1)^2 + (\varphi_2^* - \varphi_2)\right]^2 du = s^2$ gives

$$\left[(\varphi_1^* - \varphi_1)^2 + (\varphi_2^* - \varphi_2)^2\right] \le 4\alpha^2(t)\, s^2 \quad (386)$$

Substituting Equation 386 into Equation 385 and repeating the procedures n times gives

$$\int_0^T \left[(\varphi_1^* - \varphi_1)^2 + (\varphi_2^* - \varphi_2)^2\right] du \le \frac{4^n s^n A^{2n+2}}{n!} \quad (387)$$

and as $n \to \infty$ gives

$$\int_0^T \left[(\varphi_1^* - \varphi_1)^2 + (\varphi_2^* - \varphi_2)^2\right] du = 0 \quad (388)$$

therefore, $\varphi_1 = \varphi_1^*$ and $\varphi_2 = \varphi_2^*$; hence the results.

4.4. PHASE SPACE REPRESENTATION AND STABILITY ANALYSIS (THEOREM I)

Consider the system of simultaneous, nonlinear differential equations given in Equations 345. Assume that the order of the highest derivative in $L_{11}(D, t)$ is n, and that of $L_{22}(D, t)$ is m. Let

$$\begin{array}{ll} x_1 = v_1 & x_2 = v_{n+1} \\ \dfrac{dx_1}{dt} = v_2 & \dfrac{dx_2}{dt} = v_{n+2} \\ \vdots & \vdots \\ \dfrac{dx_1^{n-1}}{dt^{n-1}} = v_n & \dfrac{dx_2^{m-1}}{dt^{m-1}} = v_{n+m} \end{array} \tag{389}$$

The system of Equations 345 can be transformed to a set of (n + m) first-order equations in the following form:

$$\begin{aligned} \frac{dv_1}{dt} &= f_1(v_1, v_2, \ldots, v_n, \ldots, v_{n+m}, t) \\ \frac{dv_2}{dt} &= f_2(v_1, v_2, \ldots, v_n, \ldots, v_{n+m}, t) \\ &\vdots \\ \frac{dv_{n+m}}{dt} &= f_{n+m}(v_1, v_2, \ldots, v_n, \ldots, v_{n+m}(t)) \end{aligned} \tag{390}$$

where $\{v_i\}$, i = 1, 2, . . . , n + m, are the state variables which represent the state of the system in the phase space. Each member of the first n components represents the rate of change of the member preceding it. This is also true for the remaining m components.

The transformation of the system of Equations 345 to a set of first-order differential equations, as given in Equation 390, can be illustrated by an example. Consider the following two simultaneous equations:

$$\begin{aligned} &\frac{d^2x_1}{dt^2} + a_{11}(t)\frac{dx_1}{dt} + a_{12}x_1 + x_1^2 + \frac{dx_2}{dt} + \left(x_2^3\right) = \sin t \\ &\frac{dx_1}{dt} + a_{21}(t)x_1 + e^{x_1} + \frac{d^2x_2}{dt^2} + a_{22}(t)\frac{dx_2}{dt} + a_{23}(t)x_2 = \cos t \end{aligned} \tag{391}$$

Let n = 2 and m = 2. Then let

$$x_1 = v_1 \qquad x_2 = v_3$$

$$\frac{dx_1}{dt} = v_2 \qquad \frac{dx_2}{dt} = v_4$$

Now, Equation 391 can be put into the following form:

$$\begin{aligned}
\frac{dv_1}{dt} &= v_2 && = f_1(v_1, v_2, \ldots, v_4)\\
\frac{dv_2}{dt} &= \sin t - v_3^{\,2} - v_4 - v_1^{\,2} - a_{12}(t)v_1 - a_{11}(t)v_2 && = f_2(v_1, \ldots, v_4)\\
\frac{dv_3}{dt} &= v_4 && = f_3(v_1, \ldots, v_4)\\
\frac{dv_4}{dt} &= \cos t - a_{33}v_3 - a_{22}(t)v_4 - a_{21}(t)v_1 - v_2 - e^{v_1} && = f_4(v_1, \ldots, v_4)
\end{aligned} \tag{392a}$$

The set of Equations 392a can also be represented in the following matrix form:

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_{12(t)} & -a_{12}(t) & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -a_{21}(t) & -1 & -a_{23}(t) & -a_{22}(t) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \begin{bmatrix} 0 \\ N_2 \\ 0 \\ N_1 \end{bmatrix} \tag{392b}$$

where $N_2 = \sin t - \dot{v}_3^{\,2} - v_1^{\,2}$

$N_1 = \cos t - e^{v_1}$

In order to investigate stability, as we did for the single-degree-of-freedom case presented in Section 3, we confine ourselves to the free system, represented by Equation 345 in which $g_1(t) = g_2(t) = 0$. We further assume that the origin of the phase space for the free system is an equilibrium position of uncertain stability. In other words, it is assumed that the function f_i (Equation 390) for i = 1, 2, . . . , n + m is identically zero at $\overline{v} = \overline{0}$ and for all $t \geq t_0 = 0$. In order to study the properties of the state variables $v_i(t)$ of the free system, the system must satisfy a set of conditions which are not necessary, but sufficient and suitable for the present analysis.

The required system restrictions are (some proofs similar to those presented in Section 3 are omitted to avoid repetition):

(1) $x_{i0}^{(\mu)} \in L_2[0, T]$ and are assumed bounded over the closed interval [0, T]

$$0 < t \le +\infty \tag{393}$$

where (μ) = the μ-th derivative of $x_{i0}(t)$ with respect to time t, and

$$\mu = \begin{cases} 0, 1, \ldots, n-1 \text{ for } i = 1 \\ 0, 1, \ldots, m-1 \text{ for } i = 2 \end{cases}$$

(2) In a given domain $D \equiv \left[|x_1| \le d_1(t),\ |x_2(t)| \le d_2(t),\ 0 \le \mu \le t \le T \right]$, the μ-th derivative with respect to t of the weighted nonlinear functions, defined as $F_{ik}^{(\mu)}[t, u, Z(u)]$, is required to satisfy the following conditions:

(a) If the two triplets (t, u, z_1) and (t, u, z_2) are in D, then

$$|F_{ik}^{(\mu)}(t, u, z_1) - F_{ik}^{(\mu)}(t, u, z_2)| \le K_{ik_\mu}(t, u)|z_1 - z_2| \tag{394}$$

(b)

$$\int_0^t F_{ik}^{(\mu)}[t, u, x_{k0}(t)]\,du \le n_{ik}(t) \tag{395}$$

where $K_{ik_\mu}(t, u) \in L_2(0, T)$, that is,

$$\int_0^t K_{ik_\mu}^{\ 2}(t, u)\,du \le \alpha_{ik_\mu}^{\ 2}(t) \le \alpha_r^{\ 2}(t) \in L_2[0, T) \tag{396}$$

$$n_{ik}(t) \in L_2[0, T]$$

The functions α_{ik_μ}, α_r, and n_{ik} are continuous and bounded over [0, T] .

i = 1 or 2, k = 1 or 2, and μ is as defined above.

(3) Condition 358 is still required; that is,

$$|x_{i0}(t)| \le d_i(t)$$
$$|q_i(t)| \le d_i(t) \tag{397}$$

where i = 1 or 2

(4) The time-varying parameters and the nonlinear functions $|N_{ik}(z)|$ are assumed bounded over [0, T]. (398)

The physical meaning of the above restrictions is understood to be the same as in the previous analysis. Now, by differentiating Equations 347 μ times and by using the above restrictions and Equations 372, one can proceed, in a manner similar to that used to prove Theorem E, to find the following equations:

$$\left(x_{1n+1}^{(\mu)}(t) - x_{1n}^{(\mu)}(t)\right)^2 \leq 4\alpha_r^2(t)\left(\frac{2^2(n)\ C^2 A^{2(n-1)}}{(n - 2)!}\right)$$
$$\left(x_{2n+1}^{(\mu)}(t) - x_{1n}^{(\mu)}\right) \leq 4\alpha_r^2(t)\left(\frac{2^2(n)\ C^2 A^{2(n-1)}}{(n - 2)!}\right) \tag{399}$$

where α_r^2 is as defined by Equation 395 and A^2 and C^2 are as defined by Equations 356 and 357, respectively. Equation 398 can be used to show that, for the class of systems defined by the above restrictions, the properties of the state variables can be summarized as follows.

Theorem I

Under the restrictions stated by Equations 393 through 398 for the system of two simultaneous, nonlinear differential equations given in Equations 345, the following properties of the state variables $v_i(t)$, $i = 1, 2, \ldots, n + m$, are true:

(1) $v_i(t) \in L_2[0, T]$

(2) $v_i(t)$ are bounded by corresponding $L_2[0, T]$ functions $\beta_i(t)$, as given by

$$\beta_i(t) = \begin{cases} |x_{10}^{(i-1)}(t)| + |n_{11_{i-1}}(t)| + |n_{12_{i-1}}(t)| + m|\alpha_r(t)| & \text{for } i = 1, 2, \ldots, n \\ |x_{20}^{(i-n-1)}(t)| + |n_{21_{i-n-1}}(t)| + |n_{22_{i-n-1}}(t)| + m|\alpha_r(t)| & \text{for } i = n+1, \ldots, n + m \end{cases} \tag{400}$$

where

$$m = 4C\left(1 + \sum_{n=2}^{\infty} \frac{(2A)^{n-1}}{\sqrt{(n - 2)!}}\right) \tag{401}$$

and C, A, and $\alpha_r(t)$ are as defined before.

(3) $v_i(t)$ are uniformly continuous over $[0, T]$

(4) If $T \to +\infty$, then it follows that

$$\lim_{t\to\infty} v_i(t) = 0 \tag{402}$$

From Theorem I it can be concluded that at any time t for which $0 \leq t \leq T$, the following inequality is true:

$$v_i(t) \leq \beta_i(t) \qquad i = 1, 2, \ldots, n + m \tag{403}$$

where $\beta_i(t)$ are as defined in Equations 400. But $v_i(t)$ are the limit of sequences $\{v_{in}(t)\}$ where $i = 1, \ldots, n + m$ and $n = 1, 2, \ldots, \infty$; which belong to $L_2[0, T]$; therefore, we deduce that the L_2 norms $||v_i(t)||$ are bounded. This can be concluded from the following corollary:

Corollary: The norms of the elements of a convergent sequence in L_2 are bounded [30].

However, for the class of system under consideration, the upper-bound functions $\beta_i(t)$ can readily give the required upper-bound norms of the state variables so that:

$$||v_i(t)|| \leq ||\beta_i(t)|| \qquad i = 1, 2, \ldots, n + m \tag{404}$$

However, it should be noted that where $\bar{\beta}(t) = \{\beta_i(t)\}$, the upper-bound norms $||\bar{\beta}(t)||$ are functions of the initial state $\bar{v}_0 = \{v_{i0}\}$ at $t = 0$ and $i = 1, 2, \ldots, n + m$, so that an exact definition of boundedness can be stated as follows.

Definition [31, p. 388]:

A motion is said to be bounded for every $(\bar{v}_0, t_0)$ if there is some constant $\beta(\bar{v}_0, t_0)$ so that $||\bar{v}(t; \bar{v}_0, t_0)|| \leq \beta$ for all $t \geq t_0$. The motion is equibounded if $\beta(\bar{v}_0, t_0) \leq \beta(r, t_0)$ for all $||\bar{v}_0|| \leq r$.

Now, by using the same two definitions of the norms that were used in Section 3:

$$\sum_{i=1}^{n+m} v_i^2$$

and (405)

$$\int_0^{\infty} \left(\sum v_i^2(t)\, dt\right)$$

it can be concluded, just as it was for the single degree-of-freedom case, that the class of nonlinear systems considered here is asymptotically stable.

4.5. EXAMPLE CALCULATIONS

As an illustration of the method presented in this section, consider the multiloop system shown in Figure 27. In this system there is only one input $g_1(t)$ and two outputs $x_1(t)$ and $x_2(t)$. The two minor loops are coupled together by two nonlinear elements, each of which has the form $\frac{1}{2}x^2$. The boxes marked "b" (b > 0) represent variable gain amplifiers with amplifier

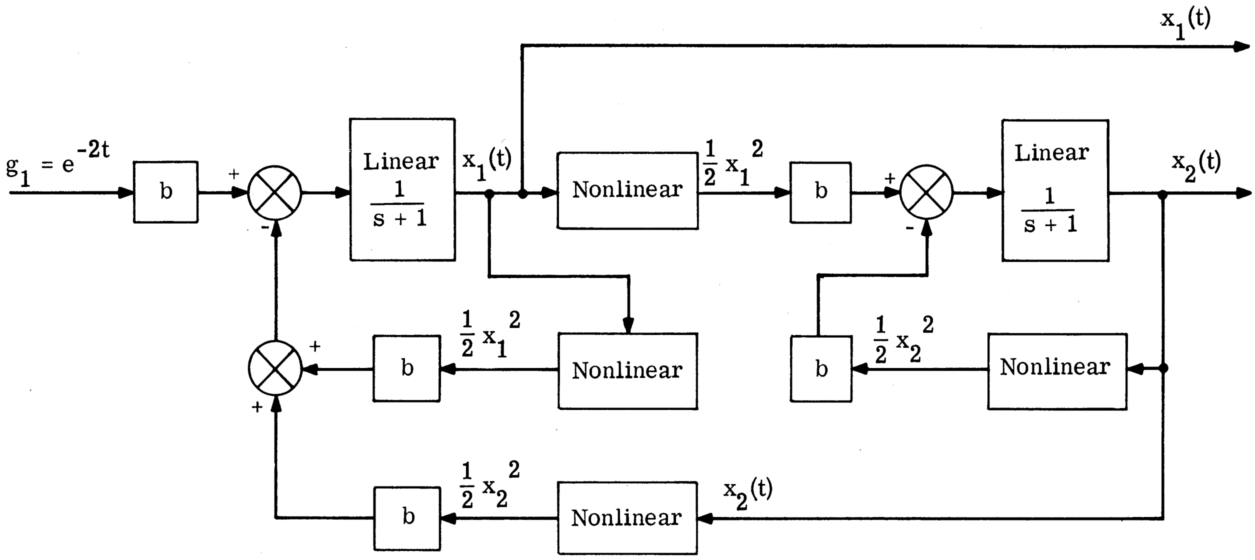

FIGURE 27. A MULTILOOP NONLINEAR FEEDBACK SYSTEM WITH FOUR NONLINEAR ELEMENTS, TWO LINEAR ELEMENTS, AND FIVE AMPLIFIER GAINS

gains b. The problem is to select the value of the amplifier gains b, so that the outputs $x_1(t)$ and $x_2(t)$ of the multiloop system do not exceed a certain specified limit during an interval of operation T. The system shown in Figure 28 is equivalent to that shown in Figure 27, consequently, the analysis is applicable to either system. In addition, one can easily show that Figures 27 and 28 are also equivalent to the nonlinear two-terminal pair network of Figure 29.

Analysis

The system of Figures 27 and 29 can be represented by the following two simultaneous, nonlinear differential equations:

$$\left(\frac{dx_1}{dt} + x_1\right) + \frac{b}{2}\left(x_1^2 + x_2^2\right) = be^{-2t}$$
$$\left(\frac{dx_2}{dt} + x_2\right) + \frac{b}{2}\left(x_1^2 - x_2^2\right) = 0 \qquad (406)$$

Equations 406 can be partitioned at the linear terms x_1 and x_2 as shown below:

$$\frac{dx_1}{dt} + x_1 = be^{-2t} - \frac{b}{2}x_1^2 - \frac{b}{2}x_2^2$$
$$\frac{dx_2}{dt} + x_2 = 0 + \frac{b}{2}x_1^2 - \frac{b}{2}x_2^2 \qquad (407)$$

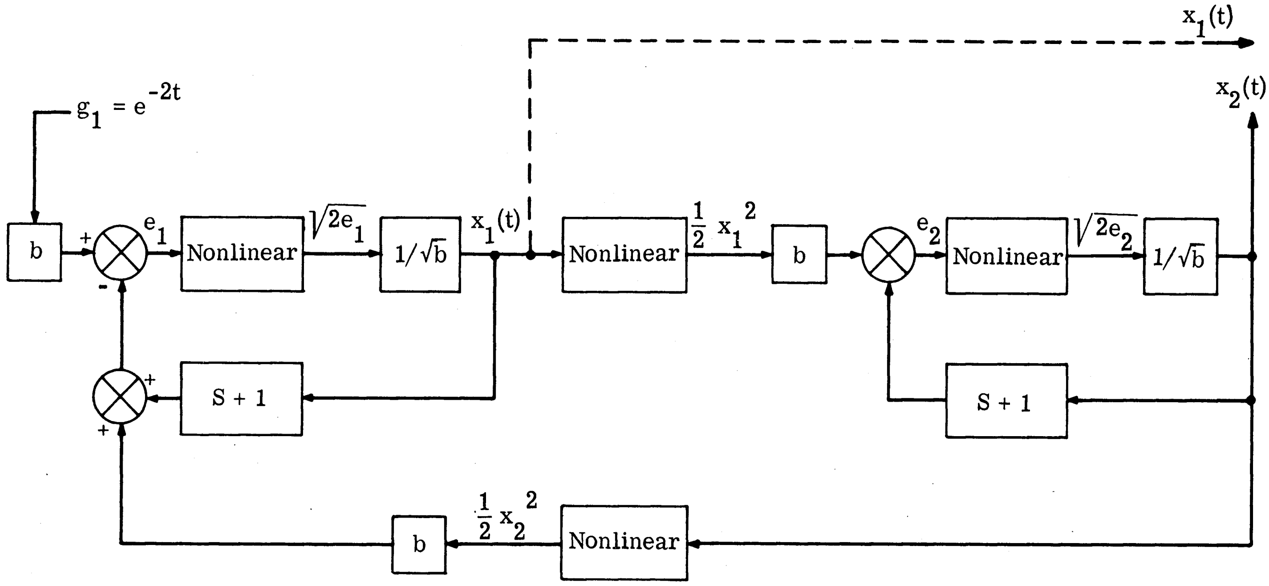

FIGURE 28. A MULTILOOP NONLINEAR CIRCUIT EQUIVALENT TO THAT OF FIGURE 27

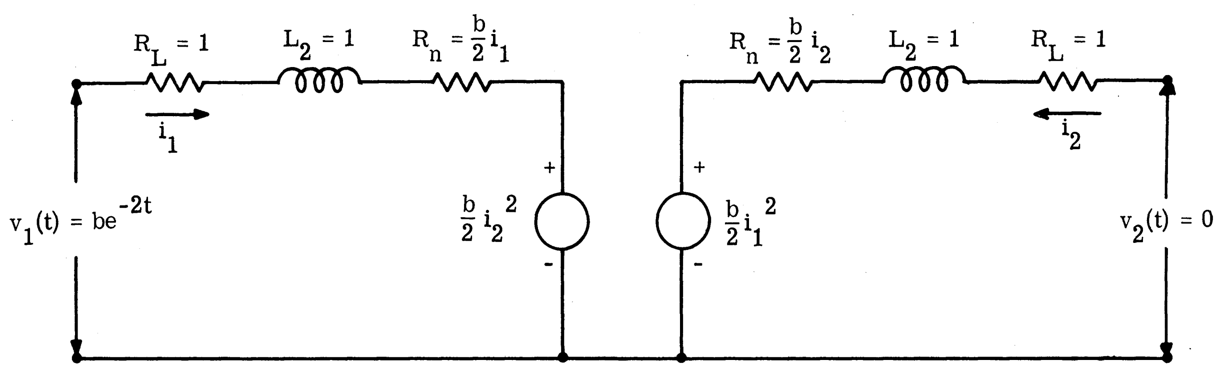

FIGURE 29. A TWO-TERMINAL PAIR NETWORK WITH NONLINEAR RESISTORS AND NONLINEAR COUPLING

The eight definitions introduced in Section 4.2 are evaluated for this particular system as follows:

$$\begin{aligned}
&W_{11}(t, u) = e^{-(t-u)} \\
&W_{22}(t, u) = e^{-(t-u)} \\
&x_{10}(t) = e^{-t} + b(e^{-t} - e^{-2t}), \ [\text{at } t = 0 \ x_{10}(0^+) = 1] \\
&x_{20}(t) = e^{-t}, \ [\text{at } t = 0 \ x_{20}(0^+) = 1] \\
&F_{11}[t, u, x_1(u)] = \frac{b}{2} e^{-(t-u)} x_1^{\ 2}(u) \\
&F_{12}[t, u, x_2(u)] = \frac{b}{2} e^{-(t-u)} x_2^{\ 2}(u) \\
&F_{22}[t, u, x_2(u)] = \frac{b}{2} e^{-(t-u)} x_2^{\ 2}(u) \\
&F_{21}[t, u, x_1(u)] = \frac{b}{2} e^{-(t-u)} x_1^{\ 2}(u)
\end{aligned} \tag{408}$$

