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DETECTION OF A SIGNAL SPECIFIED EXACTLY WITH A NOISY STORED REFERENCE SIGNAL

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FOREWORD

In this paper Mr. Birdsall has undertaken the study of a problem the solution of which promises to extend the usefulness of the theory of signal detectability as a tool in psychophysical investigations. The problem is that of the behavior of a receiver which has a noisy memory: it "knows" the signal it is looking for only approximately, not precisely.

Up to the present time, mathematical studies of the detectability of signals have assumed perfect memories for the receivers. These studies consider cases wherein the receiver "acts" as if it could reproduce a template of a signal specified exactly at the transmitter. This template agrees with the transmitted signal in every detail. The waveform recorded on the template, its amplitude, starting time, and phase all agree precisely with the signal. The receiver can then compute a cross-correlation between the template and the input waveform, which may consist of either noise alone or signal plus noise. This cross-correlation constitutes the datum necessary for making the optimum decision regarding that waveform: either it contains a signal or it does not.

It should be obvious to all familiar with statistics that, if the template were an inexact copy, or if it had noise added to it, a lower correlation would result on the average, and performance of the receiver would suffer. In this case both variables have an error component, while in the cases previously studied, one of the variables, the template recording, is noiseless.

One might say that Mr. Birdsall is investigating the case of the noisy template. If one knows that the receiver template is noisy, how should the inputs be processed? Should one compute cross-correlations, or should a different analysis be made? It turns out that if the template is not very noisy, the receiver should compute the cross-correlation, while if it is very noisy it should integrate the energy at the input.

While the study has implications in many fields, it is only the applications to psychophysics that are discussed here. It is obvious that human beings do not have perfect memories. They all have noisy templates. Mr. Birdsall's paper is directed toward the understanding of receivers with noisy templates, and since the human observer falls within that class it is hoped that the results obtained will help account for the form human data assume.

One important conclusion evolving from this study is that memory noise cannot be treated as noise added to the input. It has a nonlinear effect which must be taken into account in the interpretation of psychophysical data.

It is true that the scope of the specific example treated in the paper is limited. Even so, the resulting curves will aid in interpreting data and in achieving a better understanding of the problems encountered in psychophysical experiments.

Wilson P. Tanner, Jr.

Ann Arbor, Michigan
October 15, 1959

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ACKNOWLEDGEMENT

The problem treated in this report evolved in discussions with Wilson P. Tanner, Jr. about the effect of poor memory on detectability. The crystalization of the general memory problem to this specific "first special case" was a joint effort. The author is further indebted to Mr. Tanner for his critical review of the deviations of the likelihood ratio receiver and its performance.

ABSTRACT

This report treats the optimization problem of detecting the presence of a signal in a background of white Gaussian noise, under the restriction that the signal is specified exactly but the receiver memory contains only a noisy version of the signal. The optimum receiver is specified. The performances of both the optimum receiver and the crosscorrelation receiver with a noisy memory are calculated and compared for a special case.

DETECTION OF A SIGNAL SPECIFIED EXACTLY WITH A NOISY STORED REFERENCE SIGNAL

1. INTRODUCTION

This analysis deals with the detection of a signal specified exactly (SSE) but not known exactly by the receiver. From the presentation standpoint the problem is identical to the elementary signal known exactly (SKE), but from the receiver standpoint the expected signal is distributed. The specific case analyzed is that wherein the receiver has a stored signal s(t) which differs from the true signal S(t) by band-limited white Gaussian noise. The reception is similarly corrupted by band-limited white Gaussian noise, which is independent of the internal storage noise. The analysis of the optimum receiver is made in Section 2. Evaluation of the receiver is made in Section 3 for the special case of 2WT = 1, and in Section 4 the cross-correlation receiver is evaluated and compared to the ideal for this special case. Section 7 is a discussion of the implications of the results.

2. ANALYSIS

This analysis is based on the theory of signal detectability.[1,2] The observation is of finite time duration, T sec, and all signals and

1. W. W. Peterson and T. G. Birdsall, "The Theory of Signal Detectability, Part I. The General Theory, Part II. Applications with Gaussian Noise," Electronic Defense Group Technical Report No. 13, The University of Michigan Research Institute, Ann Arbor, Michigan, June 1953.
2. "The Theory of Signal Detectability," W. W. Peterson, T. G. Birdsall, and W. C. Fox, Transactions of the IRE, PGIT-4, Sept. 1954, pp. 171-212.

noise waveforms have a finite series-bandlimit , W cps, on that time interval. The external noise is white and Gaussian with noise power per cycle N_o, and the internal storage noise is also white and Gaussian with noise power per cycle $\lambda N_o = n_o$.

The first step in the analysis is to derive the likelihood ratio. The receiver (or observer) must base its response on the total input, which is both the observation x(t) and the stored signal s(t).

