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AN APPLICATION OF GAME THEORY TO SIGNAL DETECTABILITY

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TABLE OF CONTENTS

ABSTRACT

The problem of unknown a priori probability P(SN) is treated by considering a zero-sum two-person game, where the payoff to the receiver player depends on correctly detecting signals in noise, his opponent choosing the average "on the air" time, and solutions for both players are determined. Completely unknown, known within a range and randomly distributed a priori probability cases are solved. Particular emphasis is placed on the solutions for the receiver. In all instances these specify that the receiver should be of the same type as that specified by EDG Technical Report No. 13 The Theory of Signal Detectability which assumes the a priori probability of a signal's presence is known. The correct operating points of that receiver are specified. [redacted]

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APPLICATION OF GAME THEORY TO SIGNAL DETECTABILITY

1. INTRODUCTION

1.1 Remarks

This report is supplementary to Electronic Defense Group Technical Report No. 13, The Theory of Signal Detectability, Part I, in which the general theory of signal detectability is treated. In establishing a definition of optimum in that report, it was assumed whenever necessary that the a priori probability that a signal be transmitted, P(SN), is known. If this assumption does not hold, certain applications cannot be made directly. Several alternative assumptions can be introduced to replace exact knowledge of P(SN). Each of these is based on the idea of weighting the errors and correct responses of the receiver and maximizing in some manner the average or total expected return from these responses.

1.2 Purpose of a Game-Type Solution

The amount of basic theory drawn from the theory of games for this report is very small. Primarily, only the definition of a solution of a two-person zero-sum game is used. It is this definition and its practical value that will be discussed briefly here.

A two person game consists of the set of rules as to how each player may move, and the rule for paying off at the end of each play of the game. We shall consider games in which each player picks an overall strategy, and the reward to each of the players is given by a payoff function defined for each possible pair of strategies. The game is zero sum when the payoff to one player is the negative of the payoff to the other.

Consider the situation in which a particular player's objective can be reduced to the desire to maximize a real function of two variables by controlling the value of only one of these variables. This real function can be considered as his "payoff."

The value of the second variable is unknown to him and may be controlled by someone else. It is convenient to assume the latter, and therefore the person in question is confronted with a hypothetical or real opponent, who controls one of the variables of the payoff.

For example, let us assume that player A wishes to maximize the value of the payoff $V(a,p)$, and his opponent P wishes to minimize it. For each possible value of p, A can pick his variable "a" so that V is maximized. In general, this value of a is a function of p, that is, $a = a(p)$. After this maximization the payoff is solely a function of p, denoted by either $V(a(p),p)$ or $\max V(a,p)$. Player P can then decide which p to pick such that $V(a(p),p)$ is a minimum. Player P does this, and fixes his variable at the value p^* accordingly. One can evaluate the payoff $V(a(p^*),p^*)$; if player A chooses the variable $a = a(p^*)$ this will be the payoff, and if player A uses any other value of a, the value of the payoff will not exceed $V(a(p^*),p^*)$ since by definition $a(p^*)$ yields a maximum. This can be condensed into the equation

$$V(a,p^*) \leq V(a(p^*), p^*) = \min_p \max_a V(a,p) \qquad (1.1)$$

If the above procedure is reversed with player P picking the p(a) such that p(a) minimizes V(a,p) for each fixed a, and then player A maximizes over these values choosing a value a*, the result will be

$$\max_{a} \min_{p} V(a,p) = V(a^*, p(a^*)) \leq V(a^*,p) \tag{1.2}$$

For the problem considered in this paper; a* = a(p*) and p* = p(a*), the result is

$$V(a,p^*) \leq V(a^*,p^*) \leq V(a^*,p) \tag{1.3}$$

This is interpreted as follows: calling (a*,p*) the solution, if A plays at the solution and P does not, the value of the payoff is greater than at the solution, and therefore P should have played at p* in order to minimize the value V; conversely, if P plays at the solution and A does not, the payoff will be smaller, and thus A should have played at a*. The solution is an equilibrium point; that is, if both players play at the solution, neither will choose to change. It is called an equilibrium point in order to contrast it with the jockeying for advantage that often occurs: first player A chooses a playing point, then player P chooses his playing point with this in mind, then A changes his playing point to take advantage of P's choice, ad infinitum.

Viewed by player A, playing at the solution guarantees him at least a minimum payoff, and he may have even a larger payoff if player P does not choose the solution value. It may well be the case that not only is P at some non-solution value, but that if A knew this and acted accordingly he could realize a tremendously larger payoff. However, gambling on this latter without real knowledge of P's choice may have the effect of pulling the floor out from under A; that is, he no longer can count on a guaranteed minimum and is liable to much lower payoff value.

The application in Section 2 is to let a player P, either real or hypothetical, choose the a priori probability of there being a signal present. The natural bounds on p are $0 \le p \le 1$ but any smaller bounds can also be chosen (see Section 3).

1.3 The Problem

Assume an operator in the field has available a very versatile search receiver, and he is faced with the task of maximizing the return from his operation of that receiver to detect signals of known character in the presence of noise. It will be shown that this operator will require the same type of equipment as an operator who knows the a priori probability in the same situation.

Values are attached to each possible action of the receiver operator, and he wishes to obtain the largest average (or total) value. If the following symbols are used to describe events:

SN - There is signal and noise present at the receiver input

N - There is only noise present at the receiver input (1.4)

A - The receiver operator says there is a signal present

CA - The receiver operator says there is only noise present

then SN•A will represent the event of correct detection, and SN•CA, a miss; N•A, a false alarm; and N•CA, a correct guess that no signal was present. These are the four events that can occur, and to each is attached a value to the receiver player as follows:

$V_{SN \cdot A}$ = Value of correctly detecting a signal's presence

$V_{N \cdot CA}$ = Value of correctly detecting that no signal was present

$-K_{SN \cdot CA}$ = Value of the error of missing a signal [1]

$-K_{N \cdot A}$ = Value of the error of falsely saying a signal was present

The average or expected value obtained by the receiver operator will then be the sum of these values, weighted by the probability of the respective events. This average is called simply the payoff value V, and

$$V = V_{SN \cdot A} P(SN \cdot A) + V_{N \cdot CA} P(N \cdot CA) - K_{SN \cdot CA} P(SN \cdot CA) - K_{N \cdot A} P(N \cdot A) \quad (1.6)$$

2. THE GAME SOLUTION

2.1 The Game Defined

The receiver operator will consider himself as a player in a zero-sum, two-person game. His opponent is equipped with a band-limited transmitter, and he uses the receiver as a detection device. This is enough to allow him to use the general theory presented in Technical Report EDG-13, The Theory of Signal Detectability. Each move of the game consists of a transmission or no transmission for time T by the transmitter of one of a set of possible signals, and the decision by the receiver player in sufficiently small delay time after this so that the values specified are independent of the delay. The game consists of a large number of these moves. For example, T might be a second if listening for Morse code, or a millisecond if looking for radar pulses, and one play of the game would last for a watch, or for the duration of an attack by guided equipment.

[1] The peculiar form -K as the value of an error is to emphasize that it is really a loss. For example, if the values are in dollars and cents, and if the cost to the receiver operator is \$100 for each miss, then

$$K_{SN \cdot CA} = \$100 \text{ or } -K_{SN \cdot CA} = -\$100$$

The strategies are as dissimilar as the equipment. The player with the transmitter may choose his average transmitting time, which in the receiver's language is P(SN), the a priori probability that a signal is transmitted. The player with the receiver chooses the subset A of the receiver inputs that he will call signal plus noise, that is, he chooses the "criterion" A. The entire play is submerged in noise that is known equally well to both players by its p probability density function $f_N(X)$.[1] Because the transmitter player chooses his a priori probability only, the signal-plus-noise density function $f_{SN}(X)$[1] is made known to both players. In order that the game be realistic, the trivial assumption is made that the receiver operator gains less for either error than he gains from a correct answer. This can be expressed as $-K < V$ for either K and either V, or $0 < K + V$.

The expected payoff to the receiver player is the average value V, and to his opponent is -V.

$$V = V_{SN\cdot A}\, P(SN\cdot A) + V_{N\cdot CA}\, P(N\cdot CA) - K_{SN\cdot CA}\, P(SN\cdot CA) - K_{N\cdot A}\, P(N\cdot A) \qquad (1.6)$$

Eq (1.6) can be greatly simplified by the use of conditional probabilities and the fact that SN and N, A and CA are complimentary; that is

$$P(N) = 1 - P(SN) \quad \text{and} \quad P(A) = 1 - P(CA) \qquad (2.1)$$

Simply substituting so that the terms are in the same order as in (1.6) we have

$$V = V_{SN\cdot A}\, P(SN)\, P_{SN}(A) + V_{N\cdot CA} \left[1 - P(SN)\right] \left[1 - P_N(A)\right] - K_{SN\cdot CA} \left[1 - P_{SN}(A)\right]$$
$$- K_{N\cdot A} \left[1 - P(SN)\right] P_N(A) \qquad (2.2)$$

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[1] The existance of the density function is a basic assumption.