If we choose a domain $D \equiv [|x_1| \leq 1, \ |x_2| \leq 1, \ 0 \leq u \leq t \leq 1]$, condition 354 will give

$$K_{ij} = b \, e^{-(t-u)} \qquad \begin{aligned} &\text{for } i = 1 \text{ or } 2 \\ &\text{for } j = 1 \text{ or } 2 \end{aligned} \tag{409}$$

$$\alpha_{ij}^{\ 2} = \int_0^t K_{ij}^{\ 2}(t, u)\, du = \frac{b^2}{2}(1 - e^{-2t}) = \alpha^2(t) \tag{410}$$

$$A_{ij}^{\ 2} = \int_0^t \alpha_{ij}^{\ 2}(t)\, dt = \frac{b^2}{2}\left(\frac{1}{2} + \frac{e^{-2}}{2}\right) = 0.283b^2 \tag{411}$$

Thus

$$A_{ij} = A = 0.53b$$

Applying conditions 355 and 357 gives

$$n_{21}(t) = n_{11}(t) = \int_0^t F_{11}[t, u, x_{10}(u)]\, du$$

$$= \frac{b}{2}\left[\left(1 + b + \frac{b^2}{3}\right) e^{-t} - (1 + b)^2 e^{-2t} + b(1 + b)\, e^{-3t} - \frac{b^2}{3} e^{-4t}\right] \tag{412}$$

$$n_{22} = n_{12}(t) = \int_0^t F_{22}[t, u, x_{20}(u)]\, du = \frac{b}{2}(e^{-t} - e^{-2t}) \tag{413}$$

$$C_{21}^{\ 2} = C_{11}^{\ 2} = \int_0^1 n_{11}^{\ 2}(t)\, dt = \frac{b^2}{4}(1 + b)\left[0.432 - (2/3)(1 + b)^2 + \frac{(1 + b)^3}{4}\right] \tag{414}$$

$$C_{22}^{\ 2} = C_{12}^{\ 2} = \int_0^1 n_{22}^{\ 2}(t)\, dt = 0.004b^2 \tag{415}$$

Since for all t, $0 \leq t \leq 1$, $|x_{10}(t)|$ should be less than or equal to one, b can be chosen less than one. Thus, b can be chosen to satisfy the inequality $0 < b \leq 1$. Take $b = 0.5$ as a representative value. For this particular value of b, it is possible to choose $C^2 = C_{22}^{\ 2} = 0.004b^2 = 0.0002$ or $C = 0.014$. It also follows that $2A = 0.53$. A check of inequalities 358 shows that they are satisfied. Now, use Equation 400 and notice that an upper-bound function for $x_1(t)$ is obtained for $i = 1$, and an upper-bound function for $x_2(t)$ is obtained for $i = n + 1 = 2$. It can be shown that for the system under consideration and in the given interval, the required solutions $x_1(t)$ and $x_2(t)$ can never exceed the upper-bound functions $\beta_1(t)$ and $\beta_2(t)$ as given by

$$\begin{aligned} |x_1(t)| \leq \beta_1(t) &= |x_{10}(t)| + |n_{11}(t)| + |n_{12}(t)| + m|\alpha_1(t)| \\ |x_2(t)| \leq \beta_2(t) &= |x_{20}(t)| + |n_{21}(t)| + |n_{22}(t)| + m|\alpha_2(t)| \end{aligned} \tag{416}$$

where m is given by

$$m = 4C \sum_{k=0}^{\infty} \frac{(2A)^k}{\sqrt{k!}} \tag{417}$$

In our case, $m = 0.003$ and

$$\begin{aligned} |x_1(t)| \leq \beta_1(t) &= 2e^{-t} - e^{-2t} + 0.01(1 - e^{-2t})^{1/2} \\ |x_2(t)| \leq \beta_2(t) &= e^{-t} + .5(e^{-t} - e^{-2t}) + 0.1(1 - e^{-2t})^{1/2} \end{aligned} \tag{418}$$

for $0 \leq t \leq 1$.

In order to check the validity of inequalitiies 418, it is necessary to find the exact solutions $x_1(t)$ and $x_2(t)$ for the system under consideration.[34] The method of analysis used in this section readily leads to an approximation of the exact solutions of the system to any degree of accuracy by use of numerical techniques programmed for a digital computer. Thus, by substituting from definitions 408 into Equations 362a, the following recurrent relations can be obtained:

$$x_{1n+1}(t) = e^{-t} + 0.5(e^{-t} - e^{-2t}) - \frac{1}{4}e^{-t}\int_0^t e^u\left[x_{1n}^{\ 2}(u) + x_{2n}^{\ 2}(u)\right]du$$

$$x_{2n+1}(t) = e^{-t} + \frac{e^{-t}}{4}\int_0^t e^4\left[x_{1n}^{\ 2}(u) - x_{2n}^{\ 2}(u)\right]du \qquad (419)$$

Since Theorem H proves that the recurrent relations 419 will yield the exact and unique solution as $n \to \infty$, the convergence of the process is guaranteed. However, by stopping at the n-th iteration, it is easy to prove that the error δ_n, in the mean square sense, for the given class of multiloop system can, in general, be given by the following equation:

$$\delta_n^{\ 2} = C^2\frac{(2A)^{2n}}{(n-1)!}\left[\sum_{k=0}^{\infty}\frac{(2A)^k}{\sqrt{k!}}\right]^2 \qquad (420)$$

where

$$\delta_n^{\ 2} = \int_0^1 (x - x_n)^2\,dt \qquad (421)$$

and C^2 and A^2 are as defined before. The derivation of Equation 420 follows essentially the same procedure used to obtain Equation 170. Substituting for C^2 and A^2 in Equation 420 gives

$$\delta_n^{\ 2} = 0.00063\frac{(0.53)^{2n}}{(n-1)!} \qquad (422)$$

and for n = 1

$$\delta_1^{\ 2} = 0.00017 \qquad (423)$$

[34] The inequalities can easily be proven true because, by inspection, Equations 406 for b = 0.005 has solutions $x_1(t) = x_2(t) = e^{-t}$.

Thus the solutions can be approximately given by $x_{11}(t)$ and $x_{22}(t)$ as follows:

$$x_1(t) \cong x_{11}(t) = x_{10} - n_{11}(t) - n_{12}(t)$$
$$x_2(t) \cong x_{21}(t) = x_{20} + n_{21}(t) - n_{22}(t) \tag{424}$$

Substituting for the quantities in Equation 424 gives

$$x_1(t) \cong x_{11}(t) = 0.85\,e^{-t} + 0.318\,e^{-2t} - 0.189\,e^{-3t} + 0.021\,e^{-4t}$$
$$x_2(t) \cong x_{21}(t) = 1.15\,e^{-t} - 0.318\,e^{-2t} + 0.189\,e^{-3t} - 0.021\,e^{-4t} \tag{425}$$

for $0 \leq t \leq 1$.

Note that the approximate solution 425 is valid only for the finite interval $0 \leq t \leq 1$. Thus, the problem cannot be worked for the infinite interval of time (for the same reasons that apply to example one, Section 2). If an analysis of the problem is required for the infinite interval of time, some modifications are necessary. For example, $d_1(t)$ and $d_2(t)$ can be chosen so that $d_1(t) = e^{-a_1 t}$ and $d_2(t) = e^{-a_2 t}$, where a_1 and a_2 are finite positive numbers. This has been shown in detail in example three, Section 2.

4.6. CONCLUSIONS AND REMARKS

The analysis developed in Sections 2 and 3 was extended to include systems whose dynamic behavior can be represented by a set of simultaneous nonlinear differential equations that are difficult to reduce to the canonic form of a single-degree-of-freedom nonlinear equation. A class of multiloop systems whose exact analytic solutions can be obtained to any degree of accuracy within a prescribed error was found to exist. The type of error used to estimate the system response was the mean square error, since it was best suited to the method of analysis. The mean square error can be used when the instantaneous value of the error in the approximate system response is not as important as, for example, the value of the integrated square error in the interval of interest.

By placing suitable restrictions on the different terms appearing in the system equations, it was found that the state variables $[v_i(t),\ i = 1, 2, \ldots, n + m]$ of the system belong to an L_2 space, and that each of these state variables is the limit of a sequence of iterates which also belongs to L_2 space. Convergence of these iterates within the given restrictions was satisfied in two cases: (1) the mean and (2) the ordinary sense. Note, however, that convergence in the mean of a sequence of functions which belong to L_2 space does not imply convergence in the ordinary sense. The converse is also true. (This fact is made clear by examples in Appendix I.)

Because the norms of a convergent series in L_2 are bounded, one can be certain that a system which operates within the given conditions will have a response which is bounded. If the interval of interest T is taken as infinity, and if the system satisfies the given conditions in that interval, the system will return to the equilibrium position—assumed to be the origin of the phase space—and give rise to asymptotically stable systems, in the sense of Lyapunov. As in the case of the single-degree-of-freedom system analyzed in Section 3, two definitions of the norm of the response functions were used to investigate the behavior of the system response at any instant and to estimate, on the average, the upper-bound functions that belong to L_2 space. As a by-product of this work, some interesting relations between n-terminal pair circuit configurations and multiloop feedback system configurations were developed and fully explained. Also, the output responses of the feedback systems were shown to be equivalent to the responses of the currents in the circuits. Finally, an example was given to illustrate the method of the analysis presented.

5
SYSTEM ANALYSIS BY NUMERICAL TECHNIQUES

The preceding three sections presented an analysis of a class of physical nonlinear systems represented by either a single-degree-of-freedom equation or a set of simultaneous, nonlinear differential equations. Attention was focused upon the properties of the system response when certain conditions were satisfied and upon the problem of how to choose system parameters to yield certain requirements. These considerations are an important step toward solving problems of synthesis. The exact analytical solution for the class of systems under study was shown to be obtainable from Picard iterates, and the convergence of these iterates was assured. After a finite number of iterations, say N, is obtained from an appropriate formula, then the N-th iterates will provide the solution for a predetermined degree of accuracy. This section discusses the problem of obtaining a numerical solution for the class of systems under consideration, by using a digital computer to numerically evaluate the iterations. This reduces the complexity of a nonlinear system calculation almost to that of numerical integration. However, the accuracy of the method depends on the accuracy with which the superposition integrals appearing in the iterates are evaluated. Therefore, the problem now is to obtain approximations for integrals, derivatives, and intermediate values (interpolations) of the continuous function x(t), at any time t, when given only a table of values x_s at discrete stationary points $t = t_s$. This necessitates a discussion of interpolation, numerical integration, and differentiation. However, discussion here treats only the situation in which the spacing h between station points $(h = t_{s+1} - t_s)$ is uniform, i.e., h is constant and independent of s. Although there are various methods for solving ordinary differential equations numerically, a comprehensive treatment of each is beyond the scope of this work. Therefore, only the most pertinent are discussed; in particular, the fourth-order Runge-Kutta method will be used. This section also discusses theoretical considerations of the accuracy of the iterative procedure and a method that has been devised for estimating "inherited" errors in terms of "truncation errors." This, of course, enables one to choose an interval h so that a predetermined accuracy can be maintained.

5.1. INTRODUCTION

By following Albert A. Bennett's classification [6, p. 65], approximate methods of integration of differential equations can be separated into two main types.

Type 1

The desired region of definition is assigned in advance or made as large as possible. For this region an assumed type of solution with infinitely many parameters is progressively evolved, until a satisfactory preassigned degree of agreement is assured. This method includes, for example, power-series solutions by undetermined coefficients, if convergence through the region can be proved or is assumed. It also includes the Picard method of successive substitutions.

Type 2

An approximate solution, which is usually a polynomial of not more than sixth order, is chosen in a form suitable for each interval of not too great extent. The parameters in the solution and the length of an interval starting at the initial point are chosen so that a satisfactory preassigned degree of agreement is assumed in this interval; then, this procedure is repeated for each interval until the desired preassigned total region in the real axis is covered. Such a method can be called a step-by-step method, in the restricted sense. The use of quadrature formulas, expressed in terms of differences, to proceed from one step at t_s to the next at t_{s+1} is essentially a type 2 method. Adam's method, Milne's method, and Multon's method are all typical examples [27]. One other interesting method that belongs to type 2 but differs from the processes which use quadrature formulas is the Runge-Kutta method [13].

This section deals principally with the type 1 method: the use of iterative procedures for the systematic approximation of the partition method used to analyze the class of nonlinear systems under study. A comparison with similar methods of analysis developed by Pipes [32], Stout [36], and Ku, Wolf, and Dietz [42] for the approximate solution of similar nonlinear integral equations of the convolution type is given. The accuracy of most of the above methods depends chiefly on the accuracy of the numerical evaluation of certain integrals and derivatives. (A brief discussion of some of the problems involved in handling discrete functions is presented in Appendix J and, a more detailed discussion, in References 9, 27, and 34.) Some of the errors that can occur in the analysis are truncation errors, round-off errors, and discretization errors. Round-off and discretization errors are defined and discussed in Section 5.4 and truncation errors are defined in Appendix J.

5.2. THE ESTIMATION OF TRUNCATION ERRORS BY TWO MEASUREMENTS

It is not always possible for a computer to have an explicit formula which gives the value of the truncation error committed in the evaluation of a given result. Since truncation errors are important in providing tentative preliminary estimates of actual error propagation, a fairly simple method is given for the determination of truncation errors and the determination of the exact value of a given quantity in the computation. The method is applicable to the computation of definite integrals and to other computations as well.

The Method

Let a quantity I be approximated by a quantity R_1 with a truncation error E_1, and by a quantity R_2 with a truncation error E_2, in such a way that

$$E_2 = K E_1 \tag{426}$$

where K is any given constant number. Then,

$$I = \frac{R_1 - KR_2}{1 - K} \tag{427}$$

and

$$E_1 = \frac{R_1 - R_2}{\frac{1}{K} - 1} \tag{428}$$

The Proof

The proof of Equations 427 and 428 proceeds as follows.

$$I = R_1 + E_1 = R_2 + E_2 \tag{429}$$

Using Equations 426 and 429 gives

$$(I - R_1) = K(I - R_2)$$

or

$$I(1 - K) = R_1 - KR_2$$

which proves Equation 427. Also,

$$E_1 = I - R_1$$

$$= \frac{R_1 - KR_2}{1 - K} - R_1$$

$$E_1 = \frac{R_1 - KR_2 - R_1 + KR_1}{1 - K} = \frac{R_1 - R_2}{\frac{1}{K} - 1}$$

which proves Equation 428.

Equations 427 and 428 will now be used to obtain information about the truncation errors in the following three cases.

(1) Application to the Trapezoidal Rule

Let a definite integral I be estimated by the two quantities R_1 and R_2 by using the trapezoidal rule and two different values of h in each computation. Equation 547 of Appendix J shows that the truncation error is of the order of h^2; hence, $h_2 = 1/2\ h_1$, corresponds to k = 1/4 and, by its substitution into Equations 427 and 428, gives

$$I = \frac{R_1 - R_2/4}{1 - 1/4} = \frac{4R_1 - R_2}{3} \tag{430}$$

$$E_1 = \frac{R_1 - R_2}{4 - 1} = \frac{R_1 - R_2}{3} \tag{431}$$

(2) Application to Simpson's Rule

Equation 547 of Appendix J shows that the truncation error in Simpson's rule is of the order of h^4. Therefore, two measurements with $h_2 = 1/2\ h_1$ give a value of k = 1/16. Now, substitution of this value into Equation 427 and 428 gives

$$I = \frac{R_1 - \frac{1}{16} R_2}{1 - \frac{1}{16}} = \frac{16R_1 - R_2}{15} \tag{432}$$

$$E_1 = 16E_2 = \frac{R_1 - R_2}{16 - 1} = \frac{R_1 - R_2}{15} \tag{433}$$

Equation 433 is in agreement with the result obtained in Reference 35 (p. 175).

(3) Application to the Runge-Kutta Process

The Runge-Kutta method of the fourth order[35] uses the first five terms of the Taylor series and, hence, includes an error of the order of h^5. If two measurements of the ordinate x_1 (as defined in Figure 30 are obtained, first in one step through an interval h_2 and second in two steps through two intervals, each of length h_1, so that $h_2 = 2h_1$, the result will be a value of k = 1/16. Therefore, substitution of this value into Equations 427 and 428 gives:

$$I = X_1 = \frac{R_1 - \frac{1}{16} R_2}{1 - \frac{1}{16}} = \frac{16R_1 - R_2}{15} \tag{434}$$

$$E_1 = \frac{R_1 - R_2}{K - 1} = \frac{R_1 - R_2}{15} \tag{435}$$

where R_1 and R_2 are the two computed values of X_1. Equation 435 agrees with the result obtained by Ince in Reference 13 (Appendix B). This equation was used to solve a nonlinear problem on the digital computer by means of the Runge-Kutta process, and its use was helpful in obtaining satisfactory results as well as in providing a means for checking the computed results during the computation process.

Since the formulas, methods, and definitions usually associated with numerical data have now been explained, the rest of this section will present the application of these rules to the numerical solution of problems pertinent to this report.

FIGURE 30. THE MEASUREMENT OF x_1. (a) Two steps each of interval length h_1. (b) A single step of interval length $h_2 = 2h_1$.

5.3. THE SYSTEMATIC APPROXIMATION TO THE PARTITION METHOD

First, consider the nonlinear system defined by Equation 96 together with the restrictions 102 through 109. For this class of systems, the solution was shown to be given by the following

---

[35] $x_{n+1} = x_n + h x'_n + \frac{h^2}{2!} x''_n + \frac{h^3}{3!} x'''_n + \frac{h^4}{4!} x''''_n + O(h^5)$

iterative procedure:

$$x_{m+1}(t) = x_0(t) - \int_0^t W(t, u)\, N\left(x_m, x_m^{(1)}, x_m^{(2)}, \ldots, u\right) du \tag{436}$$

where

$$m = 0, 1, 2, \ldots, p, \ldots;\ 0 \leq t \leq T$$

For a given mean square error q^2, the p-th iterate was shown to approximate the system solution in the mean square sense where the value of p, when N is a function of x alone, is obtained by solving for m as follows:

$$q^2 = \frac{C^2 A^{2m}}{(m - 1)!}\left(\sum_{k=0} \frac{A^k}{\sqrt{k!}}\right)^2 \tag{437}$$

and C^2 and A^2 have the meanings explained in Section 2.

To calculate the integrand of Equation 436 for each step of iteration, further expressions are needed for $\dot{x}_m, \ddot{x}_m, \ldots,$ and $x_m^{(n-1)}$, where n is the order of the system equation. Therefore, Equation 436 can be replaced by its equivalent relation

$$x_{m+1}^{(i)}(t) = x_0^{(i)}(t) - \int_0^t W_{(t, u)}^{(i)}\, N\left(x_m, x_m^{(1)}, \ldots, u\right) du \tag{438}$$

$$i = 0, 1, \ldots, \leq n - 1$$

as was shown in Section 3 for a similar equation.