The receiver has error-free storage of the parameters N_o, W, T, and λ and on each observation can perform error-free operations on the specific x(t) received and s(t) stored. The receiver does not have access to the true signal S(t) other than through the above listed items.

The probability density function of this input is found as follows. The noise density function for the observation is[1]

$$f_N[x(t)] = \left(\frac{1}{2\pi N_o W}\right)^{WT} \left[\exp - \frac{1}{N_o} \int_0^T x^2(t)dt\right] , \quad (1)$$

and in similar fashion, the density function for the observation when signal is present is

$$f_{SN}\left[x(t)\right] = \left(\frac{1}{2\pi N_o W}\right)^{WT} \exp\left[- \frac{1}{N_o} \int_0^T \left\{ x(t) - S(t)\right\}^2 dt\right] . \quad (2)$$

The density functions for the storage signal s(t) are the same for both hypotheses,

$$f(s(t)) = \left(\frac{1}{2\pi\lambda N_o W}\right)^{WT} \exp \left[- \frac{1}{N_o\lambda} \int_0^T \left\{ s(t) - S(t)\right\}^2 dt\right] . \quad (3)$$

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1. See Peterson, Birdsall, and Fox, Eq (48); or Technical Report No. 13, Eq (3.2b).

Now from the receivers standpoint Eq. (3) specifies the distribution of the true signal $S(t)$ about the stored signal $s(t)$. If $S(t)$ were known, the likelihood ratio would be the ratio of Eq. (2) to Eq. (1). In the case of a distributed signal this ratio must be averaged with respect to the distribution of the signal[1]

$$\ell[x(t)] = \int_S \exp\left[-\frac{1}{N_0}\int_0^T S^2(t)dt\right] \exp\left[\frac{2}{N_0}\int_0^T x(t)S(t)dt\right] dP(S) \quad (4)$$

where, of course, for a fixed stored signal $s(t)$

$$dP(S) = f[s(t)]dS . \quad (5)$$

It is shown in the appendix that the evaluation of Eq. (4), simplified, implies that the likelihood ratio is <u>strictly monotone increasing</u> with the quadratic form

$$\ell[x(t)] \text{ -smi- } \int_0^T x^2(t) + \frac{2}{\lambda} x(t)s(t) - \frac{1}{\lambda} s^2(t)dt . \quad (6)$$

Thus the receiver which computes the above quadratic will be an ideal receiver under the conditions of the problem. Several equivalent forms are given in Eqs. (7) and (8).

$$\ell[x(t)] \text{ -smi- } \int_0^T \left[x(t) + \frac{1}{\lambda} s(t)\right]^2 dt - \frac{1+\lambda}{\lambda^2} \int_0^T [s(t)]^2 dt \quad (7)$$

$$\ell[x(t)] \text{ -smi- } \int_0^T \left[\lambda x(t) + \frac{\lambda s(t)}{\sqrt{1+\lambda} -1}\right] \left[x(t) - \frac{s(t)}{\sqrt{1+\lambda} +1}\right] dt \quad (8)$$

The receivers (all optimum) based on Eqs. (6), (7), and (8) are given in Figs. 1, 2, and 3.

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1. See Peterson, Birdsall, and Fox, Eq (56); or Technical Report No. 13, Eq (3.7b).