It is convenient to split V into two payoff functions:

$V_o(A) = V$ when $P(SN) = 0$ $\qquad$ $V_1(A) = V$ when $P(SN) = 1$

$$V_o(A) = V_{N \cdot CA} \left[1 - P_N(A) \right] - K_{N \cdot A} \; P_N(A) \qquad (2.3)$$

$$V_1(A) = V_{SN \cdot A} \, P_{SN}(A) - K_{SN \cdot CA} \left[1 - P_{SN}(A) \right] \qquad (2.4)$$

Under the restriction that each value -K is less than each value V, neither of these payoffs can dominate the other for all A. This is evident if we consider first $A = \emptyset$, the empty set[1]:

$V_1(\emptyset) = -K_{SN \cdot CA}$ $\qquad$ $V_o(\emptyset) = V_{N \cdot CA}$ $\qquad$ thus $V_o(\emptyset) > V_1(\emptyset)$

and second, consider A as all receiver inputs, $A = R$:

$V_1(R) = V_{SN \cdot A}$ $\qquad$ $V_o(R) = -K_{N \cdot A}$ $\qquad$ thus $V_o(R) < V_1(R)$.

The payoff can then be written

$$V = P(SN) \, V_1(A) + \left[1 - P(SN) \right] V_o(A) \qquad (2.5)$$

2.2 A Numerical Illustration

2.2.1 The Payoff. The ideas of payoff functions and min-max solutions may be made much clearer by treating a particular numerical example before considering general solutions. Therefore for the sake of example let us pick values for the payoff,

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1 To choose the empty set $\emptyset$ as a criterion means that the receiver operator will never say there is a signal present, i.e., will always say there is noise alone present. The reverse extreme is to choose the whole space R as a criterion.

$$V_{SN \cdot A} = 7 \qquad -K_{SN \cdot CA} = -8$$

$$-K_{N \cdot A} = -6 \qquad V_{N \cdot CA} = 2$$

From (2.3) and (2.4) V_o and V_1 are computed

$$V_o(A) = 2\left[1 - P_N(A)\right] - 6P_N(A) = 2 - 8P_N(A) \qquad (2.3n)$$

$$V_1(A) = 7P_{SN}(A) - 8\left[1 - P_{SN}(A)\right] = 15\,P_{SN}(A) - 8 \qquad (2.4n)$$

The first forms above emphasize

$$2 \geq V_o(A) \geq -6 \qquad 7 \geq V_o(A) \geq -8$$

We know from the general theory that for each value of $P_N(A)$, all possible values of the probability of detection, $P_{SN}(A)$, lie in an interval, the bounds depending on the types of signals and the noise. Therefore, further assume, again for the sake of example only, that the signals and noise are such the points in the cross-hatched area of Fig. 1 represents all of the possible combinations of the probability of detection $P_{SN}(A)$ and probability of false alarm $P_N(A)$.

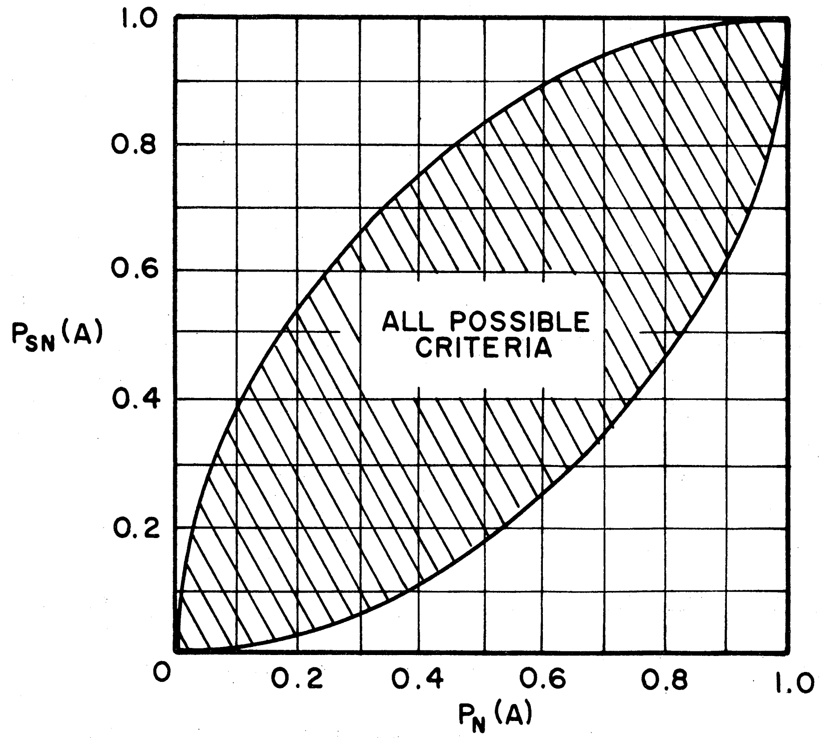

FIG 1
REGION OF ALL POSSIBLE CRITERIA
IN A TYPICAL DETECTION CASE

For the moment consider only those criteria A with a given false alarm probability, say 40 per cent. From Fig. 1 the limits on probability of detection are seen to range roughly from 11 per cent to 76 per cent. $V_o(A)$ depends only on the probability of false alarm and is this case $V_o(A) = -1.20$ for all the criteria with 40 per cent false alarm. However, $V_1(A)$ ranges from -6.35 to 3.40. Two criteria of interest are those that take on these values:

$$P_{SN}(A_\alpha) = .11, \quad P_N(A_\alpha) = .40 \text{ and}$$

$$P_{SN}(A_\beta) = .76, \quad P_N(A_\beta) = .40 \ .$$

By restricting our attention to these special criteria A_α and A_β , we can plot the payoff V as a function of the probability of false alarm and the a priori probability P(SN). Eq (2.5) shows that V is a linearly combination of $V_o(A)$ and

$V_1(A)$. Plotting $V_o(A)$ against $P_N(A)$ yields a straight line from $V_{N \cdot CA}$ at $P_N(A) = 0$ to $-K_{N \cdot A}$ at $P_N(A) = 1$. Plotting $V_1(A_\alpha)$ and $V_1(A_\beta)$ against $P_N(A)$ yields 2 curves, $V_1(A_\alpha)$ below $V_1(A_\beta)$, both from $-K_{SN \cdot CA}$ when $P_N(A) = 0$ to $V_{SN \cdot A}$ when $P_N(A) = 1$. If these two plots are made on parallel planes and corresponding parts (same $P_N(A)$) connected by straight lines, the resulting surface will enclose a volume representing all possible values of the payoff V.

Because the receiver operator can always operate on the upper surface of this payoff volume by using A_β type criteria, he will certainly do so in order to maximize the payoff. This upper surface will be referred to as the receiver's upper payoff surface, and for the numerical case in question is shown in Fig. 2. Figure 2a is a general view of the upper payoff surface. Four curves of special interest are indicated on the orthogonal projections. In Figures 2c and 2d it is apparent that there is a horizontal line on the the surface parallel to the P(SN) axis; this is marked (1) in Figures 2b, 2c, and 2d. The curve marked (2) has constant P(SN) and is a maximum when it crosses curve (1). Later, Fig. 3 on page 12 will show the maximum value of V for each fixed value of P(SN); the maximum values occur along the dotted curve marked (3) in Figure 2b. Figure 3 is actually this curve as it would appear in projection 2c. The final curve consists of the horizontal line (1) and the 2 pieces of curves marked (4); along this curve is the minimum value of V for each fixed value of $P_N(A)$, the analog of curve (3).

<u>2.2.2 Solution of the Numerical Case.</u> We have argued in 2.2.1 that the receiver operator would use criteria on the upper curve of Fig. 1. The