The solution of Equation 96 requires the evaluation of $x_0^{(i)}(t)$ and the substitution of these functions in Equation 438 to obtain a first approximation for the solution $x_1^{(i)}$, and so on for further approximations. This process is greatly expedited by using a digital computer. Therefore, the interval in which the solution x(t) is desired is divided into continuous disjoint subintervals of equal length h. The solution, then, has the following approximate values: $x_m(t)$, $x_m^{(1)}(t)$, $x_0(t)$, $x_0^{(1)}(t)$, etc., for $t = t_k = kh$; these are denoted by x_{mk}, $x_{mk}^{(1)}$, x_{0k}, $x_{0k}^{(1)}$, etc. In particular, W(t, u) and $N\left(x_m, x_m^{(1)}, x_m^{(2)}, \ldots, t\right)$ are defined as follows:

$$W(t, u) = W_{ks}, \text{ where } \left(0 \leq s \leq k \leq \frac{T}{h}\right) \tag{439}$$

$$N_m(t) = N\left[x_m(t), x_m^{(1)}(t), x_m^{(2)}(t), \ldots, t\right] = N\left(x_{mk}, x_{mk}^{(1)}, x_{mk}^{(2)}, \ldots, t_k\right) = N_{mk} \tag{440}$$

Then, for $t = t_k$, Equation 438 becomes

$$x_{m+1}^{(i)}(t_k) = x_0^{(i)}(t_k) - \int_0^{t_k} W^{(i)}(t_k, u)\, N_m(u)\, du \tag{441}$$

If the superposition integral appearing in Equation 441 is approximated by any particular rule of numerical integration (the general quadrature formulas are explained in Appendix J), Equation 441 can be written approximately as follows:

$$x_{(m+1)k}^{(i)} = x_{0k}^{(i)} - h \sum_{s=0}^{k} \gamma_s W_{ks}^{(i)} N_{ms} \tag{442}$$

where γ_s is a constant determined by the particular integration rule employed. (See Equations 541 through 543 for the special cases of the trapezoidal rule, Simpson's rule, and Weddle's rule.) For a more convenient representation, Equation 442 is given in the matrix form:

$$\begin{bmatrix} x_{(m+1)k} \\ x_{(m+1)k}^{(1)} \\ \vdots \\ x_{(m+1)k}^{(n-1)} \end{bmatrix} = \begin{bmatrix} x_{0k} \\ x_{0k}^{(1)} \\ \vdots \\ x_{0k}^{(n-1)} \end{bmatrix} - h \begin{bmatrix} W_{k0} & W_{k1} & \cdots & W_{kk} \\ W_{k0}^{(1)} & W_{k1}^{(1)} & \cdots & W_{kk}^{(1)} \\ \vdots & \vdots & & \vdots \\ W_{k0}^{(n-1)} & W_{k1}^{(n-1)} & & W_{kk}^{(n-1)} \end{bmatrix} \begin{bmatrix} \gamma_0 N_{m0} \\ \gamma_1 N_{m1} \\ \vdots \\ \gamma_k N_{mk} \end{bmatrix} \tag{443a}$$

where each term is as defined before.

Equation 443a gives the (m + 1)-th iterate in terms of the previous and known iterate of order m and gives the functional values for each discrete value of K, (k = 0, 1, . . . , T/h), where T is the period of interest. For example, the first iteration can be obtained in terms of the known discrete values $x_{0k}, x_{0k}^{(1)}, \ldots, x_{0k}^{(n-1)}$ in the following first-order iterative matrix:

$$\begin{bmatrix} x_{1k} \\ x_{1k}^{(1)} \\ \vdots \\ x_{1k}^{(n-1)} \end{bmatrix} = \begin{bmatrix} x_{0k} \\ x_{0k}^{(1)} \\ \vdots \\ x_{0k}^{(n-1)} \end{bmatrix} - h \begin{bmatrix} W_{k0} & W_{k1} & \cdots & W_{kk} \\ W_{k0}^{(1)} & W_{k1}^{(1)} & \cdots & W_{kk}^{(1)} \\ \vdots & \vdots & & \vdots \\ W_{k0}^{(n-1)} & W_{k1}^{(n-1)} & \cdots & W_{kk}^{(n-1)} \end{bmatrix} \begin{bmatrix} \gamma_0 N_{00} \\ \gamma_1 N_{01} \\ \vdots \\ \gamma_k N_{1k} \end{bmatrix} \tag{443b}$$

Then, from this matrix, a second-order iterative matrix can be obtained:

$$\begin{bmatrix} x_{2k} \\ x_{2k}^{(1)} \\ \vdots \\ x_{2k}^{(n-1)} \end{bmatrix} = \begin{bmatrix} x_{0k} \\ x_{0k}^{(1)} \\ \vdots \\ x_{0k}^{(n-1)} \end{bmatrix} - h \begin{bmatrix} W_{k0} & W_{k1} & \cdots & W_{kk} \\ W_{k0}^{(1)} & W_{k1}^{(1)} & \cdots & W_{kk}^{(1)} \\ \vdots & \vdots & & \vdots \\ W_{k0}^{(n-1)} & W_{k1}^{(n-1)} & \cdots & W_{kk}^{(n-1)} \end{bmatrix} \begin{bmatrix} \gamma_0 N_{10} \\ \gamma_1 N_{11} \\ \vdots \\ \gamma_k N_{1k} \end{bmatrix} \qquad (443c)$$

and so on, until the p-th order matrix which closely approximates the required solution to a given degree of accuracy is obtained. Obviously, the column matrix $\left[x_{0k}^{(i)}\right]$ and the square matrix $[W_{ks}]$ appearing in the right-hand side of Equation 436 remain the same in each iteration.

5.4. THE ACCURACY OF THE APPROXIMATE SOLUTION

The accuracy of the method described above depends largely on the particular method of integration adopted and on the particular problem being solved. The errors in the quantities $x_{mk}^{(i)}$, which are the i-th derivative of x evaluated at the time t_k in the m-th iteration, are defined by

$$\varepsilon_{(i,k)_m} = x_{mk}^{(i)} - x_m^{(i)}(kh) \qquad (444)$$

These errors are due to the truncation errors in the approximate evaluation of different integrals in the k-th step and the "inherited" errors from similar approximations in the previous steps. The error defined in Equation 444 is sometimes called the discretization error [9, p. 16], a name that was used by Crandall. The discretization error describes the actual propagation of the error at each discrete time in the calculation. Frequently, the discretization error is difficult to evaluate; consequently, the truncation error is sometimes used as a crude estimate for the discretization error. A qualitative argument [9, p. 170] is used for its justification (for example, see Reference 36). Generally, it can be said that at the end of n steps of calculations the total error should be of the order of n times the truncation error of a single step [9, p. 167]. It is known that, for a given fixed total interval, the number of steps n is inversely proportional to h.[36] This will reduce the order of dependence of the total error at the end of the interval on

[36] Actually, $n = \frac{T}{h} = \frac{\text{the total interval of interest}}{\text{the subinterval width}}$.

the size of the increment h. The following example illustrates this. Assume the use of the simplest form of Euler methods to obtain the numerical integration of the first-order differential equation $\frac{dx}{dt} = f(x, t)$. For a given reasonable value of h, we have

$$x_{n+1} \cong x_n + h\left(\frac{dx}{dt}\right)_{t=nh} + 0(h^2) \tag{445}$$

where $0(h^2)$ means that the added terms are of the order of h^2. From Equation 445 it can be seen that the truncation error in each step of the calculation is of the order of h^2. At the end of $\frac{T}{h}$ subintervals, the error will be

$$\frac{h^2}{2} \cdot \left(\frac{d^2x}{dt^2}\right) \cdot \frac{T}{h} = h \cdot \frac{T}{2} \cdot \left(\frac{d^2x}{dt^2}\right) \tag{446}$$

which is of the order of h only. An exact estimate of the discretization error ε_n [9, p. 168] was found to be

$$\varepsilon_n = -\frac{h}{2p}\left(fp + \frac{\partial f}{\partial t}\right)_{avg.} \left[e^{pT} - 1 - \frac{hp}{2}(pTe^{pT}) + 0(h^2p^2)\right] \tag{447}$$

where $p = \frac{\partial f}{\partial x}$.

Examination of this equation shows that if the value of hp is small, the discretization error ε_n is proportional to h, and this agrees with the previous intuitive argument. Now, we can conclude that the discretization error can be decreased by using smaller increments of h, but the number of steps and hence the amount of computation required to cover a given interval are increased. When a large number of steps is required, there is a danger that round-off errors[37] will build up to substantial proportions. However, since the magnitude of the round-off errors can be controlled by the number of decimal places to which the computation is carried, we can consider round-off errors negligible in comparison with the other errors mentioned above. Note that in the last example, and in the special case in which the maximum possible round-off error ε_r at the end of a fixed interval is inversely proportional to h and the maximum of the discretization error ε_n is directly proportional to h (Figure 31), the minimum of the maximum of the total error $\varepsilon = \varepsilon_r + \varepsilon_n$ occurs when the round-off error E_r and the truncation error E_T of a single step are of equal magnitude; that is, when [9, p. 169] $E_r = E_T$. Note that E_r equals the greatest possible round-off error at each step and E_T equals the greatest truncation error at each step. Thus when the round-off errors are neglected, the truncation errors are important in the evaluation of the discretization errors $\varepsilon_{(ik)_m}$, as defined by Equation 444. There-

[37]Round-off errors are errors committed in rounding off results to a finite number of digits.

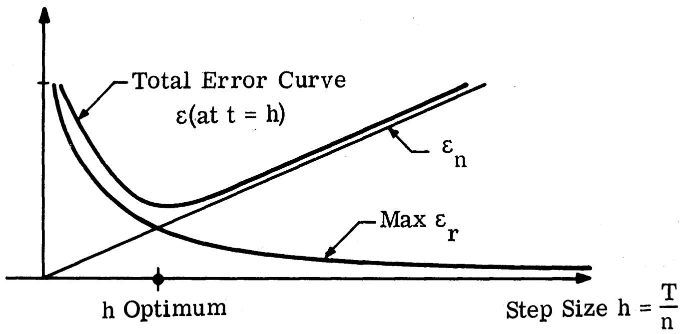

FIGURE 31. A PLOT OF ε_r AND ε_n VERSUS h

fore, the maximum absolute value of the truncation error for each step in the solution must be much less than the actual maximum discretization error in the solution. However, a preliminary value of h can be derived from an estimate of the truncation error.

In the demonstration that follows, the trapezoidal rule of integration is used. Although it is less accurate than other methods, it has been found to yield satisfactory results [36, 42]. Equation 545 is used for the truncation error evaluation and in each iteration there are integrands z of the form:

$$z = W^{(i)}(t, u)\, N_m(u), \qquad \text{where } i = \frac{d}{dt} \tag{448}$$

If one takes the first derivative of Equation 448 with respect to the variable u,

$$\frac{dz}{du} = W^{(i)}(t, u)\frac{dN_m(u)}{du} + N_m(u)\frac{d\, W^{(i)}(t, u)}{du} \tag{449}$$

lets t = nh, substitutes u = nh and u = 0 in Equation 449, and uses Equation 545, one obtains the following expression for the truncation error:

$$\begin{aligned} E_T\left[x_{m+1}{}^{(i)}(nh)\right] \leq \frac{h^2}{12}\Big\{&W^{(i)}(nh, nh)\, N'_m(nh) + N_m(nh)\,[W^{(i)}(nh, mh)]' \\ &- W^{(i)}(nh, 0)\, N_m(0) - N'_m(0)\left[W^{(i)}(nh, 0)\right]'\Big\} \end{aligned} \tag{450}$$

where the primes mean differentiation with respect to u. (Note that the truncation error results from an approximation of the superposition integrals appearing in Equation 441 when the (m + 1)-th iterate is evaluated using the trapezoidal rule.) For the special condition in which we have a nonlinearity N which is a function of x only (therefore, only the iterates of x, and not its derivatives, are required) and a linear part with a constant coefficient (therefore, W[t, u] = W[t - u), Equation 450 reduces to:

$$E[x_{m+1}(nh)] \leq \frac{h^2}{12} [W(0)N_m(nh) - N'_m(nh)W'(0) - W(nh)N'_m(0) + N_m(0)W'(nh)] \quad (451)$$

Equation 451 is in agreement with the one derived by T. M. Stout in Reference 36. Of course, the calculation of the derivatives of $N_m(u)$ necessary for the evaluation of Equations 450 and 451 can be obtained to any degree of accuracy by the methods described in Appendix J. However, a satisfactory estimate of the derivatives can be obtained from the following equation:

$$N'_m(nh) = \frac{N_m(n + 1)h - N_m(n - 1)h}{2h} + 0\,(h^2) \quad (452)$$

if the necessary values are available. The error in Equation 452 is of the order of h^2.

Now that the truncation errors in the approximations are known, the propagation of the discretization errors as given previously by Equation 444 can be studied. The truncation errors $E_T[x_m^{(i)}(nh)]$ are denoted by $E_{(i,k)_m}$, so that the exact (m + 1)-th iterates of the differential Equation 96 at t = kh can be obtained by applying the following equation:

$$x_{m+1}^{(i)}(kh) = x_0^{(i)}(kh) - h \sum_{s=0}^{k} \gamma_s W_{ks}^{(i)} N_m(sh) + E_{(i,k)_m} \quad (453)$$

Substituting Equations 453 and 442 into Equation 444 gives

$$\varepsilon_{(i,k)_{m+1}} = \left(\left[x_{0k}^{(i)} - x_0^{(i)}(kh) \right] - h \left\{ \sum_{s=0}^{k} \gamma_s W_{ks} [N_{ms} - N_m(sh)] \right\} - E_{(i,k)_m} \right. \quad (454)$$

where, as before, the γ's depend on the particular rule of integration used and i < n = the order of system.

Equation 454 gives a recurrent relation that represents the propagation of the errors in the discrete values of any of the iterates, and that greatly controls the accuracy of the method presented. The method is applicable when the forcing function g(t) is specified graphically, tabularly, or by an explicit mathematical function. However, when g(t) is given by an explicit mathematical function, the accuracy of the method can be improved by using the exact response of the linear partitioned terms $x_0^{(i)}(kh)$, so that

$$\left[x_{0k}^{(i)} - x_0^{(i)}(kh) \right] = 0 \quad (455)$$

and Equation 454 reduces to

$$\varepsilon_{(i,k)_{m+1}} = -h\left\{\sum_{s=0}^{k} \gamma_s W_{ks}[N_{ms} - N_m(sh)]\right\} - E_{(i,k)_m} \tag{456}$$

The errors in the discrete values of the nonlinear function represented by the terms $[N_{ms} - N_m(sh)]$ are nonlinear functions of the errors $\varepsilon_{(i,k)_m}$. Equation 456 can be simplified if the approximate discrete values of the nonlinear functions N_{ms} are expanded in a Taylor series about the exact values of x(t) and its derivatives appearing in the function. Therefore, we have

$$N_{mk} = N_m(kh) + \sum_J A_{J_m} \varepsilon_{(J,k)_m} \tag{457}$$

in which

$$A_{J_m} = \left[\frac{\partial N_m(t)}{\partial x^{(J)}(t)}\right]_{t=kh} \tag{458}$$

so that

$$[N_{mk} - N_m(kh)] = \sum_J A_{J_m} \varepsilon_{(J,k)_m} \tag{459}$$

where J extends over the orders of all the derivatives of x(t) contained in the nonlinear function $N_m(t)$. The quantities A_{J_m} usually are functions of $x_m^{(j)}(kh)$; but constant values for A_J can be assigned over a limited range of the expansion. When this is done, Equation 456 reduces to

$$\varepsilon_{(i,k)_{m+1}} = -h\left\{\sum_{s=0}^{k} \gamma_s \left[\sum_J A_J \varepsilon_{(J,s)_m}\right] W_{ks}^{(i)}\right\} - E_{(i,k)_m} \tag{460}$$

where $m + 1 = 1, 2, 3, \ldots, q$, (q is the q-th iterate)

$i = 0, 1, \ldots, n - 1$, (n is the order of the system)

$k = 0, 1, \ldots, \frac{T}{h}$ (T is the total interval)

Again, $\varepsilon_{(i,k)_{m+1}}$ denotes the discretization error in the (m + 1)-th iterate of the i-th derivative of the solution x(t), at the k-th discrete station point. The truncation error $E_{(i,k)_m}$ varies, of course, at each point $t_k = kh$ for each value of m, but a further simplification can be obtained by assigning a value $E_i(kh)$ for each of the quantities $E_{(i,k)_m}$ for all m. In this case Equation 460 takes the more simple form

$$\varepsilon_{(i,k)_{m+1}} = -h\left\{\sum_{s=0}^{k} \gamma_s \left[\sum_J A_J \varepsilon_{(J,s)_m}\right] W_{(ks)}^{(i)}\right\} - E_{ik} \tag{461}$$

Changing the order of summation in Equation 461 and approximating the sum over s by an intergration process reduces this equation to the new form

$$\varepsilon_{i_{(m+1)}}(t) = [-E_i(t)] - \left[\sum_{J=0}^{J\leq n-1} A_J \int_0^t W^{(i)}(t, u) J_m(u)\, du\right] \tag{462}$$

Letting

$$A_J W^{(i)}(t, u) = W_{iJ}(t, u) \tag{463}$$

reduces Equation 463 to the form

$$\varepsilon_i(t)_{(m+1)} = [-E_i(t)] - \sum_{J=0}^{J\leq n-1} \int_0^t W_{iJ}(t, u)\, \varepsilon_{J_m}(u)\, du \tag{464}$$

It was proved in Section 4 that at the limit at which n goes to infinity, Equation 464 yields the following equation:

$$\varepsilon_i(t) = [-E_i(t)] - \sum_{J=0}^{J\leq n-1} \int_0^t W_{iJ}(t, u)\, \varepsilon_J(u)\, du \tag{465}$$

where $i = 0, 1, 2, \ldots, \leq n - 1, 0 \leq t \leq T$

Equation 465 is know as a linear system of Volterra's equation of the second type [24, p. 17]. Its solution represents the error propagation at the end of a sufficiently large number of iterations. For simplicity, the solutions can be assumed to be adequate representations of the discretization error propagation at the end of the required number of p iterations necessary to approximate the system solution by the method presented. For the special condition in which the nonlinearity N is a function of x only, Equation 465 reduces to

$$\varepsilon(t) - \lambda \int_0^t W(t, u)\, \varepsilon(u)\, du = -E(t),\ 0 \leq t \leq T \tag{466}$$

where

$$\lambda \triangleq -A_0 \triangleq \frac{\partial N[x(t)]}{\partial x(t)} \tag{467}$$

Equation 466 is a linear Volterra equation of the second type,[38] the unique solution of which can be given at once by the formula:

$$\varepsilon(t) = [-E(t)] + \int_0^t H(t, u, \lambda)\, E(u)\, du \tag{468}$$

[38] If E(t) = 0, Equation 466 reduces to an equation of the first type [38, p. 10]. For the purposes of this report, it is assumed that $E_i(t)$ will always have a certain value.

where the "resolvent kernel" $H(t, u, \lambda)$ is given by the series of iterated kernels as follows:

$$-H(t, u, \lambda) = \sum_{n=0}^{\infty} \lambda^n W_{n+1}(t, u) \tag{469}$$

Series 469 converges almost everywhere. The resolvent kernel is known to satisfy the integral equation:

$$W(t, u) + H(t, u, \lambda) = \lambda \int_0^t W(t, z)\, H(z, u, \lambda)\, dz$$

$$= \lambda \int_0^t H(t, z, \lambda)\, W(z, u)\, dz \tag{470}$$

Note that the iterated kernels $W_{n+1}(t, u)$ are defined as

$$W_{n+1}(t, u) = \int_0^t W(t, z)\, W_n(z, u)\, dz \qquad n = 1, 2, 3, \ldots \tag{471}$$

where

$$W_1(t, u) = W(t, u) \tag{472}$$

From the error Equation 468, it can be concluded that the error in the solution by the procedure presented depends primarily on the truncation error $E(t)$, and secondarily on the nature of the particular problem in the region of interest. The dependence on the region of interest is represented by the λ term[39] in Equation 468. If the truncation errors are reduced significantly—and this can be done by using more accurate rules of integration—then the effect of the λ term will be negligible. For the general case, Equation 461, 464, or 465 can be used to give information about the error propagation. The choice of the particular equation depends on what is required: accuracy or simplicity of analysis.