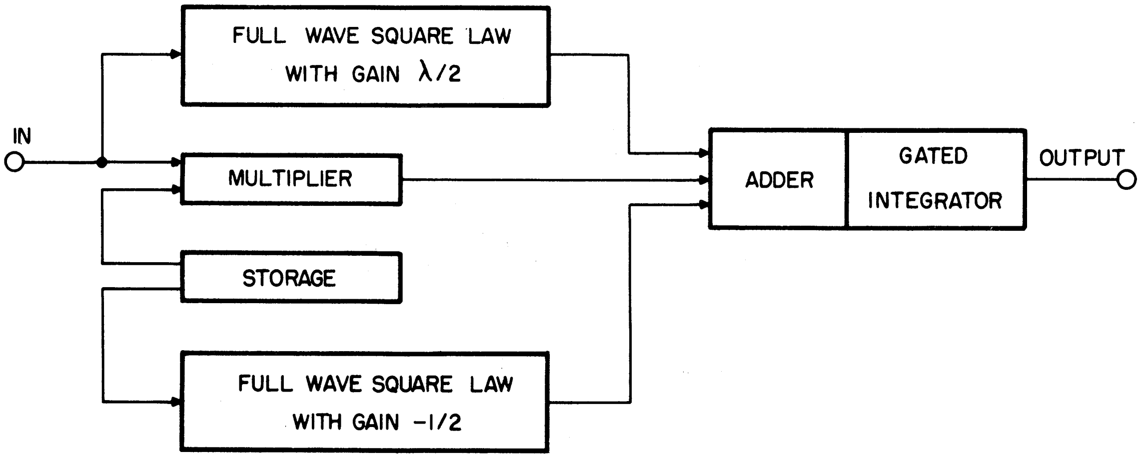

FIG.1 RECEIVER OF EQUATION 6

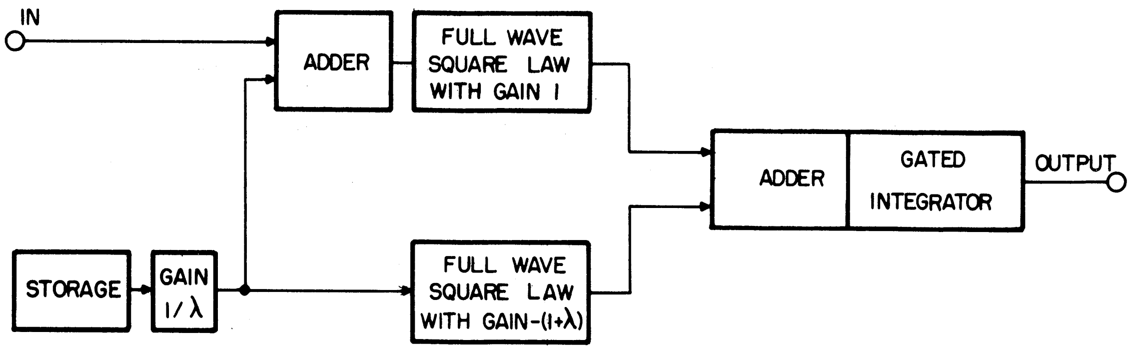

FIG. 2 RECEIVER OF EQUATION 7

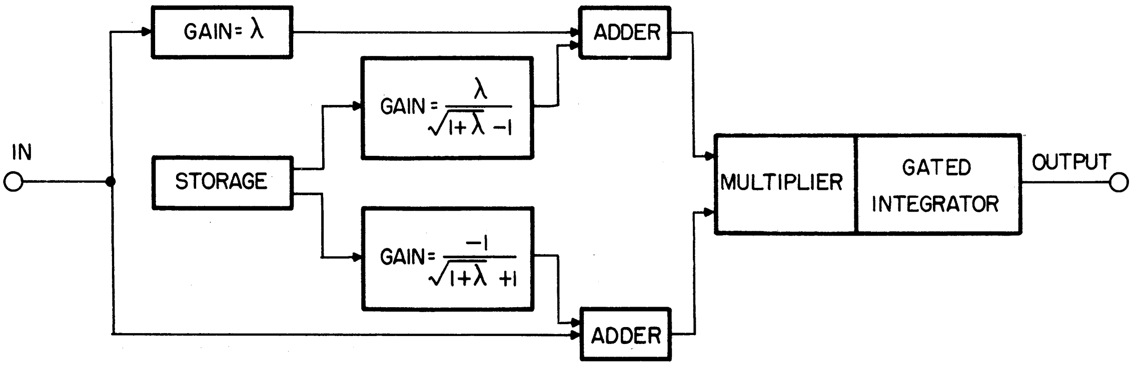

FIG 3a

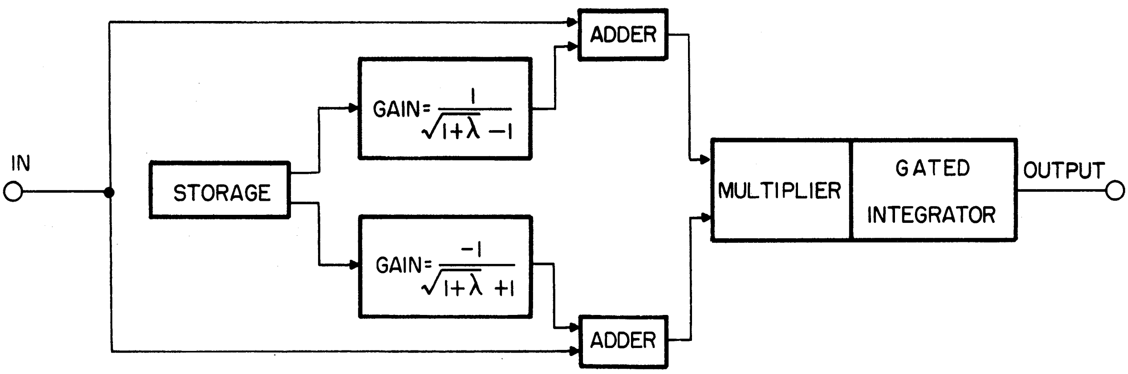

FIG. 3b

FIG. 3 RECEIVER OF EQUATION 8

(a) LOW OR MEDIUM INTERNAL NOISE VERSION

(b) HIGH INTERNAL NOISE VERSION

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3. EVALUATION OF IDEAL, 2WT = 1

The evaluation of the ideal should be carried out with an ROC curve analysis, i.e., a comparison of the distributions of either the likelihood ratio or any monotone function of the likelihood ratio, under the two conditions of noise alone and signal plus noise.