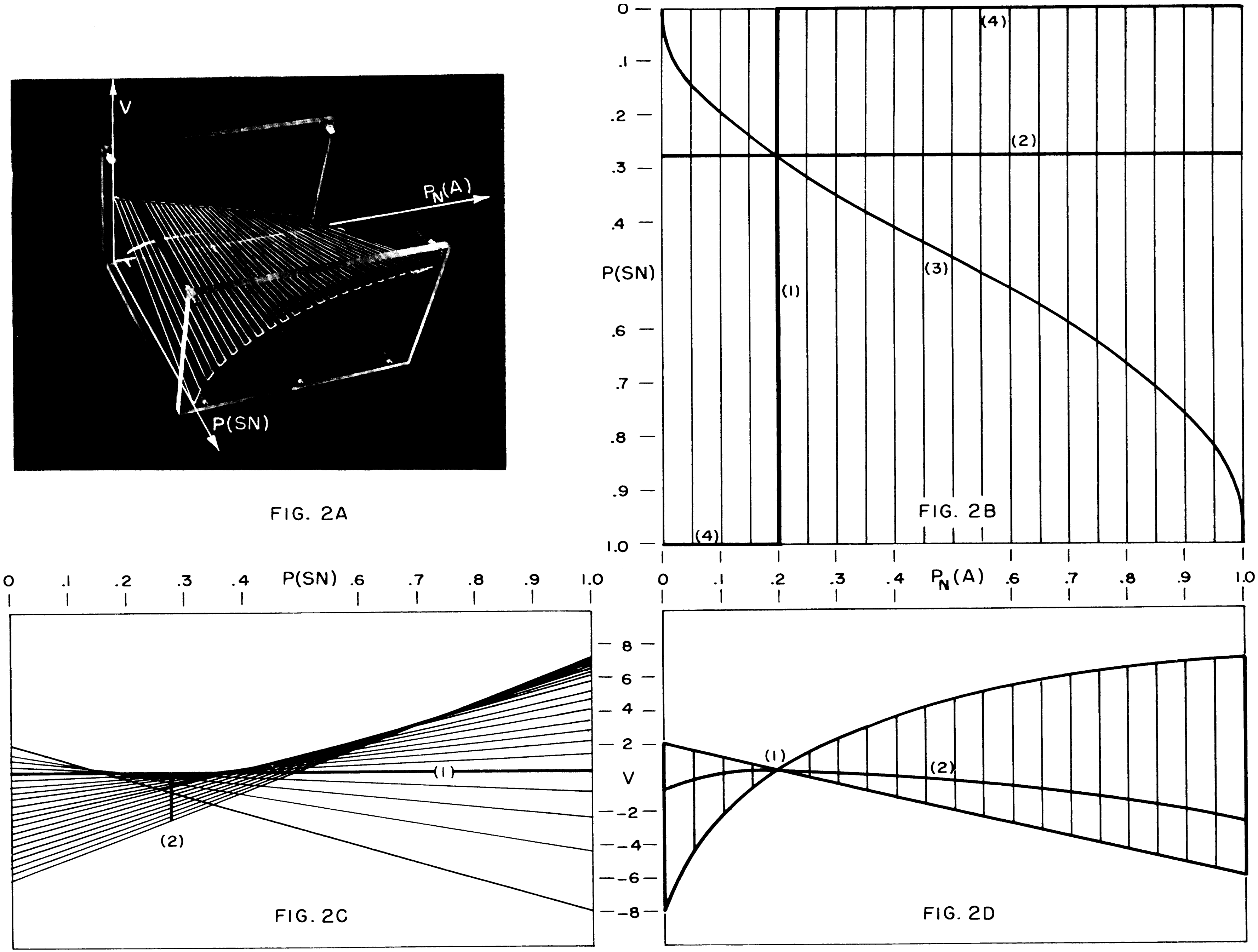

FIG. 2. AN UPPER PAYOFF SURFACE.

question of which point on this upper curve to use would be answered by equation (1.6) of <u>The Theory of Signal Detectability</u> if P(SN) were known: use that point with slope[1]

$$\beta = \frac{1 - P(SN)}{P(SN)} \; \frac{V_{N \cdot CA} + K_{N \cdot A}}{V_{SN \cdot A} + K_{SN \cdot CA}} = \frac{8}{15} \; \frac{1-P(SN)}{P(SN)}$$

Using this equation, Fig. 1, and the equations for V_o and V_1 $-(2.3n)$ and $(2.3n) - V$ can be computed as a function of P(SN) only. This is shown in Fig. 3. The minimum is at approximately $P(SN) = .28^-$, $V = .40$. The receiver solution can then be obtained by determining β, $\beta = \frac{8}{15} \frac{.72^+}{.28^-} = 1.4$, and from Fig. 1 the point on the upper bound with slope 1.4 is $P_N(A) = .20$ and $P_{SN}(A) = .56$.

This graphical method of solution is straight forward, but is useful only for particular solutions. Section 2.3 and 2.6 derive general solutions that are easily applied.

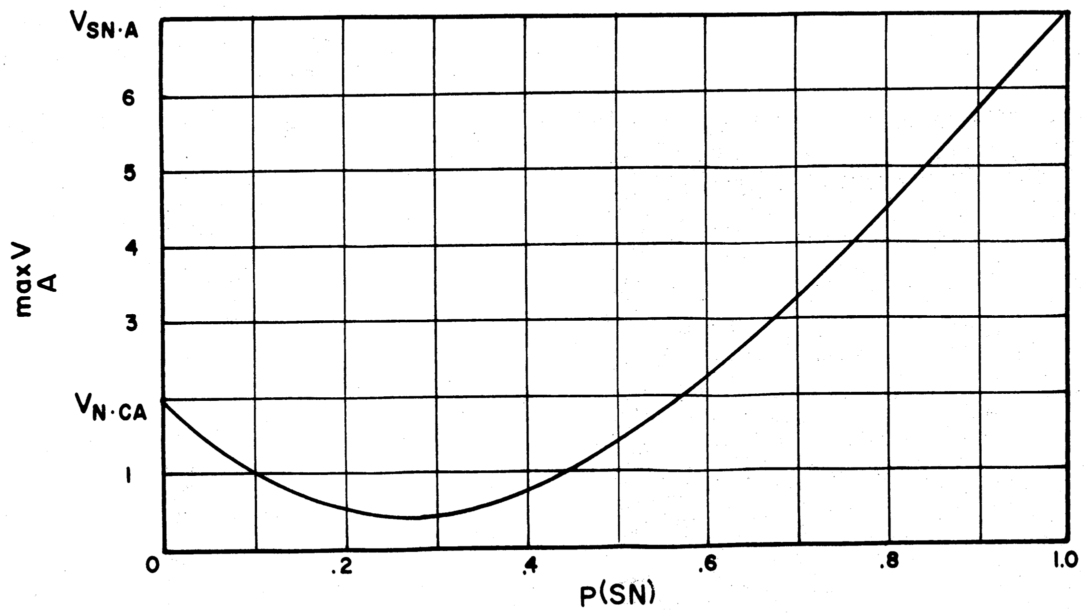

FIG 3

THE MAXIMUM VALUE OF THE PAYOFF V
FOR KNOWN A PRIORI PROBABILITY P(SN)

[1] This equation is introduced in EDG TR-13 in section 1.4, Theorems 1 and 7 prove that it applies to the upper curve, and Theorem 8 proves that β is the slope of the tangent.

2.3 The General Receiver Solution

2.3.1 Derivation and Proof of Solution. The solution of the game is defined by the min-max relation: if P(SN) = p* and criterion A* are solutions, then

$$V(A^*,p^*) = \min_{P(SN)} \max_{A} V = \max_{A} \min_{P(SN)} V.$$

If we choose to minimize over P(SN) = p first, we observe from (2.5) that for any fixed A the payoff is linear in p and therefore $\partial V/\partial p$ is a constant. Since the transmitter wishes to minimize V by choosing the proper p for each A, he will choose p either 0 or 1 except for those A such that V is independent of P(SN), i.e., unless $\left.\frac{\partial V}{\partial p}\right|_A = 0$.

If p* = 0 were the solution, Eq (1.3) would be

$$V_0(A) \leq V_0(A^*) \leq V(A^*, p) \text{ for all } p$$

where A* is the corresponding solution for the receiver. The left hand inequality implies that A* = $\emptyset$, the empty set . If p = 1 is used for the right hand inequality, this becomes

$$V_0(\emptyset) \leq V_1(\emptyset)$$

which is simply

$$V_{N\cdot CA} \leq -K_{SN\cdot CA}$$

contrary to that assumption that $V + K \geq 0$. A similar contradiction arises if p* = 1 is tried as a solution. Therefore, a solution can be obtained only if $\frac{\partial V}{\partial p} = 0$. By inspection of (2.5) this is simply the equality of V_0 and V_1. The above argument can be summarized formally.

Theorem 1. A necessary condition that A be an optimum criterion in this game is

$$V_{SN\cdot A}\, P_{SN}(A) - K_{SN\cdot CA}\left[1 - P_{SN}(A)\right] = V_{N\cdot CA}\left[1 - P_N(A)\right] - K_{N\cdot A}\, P_N(A). \quad (2.6a)$$

An equivalent formulation is

$$P_{SN}(A)\left[V_{SN\cdot A} + K_{SN\cdot CA}\right] + P_N(A)\left[V_{N\cdot CA} + K_{N\cdot A}\right] = V_{N\cdot CA} + K_{SN\cdot CA} \qquad (2.6b)$$

Theorem 1 gives only a necessary condition for a solution; there may be many criteria for which Eq (2.6) holds. The optimum criterion satisfies this condition with the maximum value of V, that is, with the largest common value $V_o = V_1$. The problem is to determine the criterion A such that (2.6) holds and that A maximizes $P_{SN}(A)\left[V_{SN\cdot A} + K_{SN\cdot CA}\right] - K_{SN\cdot CA}$. The assumption that $\left[V_{SN\cdot A} + K_{SN\cdot CA}\right] > 0$ simplifies this to the requirement that A maximize $P_{SN}(A)$. Thus the optimum criterion satisfies (2.6) and has maximum $P_{SN}(A)$ with respect to all other others that satisfy (2.6).

Because all of the bracketed quantities in (2.6b) are positive, it is convenient to introduce two parameters, α_1 and α_2, where

$$\alpha_1 = \left[V_{N\cdot CA} + K_{N\cdot A}\right] \Big/ \left[V_{SN\cdot A} + K_{SN\cdot CA}\right] \qquad (2.7)$$

$$\alpha_2 = \left[V_{N\cdot CA} + K_{SN\cdot CA}\right] \Big/ \left[V_{SN\cdot A} + K_{SN\cdot CA}\right]$$

and (2.6b) becomes

$$P_{SN}(A) + \alpha_1 P_N(A) = \alpha_2 \qquad (2.6c)$$

The general theory of signal detectability specifies that the optimum receiver for the three general definitions of "optimum" considered in Technical Report EDG-13 can be achieved by a receiver that has as its output the likelihood ratio of its input, and that if the noise is analytic the only way to achieve the optimum is for the output to be the likelihood ratio of the input, or some strictly monotone function of the likelihood ratio. For each input x(t) the likelihood ratio is defined as the ratio of the probability density functions

$$\ell(x) = f_{SN}(x) / f_N(x) \qquad (2.8)$$

and the operator will say that there is signal plus noise present whenever the output is above a predetermined operating level β.