5.5. AN EXAMPLE CALCULATION USING THE DIGITAL COMPUTER

The use of the numerical analysis technique given in Section 5 can be illustrated by the following equation, in which the technique is applied to a nonlinear system (given in Section 2):

$$\frac{dx}{dt} + x + x^2 = e^{-2t} \qquad \text{where } x = -1 \text{ at } t = 0 \tag{473}$$

[39] See Equation 467.

Partitioning this equation at the x^2 term and using previous notations and definitions, adopted in Section 2, results in the following recurrent relation:

$$x_{n+1}(t) = x_0(t) - \int_0^t e^{-(t-u)} x_n^2(u)\,du \qquad (474)$$

where

$$x_0(t) = -e^{-2t} \qquad (475)$$

Now, the values given in Section 2 for C^2 (0.025) and A^2 (1.14) can be substituted into Equation 170 to produce the following equation:

$$q^2 = \frac{(0.025)(9.6)(1.14)^n}{(n-1)!} \qquad (476)$$

Equation 476 is used to construct Table IV, which gives the required number of iterations n for a given mean square error q^2. Table IV shows that it is reasonable to perform the first seven iterations of Equation 474 on the digital computer. If one assumes that the trapezoidal rule is used to evaluate the different integrals, Equation 451 can be used to give an estimate of the truncation error committed in evaluating the integral appearing in the first iterate at the end of the interval of interest T, which is assumed to be unity. For this purpose the following quantities are needed:

$$W(t) = e^{-t} \quad N_1(nh) = x_0^2(t) = e^{-4t}$$
$$W'(t) = -e^{-t} \quad N'_1(nh) = -4e^{-4t} \qquad (477)$$

TABLE IV. THE MEAN SQUARE ERROR q^2 FOR DIFFERENT VALUES OF ITERATIONS n

n = the number of iterations	q^2 = the mean square error
2	0.31
3	0.118
4	0.0635
5	0.018
6	0.004
7	0.00075

Evaluation of these quantities at t = 0 and t = 1 gives

$$\begin{aligned} W(0) &= 1 & W(1) &= 1/e \\ W'(0) &= -1 & W'(1) &= -1/e \\ N_1(0) &= 1 & N_1(1) &= 1/e^4 \\ N'_1(0) &= -4 & N'_1(0) &= -4/e^4 \end{aligned} \tag{478}$$

Choosing h = 0.05 and substituting this together with the quantities of Equation 478 into Equation 451 gives

$$E_1[x(1)] \leq 0.000218 \tag{479}$$

After the required number of iterations, an adequate estimate of the propagation of errors can be obtained by considering Equation 466 and assigning reasonable values for E(t) and λ. Thus,

$$N(x) = x^2 \tag{480}$$

and

$$-A_0 = \lambda = -\frac{\partial N(x)}{\partial x} = -2x(t) \tag{481}$$

Although λ varies with time, a constant value is assigned over the limited range of the solution. Also, E(t) = E(kh) varies at each point in the solution so that E(0) = 0 (since the initial point is known exactly from the initial conditions). Thus, a function E(t) can be obtained if the truncation error is known at each discrete point, in the first iteration, for example, and is considered as the required function E(t) over the required number of iterates to be substituted in the error Equation 466. However, the upper bound of the error $\varepsilon(t)$ can be obtained quickly by assigning to E(t) a constant value equal to its worst value, such as that given by Equation 479. Therefore, the actual error in the solution of the given example will be bounded by the solution of the following equation (obtained from Equation 466), after the appropriate substitutions for W(t, u) and E(t):

$$\varepsilon(t) + A_0 \int_0^t e^{-(t-u)} \varepsilon(u) = -0.000218 \tag{482}$$

where λ is a constant parameter. Equation 482 can easily be transformed to a linear differential equation. Let

$$\begin{aligned} \varepsilon_1(t) &= \frac{\varepsilon(t)}{e^{-t}} = \varepsilon(t)\, e^t \\ f_1(t) &= \frac{-0.000218}{e^{-t}} = -0.000218\, e^{+t} \end{aligned} \tag{483}$$

Dividing both sides of Equation 482 by e^{-t} and making use of Equations 483 gives

$$\varepsilon_1(t) + A_0 \int_0^t \varepsilon_1(u)\,du = -0.000218\ e^{+t} \tag{484}$$

Differentiating both sides of Equation 484 gives

$$\frac{d\varepsilon_1(t)}{dt} + A_0\ \varepsilon_1(t) = -0.000218\ e^{+t} \tag{485}$$

In differential notations, Equation 485 takes the form

$$(D + A_0)\ \varepsilon_1(t) = -0.000218\ e^{+t} \tag{486}$$

Equation 486 has the following general solution:

$$\varepsilon_1(t) = C_1\ e^{-A_0 t} - 0.000218\ \frac{e^t}{D + A_0}, \qquad A_0 \neq -1 \tag{487}$$

where C_1 is an arbitrary constant determined from the requirement that at $t = 0$, $\varepsilon_1 = \varepsilon = 0$. Again, by using Equation 483, Equation 487 can be written in the following form:

$$\varepsilon(t) = C_1 e^{-(1+A_0)t} - 0.000218\ e^{-t}\left[\frac{1}{D + A_0}\ e^t\right], \qquad (0 \leq t \leq 1),\ A_0 \neq 1 \tag{488}$$

The information obtained from Equation 488 about the actual error $\varepsilon(t)$ depends on the choice of the value of the parameter A_0. From the physics of the problem and a consideration of Equation 481, we can conclude that A_0 is ≤ 0, since $x(t) \leq 0$. Figure 32 shows different solutions for different reasonable values of A_0. It indicates that the value of the error $\varepsilon_{0,k} < 1.7\ E_1[x(1)] = 0.00037$, which gives results correct to the first three digits.

The method of successive approximation prescribed above is programmed for the 707 digital computer, which solves the problem by the steps shown in Figure 33. The results are given in Table V which also contains the exact solution so that one can calculate the true errors resulting from the approximations. The true errors are also plotted in Figure 32. The maximum value of the error is 0.000083 which is much less than 0.00037; this agrees with the predicted results.

Figure 34, which gives the different approximations of the actual solution, shows that the curve representing the fourth iteration approaches the exact solution of the nonlinear system in such a way that the curves representing the remaining iterations, corresponding to n = 5, n = 6, and

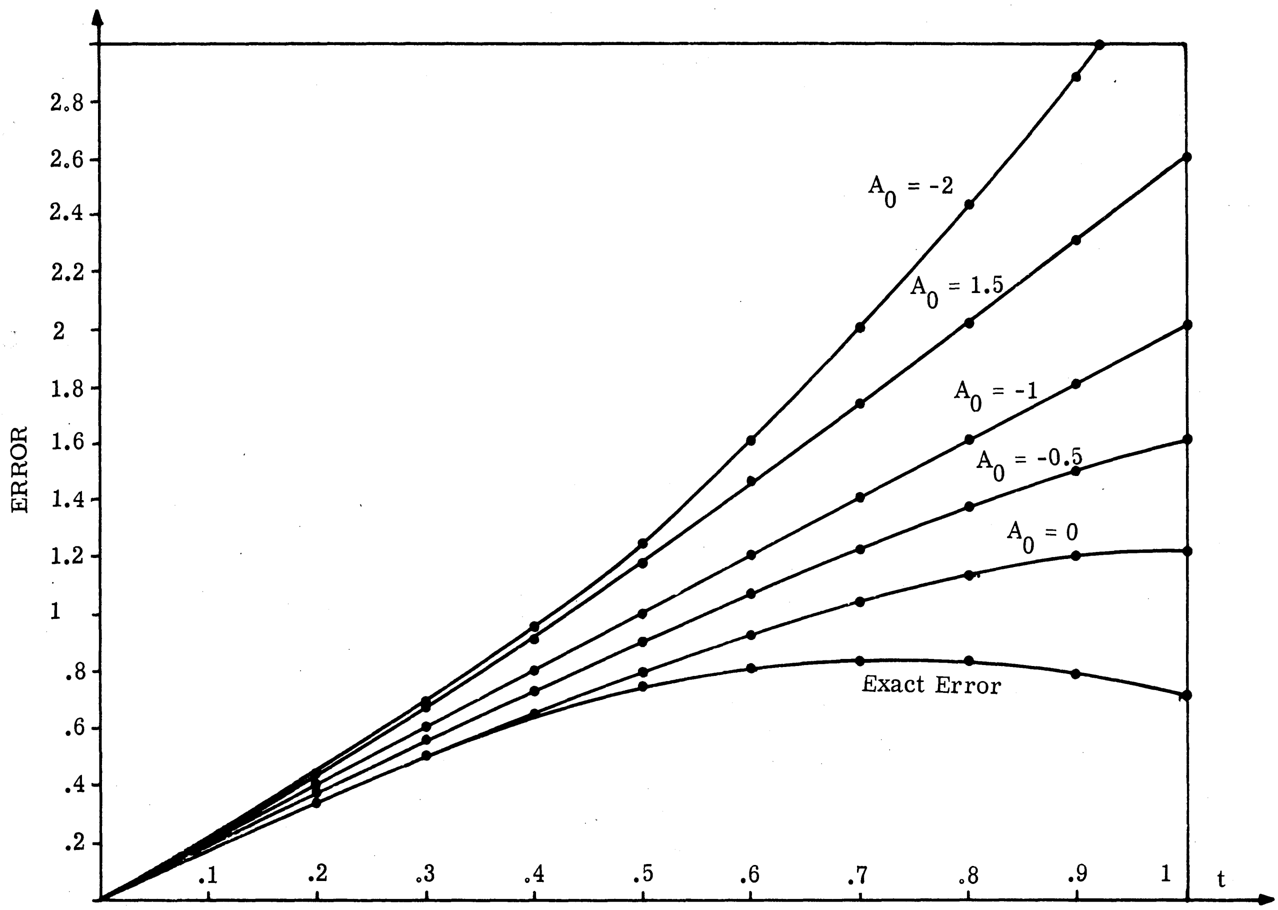

FIGURE 32. A PLOT OF THE ACTUAL ERROR PROPAGATION AND THE THEORETICAL ERROR PROPAGATION FOR DIFFERENT VALUES OF A_0

n = 7, almost coincide. Obviously, the curve corresponding to n = 0 gives the solution of the linear system:

$$\frac{dx}{dt} + x = e^{-t} \tag{489}$$

so that t = 0, x = 1.

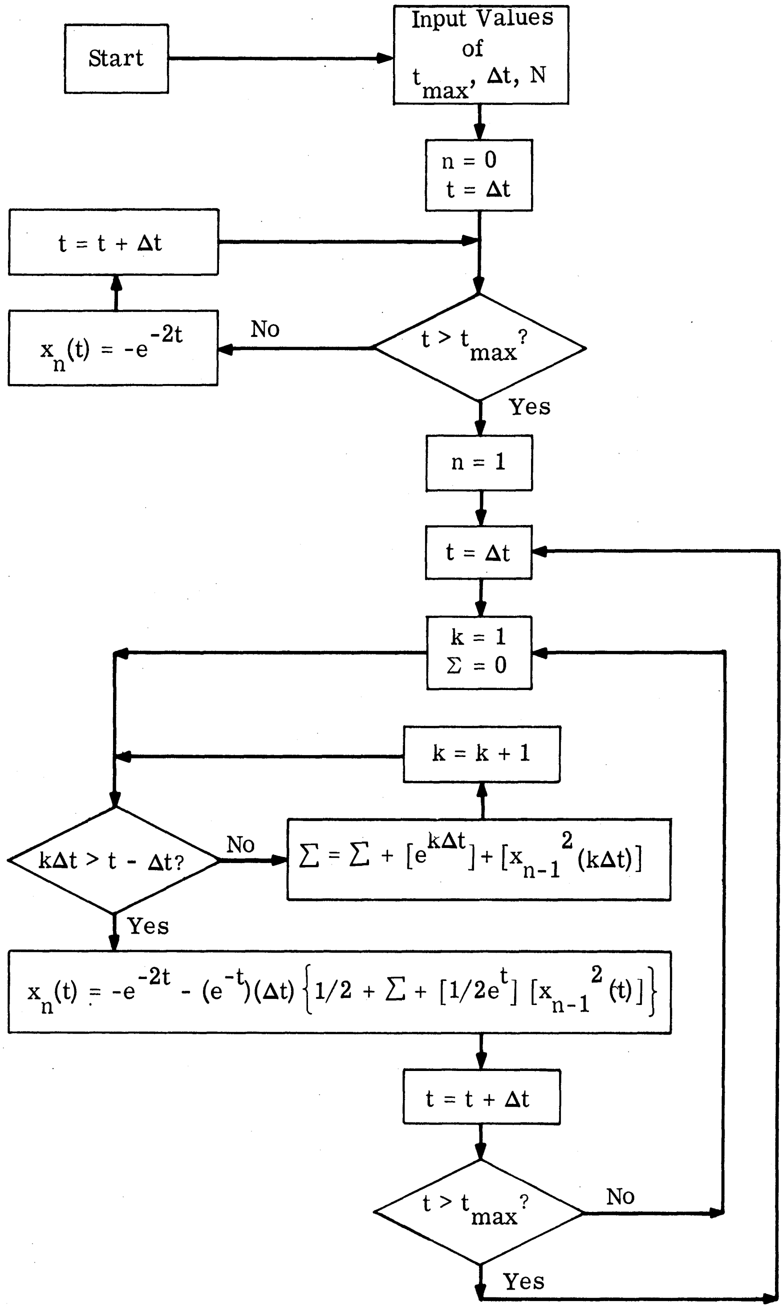

FIGURE 33. A TYPICAL FLOW CHART FOR THE SOLUTION BY THE METHOD OF SUCCESSIVE APPROXIMATION

TABLE V. NUMERICAL RESULTS FROM THE DIGITAL COMPUTER

N	0	1	2	3	4	5	6	Answer by the Method of Approximation	Exact Answer $x = -e^{-t}$	Errors due to Successive Approximation	Runge-Kutta Method
0.05	-0.904837	-0.949086	-0.951137	-0.951235	-0.951239	-0.951240	-0.951240	-0.951240	-0.951229	-0.000011	-0.951229
0.10	-0.818731	-0.897050	-0.904311	-0.904823	-0.904855	-0.904857	-0.904857	-0.904857	-0.904837	-0.000020	-0.904837
0.15	-0.740818	-0.844979	-0.859211	-0.860615	-0.860728	-0.860736	-0.860737	-0.860737	-0.860708	-0.000029	-0.860707
0.20	-0.670320	-0.793685	-0.815666	-0.818463	-0.818742	-0.818766	-0.818768	-0.818768	-0.818731	-0.000037	-0.818730
0.25	-0.606531	-0.743761	-0.773598	-0.778232	-0.778786	-0.778841	-0.778845	-0.778846	-0.778801	-0.000045	-0.778800
0.30	-0.548812	-0.695628	-0.732983	-0.739802	-0.740751	-0.740859	-0.740870	-0.740871	-0.740818	-0.000053	-0.740818
0.35	-0.496585	-0.649569	-0.693831	-0.703073	-0.704537	-0.704725	-0.704745	-0.704747	-0.704688	-0.000059	-0.704688
0.40	-0.449329	-0.605763	-0.656162	-0.667958	-0.670048	-0.670345	-0.670381	-0.670385	-0.670320	-0.000065	-0.670320
0.45	-0.406570	-0.564308	-0.620003	-0.634384	-0.637195	-0.637634	-0.637691	-0.637698	-0.637628	-0.000070	-0.637628
0.50	-0.367879	-0.525239	-0.585373	-0.602292	-0.605897	-0.606508	-0.606594	-0.606605	-0.606531	-0.000074	-0.606530
0.55	-0.332871	-0.488545	-0.552285	-0.571625	-0.576076	-0.576888	-0.577011	-0.577028	-0.576950	-0.000078	-0.576949
0.60	-0.301194	-0.454178	-0.520740	-0.542337	-0.547661	-0.548701	-0.548869	-0.548892	-0.548812	-0.000080	-0.548811
0.65	-0.272532	-0.422069	-0.490729	-0.514383	-0.520588	-0.521875	-0.522096	-0.522128	-0.522046	-0.000082	-0.522047
0.70	-0.246597	-0.392128	-0.462233	-0.487721	-0.494794	-0.496345	-0.496625	-0.496668	-0.496585	-0.000083	-0.496585
0.75	-0.223130	-0.364254	-0.435223	-0.462311	-0.470223	-0.472048	-0.472394	-0.472450	-0.472367	-0.000083	-0.472366
0.80	-0.201897	-0.338341	-0.409663	-0.438112	-0.446819	-0.448924	-0.449341	-0.449411	-0.449329	-0.000082	-0.449328
0.85	-0.182684	-0.314277	-0.385509	-0.415085	-0.424533	-0.426916	-0.427409	-0.427495	-0.427415	-0.000080	-0.427414
0.90	-0.165299	-0.291951	-0.362712	-0.393190	-0.403316	-0.405973	-0.406543	-0.406647	-0.406570	-0.000077	-0.406569
0.95	-0.149569	-0.271253	-0.341220	-0.372384	-0.383120	-0.386043	-0.386692	-0.386815	-0.386741	-0.000074	-0.386741
1.00	-0.135335	-0.252075	-0.320978	-0.352629	-0.363903	-0.367077	-0.367807	-0.367949	-0.367879	-0.000070	-0.367870