To date, the author has been unable to obtain these distributions in closed form. In all of the detection literature the performance of optimum receivers has been obtained in closed form for only a handful of cases. In the others, approximations or analogs have been used to obtain numerical results. Such techniques could also be applied to this optimum receiver. A less direct approach was taken for this problem; namely, the performance in a two-alternative-choice-in-time has been determined and is given in this section. It should be mentioned that although such an evaluation is not as complete as an ROC analysis, it usually agrees with the results of an ROC analysis in the medium probability range (1% to 50% false alarm range).

The evaluation assumes that the stored signal is the same for two observation intervals, $x_1(t)$ and $x_2(t)$, one which is due to noise alone and one which is due to signal plus noise. The receiver operation is to compare the outputs corresponding to the observations and indicate the interval with the larger output. The evaluation determines the signal strength necessary to obtain performance equivalent to that which would have been obtained had the signal been known exactly.

Equation (7) is the easiest relation to evaluate in this situation. Because it is assumed that the stored signal s(t) is the same for both observations, one concludes that

$$\ell(x_1(t)) > \ell(x_2(t)) \Longleftrightarrow \int_0^T \left[x_1(t) + \frac{1}{\lambda} s(t)\right]^2 dt > \int_0^T \left[x_2(t) + \frac{1}{\lambda} s(t)\right]^2 dt. \quad (9)$$

It was assumed that 2WT = 1, and hence each function of time can be represented by a single sample, and that the integrals will be equal to 1/2W times the integrand at the sampled points. Hence, if the samples are denoted by dropping the "t",

$$\ell\left[x_1(t)\right] > \ell\left[x_2(t)\right] \Longleftrightarrow (x_1 + \frac{1}{\lambda} s)^2 > (x_2 + \frac{1}{\lambda} s)^2 \quad (10)$$

$$\Longleftrightarrow (x_1 + \frac{1}{\lambda} s)^2 - (x_2 + \frac{1}{\lambda} s)^2 > 0 \quad (11)$$

$$\Longleftrightarrow (x_1 - x_2)(x_1 + x_2 + \frac{2}{\lambda} s) > 0 \quad (12)$$

$$\Longleftrightarrow \begin{cases} (x_1 - x_2) > 0 \text{ and } (x_1 + x_2 + \frac{2}{\lambda} s) > 0 \\ \text{or} \\ (x_1 - x_2) < 0 \text{ and } (x_1 + x_2 + \frac{2}{\lambda} s) < 0 . \end{cases} \quad (13)$$

Now if $x_1(t)$ is due to signal plus noise and $x_2(t)$ due to noise alone, the probability that the above inequalities hold is the probability of a "correct decision", i.e., in this case, that the output of observation number one is greater than that of observation number two. It is obvious that the situation is completely symmetric and that the probability of a correct decision is the same as if the hypothesis had been reversed.

The variables $(x_1 - x_2)$ and $(x_1 + x_2 + \frac{2}{\lambda} s)$ are independent Gaussian variables. The means and variances are as follows:

$$\text{Normalize so that } \sigma^2(x_1) = 1.$$

$$\text{Then} \quad \sigma^2(s) = \lambda,$$

$$\text{and} \quad \mu(x_1) = S \tag{14}$$

$$\mu(x_2) = 0$$

$$\mu(s) = S.$$

Hence $\mu(x_1 - x_2) = S - 0 = S$

$$\sigma(x_1 - x_2) = \sqrt{1+1} = \sqrt{2} \tag{15}$$

so that

$$\Pr(x_1 - x_2 > 0) = \Phi\left(\frac{S}{\sqrt{2}}\right)$$

and (16)

$$\Pr(x_1 - x_2 < 0) = \Phi\left(-\frac{S}{\sqrt{2}}\right),$$

where Φ is the normal, or Gaussian, distribution function.

Similarly,

$$\mu(x_1 + x_2 + \frac{2}{\lambda} s) = S + 0 + \frac{2}{\lambda} S = S\,\frac{\lambda+2}{\lambda}$$

(17)

$$\sigma(x_1 + x_2 + \frac{2}{\lambda} s) = \sqrt{1 + 1 + \frac{4}{\lambda^2}\lambda} = \sqrt{\frac{2(\lambda+2)}{\lambda}}$$

so that

$$\frac{\mu}{\sigma} = S\sqrt{\frac{\lambda+2}{2\lambda}}, \tag{18}$$

$$\Pr(x_1 + x_2 + \frac{2}{\lambda} s > 0) = \Phi\left(S\sqrt{\frac{\lambda+2}{2\lambda}}\right),$$

and (19)

$$\Pr(x_1 + x_2 + \frac{2}{\lambda} s < 0) = \Phi\left(-S\sqrt{\frac{\lambda+2}{2\lambda}}\right).$$