Because the definition of "optimum" is slightly different in this game case from those considered previously the theory cannot immediately be applied to show that such a receiver is the optimum for the game. However, such a receiver is worthy of special consideration as a trial solution, and criteria associated with it are given the special notation $A(\beta)$, where β is the operating level. In a case where the noise is analytic, the set of points with likelihood ratio exactly equal to the boundary value β is a set of probability zero, and therefore of no consequence. A more general consideration than analytic noise may require consideration of the boundary. For this reason any criterion A which is like the set $A(\beta)$ but includes either all, part or none of the boundary, is called an $A_1(\beta)$. Obviously every $A(\beta)$ is an $A_1(\beta)$ but not conversely.

The solution to the game is found by proving the following statement.

Theorem 2 If there is a value of β such that some criterion A is both an $A_1(\beta)$ and satisfies (2.6), then A is a receiver solution.

This is seen if any other criterion B that satisfies the necessary condition (2.6) is compared with A. Split both A and B into their common part and the remainders:[1]

$$B = (B \cap A) \cup (B - A)$$

$$A = (B \cap A) \cup (A - B)$$

For the common part and for A-B, since both are subsets of A, $\ell(x) \geq \beta$; in B - A, $\ell(x) \leq \beta$. From (2.6c) then we have

$$\alpha_2 = P_{SN}(B \cap A) + P_{SN}(B - A) + \alpha_1 P_N(B \cap A) + \alpha_1 P_N(B - A)$$

[1] $C \cap D$ is the common part of criteria C and D. $C \cup D$ is the criterion consisting of both C and D.

$$\alpha_2 = P_{SN}(B \cap A) + P_{SN}(A - B) + \alpha_1 P_N(B \cap A) + \alpha_1 P_N(A - B).$$

Equating and canceling the effect of the common part,

$$P_{SN}(B - A) + \alpha_1 P_N(B - A) = P_{SN}(A - B) + P_N(A - B) \quad (2.9)$$

In integral form this is

$$\int_{B - A} \left[f_{SN}(x) + \alpha_1 f_N(x) \right] dx = \int_{A - B} \left[f_{SN}(x) + \alpha_1 f_N(x) \right] dx ,$$

and factoring out $f_N(x)$ in the integrand

$$\int_{B - A} f_N(x) \left[\ell(x) + \alpha_1 \right] dx = \int_{A - B} f_N(x) \left[\ell(x) + \alpha_1 \right] dx .$$

Now on the left, $\ell(x) + \alpha_1 \leq \beta + \alpha_1$, and on the right $\ell(x) + \alpha_1 \geq \beta + \alpha_1$. Therefore substitute $\beta + \alpha_1$ for $\ell(x) + \alpha_1$ on both sides (and then cancel this constant factor); the left side dominates the right.

$$\int_{B - A} f_N(x)\, dx \geq \int_{A - B} f_N(x)\, dx$$

These integrals are the false alarm probabilities on the difference sets B - A and A - B.

$$P_N(B - A) \geq P_N(A - B). \quad (2.10)$$

This result and equation (2.9) yield

$$P_{SN}(B - A) \leq P_{SN}(A - B)$$

and the proof is completed by adding $P_{SN}(B \cap A)$ to both sides to obtain the inequality

$$P_{SN}(B) \leq P_{SN}(A) \quad (2.11)$$

Thus the receiver solution is the $A_1(\beta)$ that satisfies (2.6), and no other criterion satisfying (2.6) has a larger value of $P_{SN}(A)$, and therefore no other criterion satisfying Theorem 1 has a larger payoff V.

Since $A_1(\beta)$ also lead to the upper payoff surface, the argument of Section 2.2 is now completely justified.

2.3.2 Existence. If it could be shown that for all possible pairs of positive numbers (α_1, α_2) the hypothesis of Theorem 2 is met then the solution would be complete. We can restrict our attention to

$$0 \leq \alpha_2 \leq 1 + \alpha_1 \tag{2.12}$$

by definition of α_1 and α_2.

Consider: $$L(\beta) = P_{SN}(A(\beta)) + \alpha_1 P_N(A(\beta)) \tag{2.13}$$

If $\beta_1 < \beta_2$ then $L(\beta_1) \geq L(\beta_2)$. This is true because all of the points with $\ell(x) \geq \beta_2$ have $\ell(x) > \beta_1$, and additional points do not decrease L. If there is no β such that $L(\beta) = \alpha_2$[1], (this is Eq (2.6c)), there will be some value of β, call it β^*, such that $L(\beta^*) > \alpha_2$ and for all larger values $L(\beta) < \alpha_2$. This situation is sketched in Fig. 4.

The drop in $L(\beta)$ is due to those points with likelihood ratio equal to β^*. Since for any $\beta > \beta^*$, the criterion $A(\beta)$ is so small that $L(\beta) < \alpha_2$ we use an $A_1(\beta)$ such that $L(\beta^*) = \alpha_2$, that is, since $L(\beta^*) > \alpha_2$, we remove just enough points of likelihood ratio β^* from $A(\beta^*)$. We can certainly do this whenever probability density functions exist.[2] Thus, if there is no β such that an $A(\beta)$ satisfies (2.6c), there is a β such that a slight modification does.

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[1] This trouble cannot arise if the noise is analytic. See EDG Technical Report No. 13, Appendix B.

[2] See EDG Technical Report No. 13, Part I, Lemma 4, p. 40.

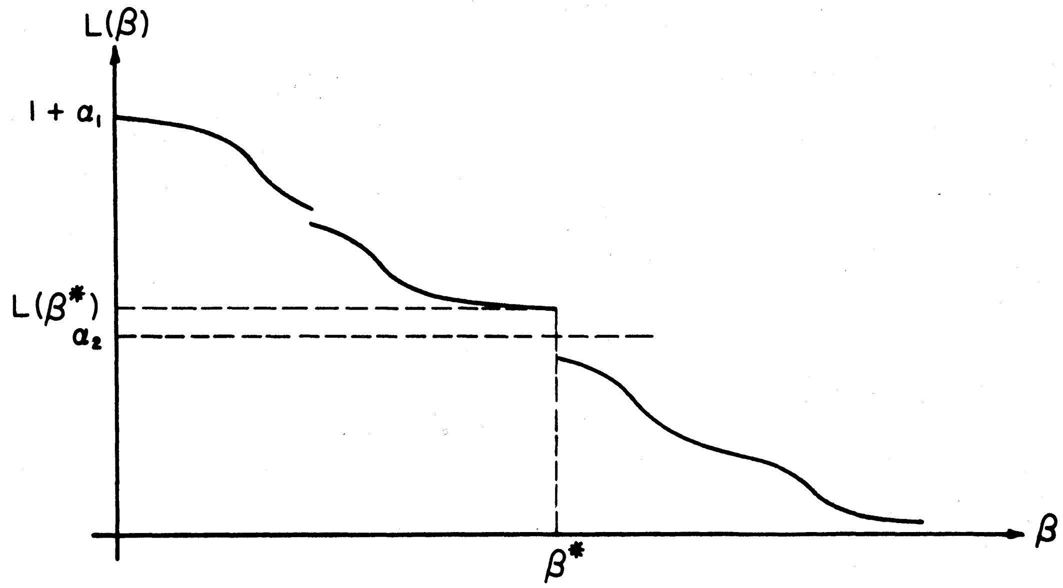

FIG 4
EXAMPLE OF DISCONTINUOUS L(β)

2.3.3 Summary. The receiver solution specifies that the receiver should have as its output the function $\ell(x)$, the likelihood ratio of its input x. If we compare this game with the first type of optimum in The Theory of Signal Detectability which arose from the same payoff but for known a priori probability P(SN), we have a good example of how game-theory and probability can complement each other in the treatment of a problem.

When P(SN) is known, the payoff is maximized by using the bias level

$$\beta = \frac{1 - P(SN)}{P(SN)} \; \frac{V_{N\cdot CA} + K_{N\cdot A}}{V_{SN\cdot A} + K_{SN\cdot CA}} = \frac{1 - P(SN)}{P(SN)} \alpha_1 \qquad (2.14)$$

Obviously, knowledge of P(SN) is paramount. Lacking that knowledge, a best "safe" value of β is yielded by the game: β such that

$$P_{SN}(A_1(\beta)) + \alpha_1 P_N(A_1(\beta)) = \alpha_2$$

where α_1 and α_2 are defined by Eq (2.7).

It is "safe" because the payoff is made independent of the value of P(SN), and it is the best because it yields the maximum payoff over all other safe values. For any different criterion the opponent can lower the payoff below this "best safe" value. However, if P(SN) is known, playing the game solution will not be better than the solution given by (2.14) and will be worse unless the β's luckily coincide.

Thus, the receiver is always designed to transform each input x = x(t) into the output $\ell(x)$. Then either probability or game solutions determine the level β depending on knowledge or lack of knowledge of P(SN).