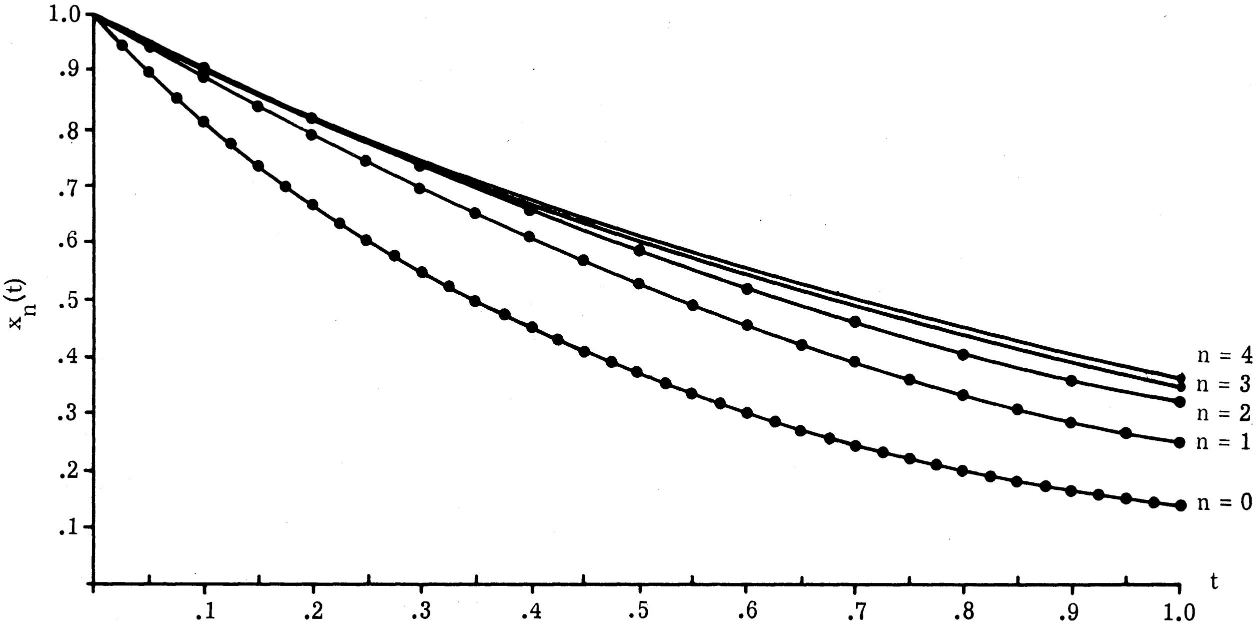

FIGURE 34. DIFFERENT APPROXIMATION FOR THE ACTUAL SOLUTION

5.6. COMPARISON WITH SIMILAR METHODS

Similar methods of analysis have been developed by Stout [36] and Wolf, Ku, and Dietz [42]. These methods require the conversion of a nonlinear differential equation to a nonlinear integral equation of the convolution type. In the notations used thus far, it has the form

$$x(t) = x_0(t) + \int_0^t W(t - u)\, N(x, \dot{x}, \ldots)\, du \tag{490}$$

in which $x_0(t)$ contains the response of the linear portion of the system to both the forcing function and the initial conditions. A step-by-step method of analysis is then used to approximate the integrals. This method results in algebraic equations which, in general, are nonlinear and which must be solved for each discrete value of x_k. Stout solved these nonlinear algebraic equations graphically; however, this method does not provide the means for eliminating extraneous solutions which can occur for certain nonlinearities. Wolf, Ku, and Dietz solved these equations analytically and suggested a method for eliminating the extraneous solutions by differentiating the original system equation. This, of course, increases the complexity of the problem. The difference between the methods of Stout and Wolf, Ku, and Dietz is that the former evaluated the integral between the limits t = 0 and t = kh, and the latter used the "m-point" integration rules in which each step in the solution results in a new equation with a new set of initial conditions; this permits the computer to use more accurate rules of integration. Stout was unable to calculate the discretization errors in the solutions because the graphical procedures he used restricted him to working with a nonlinearity which was a function of either x or $\dot{x}$, only. The method of Wolf, Ku, and Dietz requires a starting procedure to calculate the first (M - 1) discrete values of the solution necessary to carry on the computations. The numerical analysis method presented herein differs from these other methods by using successive approximations of integral equations of the superposition type, e.g.,

$$x(t) = x_0(t) + \int_0^t W(t, u)\, N(x, \dot{x}, \ldots)\, du \qquad (491)$$

and by including, in the analysis, systems with time-varying parameters. The integrals of different iterations are evaluated by a numerical procedure carried out on a general purpose digital computer. While investigating electric circuits, L. A. Pipes [32] adopted the same method for the approximation of integrals of the convolution type represented by Equation 490. This requires analytic expressions for the nonlinear function so that the integrand increases in complexity even under the best conditions. Thus, the method of Pipes is only suitable for very small nonlinearities which require one or two iterations, and does not provide the means to judge the accuracy of the solution obtained. In this report there is no complexity of integrals since numerical techniques are employed, and the errors committed during the solution can be evaluated. Thus, the solution can be obtained to any degree of accuracy. The need for a starting procedure and the possibility of obtaining extraneous solutions are eliminated. Further, the M-point integration rule, described by Wolf, Ku, and Dietz, can also be employed to make possible the use of more accurate rules of integration.

5.7. CONCLUSIONS AND REMARKS

A systematic method has been presented for extending the general partition method to the numerical analysis of a broad class of physical system described by higher order differential equations. The forcing function, the nonlinear function, and the time-varying parameters can be specified graphically, tabularly, or by explicit mathematical functions. A means of estimating the errors in the approximate solution has been devised so that the accuracy of the results can be determined and the appropriate interval selected. Although developed for the analysis of systems described by a single equation, the method can be extended to the analysis of systems described by simultaneous equations. An example calculation using the digital computer was given, and the results compared satisfactorily with the exact results.

An important property of the analysis presented is that, for a given nonlinear term N, the complexity of the analysis does not increase greatly as the order of the system n gets higher. If the system is of a higher order, then the numerical solution of a linear equation of higher order should be found, a relatively simple procedure. This makes the method convenient and practical for the analysis of higher order systems.

6
CONCLUSIONS, CRITICISMS, AND SUGGESTIONS FOR FUTURE STUDY

6.1. SUMMARY

The use of the partitioning technique in nonlinear analysis was briefly reviewed, and it was stated that under a broad class of conditions, properly imposed on the linear, nonlinear, and forcing function terms, a class of systems exists for which exact and unique solutions can be found. Mathematically, the class of systems of interest can be represented by an equation, such as Equation 1, which satisfies the conditions of Equations 2 through 4. The types of nonlinearities allowed include algebraic nonlinearities in the form of polynomials of x and its admissible derivatives, and transcendental nonlinearities restricted to functions of x(t) whose inverse has a derivative which is a rational function. In the author's opinion, the present method will prove most useful for nonlinear synthesis rather than nonlinear analysis. A complete synthesis theory could be developed by using the system shown in Figure 6 as a basis. Then, by appropriate partitioning, almost any desired configuration could be obtained for single and multiloop feedback systems. Arguments both for and against the method of analysis were given in detail. Note, however, that the analysis given in Section 1 is confined to physical systems but does not include all physical systems. This is evident at once from the use of the Lipschitz condition (Equation 4) which is a sufficient condition for uniqueness only. Thus, any other condition which would assure uniqueness could have been selected, and from that an analysis could have been made. Hence, in Section 2, the analysis was extended to study the behavior of a class of systems that can be described by a nonlinear differential equation, with time-varying parameters, of the form of Equation 96. By placing certain restrictions on the linear, nonlinear, and forcing function terms, the equation was found to have a unique solution which is the limit of a sequence of iterates $\{x_n\}$, each of which together with the system solution belong to an L_2 space.[40] Under the conditions given, uniform convergence of the sequence $\{x_n\}$ to the solution x(t) is guaranteed in the mean sense and the ordinary sense.[41] The solution

[40] Appendix E shows that uniqueness does not imply the convergence of the iterates $\{x_n\}$; the converse is also true.

[41] Appendix I shows that convergence in the mean does not imply convergence in the ordinary sense.

x(t) was shown to remain less than some upper-bound function $\beta(t)$ which also belongs to L_2 space in the period of interest $0 \leq t \leq T$. The upper-bound function $\beta(t)$, which can easily be obtained, was shown to be a linear combination of some functions in $L_2[0, T]$ which are defined from the system restrictions and which depend on the system parameters. Thus, the control engineer can gain further insight into his system by examining the conditions imposed on the system equation; hence, this technique should help solve problems of nonlinear synthesis.

A study of the behavior of the admissible system with respect to initial conditions and the system parameters also was presented. The study showed that the system response x(t) is uniformly continuous with respect to both initial conditions and system parameters. Equation 170 gives an upper bound for the error when the actual solution x(t) is approximated by the n-th iterate x_n in the sense of mean square. The number of iterations n required for this approximation was shown to depend on how the system equation is partitioned. Since the L_2 space is complete, the solution x(t) also will belong to L_2 space. This is not true for the case studied by A. A. Wolf [47]. Wolf demonstrated that the limiting process of a sequence of functions of zero order and exponential type can give rise to functions of higher order that are not of exponential type.

In Section 3 the partitioning technique and the state-variable approach were used to study the bahavior of the system Equation 96 when it was governed by certain conditions. Under these conditions, the state variables $(v_1, v_2, \ldots, v_n)$, which represent the state of the system in the phase space, were shown to belong to L_2 space. By studying the definition of asymptotic stability in the sense of Lyapunov, it was found that the system, as restricted, satisfied that definition. Hence, the restrictions placed on the system were sufficient not only to prove uniqueness of a solution that belongs to L_2 space, but also to guarantee, in the sense of Lyapunov, asymptotic stability. By selecting two different definitions of the norm, the behavior of the system trajectory was examined for two cases: (1) at any time t and (2) on the average during the interval of interest. In many applications, studying the system trajectory on the average is usually unsatisfactory, since the average value may not exceed a certain specified value whereas the actual system response may assume the undesirable shape given in Figure 23. For this reason an expression for an upper-bound state vector inside which the system will always remain during operation was obtained. The upper-bound state vector was found to depend only on the restrictions placed on the system; hence, it can be changed to suit the specific application by changing the system parameters. Although matrices are valuable tools for simplifying the description and study of differential equations, they were not used in this work because all the essential ideas and techniques of the previous analysis could be presented adequately without them. Further, in most engineering applications the common practice is to reduce each problem to the form of equations previously analyzed; therefore it was more useful to impose the restrictions on the system equations as they were. (This procedure becomes clearer in the numerical analysis of Section 4.)

In Section 4, the analysis presented in Sections 2 and 3 was extended to include systems whose dynamic behavior can be represented by a set of n simultaneous nonlinear equations. Sometimes, it is difficult to reduce these equations to the canonic form of a single-degree-of-freedom nonlinear equation of the form studied in Sections 2 and 3. Section 4 showed that a class of multiloop systems exists for which the exact analytic solution can readily be obtained to any degree of accuracy within a prescribed error. This class of systems is defined by the restrictions given in Section 4, and the stability and boundedness of the solution are guaranteed under the assumed conditions that are given there. As an incidental result of the work in that section, the relations between n terminal-pair circuit configurations and multiloop feedback system configurations were investigated and explained. The outputs of the feedback systems were shown to be equivalent to the current response of the circuits, and an example was given to illustrate the ideas and methods of analysis presented.

In Section 5, a systematic procedure for extending the general partitioning method to numerical analysis was presented. The forcing functions, the nonlinear functions, and the time-varying parameters can be specified graphically, tabularly, or by explicit mathematical functions. Equations representing the propagation of the discretization error equations in the approximate solutions were developed. This development made it possible to judge the accuracy of the results and to select the appropriate interval. The method is applicable to the analysis of systems described by a single equation, and can be extended to the analysis of systems described by simultaneous equations. A detailed discussion comparing the given method of analysis with similar methods by Stout and Wolf, Ku, and Dietz was given, and arguments were presented in support of the various approaches.

6.2. QUESTIONS FOR FUTURE WORK

This report has discussed only a few of the many aspects of the analysis of nonlinear systems. Let us cite some of the aspects which still should be investigated.

(1) Exploration of other criteria to include a new class of nonlinear systems. For example, the system restrictions can be chosen so that the solution belongs to L_p space for $p \geq 1$. The norm of the solution x(t), defined as $\|x\|_p = \left\{\int_0^T [x_0(t)]^p dt\right\}^{1/p}$, is to be less than or equal to a given number. Reference 18 gives applications and physical interpretations of the meanings

associated with these norms. They constitute an important class of constraints for the input or the output of a given system, a fact of paramount importance in many engineering problems today. We observe here that when $p = 1$, $p = 2$, and $p = \infty$, the norm of x(t) is equal, respectively, to the area of x(t), the square root of the energy of x(t), and the maximum magnitude of x(t) in the interval [0, T]. (Actually, when $p = \infty$, $\|x\|_p$ is equal to the essential supremum of x(t) in the interval [0, T].)

In order to help the reader follow the analysis when p = 1, it is convenient to make the following assumptions, suggested by H. T. Davis in Reference 11.

(a) The function $x_0(t)$ is integrable and bounded in the interval $0 \leq t \leq T$, where $x_0(t)$ has the same meaning as in our analysis.

(b) The weighted nonlinear function F(t, u, v) satisfies the following two conditions:

(i) F(t, u, v) is integrable and bounded, that is, $F(t, u, v) \leq M$ in a given domain, when M is a finite number.

(ii) F(t, u, v) satisfies the Lipschitz condition $|F(t, u, v_1) - F(t, u, v_2)| \leq L|v_1 - v_2|$ within its domain of definition.

(2) Expansion of the solution x(t), since $x(t) \in L_2[0, T]$, by a linear combination: $c_1\varphi_1(t) + c_2\varphi_2(t) + \ldots$ of an orthonormal system of functions $\varphi_i(t)$ over [0, T], and development of methods for finding recurrent relations for the c's and the general solution of these recurrent relations.

(3) Development of procedures for solving systems such as those discussed in this report, but with random excitations.

(4) Development of a stability theory along other lines of reasoning than those presented here.

(5) Extension of the discretization error equations of Section 5 to include systems described by simultaneous differential equations.

(6) Application of partitioning methods to solve systems whose linear parts consist of both lumped and distributed parameters which can be described by a lumped-distributed parameter nonlinear system as follows: $L_1(D, t)x(t) + L_2(D, t)x(t - u) + N(x) = g(t)$.

Appendix A
THE EXACT SOLUTION OF EQUATION 30

Given:

$$\frac{dx}{dt} + \delta x + \alpha x^2 = \beta \qquad (492)$$

so that t = 0, x = 0.

Separating the variables in Equation 492 gives

$$\frac{dx}{\alpha x^2 + \delta x - \beta} = -dt \qquad (493)$$

Integrating both sides of Equation 493 gives

$$\frac{2}{\sqrt{\delta^2 + 4\alpha\delta}} \log \frac{(\delta + 2\alpha x) - \sqrt{\delta^2 + 4\alpha\beta}}{(\delta + 2\alpha x) + \sqrt{\delta^2 + 2\alpha\beta}} = -t + C \qquad (494)$$

where C is a constant of integration.

Applying the initial conditions gives

$$C = \frac{2}{\sqrt{\delta^2 + 4\alpha\beta}} \log \left[\frac{\delta - \sqrt{\delta^2 + 4\alpha\beta}}{\delta + \sqrt{\delta^2 + 2\alpha\beta}}\right] \qquad (495)$$

Substituting for C in Equation 494 and solving for x gives

$$x = \left[\frac{1}{2\alpha}\sqrt{\delta^2 + 4\alpha\beta}\left\{\frac{1 + \dfrac{\delta - \sqrt{\delta^2 + 2\alpha\beta}}{\delta + \sqrt{\delta^2 + 2\alpha\beta}}\, e^{-at}}{1 - \dfrac{\delta - \sqrt{\delta^2 + 4\alpha\beta}}{\delta + \sqrt{\delta^2 + 4\alpha\beta}}\, e^{-at}}\right\} - \delta\right] \qquad (496)$$

where

$$a = +1/2\sqrt{\delta^2 + 4\alpha\beta} \qquad (497)$$

is the characteristic exponent.

Appendix B
A SUMMARY OF THE L_2 SPACE

Definition (1)

In referring to an L_2 space, sometimes called the Hilbert space, we mean the totality of quadratic integrable functions f(t) defined in the closed interval [0, T] for which the norm, or the length $\|f\|$ as given by $\|f\|^2 = \int_0^T f^2(x)\,dx$ exists and is finite.

Definition (2)

The kernel g(t, u) defined in the square $(0 \leq t \leq T, 0 \leq u \leq T)$ belongs to an L_2 space if the norm, as given by $\|g\|^2 = \int_0^T \int_0^T g^2(t, u)\,dt\,du$ exists and is less than a certain constant N^2.

Fubines Lemma

For the kernel g(t, u) as defined in Definition (2), the functions $A^2(t) = \int_0^T g^2(t, u)\,du$ and $B^2(u) = \int_0^T g^2(t, u)\,dt$ exist almost everywhere for $(0 \leq t \leq T, 0 \leq u \leq T)$ and belong to an L_2 space. Also, $g^2 = \int_0^T A^2(v)\,dv = \int_0^T B^2(v)\,dv$.

Statement

If f is in L_2 and g is in L_2, then $(f \pm f)$ are in L_2. In general, any finite linear combination of functions that are in L_2 belong to L_2. Also, $\|f+g\| \leq \|f\| + \|g\|$; that is, the triangle inequality holds.

Statement

If f is in L_2 and g is in L_2, then the product fg is absolutely integrable and Schwarz's inequality is given by $\left[\int_0^T (fg)\,du\right]^2 \leq \int_0^T f^2(u)\,du \int_0^T g^2(u)\,du$

Statement

If $f(x)^2$ is integrable over a finite interval, then f(x) also is integrable over a finite interval; but the converse is not true. (Note that this complete statement is not true for infinite intervals.)

Theorem

Given a sequence $\{f_n\}$ of L_2 functions, if $\{f_n\}$ converges in the mean to the L_2 function f, that is, $\lim_{n \to \infty} \int_0^T [f_n(t) - f(t)]^2\,dt = 0$, it always follows that $\lim_{\substack{n \to \infty \\ m \to \infty}} \int_0^T (f_m - f_n)^2\,dt = 0$ for $m > n$. The converse is also true.

H. Weyl's Lemma

Given a sequence $\{f_n\}$ of L_2 functions in an interval $(0 \le t \le T)$, if $\underset{n\to\infty}{\text{Lim}} f_n = \varphi(t)$ (convergence in the mean) and $\lim_{n\to\infty} f_n = \psi(t)$ (convergence in the ordinary sense), then $\varphi(t) = \psi(t)$ almost everywhere.

Appendix C[42]

THE IMPULSE RESPONSE OF TIME-VARYING LINEAR SYSTEMS AND THE EXTENSION OF THE ANALYSIS OF SECTION 2 TO SITUATIONS IN WHICH THE FORCING FUNCTION IS $g_1(t) = K(D, t)g(t)$ WHERE K(D, t) IS A LINEAR TIME-VARYING OPERATOR

The behavior of linear time-varying systems can be described in several ways. Usually, the behavior is described implicitly by means of a differential equation relating the input g(t) to the output x(t). This equation is, in general, of the form

$$\left[a_n(t)D^n + \ldots + a_1(t)D + a_0(t)\right]x(t) = \left[b_m(t)D^m + \ldots + b_1(t)Dt + b_0t\right]g(t) \tag{498}$$

where D = d/dt and the a's and b's are known functions of time.

Equation 498 can be written more compactly as

$$L(D, t)x(t) = K(D, t)g(t) \tag{499}$$

where L(D, t) and K(D, t) represent the left-hand and right-hand operators, respectively, of Equation 498. A commonly used form of explicit description of the behavior of system 498 is based on the concept of the impulse response of the system. The impulse response W(t, u) is the solution of the equation:

$$L(D, t)W(t, u) = K(D, t)\delta(t - u) \tag{500}$$

subject to the homogeneous boundary conditions at t = u or, equivalently, to the following conditions: $W^{(\lambda)}(t, u) = 0$ for $\lambda = 0, 1, 2, \ldots, (n - 2)$, and $W^{(n-1)}(t, u) = \frac{1}{a_n(u)} \neq 0$ where $\delta(t - u)$ represents a unit impulse at t = u. The salient property of W(t, u) is that the response to any given input g(t) can be expressed in terms of W(t, u) and g(t) through the superposition integral

$$x(t) = \int_{-\infty}^{\infty} W(t, u)g(u)\,du \tag{501}$$

[42] Most of the material given here was obtained from Reference 49.

If it is assumed that $g(t) = 0$ with $t < 0$ and that a physically realizable system, that is, $W(t, u)$ is always zero for $t < u$, Equation 501 reduces to

$$x(t) = \int_0^t W(t, u)g(u)\,du \tag{502}$$

In the analysis presented in Section 2, a special case of Equation 499 was considered, namely, equations of the form

$$L(D, t)x(t) = g(t) \tag{503}$$

By definition, the impulse response of Equation 503 is identically equal to the Green's function of Equation 499, which is denoted by $G(t, u)$.

Obviously, $G(t, u)$ should have the same homogeneous boundary conditions given to Equation 500. Note that $W(t, u)$ and $G(t, u)$ are closely related to each other (as shown in the following equations).