Combining Eqs. (16) and (19) as indicated in Eq. (13),

$$\Pr(\text{correct}) = \Phi\left(\frac{S}{\sqrt{2}}\right)\Phi\left(S\sqrt{\frac{\lambda+2}{2\lambda}}\right) + \Phi\left(-\frac{S}{\sqrt{2}}\right)\Phi\left(-S\sqrt{\frac{\lambda+2}{2\lambda}}\right). \tag{20}$$

Had there been no storage corruption, the situation would have been labeled "SKE", and the corresponding performance would have been obtained from Eq. (20) by letting $\lambda \to 0$.

$$\text{SKE} \quad \Pr(\text{correct}) = \Phi\left(\frac{S}{\sqrt{2}}\right) \tag{21}$$

Figure 4 presents Eq. (20) for no internal noise ($\lambda = 0$), for as much internal as external noise ($\lambda = 1$), and the limiting performance as the internal noise increases ($\lambda = \infty$).

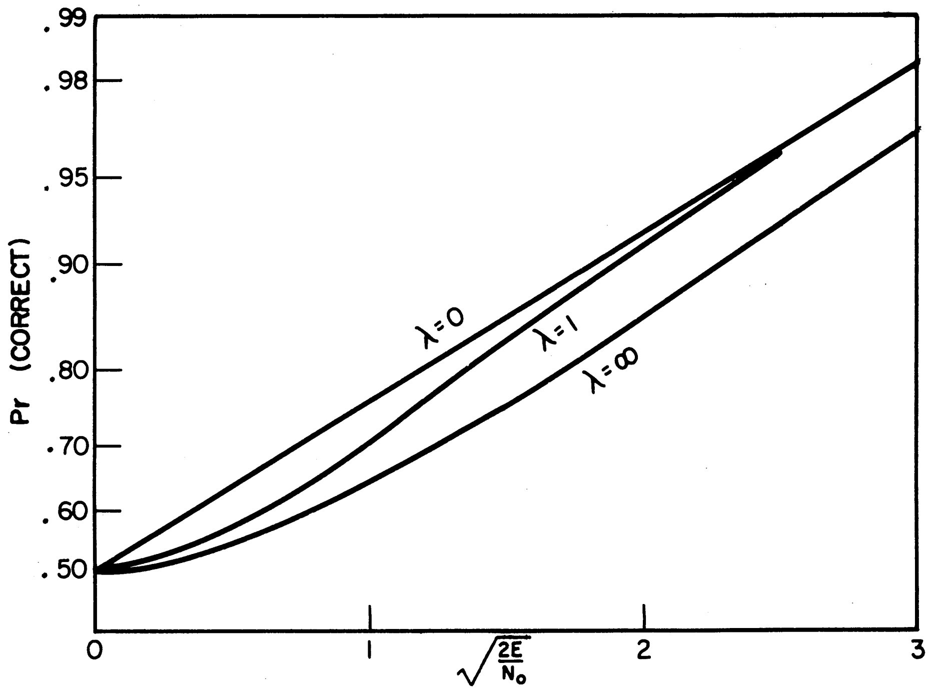

FIG. 4. IDEAL RECEIVER PERFORMANCE, TWO ALTERNATIVES FORCED CHOICE, 2WT = 1

To complete this analysis, the efficiency of this receiver should be computed.[1] The computations were carried out for efficiencies above 0.10 for constant values of λ, and are shown in Fig. 5. Lines of constant performance are indicated on this figure to show the regions that would be encountered in normal psychophysical experimentation.

1. W. P. Tanner, Jr., and T. G. Birdsall, "Definitions of d' and η as Psychophysical Measures," Electronic Defense Group Technical Report No. 80, The University of Michigan Research Institute, Ann Arbor, Michigan, March, 1958, and JASA, Vol. 30. No. 10, October 1958, pp. 922-928.