2.3.4 Use of Receiver Operating Characteristics Curves to Obtain Particular Solutions. For those who are not familiar with the notion of a receiver operating characteristic the following may help to explain it and its usefulness. The ROC curve is a graph of $P_{SN}(A)$ vs $P_N(A)$ for all $A_1(\beta)$ criteria. It is often drawn together with the graph of its slope. A typical ROC curve is (1) in Fig. 5a; the diagonal marked (2) represents the effect of ignoring the receiver and guessing; the curve (3) is simply the first curve rotated about the (.5, .5) point and is useful in theory because the area between (1) and (3) inclusive contains all of the possible combination of $P_{SN}(A)$, $P_N(A)$ for any particular situation (i. e., fixed types of signals and noise and their energies, fixed observation time, etc.) and therefore any receiver of any type is represented by a curve in this area.

Equation (2.6c) can be plotted on the R.O.C.; it is a straight line from (0, α_2) with slope $-\alpha_1$. Thus all of the points on this straight line and between or on curves (1) and (3) represent possible receivers that satisfy the necessary condition

$$P_{SN}(A) + \alpha_1 P_N(A) = \alpha_2 . \qquad (2.6c)$$

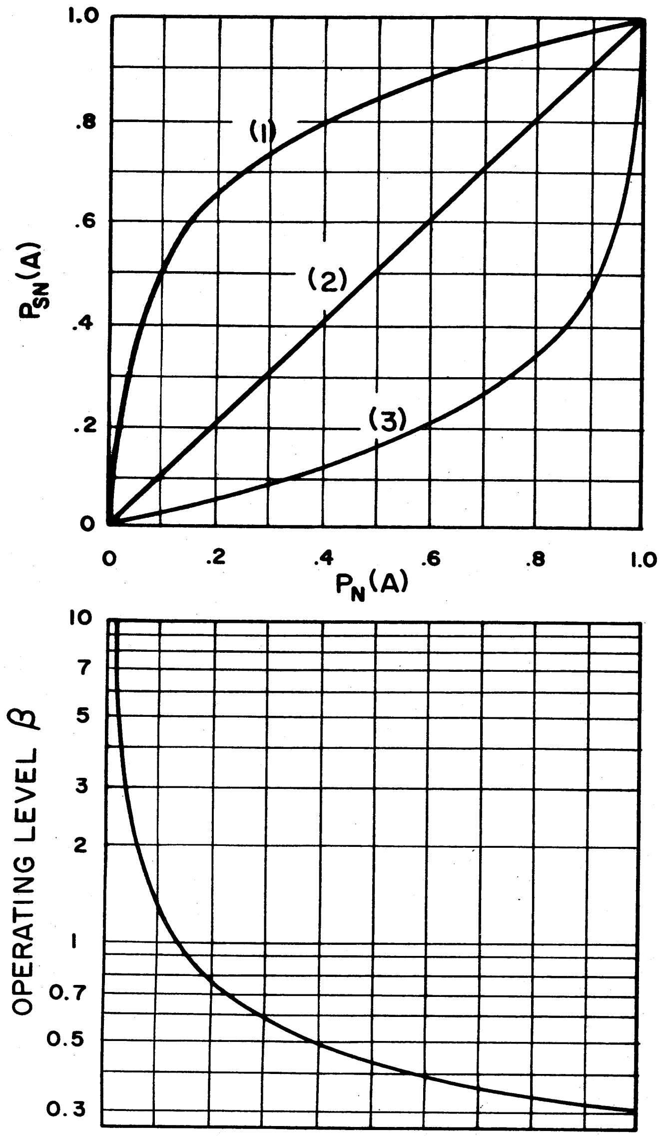

FIG 5a
TYPICAL RECEIVER OPERATING CHARACTERISTIC

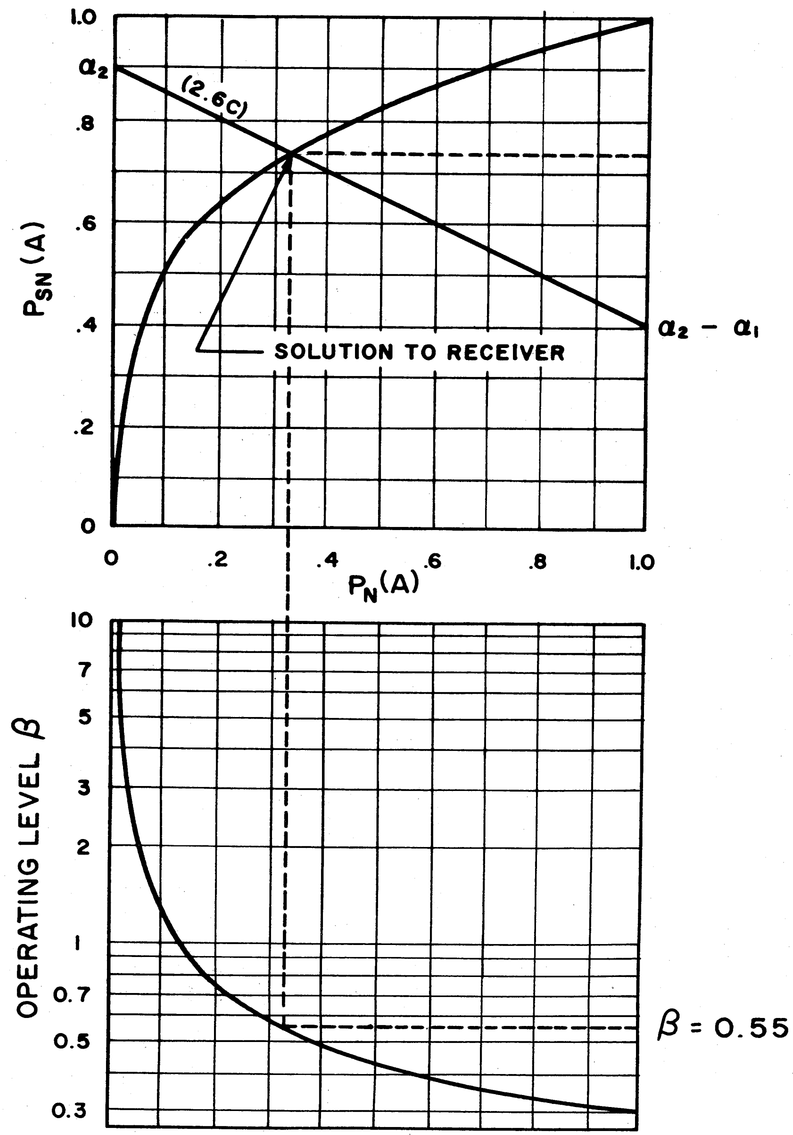

FIG 5b
TYPICAL RECEIVER OPERATING CHARACTERISTIC WITH GAME SOLUTION FOR $\alpha_1 = .5$, $\alpha_2 = .9$

Obviously the intersection with curve (1) yields a maximum value of $P_{SN}(A)$ among these. For example, if the ROC is as in Fig. 5, and if $\alpha_2' = .9$, $\alpha_1 = .5$, the solution is obtained in Fig. 5b. From that graph we read $P_N(A) = .33$, $P_{SN}(A) = .74$, and $A = A(.55)$, i.e., the optimum receiver which generates likelihood ratio should be set at an output bias level of .55.

2.4 The General Transmitter Solution

2.4.1 A Point Solution. A solution can be obtained for the transmitter if we maximize first over A and then minimize with respect to P(SN). The maximization restricts the receiver so that $P_{SN}(A)$ and $P_N(A)$ lie along the receiver operating characteristic with the operating level a function of P(SN), Eq (2.14). Further, if $P_N(A)$ is considered as the independent variable, the slope of the curve at any point is the value of the operating level associated with that point. Call the solution for the receiver operating level β_S. The existence of a min-max solution requires that this β_S maximize V for P at his solution, and therefore the derivative of V with respect to the receiver will be zero at the solution. Recalling

$$V = P(SN)\, V_1 + (1 - P(SN))\, V_o \ , \qquad (2.5)$$

$$V_o = V_{N\cdot CA} - \left[V_{N\cdot CA} + K_{N\cdot A}\right] P_N(A) \ , \qquad (2.3)$$

$$V_1 = \left[V_{SN\cdot A} + K_{SN\cdot A}\right] P_{SN}(A) - K_{SN\cdot A} \ , \qquad (2.4)$$

we form

$$\frac{\partial V}{\partial P_N(A)} = P(SN) \left[V_{SN\cdot A} + K_{SN\cdot A}\right] \beta - (1 - P(SN)) \left[V_{N\cdot CA} + K_{N\cdot A}\right] \qquad (2.15)$$

and letting $\frac{\partial V}{\partial P_N(A)} = 0$, $\beta = \beta_S$ and

$$P(SN) = \frac{V_{N\cdot CA} + K_{N\cdot A}}{\left[V_{SN\cdot A} + K_{SN\cdot A}\right] \beta_S + \left[V_{N\cdot CA} + K_{N\cdot A}\right]} = \frac{\alpha_1}{\beta_S + \alpha_1} \qquad (2.16)$$

Note that if (2.16) is solved for β_S that

$$\beta_S = \frac{1 - P(SN)}{P(SN)} \alpha_1$$

checking the fact that (2.16) yields an equilibrium point, for if the transmitter P chooses $P(SN) = \frac{\alpha_1}{\beta_S + \alpha_1}$ and tells the receiver, the receiver will not change from the point at which he is already operating (based on no knowledge of P(SN)).