Treating $K(D, t)\delta(t - u)$ as the input to the system Equation 503 gives

$$W(t, u) = \int_{-\infty}^{\infty} G(t, \tau)\,[K(D, \tau)\,\delta(\tau - u)\,d\tau \tag{504}$$

where τ is the variable of integration and $D = d/d\tau$. Making use of the identity

$$\int_{-\infty}^{\infty} f(\tau)\,D^{(k)}\,\delta(\tau - u)\,d\tau = (-1)^k f^{(k)}(u) \tag{505}$$

permits Equation 504 to be reduced to the desired relation, namely:

$$W(t, u) = b_0(u)\,G(t, u) - \frac{d}{du}\left\{b_1(u)\,[G(t, u)]\right\} + \ldots + (-1)^m \frac{d^m}{d_u^{\,m}}[b_m(u)\,G(t, u)] \tag{506}$$

so that relation 506 can be expressed more compactly as

$$W(t, u) = K^*(D, u)\,[G(t, u)] \tag{507}$$

where D should be interpreted as d/du, and the operator $K^*(D, u)$ is the adjoint of $K(D, u)$. Equation 507 shows immediately that the analysis in Section 2 easily can be extended to a new type of forcing functions of the form:

$$g_1(t) = K(D, t)g(t) \tag{508}$$

It is only necessary to make a simple modification so that the impulse response defined by Equation 98 is operated on by the operator $K^*(D, u)$ and made to yield the new weighting function used in the analysis. No other changes are necessary.

Notice, also, that Equation 507 leads to another conclusion which is characteristic of time-varying linear systems; namely, that the impulse response W(t, u) of the system Equation 498 can be obtained by making use of the impulse response G(t, u) of the simpler system Equation 503.

Appendix D
A PROOF FOR EQUATION 133

It was proved that

$$(x_2 - x_1)^2 < C^2 \alpha^2(t) \tag{509}$$

Substituting Equation 509 into Equation 132 when n = 1 gives

$$(x_3 - x_2)^2 \leq C^2 \alpha^2(t) \int_0^t \alpha^2(u)\, du \tag{510}$$

Repeating the same procedure using Equation 510 and n = 2 gives

$$(x_4 - x_3)^2 \leq C^2 \alpha^2(t) \int_0^t \alpha(u)\, du \int_0^u \alpha^2(z)\, dz \leq C^2 \alpha^2(t) \frac{A^2}{2!} \tag{511}$$

And it is known that $\int_0^t \alpha^2(t)\, du \leq \int_0^t \alpha^2(t)\, du \leq A^2$, and

$$\int_0^t \alpha^2(u)\, du \int_0^u \alpha^2(z)\, dz \int_0^z \alpha^2(\theta)\, d(\theta) \ldots \text{ n times} = \frac{A^{2n}}{n!} \tag{512}$$

Then, repeating the last two steps for n = 3, 5, . . . , gives

$$[x_{n+2}(t) - x_{n+1}(t)]^2 \leq C^2 \alpha^2(t) \frac{A^{(2n)}}{(n)!} \tag{513}$$

From Equation 513 it can be concluded that

$$\int_0^T [x_{n+1}(t) - x_n(t)]^2\, dt \leq C^2 \frac{A^{2n}}{(n!)} \tag{514}$$

hence, the result.

Appendix E
UNIQUENESS AND SUCCESSIVE APPROXIMATIONS [8, p. 53]

In order to show that the convergence of successive approximations does not imply uniqueness, let us consider the following familiar example:

$$\frac{dx}{dt} = x^{1/3} \qquad (t = 0,\ x = 0) \tag{515}$$

$$x_{n+1} = x_0 + \int_0^t \left[x_n(t)\right]^{1/3} du \tag{516}$$

Therefore,

$$x_0 = x_1 = x_2 = x_3 = \ldots = x_n = 0 \tag{517}$$

so that the successive approximations are all zero; hence, they converge to the identically zero solution. On the other hand, the function $\varphi(t)$, defined by

$$\varphi(t) = \frac{2t^{1/3}}{3} \tag{518}$$

is another solution which exists to the right of the origin.

Also, uniqueness does not imply convergence of successive approximations, as shown by the following example:

$$\frac{dx}{dt} = f(x, t) \qquad [t = 0,\ x = 0] \tag{519}$$

where

$$f(x, t) = \begin{cases} 0 & (t = 0,\ -\infty < x < +\infty) \\ 2t & (0 < t \le 1\ -\infty\ x < 0) \\ 2t - \dfrac{4x}{t} & (0 < t \le 1\ 0 \le x \le t^2) \\ -2t & (0 < t \le 1\ t^2 < x < +\infty) \end{cases} \tag{520}$$

in the region $[0 < t \le 1,\ |x| < \infty]$; the function f(x, t) is continuous and is bounded by the constant 2. Therefore, uniqueness is guaranteed since Equation 519 is linear. The successive approximations given by

$$x_{n+1} = x_0 + \int_0^t f(x_n, u)\, du \qquad (\text{for } n = 0, 1, 2, \ldots) \tag{521}$$

become

$$x_0 = 0,\ x_{2n-1}(t) = t^2,\ x_{2n}(t) = -t^2 \qquad (n = 1, 2, \ldots) \tag{522}$$

Therefore, the sequence $\{x_n(t)\}$ has two cluster values for $t \neq 0$; hence, the successive approximations do not converge. Also note that neither of the two convergence subsequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ converges to a solution because

$$\begin{aligned} x'_{2n-1} &= 2t \neq f(t, t^2) \\ x'_{2n} &= -2t \neq f(t, -t^2) \end{aligned} \tag{523}$$

Appendix F
EXTENSION OF THE ANALYSIS TO MORE COMPLEX NONLINEARITIES

In the analyses given in Sections 2, 3, and 4, the weighted nonlinear function F was assumed (in its domain of definition) to be a function of t, u, and one independent variable, say x. In order to extend the analysis to include more complex situations, that is, to include the situation in which F assumes the new form

$$F = F[t, u, x(t), x^{(1)}(t), \ldots, x^{(m)}(t)]$$

(where (m) denotes the m-th derivative of x with respect to the independent variable t, $m \leq n - 1$, and n is the order of the system), one needs more complex conditions in order to prove convergence, uniqueness, and other properties. For example, consider the conditions required for an analysis when F contains x and $x^{(1)}$. In other words, F may assume the form $F = F[t, u, x, x^{(1)}]$. Thus, we need

(1) $x_0^{(i)}(t) \in L_2[0, T]$, where i = 0 or 1, as above, (i) denotes the i-th derivative of $x_0(t)$ with respect to the time t, and $x_0(t)$ is as defined in previous analysis.

(2) In a given domain $D \equiv [|x(t)| \leq d_0(t),\ x^{(1)}t \leq d_1(t),\ 0 \leq u \leq t \leq T]$, the function $F[t, u, x(u), x^{(1)}(u)]$, and its first derivative with respect to t, may satisfy

(a) $$\left|F^{(i)}\left[t, u, z_1, z_1^{(1)}\right] - F^{(i)}\left[t, u, z_2, z_2^{(1)}\right]\right| \leq K(t, u) \sum_{k=0}^{1} \left|z_2^{(k)} - z_1^{(k)}\right|$$

(b) $$\int_0^t F^{(i)}\left[t, u, z, (u), z_1^{(1)}(u)\right] du \leq n(t)$$

where n(t) and K(t, u) are continuous, bounded, and in $L_2[0, T]$.

(3) We also need the constraint conditions:

$$\left| x_0^{(i)}(t) \right| \leq d_i(t)$$

$\left| q_i(t) \right| \leq d_i(t)$, where $q_i(t)$ is given by

$$q_i(t) = \left| x_1^{(i)}(t) \right| + C\,\alpha(t) \sum_{m=0}^{\infty} \frac{2^{(m+2)} A^m}{\sqrt{m!}}$$

where

$$C = \left[\int_0^t n^2(t)\,dt \right]^{1/2}$$

$$\alpha(t) = \left[\int_0^t K^2(t, u)\,du \right]^{1/2}$$

$$A = \left[\int_0^t \alpha^2(t)d(t) \right]^{1/2}$$

$$x_1^{(i)} = x_0^{(i)}(t) - \int_0^t F^{(i)}\left[t, u, x_0^{(u)}, x_0^{(1)}(u)\right] du$$

It is obvious from these conditions that the problem is essentially the same as that which was analyzed in detail in Section 4. Similar conditions are given by Equations 353 through 360. The results, method of analysis, and conclusions are the same.

Appendix G
A PROOF OF EQUATION 289

To prove that

$$\lim_{m \to \infty} x_m^{(k)} \rightarrow x^{(k)} = v_{k+1} \tag{524}$$

where $x_m^{(k)}$ and $x^{(k)}$ are given by

$$x^{(k)} = v_{k+1} = x_0^{(k)}(t) - \int_0^t F^{(k)}[t, u, x(u)]\,du \tag{525}$$

$$x_m^{(k)} = x_0^{(k)}(t) - \int_0^t F^{(k)}[t, u, x_{m-1}(u)]\, du \tag{526}$$

requires two steps: (1) One should prove that $\lim_{m\to\infty} x_m^{(k)}$ exists. (2) One should prove that the limit in step (1) satisfies Equation 524. For step (1), one can form the infinite series

$$x_1^{(k)} + \left(x_2^{(k)} - x_1^{(k)}\right) + \left(x_3^{(k)} - x_2^{(k)}\right) + \dots \tag{527}$$

to an infinity of terms. Note that $x_m^{(k)}$ is the m-th partial sum of the above series; thus, by making use of Equation 288, one gets

$$\left|x_m^{(k)}\right| \leq \left|x_1^{(k)}\right| + \left|\alpha_k(t)\right| C_0 + C_0 \left|\alpha_k(t)\right| \sum_{k=2}^{m} \frac{A_0^{k-1}}{\sqrt{(k-2)!}} \tag{528}$$

Now, applying the M-test on series 527 shows that as $m \to \infty$, $x_m^{(k)}$ has a limit, which proves step (1). For step (2), subtract Equation 525 from Equation 526 and square the result. This gives

$$\left[x_m^{(k)} - v_{k+1}\right]^2 = \left(\int_0^t \left\{F^{(k)}[t, u, x_{m-1}(u)] - F^{(k)}[t, u, x(u)]\right\} du\right)^2$$

$$\left[x_m^{(k)} - v_{k+1}\right]^2 \leq \left[\int_0^t K_k(t, u)\, |x_{m-1}(u) - x(u)|\, du\right]^2 \tag{529}$$

Thus:

$$\left[x_m^{(k)} - v_{k+1}\right]^2 \leq \alpha_k^2(t) \int_0^T |x_{m-1}(u) - x(u)|^2\, du \tag{530}$$

It was proved in Section 2 that

$$\lim_{m\to\infty} x_{m-1}(u) = x(t) \tag{531}$$

Therefore, from Equation 526, it can be concluded that

$$\lim_{m\to\infty} x_m^{(k)} = v_{k+1} \tag{532}$$

There is no need to prove that as $m \to \infty$, $x_m^{(k)} \to v_{k+1}$ uniquely, since the property of uniqueness was proved in Section 2 where it was shown that as m tends to infinity, $v_{m-1}(u)$ tends to $v_1(u)$, uniquely.

Appendix H

A CONTINUOUS FUNCTION WHICH BELONGS TO L_2 [0, ∞] AND DOES NOT HAVE A LIMIT AT $t = +\infty$

A continuous function, which is square integrable on $[0, +\infty)$, need not have a zero limit at $t = +\infty$. This can be seen in Figure 35. From Figure 35 we have $\int_0^\infty f^2(t)\,dt = 1/4 \sum_{n=1}^{\infty} \frac{1}{2} < \infty$. In this example the limit does not exist. In order for the limit to exist, the following lemma [50, p. 86] is needed: if f(t) is uniformly continuous[43] on $[0, +\infty)$, and if $f(t) \in L_p$ for $1 \leq p < \infty$, then $\lim_{t\to\infty} f(t) = 0$; a similar statement holds for a function of more than one variable.

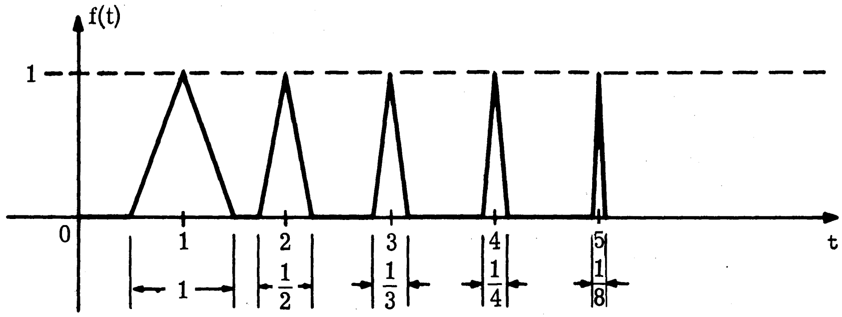

FIGURE 35. A CONTINUOUS FUNCTION f(t) WHICH BELONGS TO L_2 [0, ∞] AND DOES NOT HAVE A LIMIT AT $t = +\infty$

[43] Definition: A function f(x), defined on a set A, is uniformly continuous on the set A if, and only if, corresponding to an arbitrary positive number ε, there exists a positive number $\delta = \delta(\varepsilon)$ only, so that for any x_1 and x_2 belonging to A, $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \varepsilon$. Also, uniform continuity implies continuity; the converse is true only if A is compact.

Appendix I

CONVERGENCE IN THE MEAN AND CONVERGENCE IN THE ORDINARY SENSE

The following example shows that convergence almost everywhere does not imply convergence in the ordinary sense [30, p. 169].

Let a sequence $\{f_n(t)\}$ be defined in [0, 1] by the requirements $f_n(t) = n$ for $0 \leq t \leq \frac{1}{n}$, and $= 0$, otherwise.

For arbitrary $t \in [0, 1]$, $\lim_{n \to \infty} f_n(t) = 0$ almost everywhere; but, at the same time, $\int_0^1 f_n^2(t)\, dt = \int_0^{1/n} n^2\, dt = n$. Therefore, f_n converges to zero almost everywhere, but does not converge to zero in the mean.

The following example shows that convergence in the mean does not imply convergence everywhere [30, p. 96]:

For the half-open interval [0, 1], define the k functions $f_1^{(k)}(t), f_2^{(k)}(t), \ldots, f_k^{(k)}(t)$, for every natural number k, according to the relation

$$f_i^{(k)}(t) = \begin{cases} 1 \text{ for } t \in \left[\frac{i-1}{k}, \frac{i}{k}\right) \\ 0 \text{ otherwise} \end{cases}$$

in particular, $f_1^{(1)}(t) = 1$ on [0, 1). If all these functions are numbered with indexes 1, 2, 3, . . . , the sequence

$$\varphi_1(t) = f_1^{(1)}(t),\ \varphi_2(t) = f_1^{(2)}(t),\ \varphi_3(t) = f_2^{(2)}(t),\ \varphi_4(t) = f_1^{(3)}(t), \ldots$$

can be obtained. Now, it can easily be shown that convergence in the mean is zero and that, at the same time, $\lim_{n\to\infty} \varphi_n(t) = 0$ is fulfilled for <u>no point</u> of t in the interval [0, 1). In fact, if $t_0 \in [0, 1)$, we can find an i for every k so that

$$t_0 \in \left[\frac{i-1}{k}, \frac{i}{k}\right)$$

and $f_i^{(k)}(t_0) = 1$.

In other words, no matter how far we go in the sequence of numbers: $\varphi_1(t_0), \varphi_2(t_0), \varphi_3(t_0)$, . . . , we shall always encounter in this sequence numbers equal to one. Therefore, convergence in the mean does not necessarily imply convergence in the ordinary sense.

Appendix J

INTERPOLATION, APPROXIMATE DIFFERENTIATION, AND APPROXIMATE INTEGRATION

In calculating the derivative or the definite integral of a function by means of a set of given values of that function, the function is usually represented by an interpolation formula which can be differentiated or integrated as desired. Briefly, the general problem of interpolation is to represent a function f(t), known or unknown, in a form $\varphi(t)$, chosen in advance, with the aid of given values which the function f(t) takes for discrete values of the independent variable t. For the purpose of our work the function $\varphi(t)$ usually is chosen as a polynomial and the interpolation is known as polynomial interpolation. The justification for replacing a given function by a polynomial rests on a theorem given by Weierstrass in 1885. This theorem states:

Every function which is continuous in an interval (a, b) can be represented in that interval to any degree of accuracy by a polynomial p(t). It is possible to find this polynomial p(t) so that $|f(t) - p(t)| < \varepsilon$ for every value of t in the interval (a, b) where t is any preassigned positive quantity. This is clarified geometrically in Figure 36. The interpolation formula usually is given in terms of the first, second, and higher order differences which are obtained from a diagonal or horizontal difference table, shown in Tables VI and VII.

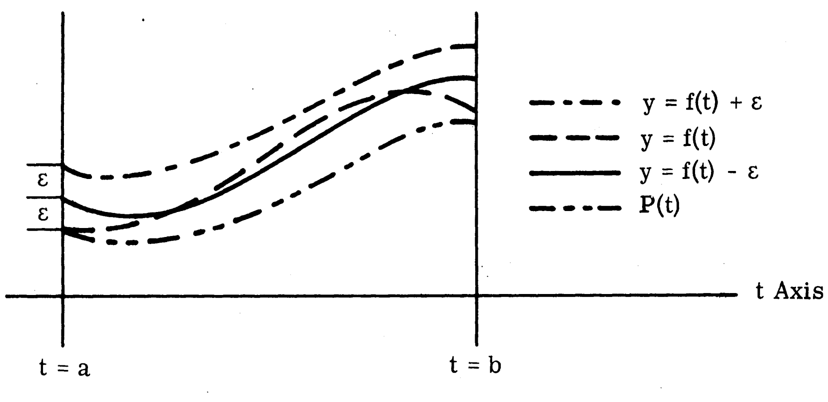

FIGURE 36. GEOMETRICAL MEANING OF THE WEIERSTRASS THEOREM.
---- y = f(t) + ε, --- y = f(t), —— y = f(t) − ε, ---- P(t).