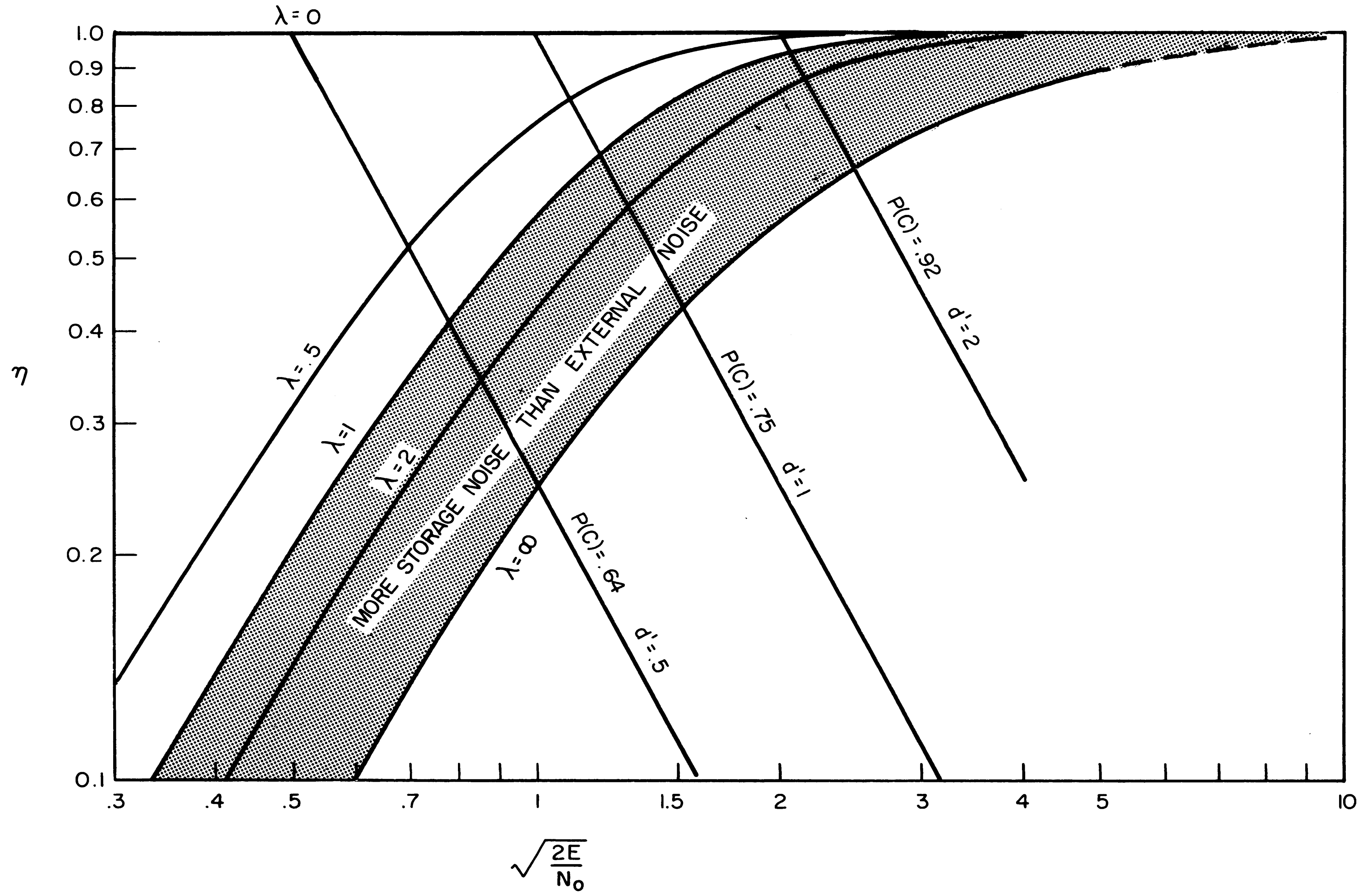

FIG. 5 RECEIVER EFFICIENCY, η, NOISY MEMORY LIKELIHOOD RATIO RECEIVER

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4. EVALUATION OF CROSS-CORRELATION, 2WT = 1

This is an extremely simple case, since the cross-correlation simply supplies the sign for comparison. Specifically,

$$\int_0^T x_1(t)s(t)dt > \int_0^T x_2(t)s(t)dt \Leftrightarrow x_1 s > x_2 s \tag{22}$$

$$\Longleftrightarrow \begin{cases} s > 0 \quad \text{and} \quad x_1 > x_2 \\ \text{or} \\ s < 0 \quad \text{and} \quad x_1 < x_2 \end{cases} \tag{23}$$

when x_1 is due to signal plus noise, the probability that $x_1 > x_2$ is the same as the probability that $x_1 - x_2 > 0$, namely,

$$Pr(x_1 > x_2) = \Phi\left(\frac{S}{\sqrt{2}}\right). \tag{24}$$

Under any conditions the probability that the stored sample is positive is

$$Pr(s > 0) = \Phi\left(\frac{S}{\sqrt{\lambda}}\right). \tag{25}$$

Hence,

$$\text{Prob(correct)} = \Phi\left(\frac{S}{\sqrt{2}}\right)\Phi\left(\frac{S}{\sqrt{\lambda}}\right) + \Phi\left(-\frac{S}{\sqrt{2}}\right)\Phi\left(-\frac{S}{\sqrt{\lambda}}\right). \tag{26}$$

By comparing Eq. (20), for the ideal, with Eq. (26) one sees that where the term $\sqrt{\frac{\lambda+2}{2\lambda}}$ appeared for the ideal, the term $\sqrt{\frac{1}{\lambda}}$ appears for the crosscorrelator. For small values of λ ($\lambda < 0.1$), these are roughly the same; for large values of λ, the term for the ideal rapidly approaches $\sqrt{\frac{1}{2}}$, while the crosscorrelator term decends toward zero. The curves of Figs. 4 and 5 apply with the following corrections.