2.4.2 Uniqueness of the Receiver Solution. In section 2.2 a solution to a game with payoff V(p) was defined as an equilibrium point (a*, p*) such that Eq (1.3) held.

$$V(a, p^*) \leq V(a^*, p^*) \leq V(a^*, p) \tag{1.3}$$

This implies that the solution shall consist of a single criterion A and a constant value of P(SN), and such solutions were obtained. Actually this is only one manner of playing the game. It is the simplest manner of play, and $V^* = V\left(A(\beta_S), \frac{\alpha_1}{\alpha_1 + \beta_S}\right)$ is the unique value of the game, since the receiver can actually attain a payoff V* by using $A(\beta_S)$ while the transmitter can act to hold the value down to this level by letting $P(SN) = \frac{\alpha_1}{\beta_S + \alpha_1}$. There may however be equally advantageous manners of playing of a more complex type; namely, where the players choose probability distributions for their respective variables instead of simple solutions. The modified form of Eq (1.3) is then

$$\overline{V}(F, G^*) \leq \overline{V}(F^*G^*) \leq \overline{V}(F^*, G) \tag{2.17}$$

where

$$\overline{V}(F, G) = \iint V(a, p)\, dF(a)\, dG(p)$$

and F*, G* are optimum distributions. This is nothing more than averaging over the various combinations that occur. It is assumed that the two variables are independent, that is, although whether the receiver operator decides to say "yes, there is signal plus noise" or not is dependent on whether a signal was actually present or not, the criterion for deciding is independent of the instantaneous (unknown to the receiver) a priori probability of these being a signal present.

One optimum distribution G* for the transmitter is that already obtained: to let P(SN) be exactly $\frac{\alpha_1}{\beta_S+\alpha_1}$. Eq (2.17) then becomes

$$\int V\left(a, \frac{\alpha_1}{\beta_S+\alpha_1}\right) dF(a) \le V^* \le \iint V(a,p)\, dF^*(a)\, dG(p) \tag{2.18}$$

Letting G(p) = G*(p) and restricting our attention to $A_1(\beta)$ type criterion, the right hand inequality becomes a necessary condition for any candidate for an optimal distribution $H^*(\beta)$ for the receiver

$$V^* \le \int V\left(A_1(\beta), \frac{\alpha_1}{\beta_S+\alpha_1}\right) dH^*(\beta) \quad . \tag{2.19}$$

If the value P(SN) = $\frac{\alpha_1}{\beta_S+\alpha_1}$ is used in Eq (2.15) for the derivative of V with respect to the false alarm rate the result simplifies to

$$\frac{\partial V\left(A_1(\beta), \frac{\alpha_1}{\beta_S+\alpha_1}\right)}{\partial P_N(A)} = (\beta - \beta_S) \frac{\left[V_{N\cdot CA} + K_{N\cdot A}\right]\left[V_{SN\cdot A} + K_{SN\cdot CA}\right]}{\left[V_{SN\cdot A} + K_{SN\cdot CA}\right]\left[\beta_S + V_{N\cdot CA} + K_{N\cdot A}\right]} \quad . \tag{2.20}$$

Because

$$\frac{\partial V\left(A_1(\beta), \frac{\alpha_1}{\beta_S+\alpha_1}\right)}{\partial\beta} = \frac{\partial V\left(A_1(\beta), \frac{\alpha_1}{\beta_S+\alpha_1}\right)}{\partial P_N(A_1(\beta))} \quad \frac{d\, P_N(A_1(\beta))}{d\beta}$$

where the second factor is always negative, $\frac{\partial V}{\partial P_N(A)}$ will have the same sign as $(\beta_S - \beta)$, and V has a unique maximum of $\beta = \beta_S$. This means, of course, that

$$V^* > V\left(A_1(\beta), \frac{\alpha_1}{\beta_S + \alpha_1}\right) \text{ for } \beta \neq \beta_S$$
$$V^* = V\left(A_1(\beta), \frac{\alpha_1}{\beta_S + \alpha_1}\right) \text{ for } \beta = \beta_S \tag{2.21}$$

The only way that (2.19) can be satisfied is for $H^*(\beta)$ to be a step function at $\beta = \beta_S$, that is, for the receiver to use the criterion $A_1(\beta_S)$ and not vary from that in any manner.

2.4.3 Distribution Solutions, Non-Uniqueness of Transmitter Solution. Let us now allow the transmitter player to use a distribution for the a priori probability. Because both the value of the game V* and the receivers unique solution are known, the defining equation of a solution, Eq (2.17) is greatly simplified. The right hand inequality is automatically satisfied by the receiver solution and the left hand inequality becomes

$$\iint V(A_1(\beta), p) \, dG^*(p) dH(\beta) \leq V^* \tag{2.22}$$

where $G^*(p)$ is an optimum solution for the transmitter and $H(\beta)$ is any distribution for the receiver (not necessarily optimum). This can be simplified to requiring that

$$\int V\,(A_1(\beta), p) \, dG^*(p) \leq V^* \quad \text{for all } \beta \tag{2.23}$$

because if (2.23) holds so does (2.22); and if (2.23) does not hold, say for $\beta = \beta^*$, then $H(\beta)$ can be a step function at $\beta = \beta^*$ and (2.22) would not hold. Because for fixed β the value V is a linear combination of V_o and V_1,

$$V(A_1(\beta), p) = p V_1(A_1(\beta)) + (1 - p) V_o(A_1(\beta))$$

Eq (2.23) became

$$V_1(A_1(\beta)) \left[\int pdG^*(p) \right] + V_o(A_1(\beta)) \left[1 - \int pdG^*(p) \right] \leq V^* \text{ for all } \beta \tag{2.24}$$

In the previous section it was shown that

$$V\left(A_1(\beta), \frac{\alpha_1}{\beta_S+\alpha_1}\right) \leq V^* \qquad \text{for all } \beta \tag{2.21}$$

That is,

$$V_1(A_1(\beta)) \frac{\alpha_1}{\beta_S+\alpha_1} + V_o(A_1(\beta)) \left[1 - \frac{\alpha_1}{\beta_S+\alpha_1} \right] \leq V^* \quad \text{for all } \beta$$

Thus any distribution G*(p) with the correct mean value

$$\int pdG^*(p) = \frac{\alpha_1}{\beta_S+\alpha_1} \tag{2.25}$$

will satisfy the necessary and sufficient condition (2.23) and therefore (2.22) and is therefore an optimum solution equivalent to

$$P(SN) = \frac{\alpha_1}{\beta_S+\alpha_1} \tag{2.26}$$

2.5 The General Solutions Applied to the Numerical Illustration

Let us assume that the nature of the signal and noise are such that the receiver operating characteristic is Fig. 6, e.g., d = 1. This particular operating characteristic occurs repeatedly in detectability. For payoff values, let

$$V_{SN\cdot A} = \$7 \qquad -K_{SN\cdot CA} = -\$8$$
$$-K_{N\cdot A} = -\$6 \qquad V_{N\cdot CA} = \$2$$

per unit time T. These are the values used in constructing the model, Fig. 2.