TABLE VI. DIAGONAL DIFFERENCE TABLE

t	x	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$	$\Delta^5 x$	$\Delta^6 x$	$\Delta^7 x$	$\Delta^8 x$
t_0	x_0								
		Δx_0							
t_1	x_1		$\Delta^2 x_0$						
		Δx_1		$\Delta^3 x_0$					
t_2	x_2		$\Delta^2 x_1$		$\Delta^4 x_0$				
		Δx_2		$\Delta^3 x_1$		$\Delta^5 x_0$			
t_3	x_3		$\Delta^2 x_2$		$\Delta^4 x_1$		$\Delta^6 x_0$		
		Δx_3		$\Delta^3 x_2$		$\Delta^5 x_1$		$\Delta^7 x_0$	
t_4	x_4		$\Delta^2 x_3$		$\Delta^4 x_2$		$\Delta^6 x_1$		$\Delta^8 x_0$
		Δx_4		$\Delta^3 x_3$		$\Delta^5 x_2$		$\Delta^7 x_1$	
t_5	x_5		$\Delta^2 x_4$		$\Delta^4 x_3$		$\Delta^6 x_2$		
		Δx_5		$\Delta^4 x_4$		$\Delta^5 x_3$			
t_6	x_6		$\Delta^2 x_5$		$\Delta^4 x_4$				
		Δx_6		$\Delta^4 x_5$					
t_7	x_7		$\Delta^2 x_6$						
		Δx_7							
t_8	x_8								

From Table VI, the diagonal difference table, we obtain

First differences: $\Delta x_0 = x_1 - x_0, \Delta x_1 = x_2 - x_1, \ldots, \Delta x_{n-1} = x_n - x_{n-1}$

Second differences: $\Delta^2 x_0 = \Delta x_1 - \Delta x_0 = x_2 - 2x_1 + x_0,$

$\Delta^2 x_1 = \Delta x_2 - \Delta x_1 = x_3 - 2x_2 + x_1,$

Third differences: $\Delta^3 x_0 = \Delta^2 x_1 - \Delta^2 x_0 = x_3 - 3x_2 + 3x_1 - x_0$

$\Delta^3 x_1 = \Delta^2 x_2 - \Delta^2 x_1 = x_4 - 3x_3 + 3x_2 - x_1$

From Table VII, the horizontal difference table, we obtain

First differences: $\Delta_1 x_1 = x_1 - x_0, \Delta_1 x_2 = x_2 - x_1, \ldots, \Delta_1 x_n = x_n - x_{n-1}$

Second differences: $\Delta_2 x_2 = \Delta_1 x_2 - \Delta_1 x_1 = x_2 - 2x_1 + x_0$

$\Delta_2 x_3 = \Delta_1 x_3 - \Delta_1 x_2 = x_3 - 2x_2 + x_1$

Third differences: $\Delta_3 x_3 = \Delta_2 x_2 - \Delta_2 x_2 = x_3 - 3x_2 + 3x_1 - x_0$

$\Delta_3 x_4 = \Delta_2 x_4 - \Delta_2 x_3 = x_4 - 3x_3 + 3x_2 - x_1$

TABLE VII. HORIZONTAL DIFFERENCE TABLE

t	x	$\Delta_1 x$	$\Delta_2 x$	$\Delta_3 x$	$\Delta_4 x$	$\Delta_5 x$	$\Delta_6 x$	$\Delta_7 x$	$\Delta_8 x$
t_0	x_0								
t_1	x_1	$\Delta_1 x_1$							
t_2	x_2	$\Delta_1 x_2$	$\Delta_2 x_2$						
t_3	x_3	$\Delta_1 x_3$	$\Delta_2 x_3$	$\Delta_3 x_3$					
t_4	x_4	$\Delta_1 x_4$	$\Delta_2 x_4$	$\Delta_3 x_4$	$\Delta_4 x_4$				
t_5	x_5	$\Delta_1 x_5$	$\Delta_2 x_5$	$\Delta_3 x_5$	$\Delta_4 x_5$	$\Delta_5 x_5$			
t_6	x_6	$\Delta_1 x_6$	$\Delta_2 x_6$	$\Delta_3 x_6$	$\Delta_4 x_6$	$\Delta_5 x_6$	$\Delta_6 x_6$		
t_7	x_7	$\Delta_1 x_7$	$\Delta_2 x_7$	$\Delta_3 x_7$	$\Delta_4 x_7$	$\Delta_5 x_7$	$\Delta_6 x_7$	$\Delta_7 x_7$	
t_8	x_8	$\Delta_1 x_8$	$\Delta_2 x_8$	$\Delta_3 x_8$	$\Delta_4 x_8$	$\Delta_5 x_8$	$\Delta_6 x_8$	$\Delta_7 x_8$	$\Delta_8 x_8$

Note that $\Delta^3 x_1 = \Delta_3 x_4$ and, in general, the relations between the Δ's with exponents and those with subscripts are

$$\Delta^m x_k = \Delta_m x_{k+m} \quad \text{(going forward from } x_k\text{)}$$

or (533)

$$\Delta_m x_n = \Delta^m x_{n-m} \quad \text{(going backward from } x_n\text{)}$$

where m denotes the order of differences, and k and n denote the number of tabulated values. In the solution of differential equations, the discrete values of the functions for previous times are always known, but those for future times are always unknown. The horizontal difference table shows all orders of differences in terms of the last known values of the function; these differences can be used to compute the next value of the function. For this reason, the horizontal difference table is more convenient than the diagonal one. In situations where it is necessary to interpolate data at the beginning of the table or near the middle of the tabulated set, one usually uses formulas containing differences given in the diagonal difference table. Thus, for example, Newton's formula for forward interpolation is given by

$$\varphi(t) = \varphi(t_0 + uh) = g(u) = (1 + \Delta)^u x_0 = x_0 + u \Delta x_0 + \frac{u(u-1)}{2} \Delta^2 x_0 + \frac{u(u-1)(u-2)}{3} \Delta^3 x_0 + \ldots + \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!} \Delta^n x_0 \quad (534)$$

where x_0 = the value of the function at $t = t_0$

h = interval width = $x_{s+1} - x_s$ and is assumed uniform, i.e., a constant independent of x, and s = 0, 1, . . . , n - 1

$u = \frac{t - t_0}{h}$ or $t = t_0 + hu$

$\Delta^n x_0$ is as explained in Table VI

Equation 534 is used when the interpolation is near the point (t_0, x_0). Note that the interpolation function $\varphi(t)$ of Equation 534 is an n-th degree polynomial representation of the original function which is given in terms of the values $x_0, x_1, \ldots$, and which is the value of the function for the values $t_0, t_1, \ldots$, of the independent variable. If interpolation is required near the center of the given tabular set, a central difference formula is usually adopted. For example, we can use Bessel's central difference formula:

$$x = x_0 + u\,\Delta x_0 + \frac{u(u-1)}{2}\,\frac{\Delta^2 x_{-1} + \Delta^2 x_0}{2} + \frac{(u-1/2)(u)(u-1)}{3!}\,\Delta^3 x_1$$

$$+ \frac{u(u-1)(u+1)(u-2)}{4!}\,\frac{\Delta^4 x_{-2} + \Delta^4 x_{-1}}{2} + \frac{(u-1/2)(u)(u-1)(u+1)(u-2)}{5!}\,\Delta^5 x_{-2}$$

$$+ \frac{u(u-1)(u+1)(u-2)(u+2)(u-3)}{6!}\,\frac{\Delta^6 x_{-3} + \Delta^6 x_{-2}}{2}$$

$$+ \frac{u(u-1)(u+1)(u-2)(u+2)\ldots(u-n)(u+n-1)}{(2n)!} \times \frac{\Delta^{2n} x_{-n} + \Delta^{2n} x_{-n+1}}{2} \qquad (535)$$

where the difference appearing in Equation 535 can be obtained by a special consideration of the quantities lying as near as possible to the horizontal line drawn halfway between x_0 and x_1. These quantities are shown in Table VIII.

If u = 1/2 in Equation 535, the formula of <u>interpolating to halves</u> can be obtained:

$$x = \frac{x_0 + x_1}{2} - \frac{1}{8}\,\frac{\Delta^2 x_{-1} + \Delta^2 x_0}{2} + \frac{3}{128}\,\frac{\Delta^4 x_{-2} + \Delta^4 x_{-1}}{2} + \frac{5}{1028}\,\frac{\Delta^6 x_{-3} + \Delta^6 x_{-2}}{2} + \ldots,$$

$$+ \frac{(-1)^n[1 \cdot 3 \cdot 5 \cdot \ldots, (2n-1)]^2}{2^{2n}(2n)!}\,\frac{\Delta^{2n} x_{-n} + \Delta^{2n} x_{-n+1}}{2} \qquad (536)$$

TABLE VIII. CENTRAL DIFFERENCE TABLE

x	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$	$\Delta^5 x$	$\Delta^6 x$	$\Delta^7 x$	$\Delta^8 x$
x_{-4}								
	Δx_{-4}							
x_{-3}		$\Delta^2 x_{-4}$						
	Δx_{-3}		$\Delta^3 x_{-4}$					
x_{-2}		$\Delta^2 x_{-3}$		$\Delta^4 x_{-4}$				
	Δx_{-2}		$\Delta^3 x_{-3}$		$\Delta^5 x_{-4}$			
x_{-1}		$\Delta^2 x_{-2}$		$\Delta^4 x_{-3}$		$\Delta^6 x_{-4}$		
	Δx_{-1}		$\Delta^3 x_{-2}$		$\Delta^5 x_{-3}$		$\Delta^7 x_{-4}$	
x_0		$\Delta^2 x_{-1}$		$\Delta^4 x_{-2}$		$\Delta^6 x_{-3}$		$\Delta^8 x_{-4}$
	Δx_0		$\Delta^3 x_{-1}$		$\Delta^5 x_{-2}$		$\Delta^7 x_{-3}$	
x_1		$\Delta^2 x_0$		$\Delta^4 x_{-1}$		$\Delta^6 x_{-2}$		$\Delta^8 x_{-3}$
	Δx_1		$\Delta^3 x_0$		$\Delta^5 x_{-1}$		$\Delta^7 x_{-2}$	
x_2		$\Delta^2 x_1$		$\Delta^4 x_0$		$\Delta^6 x_{-1}$		
	Δx_2		$\Delta^3 x_1$		$\Delta^5 x_0$			
x_3		$\Delta^2 x_2$		$\Delta^4 x_1$				
	Δx_3		$\Delta^3 x_2$					
x_4		$\Delta^2 x_3$						
	Δx_4							
x_5								

If interpolation is needed near the end of a set of tabular values, or if extrapolation is required a short distance ahead, Newton's formula for backward interpolation should be used as given below (where horizontal difference tables are used):

$$\varphi(t) = \varphi(t_n + h_u) = \psi(u)$$

$$= x_n + u\,\Delta_1 x_n + \frac{u(u+1)}{2}\Delta_2 x_n + \frac{u(u+1)(u+2)}{3!}\Delta_3 x_n$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\Delta_4 x_n + \ldots + \frac{u(u+1)(u+2)\ldots(u+n-1)}{n!}\Delta_n x_n \qquad (537)$$

The relative accuracy of the interpolation formulas is investigated in detail in Reference 35.

The problem of calculating the derivatives of a function by means of a set of given values of that function is solved by representing the function by an interpolation formula such as those given by Equations 534, 535, and 536 and differentiating this formula as many times as desired. The considerations governing the choice of a formula employing differences are the same as those governing interpolation. That is, if the derivative of a function at a point near the beginning of a set of tabular values is desired, use is made of Newton's formula given in Equation 534; but, if the derivative at a point near the end of a table is desired, use is made of Newton's formula given in Equation 537. For points near the middle of the table, a central difference formula should be used, for example, Bessel's formula given in Equation 535. When the above formulas are used to obtain derivatives, one should make the following substitution: $u = \frac{t - t_0}{h}$, and $\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt} = \frac{1}{h}\left(\frac{dx}{du}\right)$. For example, using Newton's formula 534 to calculate the first few derivatives at $t = t_0$ yields

$$\begin{aligned} \left(\frac{dx}{dt}\right) &= \frac{1}{h}\left[\Delta x_0 + \frac{2u - 1}{2!}\Delta^2 x_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 x_0 + \ldots\right] \\ \left(\frac{d^2x}{dt^2}\right) &= \frac{1}{h^2}\left[\frac{2}{2!}\Delta^2 x_0 + \frac{6u - 6}{3!}\Delta^3 x_0 + \ldots\right] \\ \left(\frac{d^3x}{dt^3}\right) &= \frac{1}{h^3}\left[\frac{6}{3!}\Delta^3 x_0 + \ldots\right] \end{aligned} \tag{538}$$

At $t = t_0$, $u = 0$, Equations 538 reduce to the following:

$$\begin{aligned} \left(\frac{dx}{dt}\right)_{t_0} &= \frac{1}{h}\left[\Delta x_0 - \frac{\Delta^2 x_0}{2!} + \frac{2}{3!}\Delta^3 x_0 \ldots\right] \\ \left(\frac{d^2x}{dt^2}\right)_{t_0} &= \frac{1}{h^2}\left[\Delta^2 x_0 - \Delta^3 x_0 + \ldots\right] \\ \left(\frac{d^3x}{dt^3}\right)_{t_0} &= \frac{1}{h^3}\left[\Delta^3 x_0\right] \end{aligned} \tag{539}$$

Evidently, we can find the derivatives in exactly the same way by differentiating the other given interpolation formulas.

The problem of numerical integration, like that of numerical differentiation, is solved by representing the integrand by an interpolation formula and then integrating the formula between the desired limits. In this way a quadrature formula is derived; the process is known as mechanical quadrature if it is concerned with functions of a single variable, as is the present case. Integrating Newton's formula 534 with the limits of integration taken as $t = t_0$ and $t = t_0 + nh$ or $u = 0$ and $u = n$ gives:

$$I = \int_{t_0}^{t_0+nh} x(t)\,dt = h\int_0^n \left[x_0 + u\Delta x_0 + \frac{u(u-1)}{2}\Delta^2 x_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 x_0 + \ldots\right] du$$

or (540)

$$I = h\left[nx_0 + \frac{n^2}{2}\Delta x_0 + \frac{n^3}{3} - \frac{n^2}{2}\frac{\Delta^2 x_0}{2!} + \frac{n^4}{4} - n^3 + n^2\frac{\Delta^3 x_0}{3!} + \ldots\right]$$

From Equation 540, several well-known special formulas can be obtained: the trapezoidal rule, Simpson's third rule, Simpson's three-eighths rule, Weddle's rule, and others. In the case of the trapezoidal rule, for example, n = 1 is assumed and therefore all terms containing $\Delta^2 x_0$ and higher differences are rejected, i.e., the curve joining the two points is assumed to be a straight line. Thus, it can be shown that

$$\int_{t_0}^{t_0+h} = h\left[x_0 + \frac{1}{2}\Delta x_0\right] = h\left[x_0 + \frac{1}{2}(x_1 - x_0)\right] = \frac{h}{2}[x_0 + x_1]$$

Similarly,

$$\int_{t_0+h}^{t_0+2h} = \frac{h}{2}[x_1 + x_2], \text{ and so on}$$

Adding all such expressions gives integration from t_0 to $t_0 + nh$ (when n is even or odd), as follows:

$$\int_{t_0}^{t_0+nh} = \frac{h}{2}[x_0 + 2x_1 + 2x_2 + \ldots + x_n] = \frac{h}{2}\sum_{i=0}^{n} \gamma_i x_i \qquad (541)$$

where $\gamma_i = (1, 2, 3, \ldots)$. For Simpson's one-third rule, let n = 2 and neglect all differences beyond the second in the general formula 540. Then use three ordinates at one time to obtain

$$\int_{t_0}^{t_0+2h} x\,dt = \frac{h}{3}(x_0 + 4x_1 + x_2)$$

$$\int_{t_0}^{t_0+4h} x\,dt = \frac{h}{3}(x_2 + 4x_3 + x_4), \text{ and so on.}$$

By adding all such expressions, the following formula by Simpson is obtained:

$$I = \int_{t_0}^{t_0+nh} x(t)\,dt = \frac{h}{3}[x_0 + 4(x_1 + x_3 + . + x_{n-2}) + 2(x_2 + x_4 + \ldots + x_{n-1}) + x_n]$$

$$= \frac{h}{3}\sum_0^n \gamma_i x_i, \tag{542}$$

when $\gamma_i = 1, 4, 2, \ldots, 4, 2, 1$, and n even.

If n = 6 and all differences beyond 6 are neglected, and the six coordinates $(x_0, x_1, \ldots, x_6)$ are used at one time, Weddle's rule is obtained. For the first six intervals, this gives

$$\int_{t_0}^{t_0+6h} x\,dt = \frac{3h}{10}(x_0 + 5x_1 + x_2 + 6x_3 + x_4 + 5x_5 + x_6)$$

For the next six intervals, from x_6 to x_{12}, it gives

$$\int_{t_6}^{t_{12}} x\,dt = \frac{3h}{10}(x_6 + 5x_7 + x_8 + 6x_9 + x_{10} + 5x_{11} + x_{12})$$

Finally, adding these expressions gives

$$\int_{t_0}^{t_0+nh} x\,dt + \frac{3h}{10}\sum_0^n \gamma_i x_i \tag{543}$$

where $\gamma_i = 1, 5, 1, 6, 1, 5, 2, 5, 1, 5, 2$, etc., and n is a multiple of 6.

A computer should have some means of estimating the reliability of every computed result. There are some means of estimating the magnitude of the majority of unavoidable errors. As an example, the error committed in the evaluation of an integral by using the trapezoidal rule, usually known as the truncation error, is given by

$$E_{T_1} \leq + \frac{h^2 T}{12} x''_m \qquad (544)$$

where T = nh = the interval of integration

$x''_m(t)$ = maximum value of the second derivative of the integrand x(t)

A more optimistic estimate of the error is given by the equation

$$E_{T_1} \leq + \frac{h^2}{12} [x'(nh) - x'(0)] \qquad (545)$$

where x'(nh) and x'(0) are the first derivatives of the integrand at the end and the beginning of the interval T. Since the first derivative is the integral of the second,

$$x'(nh) - x'(0) = \int_0^{nh} x''(t)\, dt = T\, x''_{av} \qquad (546)$$

where x''_{av} denotes the average value of the second derivative of the integrand in the interval T. Therefore, Equation 545 can be written

$$E_{T_2} \leq + \frac{Th^2}{12} x''_{av} \qquad (547)$$

Equation 545 is much easier to use since it contains only the difference between two easily located first derivatives and not the maximum value of the second derivative, the determination of which is more difficult. Moreover, Equation 545 gives a fairly accurate estimate of the truncation error for reasonable values of T. For these reasons, Equation 545 was used in the calculations. Of course the first derivative can easily be obtained, to any degree of accuracy, from the previous considerations in this appendix. In estimating a given integral by Simpson's rule, the truncation errors committed are undoubtedly less than those resulting from use of the trapezoidal rule, but they are more difficult to obtain. It can be shown that the truncation error in Simpson's rule is given by the following equation [35, Chap. 8]:

$$E_s = - \frac{Th^4}{180} x^{(IV)}_{max.} \qquad (548)$$

where $x^{(IV)}_{max}$ is the maximum value of the fourth derivative of the integrand during the interval T.

It is not always possible for a computer to have an explicit formula which gives the value of the truncation error committed in the evaluation of a given result. One example may be the Runge-Kutta method. It is true that Bieberback[44] has found an expression that providés an upper-bound for the error at a given step in the Runge-Kutta process, but since this estimate depends on quantities that do not appear directly in the computation, some additional separate calculations are required. For this reason, the author suggests the use of the general method of Section 5.2 as a means for calculating truncation errors on the digital computer.

[44] L. Bieberback, Theory of Differential Equations, New Dover Publications, New York, p. 45.

REFERENCES

1. Beckenbach, E. F., Modern Mathematics for Engineers, University of California, Engineering Extension Series, McGraw-Hill, New York, N. Y., 1956, p. 406.
2. Bilal, A. Y. and Kazda, L. F., "The Analysis of a Certain Class of Nonlinear Time Varying Differential Equations," presented at the Joint Automatic Control Conference at New York University, New York, June 1962, and accepted for publication in the Trans. Am. Inst. Elec. Eng.
3. Birkhoff, G. D. and Kellogg, O. D., "Invariant Points in Function Space," Trans. Am. Math. Soc., 1922, Vol. 23, pp. 96-115.
4. Bochner, S., Lectures on Fourier Integrals, Princeton University Press, Princeton, N. J., 1959, Chap. VIII.
5. Bohn, E. V., "Concerning the Partition Theory and Associated Transform Methods," Proc. I.R.E., July 1961, Vol. 49, No. 7, p. 1215.
6. Bulletin of the National Research Council, Division of Physical Sciences, "Numerical Integration of Differential Equations," No. 92, 1953.
7. Cesari, L., "Problems of Asymptotic Behavior and Stability," Am. Inst. Elec. Eng., Applications and Industry, September 1961, Vol. 80.
8. Coddington, E. A. and Levinson, N. J., Theory of Ordinary Differential Equations, McGraw-Hill, New York, N. Y., 1955.
9. Crandall, N. Y., Engineering Analysis, A Survey of Numerical Procedures, McGraw-Hill, New York, N. Y., 1956.
10. Cunningham, W. J., Introduction to Nonlinear Analysis, McGraw-Hill, New York, N. Y., 1958, p. 122.
11. Davis, H. T., Introduction to Nonlinear Differential and Integral Equations, United States Atomic Energy Commission, 1961.
12. Ingewerson, D. R., "Principal Definitions of Stability," Am. Inst. Elec. Eng. Workshop on Lyapunov's Second Method, September 1960, L. F. Kazda (ed.), The University of Michigan, Ann Arbor, Mich.
13. Ince, E. L., Ordinary Differential Equations, Dover Publications, Inc., New York, 1956, pp. 296-403.
14. Kalman, R. E and Betram, J. E., "Control System Analysis and Design Via the Second Method of Lyapunov," Trans. ASME, Ser. D., June 1960, Vol. 82D, pp. 371-393.
15. Kamke, E., Differentialgleichungen, Lösungsmethoden und Lösungen, Leipzig, 1943, Vol. 1.
16. Kantorovitch, L. V. and Akilov, G. P., Functional Analysis in Normed Spaces, Moscow, 1959.
17. Kaplan, W., Operational Methods for Linear Systems, Addison Wesley, Reading, Mass., 1962, Chap. 8.
18. Kranc, G. M. and Sarachik, P. E., "An Application of Functional Analysis to an Optimal Control Problem," to appear in Trans. ASME, Ser. D., 1963.
19. Krylov, H. and Bogolyubov, N., Introduction to Nonlinear Mechanics, English trans. edited by S. Lefschetz, Princeton University Press, Princeton, N. J., 1943.