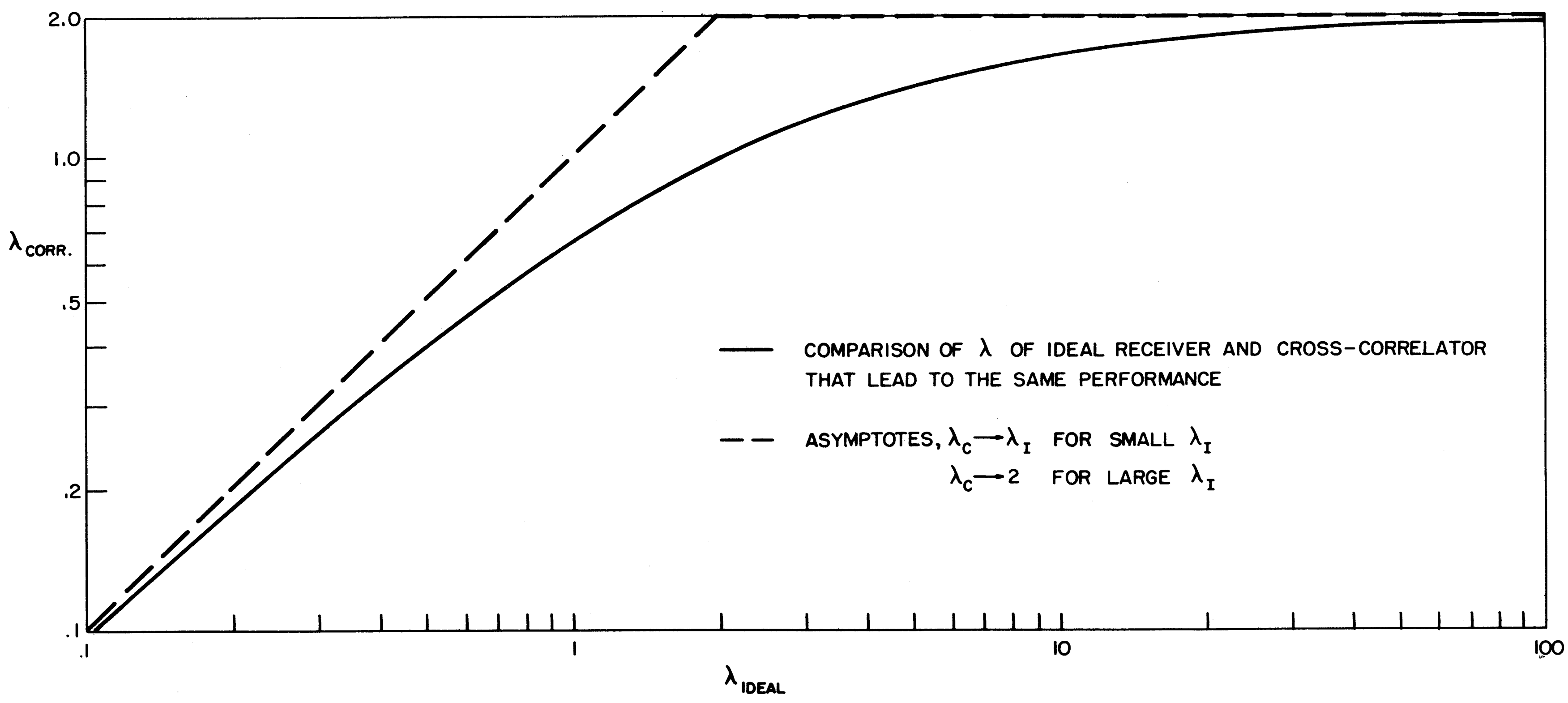

FIG. 6

λ FOR CROSSCORRELATOR VS. λ FOR IDEAL

TABLE I

λ for crosscorrelator	0	.40	.67	1.00	2.00
λ for ideal	0	.50	1.00	2.00	∞

A complete curve of this relation is given in Fig. 6. This shows the serious loss of efficiency when the cross-correlator memory is noisy, since 2 db more storage noise than external noise has the same effect as 8.5 db on the ideal (noisy-storage) receiver. Figure 7 is included for comparison with Fig. 4.