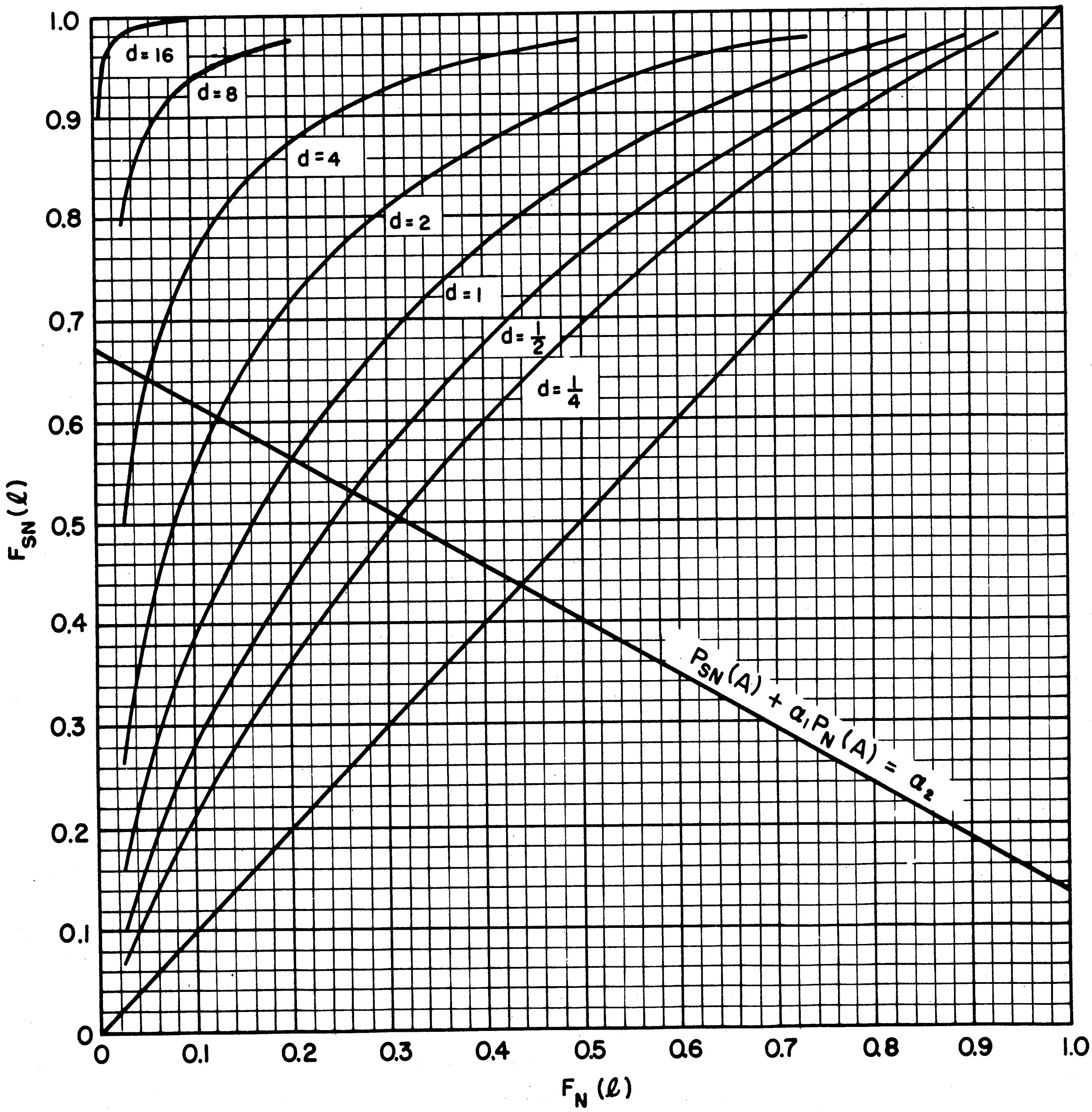

FIG. 6

RECEIVER OPERATING CHARACTERISTIC.
WITH GAME SOLUTION FOR $\alpha_1 = .53$, $\alpha_2 = .67$

($\ln \ell$ IS A NORMAL DEVIATE WITH $\sigma_N^2 = \sigma_{SN}^2$, $(M_{SN} - M_N)^2 = d\ \sigma_N^2$)

If the receiver were dead, the best possible thing for the operator to do is given by the usual game solution for independent processes found by the intersection of (2.6c) and the diagonal, and P(SN) = $\alpha_1/(1 + \alpha_1)$:

$$P(SN) = .348 \quad P(A) = .435 \quad V = -2.478$$

and the best the receiver operator can do against a smart transmitter is to say there is signal plus noise 43.5 per cent of the time and to lose \$2.48 per unit time.

However, with a receiver operating, the transmitter still uses a random process, but the receiver player is no longer independent of the transmitter. To obtain the receiver solution, compute

$$\alpha_2 = \frac{2 + 8}{7 + 8} = 2/3 \quad \frac{\alpha_2}{\alpha_1} = \frac{2 + 8}{2 + 6} = 1.25 \quad \alpha_1 = 8/15$$

Plotting (2.6c) on the ROC we can obtain the solutions from the intersection of (2.6c) and the curve d = 1, and $P(SN) = \dfrac{\alpha_1}{\beta_S + \alpha_1}$. Using the intersection of (2.6c) and d = 1 yields

$$P_{SN}(A) = .56 \qquad P_N(A) = .20$$

and the slope of the curve gives the associated β_S = 1.40. From these and $P(SN) = \alpha_1/\beta_S + \alpha_1$

$$P(SN) = \frac{.533}{1.4 + .533} = .27586 \ , \quad V = .40$$

Thus operating the receiver has forced a reduction of P(SN) from 35 per cent to 28 per cent and increased the payoff from a cost of \$2.48 per unit time to a gain of \$.40 per unit time. This change has been accomplished with very poor reception.

In a similar fashion, if the reception is characterized by d = 4, the intersection of d = 4 and (2.6c) yields

$$P_{SN}(A) = .64 \qquad P_N(A) = .056 \qquad \beta = 3$$

and therefore

$$P(SN) = .1818 \qquad V = 1.55$$

2.6 Limiting Cases as Detection Improves

It might appear from the example that as detection improves (due to a decrease of noise or increased signal energy, observation time or other reason) that false alarm probability and a priori probability both go to zero. Actually, this is always the case when $V_{SN \cdot A} > V_{N \cdot CA}$.

The solution to the receiver is obtained by the intersection of $V_o(A_1(\beta))$ and $V_1(A_1(\beta))$. Now V_o is a linear function of $P_N(A)$ decreasing from $V_{N \cdot CA}$ to $-K_{N\,A}$ independent of the perfection of the detection. On the other hand, V_1 increases monotonicly from $-K_{SN \cdot CA}$ to $V_{SN \cdot A}$ and as detection improves the rise becomes sharper.

If $V_{SN \cdot A} > V_{N \cdot CA}$, the intersection will occur at a value slightly less than that of the smaller, $V_{N \cdot CA}$, and therefore for a very small value of $P_N(A)$, and thus large values of β. The level β normally is unbounded,[1] and therefore as β becomes larger and larger, the a priori probability P(SN) will approach zero.

If $V_{SN \cdot A}$ $V_{N \cdot CA}$, the intersection will occur when V_o is slightly smaller than $V_{SN \cdot A}$. Using the linearity of V_o the $P_N(A)$ at the solution is ;ion slightly greater than

$$P_N(A) \doteq \frac{V_{SN \cdot A} - V_{N \cdot CA}}{V_{N \cdot CA} + K_{N \cdot A}}$$

1 See examples in Technical Report, EDG No. 13, Part II. All those considered require $\beta \rightarrow \infty$ for $P_N(A) \rightarrow 0$ for any degree of detection.

For any fixed false alarm rate, as detection improves, the operating level β normally approaches zero[1] and therefore P(SN) approaches one.

Obviously $V_{SN \cdot A} = V_{N \cdot CA}$ must be treated separately and more carefully because slightly to either side P(SN) goes to zero or to one. One special case of interest is that of Fig. 4.1 of EDG Technical Report No. 13 in which the ROC curves are symmetric. A straight line $P_{SN}(A) + P_N(A) = 1$ intersects all of the ROC curves at $\beta = 1$ while a straight line $P_{SN}(A) + \alpha_1 P_N(A) = 1$ intersects either increasingly larger or decreasingly smaller values of β depending on whether α_1 is greater than or less than one.

The limiting cases are summarized in Table I.

CASE	β_S	P(SN)	$P_{SN}(A)$	$P_N(A)$	V*
$V_{SN \cdot A} > V_{N \cdot CA}$, $\alpha_2 < 1$	∞	0	$\frac{V_{N \cdot CA} + K_{SN \cdot A}}{V_{SN \cdot A} + K_{SN \cdot A}}$	0	$V_{N \cdot CA}$
$V_{SN \cdot A} = V_{N \cdot CA}$, $\alpha_2 = 1$	?	?	1	0	$V_{SN \cdot A} = V_{N \cdot CA}$
$\ell n \ell$ NORMAL $K_{SN \cdot CA} < K_{N \cdot A}$	∞	0			
$\ell n \ell$ NORMAL $K_{SN \cdot CA} = K_{N \cdot A}$	1	$\frac{1}{2}$			
$\ell n \ell$ NORMAL $K_{SN \cdot CA} > K_{N \cdot A}$	0	1			
$V_{SN \cdot A} < V_{N \cdot CA}$, $\alpha_2 > 1$	0	1	1	$\frac{V_{SN \cdot A} - V_{N \cdot CA}}{V_{SN \cdot A} + K_{SN \cdot CA}}$	$V_{SN \cdot A}$

TABLE I. LIMITING VALUES AS DETECTION IMPROVES.

[1] See examples in Tech. Report. EDG No. 13, Part II, Fig. 4.5 where $\beta \rightarrow \beta_S > 0$ as $P_N(A) \rightarrow 1$ for any realizable detection, but $\beta_S \rightarrow 0$ as detection improves.

3. SOLUTIONS WHEN INFORMATION IS AVAILABLE

There may be instances in which the receiver operator has some information about the a priori probability but not complete knowledge of the value of P(SN).

3.1 A Priori Probability Restricted to a Known Interval

In Section 2 it was assumed that P(SN) could take on all values between zero and one. This may very well not be the case; in fact, the range of P(SN) may be very narrow. Therefore let us assume that

$$0 \leq p_o \leq P(SN) \leq p_1 \leq 1 \tag{3.1}$$

and a game-type solution is desired.

The solution to this is readily obtained. When $p_o = 0$ and $p_1 = 1$ we used the fact that neither $V(p_o)$ nor $V(p_1)$ dominated the other for all criteria, and therefore the solution was interior, i.e., was given by a criterion A that was neither the empty set nor the set of all inputs, and by a priori probability that was neither $p_o = 0$ nor $p_1 = 1$. Now if the game solution for the transmitter for the case when P(SN) was allowed to range from zero to one is within the limits of interest

$$p_o \leq \frac{\alpha_1}{\beta_S + \alpha_1} \leq p_1 \tag{3.2}$$

then the solutions are precisely the same to both receiver and transmitter, as in Section 2. This is apparent when we observe that at the solution in Section 2 the min-max relation held for all a priori probabilities and all criteria, and therefore it still holds for restricted values as long as the solution is among these restricted values. This applies to both types of transmitter solution.