20. Ku, Y. H., "Analysis of Nonlinear Coupled Circuits II," Trans. Am. Inst. Elec. Eng., September 1955, Vol. 74, pp. 439-43.

21. Ku, Y. H., "Analysis of Nonlinear Coupled Circuits," Trans. Am. Inst. Elec. Eng., 1954, Vol. 73, pp. 626-31.

22. Ku, Y. H. and Wolf, A. A., "Transform-Ensemble Method for the Analysis of Linear and Nonlinear Systems with Random Inputs," Proc. Nat. Electron. Conf., 1960, Vol. 15, pp. 441-55.

23. Ku, Y. H., "Theory of Nonlinear Control," Proc. First Intern. Symp. Autom. Control, Moscow, August 1960.

24. Lalesco, T., Introduction A La Théorie Des Équations Intégrals, Paris, A. Herrmann, 1912, p. 17.

25. Lanning, J. H. and Battin, R. W., Random Process in Automatic Control, McGraw-Hill, New York, N. Y., 1956, p. 123.

26. Mikhlin, S. G., Integral Equations and their Applications to Certain Problems in Mathematical Physics and Technology, transl. from the Russian by A. A. Armstrong, Program Press, New York, N. Y., 1957.

27. Milne, W. E., Numerical Solution of Differential Equations, John Wiley and Sons, Inc., New York, 1953.

28. Minorsky, N., Introduction to Nonlinear Mechanics, Edwards Brothers, New York, 1943.

29. Murray, F. J. and Miller, K. S., Existence Theorems for Ordinary Differential Equations, New York University Press, New York, 1959.

30. Natanson, I. P., Theory of Functions of Real Variables, transl. from the Russian by L. F. Boron and E. Hewitt, Fredrickingon Publishing Company, New York, Vol. 1.

31. Olmsted, J. M. H., Real Variables, An Introduction to Theory of Functions, Appleton-Century-Crofts, Inc., New York, 1956, p. 72.

32. Pipes, L. A., "Applications of Integral Equations to the Solution of Nonlinear Electric Circuit Problems," Trans. Am. Inst. Elec. Eng., 1953, Vol. 72, pp. 445-50

33. Poincaré, H., Les Méthodes Nouvelles de la Méchanique Céleste, Paris, 1892, 1893, 1899, Vols. 1-3.

34. Richardson, C. H., Introduction to the Calculus of Finite Differences, D. Van Nostrand Company, Inc., New York, 1954.

35. Scarborogh, J. B., Numerical Mathematical Analysis, 4th Edition, Johns Hopkins Press, Baltimore, Md., 1958.

36. Stout, T. M., "A Step-by-Step Method for Transient Analysis of Feedback Systems with One Nonlinear Element," Am. Inst. Elec. Eng. Applications and Industry, January 1957, Vol. 28.

37. Titchmarch, E. C., Theory of Functions, 2nd ed., Oxford University Press, London, 1939, p. 248.

38. Tricomi, F. G., Integral Equations, Interscience Publishers, New York and London, 1957, Vol. V.

39. Volterra, V., Theory of Functional and of Integral and Integro-Differential Equations, Dover, New York, 1959.

40. Wiener, N., The Fourier Integral and Certain of its Applications, Dover, New York, 1933.

41. Wolf, A. A., "On the Significance of Transients and Steady State Behavior in Nonlinear Systems," Proc. I.R.E., October 1959, Vol. 47, No. 10, p. 1786.

42. Wolf, A. A., Ku, Y. H., and Dietz, J. H., "Systematic Approximation to the Partition Method," Am. Inst. Elec. Eng., Applications and Industry, July 1960, Vol. 1, p. 49.

43. Wolf, A. A., "Recent Advances in Analysis and Synthesis of Nonlinear Systems," Am. Inst. Elec. Eng., Applications and Industry, November 1961, Vol. 37, p. 289.

44. Wolf, A. A., "Analysis of Transdental Nonlinear Systems," Trans. Am. Inst. Elec. Eng., Vol. 80, in press 1961.

45. Wolf, A. A., "Generalized Recurrence Relations in the Analysis of Physical Nonlinear Systems," Am. Inst. Elec. Eng., Communication and Electronics, September 1961, p. 383.

46. Wolf, A. A., "Recurrence Relations in the Solution of a Certain Class of Nonlinear Systems," Trans. Am. Inst. Elec. Eng., January 1960, Vol. 78, No. 46, pp. 830-34.

47. Wolf, A. A., "A Mathematical Theory of the Analysis of a Class of Nonlinear Systems," Doctoral dissertation, University of Pennsylvania, Philadelphia, Pa., June 1958.

48. Zaanen, A. C., Linear Analysis, North-Holland Publishing Company, Amsterdam, 1956, p. 86.

49. Zadeh, L. A., "Frequency Analysis of Variable Networks," Proc. I.R.E., 1950, pp. 38, 291-99.

50. Zadeh, L. A., "Determination of the Impulse Response of a Variable Network," J. Appl. Phys., July 1950, Vol. 21, No. 7.

PROJECT MICHIGAN DISTRIBUTION LIST 5
1 August 1963—Effective Date

Copy No.	Addressee
1	Commanding General U.S. Army Electronics Command Fort Monmouth, New Jersey ATTN: AMSEL-RD
2-3	Commanding General U.S. Army Electronics Command Fort Monmouth, New Jersey ATTN: AMSEL-CB
4-33	Commanding Officer U.S. Army Electronics R & D Laboratory Fort Monmouth, New Jersey ATTN: SELRA/ADT
34	Commanding General U.S. Army Electronics Proving Ground Fort Huachuca, Arizona ATTN: Technical Library
35-38	Director, U.S. Army Engineer Geodesy Intelligence & Mapping R & D Agency Fort Belvoir, Virginia
(35)	ATTN: Intelligence Division
(36)	ATTN: Research & Analysis Division
(37)	ATTN: Photogrammetry Division
(38)	ATTN: Strategic Systems Division (ENGGM-SSO)
39	Director, U.S. Army Cold Regions Research & Engineering Laboratory P. O. Box 282 Hanover, New Hampshire
40-41	Director, U.S. Army Engineers Research & Development Laboratory Fort Belvoir, Virginia ATTN: Technical Documents Center
42	Commanding Officer U.S. Army Research Office (Durham) Box CM, Duke Station Durham, North Carolina ATTN: Chief Information Processing Office
43	Assistant Commandant U.S. Army Air Defense School Fort Bliss, Texas
44	Commandant U. S. Army Engineer School Fort Belvoir, Virginia ATTN: ESSY-L
45	Commanding Officer U. S. Army Combat Development Command Intelligence Agency Fort Holabird, Baltimore 19, Maryland
46	Commanding Officer, U. S. Army Electronic Research Unit P. O. Box 205 Mountain View, California ATTN: Electronic Defense Laboratories
47	U.S. Army Research Liaison Office MIT-Lincoln Laboratory Lexington 73, Massachusetts
48-49	Office of Naval Research Department of the Navy 17th & Constitution Avenue, N.W. Washington 25, D. C.
(48)	ATTN: Code 463
(49)	ATTN: Code 461
50	The Hydrographer U. S. Navy Hydrographic Office Washington 25, D. C. ATTN: Code 1640
51-52	Director, U. S. Naval Research Laboratory Washington 25, D. C. ATTN: Code 2027
53	Commanding Officer U. S. Navy Ordnance Laboratory Corona, California ATTN: Library
54	Commanding Officer & Director U.S. Navy Electronics Laboratory San Diego 52, California ATTN: Library
55	Commander, U.S. Naval Ordnance Laboratory White Oak Silver Spring, Maryland ATTN: Technical Library
56-80	DDC (TIPCR) Cameron Station Alexandria, Virginia
81-86	Aeronautical Systems Division (ASD) Wright-Patterson AFB, Ohio
(81-83)	ATTN: ASD (ASRNOO)
(84)	ATTN: ASD (ASAPR-D)
(85-86)	ATTN: ASD (ASRNGE-1)
87-89	Commander, Rome Air Development Center Griffiss AFB, New York
(87)	ATTN: RAALD
(88)	ATTN: RAWIC
(89)	ATTN: RALSS
90-94	Central Intelligence Agency 2430 E. Street, N.W. Washington 25, D. C. ATTN: OCR Mail Room
95-96	Scientific & Technical Information Facility P. O. Box 5700 Bethesda, Maryland ATTN: NASA Representative
97-98	National Aeronautics & Space Administration Manned Space Craft Center Houston 1, Texas ATTN: Chief, Technical Information Division
99	Cornell Aeronautical Laboratory, Incorporated Washington Projects Office Falls Church, Virginia ATTN: Technical Library
100	The Rand Corporation 1700 Main Street Santa Monica, California ATTN: Library
101	Research Analysis Corporation 6935 Arlington Road Bethesda, Maryland Washington 14, D. C. ATTN: Chief, Information and Control Systems Division
102-103	Cornell Aeronautical Laboratory, Incorporated 4455 Genesee Street Buffalo 21, New York ATTN: Librarian VIA: Bureau of Naval Weapons Representative 4455 Genesee Street Buffalo 21, New York

PROJECT MICHIGAN DISTRIBUTION LIST 5 (Continued)

Copy No.	Addressee
104	U.S. Naval Photographic Interpretation Center 4301 Suitland Road Washington 25, D. C.
105	Coordinated Science Laboratory University of Illinois Urbana, Illinois ATTN: Librarian VIA: ONR Resident Representative 605 S. Goodwin Avenue Urbana, Illinois
106	The Ohio State University Research Fou 1314 Kinnear Road Columbus 12, Ohio ATTN: Security Office VIA: Commander, Wright Air Development Division Wright-Patterson AFB, Ohio ATTN: ASRKSE
107-109	Jet Propulsion Laboratory California Institute of Technology 4800 Oak Grove Drive Pasadena, California
110-111	Battelle Memorial Institute 505 King Avenue Columbus 1, Ohio ATTN: RACIC VIA: Advanced Research Projects Agency Washington 25, D. C.
112	Bureau of Naval Weapon Department of the Navy Washington 25, D. C. ATTN: Code RRRE-3
113	Commanding General U. S. Army Materiel Command Washington 25, D. C. ATTN: AMCRD-RS-PE-E
114	Commanding Officer, U. S. Army Liaison Group, Project MICHIGAN The University of Michigan P. O. Box 618 Ann Arbor, Michigan

AD Div. 15/2

Inst. of Science and Technology, U. of Mich., Ann Arbor THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS, by A. Y. Bilal. Sept. 63. 151 p. incl. illus tables, 50 refs.
(Report No. 2900-392-T)
(Contract DA-36-039 SC-78801/GP-524)
(Project No. 3D5801001) Unclassified report

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{X_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for
(over)

UNCLASSIFIED

I. Bilal, A. Y.
II. U. S. Army Electronics Command/National Science Foundation
III. Contract DA-36-039 SC-78801/GP-524
IV. Project No. 3D5801001

Defense Documentation Center
UNCLASSIFIED

AD Div. 15/2

Inst. of Science and Technology, U. of Mich., Ann Arbor THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS, by A. Y. Bilal. Sept. 63. 151 p. incl. illus tables, 50 refs.
(Report No. 2900-392-T)
(Contract DA-36-039 SC-78801/GP-524)
(Project No. 3D5801001) Unclassified report

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{X_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for
(over)

UNCLASSIFIED

I. Bilal, A. Y.
II. U. S. Army Electronics Command/National Science Foundation
III. Contract DA-36-039 SC-78801/GP-524
IV. Project No. 3D5801001

Defense Documentation Center
UNCLASSIFIED

AD Div. 15/2

Inst. of Science and Technology, U. of Mich., Ann Arbor THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS, by A. Y. Bilal. Sept. 63. 151 p. incl. illus tables, 50 refs.
(Report No. 2900-392-T)
(Contract DA-36-039 SC-78801/GP-524)
(Project No. 3D5801001) Unclassified report

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{X_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for
(over)

UNCLASSIFIED

I. Bilal, A. Y.
II. U. S. Army Electronics Command/National Science Foundation
III. Contract DA-36-039 SC-78801/GP-524
IV. Project No. 3D5801001

Defense Documentation Center
UNCLASSIFIED

AD Div. 15/2

Inst. of Science and Technology, U. of Mich., Ann Arbor THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS, by A. Y. Bilal. Sept. 63. 151 p. incl. illus tables, 50 refs.
(Report No. 2900-392-T)
(Contract DA-36-039 SC-78801/GP-524)
(Project No. 3D5801001) Unclassified report

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{X_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for
(over)

UNCLASSIFIED

I. Bilal, A. Y.
II. U. S. Army Electronics Command/National Science Foundation
III. Contract DA-36-039 SC-78801/GP-524
IV. Project No. 3D5801001

Defense Documentation Center
UNCLASSIFIED

AD

the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters. Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov. The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy. For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis. The method lends itself easily to digital computer calculations.

UNCLASSIFIED

DESCRIPTORS

Nonlinear systems
Control
Analysis
Stability
Synthesis
Differential equations

UNCLASSIFIED

AD

the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters. Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov. The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy. For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis. The method lends itself easily to digital computer calculations.

UNCLASSIFIED

DESCRIPTORS

Nonlinear systems
Control
Analysis
Stability
Synthesis
Differential equations

UNCLASSIFIED

AD

the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters. Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov. The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy. For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis. The method lends itself easily to digital computer calculations.

UNCLASSIFIED

DESCRIPTORS

Nonlinear systems
Control
Analysis
Stability
Synthesis
Differential equations

UNCLASSIFIED

AD

the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters. Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov. The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy. For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis. The method lends itself easily to digital computer calculations.

UNCLASSIFIED

DESCRIPTORS

Nonlinear systems
Control
Analysis
Stability
Synthesis
Differential equations

UNCLASSIFIED

AD Div. 15/2

Inst. of Science and Technology, U. of Mich., Ann Arbor
THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS, by A. Y. Bilal. Sept. 63. 151 p. incl. illus tables, 50 refs.
(Report No. 2900-392-T)
(Contract DA-36-039 SC-78801/GP-524)
(Project No. 3D5801001) Unclassified report

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{X_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for
(over)

UNCLASSIFIED

I. Bilal, A. Y.
II. U. S. Army Electronics Command/National Science Foundation
III. Contract DA-36-039 SC-78801/GP-524
IV. Project No. 3D5801001

Defense Documentation Center
UNCLASSIFIED

AD Div. 15/2

Inst. of Science and Technology, U. of Mich., Ann Arbor
THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS, by A. Y. Bilal. Sept. 63. 151 p. incl. illus tables, 50 refs.
(Report No. 2900-392-T)
(Contract DA-36-039 SC-78801/GP-524)
(Project No. 3D5801001) Unclassified report

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{X_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for
(over)

UNCLASSIFIED

I. Bilal, A. Y.
II. U. S. Army Electronics Command/National Science Foundation
III. Contract DA-36-039 SC-78801/GP-524
IV. Project No. 3D5801001

Defense Documentation Center
UNCLASSIFIED

AD Div. 15/2

Inst. of Science and Technology, U. of Mich., Ann Arbor
THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS, by A. Y. Bilal. Sept. 63. 151 p. incl. illus tables, 50 refs.
(Report No. 2900-392-T)
(Contract DA-36-039 SC-78801/GP-524)
(Project No. 3D5801001) Unclassified report

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{X_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for
(over)

UNCLASSIFIED

I. Bilal, A. Y.
II. U. S. Army Electronics Command/National Science Foundation
III. Contract DA-36-039 SC-78801/GP-524
IV. Project No. 3D5801001

Defense Documentation Center
UNCLASSIFIED

AD Div. 15/2

Inst. of Science and Technology, U. of Mich., Ann Arbor
THE ANALYSIS OF A CERTAIN CLASS OF NONLINEAR SYSTEMS, by A. Y. Bilal. Sept. 63. 151 p. incl. illus tables, 50 refs.
(Report No. 2900-392-T)
(Contract DA-36-039 SC-78801/GP-524)
(Project No. 3D5801001) Unclassified report

In nonlinear analysis, the partitioning technique has been used to analyze a certain class of nonlinear systems whose dynamic behavior can be represented by a nonlinear differential equation with time-varying parameters. When suitable restrictions were placed on the linear, nonlinear, and forcing function terms, the system equation presented a unique solution which existed to the right of the initial state. The system solution was given as a limit of a sequence of Picard iterates $\{X_n\}$ which are well defined in a given domain and which belong to L_2 space. A formula was developed which permits determining the number of iterates necessary for
(over)

UNCLASSIFIED

I. Bilal, A. Y.
II. U. S. Army Electronics Command/National Science Foundation
III. Contract DA-36-039 SC-78801/GP-524
IV. Project No. 3D5801001

Defense Documentation Center
UNCLASSIFIED

AD

the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters. Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov.
The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy.
For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis.
The method lends itself easily to digital computer calculations.

UNCLASSIFIED

DESCRIPTORS

Nonlinear systems
Control
Analysis
Stability
Synthesis
Differential equations

UNCLASSIFIED

AD

the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters. Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov.
The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy.
For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis.
The method lends itself easily to digital computer calculations.

UNCLASSIFIED

DESCRIPTORS

Nonlinear systems
Control
Analysis
Stability
Synthesis
Differential equations

UNCLASSIFIED

AD

the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters. Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov.
The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy.
For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis.
The method lends itself easily to digital computer calculations.

UNCLASSIFIED

DESCRIPTORS

Nonlinear systems
Control
Analysis
Stability
Synthesis
Differential equations

UNCLASSIFIED

AD

the approximation of the solution in the mean square sense. Within the restrictions imposed on the system, the system response was found to be uniformly continuous with respect to the initial conditions and the system parameters. Two different definitions of the norms were selected, and the behavior of the system trajectory was investigated for two important cases: (1) at any instant of time and (2) on the average during an interval of interest. The system (under the given conditions) was asymptotically stable in the sense of Lyapunov.
The analysis was extended to include systems whose behavior is governed by simultaneous nonlinear differential equations. It was found that a class of multiloop nonlinear systems exists for which it is possible to find the exact analytic solution to any degree of accuracy.
For situations in which the forcing function, the nonlinear functions, and the time-varying parameters can be defined graphically or tabularly, a systematic method has been presented to include numerical analysis.
The method lends itself easily to digital computer calculations.

UNCLASSIFIED

DESCRIPTORS

Nonlinear systems
Control
Analysis
Stability
Synthesis
Differential equations

UNCLASSIFIED