If the game solution for P(SN) falls outside of the range of interest, that solution can still be used to obtain a solution within the range of interest. First, we observe that at all times the receiver player will use a likelihood ratio type receiver, because the solution must be stable, and whatever the solution value p of the a priori probability the receiver solution will satisfy (2.14) that is, be a likelihood ratio type receiver with $\beta = \frac{1-p}{p}\alpha_1$. Second, we observe that there was a unique value of β such that $V_o = V_1$, namely, the game solution value given in Section 2.3 and correspondingly a unique value for the solution for the a priori probability. Thus for only this value of P(SN) does the corresponding β yield $V_o = V_1$. Because the payoff surface is continuous, in any restricted range of a priori probability not containing this unique solution either V_o dominates V_1 throughout, or V_1 dominates V_o together.

Consider an interval such as $0 \leq p \leq p_1 < \frac{\alpha_1}{\beta_S + \alpha_1}$. To determine whether V_o dominates V_1 or vice versa, we can try at any point, for example, p = 0. If p = 0, the corresponding optimum criterion is the empty set $\emptyset$, and we have previously seen that for the empty set V_o dominates V_1. Thus for any $p < \frac{\alpha_1}{\beta_S + \alpha_1}$, $V_o\left(A_1\left(\frac{1-p}{p}\alpha_1\right)\right)$ dominates $V_1\left(A_1\left(\frac{1-p}{p}\alpha_1\right)\right)$. Recalling (2.5)

$$V(A) = p\, V_1(A) + (1-p)\, V_o(A) \qquad (2.5)$$

and using the dominance $V_o(A_1(\beta)) - V_1(A_1(\beta)) > 0$ we can rewrite V as

$$V(A_1(\beta)) = V_o(A_1(\beta)) - p\left[V_o(A_1(\beta) - V_1(A_1(\beta)\right] \qquad (3.3)$$

for all β associated with $p < \frac{\alpha_1}{\beta_S + \alpha_1}$. Therefore if the range of a priori probability is restricted to

$$p_o \leq p \leq p_1 < \frac{\alpha_1}{\beta_S + \alpha_1} \qquad (3.4)$$

by using the min-max principle we can consider A as fixed and minimize first with respect to p, which in all cases will result in the choice

$$p = p_1 \tag{3.5}$$

and then maximizing over A yields correspondingly

$$\beta = \frac{1 - p_1}{p_1} \alpha_1 \tag{3.6}$$

Conversely if

$$\frac{\alpha_1}{\beta_S + \alpha_1} < p_o \leq p \leq p_1 \tag{3.7}$$

the dual argument yields

$$p = p_o \tag{3.8}$$

and correspondingly

$$\beta = \frac{1 - p_o}{p_o} \alpha_1 \tag{3.9}$$

Summarizing the arguments, since either order of maximization and minimization leads to the same result, we observe that maximizing first with respect to A would cause the receiver to be of the likelihood ratio type operating on a portion of the ROC where V_o dominated V_1 throughout (or vice versa). If the minimization over p is carried out first, for all those criteria on the range of the ROC indicated above the same extreme value p_1 (or p_o) is chosen and therefore is independent of the choice of A in the subsequent maximization.

3.2 A Priori Probability a Random Variable With Known Mean

It was shown in Section 2.4.3 that any distribution of a priori probability with the proper mean was a transmitter solution, and correspondingly the receiver has no choice but to use $A_1(\beta_S)$, the unique solution, as criteria. It may occur that the mean is known to be $\bar{p}$. If $\bar{p}$ is $\frac{\alpha_1}{\alpha_1 + \beta_S}$ then knowing this is of no value to the receiver. This is why the game solution (to the transmitter) is called "equilibrium". Suppose that $\bar{p}$ is not this equilibrium value.

The receiver wishes to maximize $\bar{V}$ $(H(\beta), G(p))$, the average payoff.

$$\bar{V}(H, G) = \iint V\Big(A_1(\beta), p\Big) \; dG(p)dH(\beta)$$

for a fixed β, as in Section 2.4.3

$$\int V(A_1(\beta),p) \; dG(p) \;=\; V_1(A_1(\beta) \int pdG(p) \;+\; V_0(A_1(\beta)) \;-\; V_0(A_1(\beta)) \int pdG(p)$$

$$= \bar{p} \; V_1(A_1(\beta)) \;+\; (1 - \bar{p}) \; V_0(\beta))$$

Because this is maximized as in Eq (2.14) by

$$\beta \;=\; \frac{1 - \bar{p}}{\bar{p}} \; \alpha_1 \tag{3.10}$$

the average payoff $\bar{V}(H,G)$ will be maximized by the trivial distribution H that has a jump at β — i.e., the receiver should use an $A_1\left(\frac{1 - \bar{p}}{\bar{p}} \alpha_1\right)$ criterion. Although this is what would be expected, it is important in that it points up the fact that the receiver operator is unable to make use of any information about the transmitters distribution other than its mean, in order to maximize the average payoff $\bar{V}(H,G)$.

4. SUMMARY OF SOLUTIONS

The optimum receiver for the cases considered is, as in EDG Technical Report No. 13, that receiver that has as its output the likelihood ratio function of its input.

1) If the a priori probability may range from zero to one, the operating level β_S of such a receiver (that is, the bias level such that if the output

exceeds the bias, it is profitable to say there was a signal present) is adjusted so that

$$P_{SN}(A_1(\beta_S)) + \alpha_1 P_N(A_1(\beta_S)) = \alpha_2 \qquad (2.6c)$$

where α_1 and α_2 are constants depending on the various values placed on the possible responses, Eq (2.7). The solution for a priori probability is

$$P(SN) = \frac{\alpha_1}{\beta_S + \alpha_1} \qquad (2.15)$$

The receiver operating characteristic, a plot of $P_{SN}(A)$ vs $P_N(A)$, is a convenient device for solving for both β_S and P(SN).

2) If the a priori probability is restricted to range from p_o to p_1 inclusive, the transmitter solution is given by (2.15) if that value is between p_o and p_1, or is as close that value as possible, and the receiver solution is accordingly

$$\beta = \frac{1 - P(SN)}{P(SN)} \alpha_1 \qquad (2.14)$$

3) If the a priori probability is a random variable with known mean value, then the receiver solution is

$$\beta = \frac{1 - \bar{p}}{\bar{p}} \alpha_1 \qquad (3.10)$$

As a consequence of the min-max principle, the optimum value of $\bar{p}$ is

$$\bar{p} = \frac{\alpha_1}{\beta_S + \alpha_1}$$

Any distribution of a priori probability with this mean value is an optimum distribution from the anti-detection viewpoint. This allows the transmitter considerable freedom to achieve other purposes.

LIST OF SYMBOLS

A	The event "The operator says there is signal plus noise present," or a criterion, i.e., the set of receiver inputs for which the operator says there is a signal present.
$A(\beta)$	The criterion of all receiver inputs with likelihood ratio not less than β.
$A_1(\beta)$	Any criterion differing from $A(\beta)$ only on receiver inputs with likelihood ratio equal to β.
$A \cap B$	The common part of criteria A and B.
A - B	All receiver inputs in A but not in B.
CA	The event "The operator says there is noise alone."
$f_N(x)$	The probability density for points x in R if there is noise alone.
$f_{SN}(x)$	The probability density for points x in R if there is signal plus noise.
F(a), F*(a)	A probability distribution function associated with the receiver's strategy.
G(p), G*(p)	A probability distribution function associated with the transmitter's strategy.
$H(\beta)$, $H^*(\beta)$	A probability distribution function associated with a likelihood ratio receivers strategy.
$K_{N \cdot CA}$	Value of the error of falsely saying a signal was present.
$K_{SN \cdot CA}$	Value of the error of missing a signal.
$\ell(x)$	The likelihood ratio for the receiver input x. $\ell(x) = \dfrac{f_{SN}(x)}{f_N(x)}$.
$L(\beta)$	$L(\beta) = P_{SN}(A(\beta)) + \alpha_1 P_N(A(\beta))$
N	The event "there is noise alone."
P(SN), p	A priori probability of a signal being transmitted.
p_o, p_1	Bounds on p.
$\bar{p}$	The average value of p.
$P_N(A)$	The probability that the operator will say there is signal plus noise if there is noise alone, i.e., the false alarm probability.

$P_{SN}(A)$	The probability that the operator will say there is signal plus noise if there is signal plus noise, i.e., the probability of detection.
R	The space of all receiver inputs; as a criterion R indicates the receiver always indicates a signal is present.
SN	The event "There is signal plus noise."
T	The unit of time of transmission; the duration of the observation.
V, V(a,p)	The payoff function for fixed strategies.
$V_o(A)$	The payoff when p = 0.
$V_1(A)$	The payoff when p = 1.
$\overline{V}(F,G)$	The payoff function for distributed strategies.
V*	The value of the payoff function at the solution.
$V_{SN \cdot A}$	Value of correctly detecting a signal's presence.
$V_{N \cdot CA}$	Value of correctly detecting that no signal was present.
x	A receiver input x(t).
α_1	$\alpha_1 = \left[V_{N \cdot CA} + K_{N \cdot A}\right] / \left[V_{SN \cdot A} + K_{SN \cdot CA}\right]$
α_2	$\alpha_2 = \left[V_{N \cdot CA} + K_{SN \cdot CA}\right] / \left[V_{SN \cdot A} + K_{SN \cdot CA}\right]$
β_S	The optimum operating level of a likelihood ratio receiver.
$\emptyset$	The empty set; as a criterion $\emptyset$ indicates the receiver never indicates a signal is present.

Note: An * usually indicates optimum or solution functions or values.

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