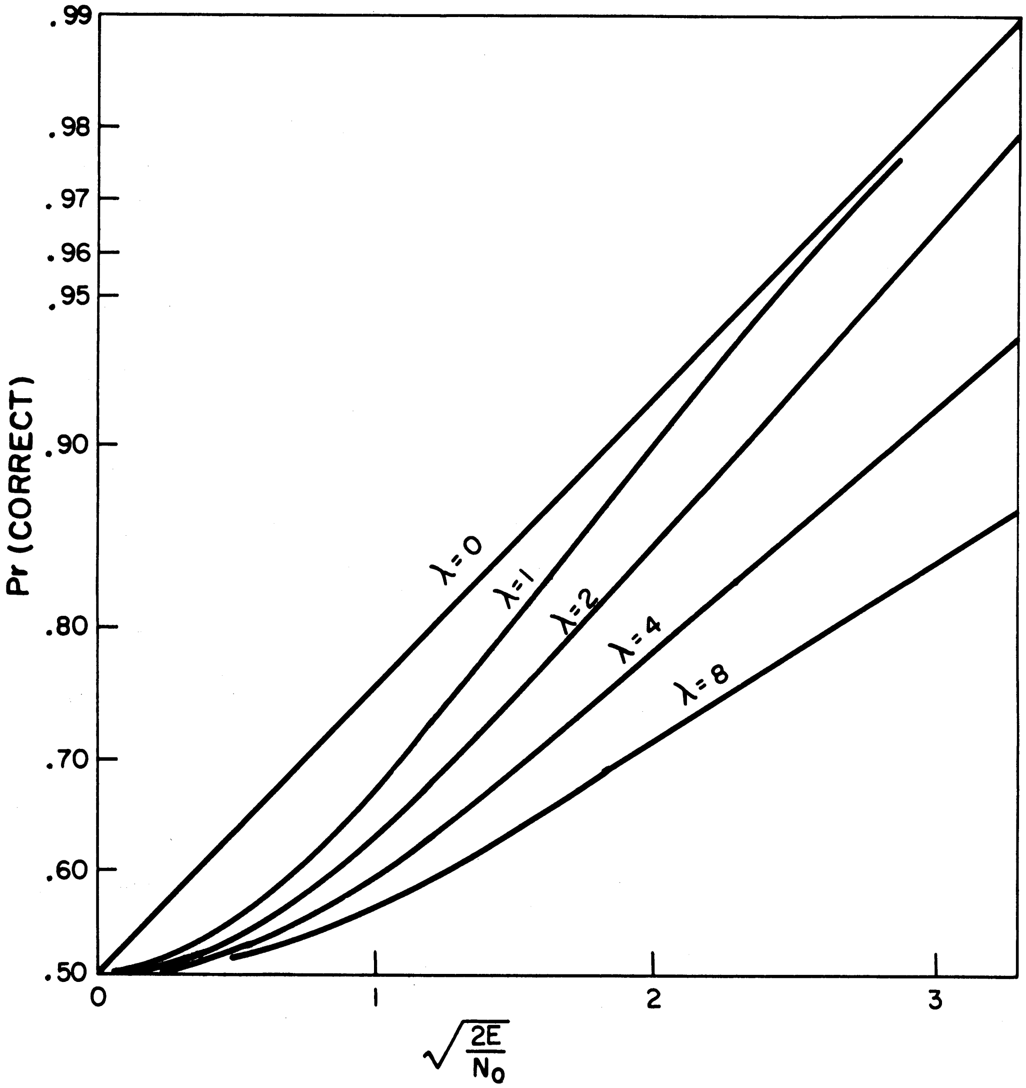

FIG. 7. CROSSCORRELATOR PERFORMANCE, TWO ALTERNATIVES FORCED CHOICE, 2WT = 1

5. COMPARISON COMPUTATIONS

Two further computations may be made for the sake of comparison. Had the signal actually been distributed on transmission with the same distribution as that in storage, the ideal receiver of Section 2 would be the true likelihood receiver for this new case, and the only differences would be where the signal variance increases, sample variances also increase; namely, for a signal in the first interval,

$$\sigma^2(x_2) = 1 \qquad \sigma^2(x_1) = 1+\lambda \tag{27}$$

Equation (20) becomes

$$\Pr(\text{correct}) = \Phi\left(\frac{S}{\sqrt{\lambda+2}}\right)\Phi\left(\frac{S}{\sqrt{\lambda}}\sqrt{\frac{\lambda^2+4\lambda+4}{\lambda^2+2\lambda+4}}\right)+\Phi\left(\frac{-S}{\sqrt{\lambda+2}}\right)\Phi\left(\frac{-S}{\sqrt{\lambda}}\sqrt{\frac{\lambda^2+4\lambda+4}{\lambda^2+2\lambda+4}}\right) \tag{28}$$

The radical $\sqrt{\frac{\lambda^2+4\lambda+4}{\lambda^2+2\lambda+4}}$ rises from 1.0 at $\lambda = 0$, to 1.15 between $\lambda = 2$ and $\lambda = 3$, then decreases slowly to 1.00 as $\lambda \rightarrow \infty$. Hence a good lower approximation is

$$\Pr(\text{correct}) = \Phi\left(\frac{S}{\sqrt{\lambda+2}}\right)\Phi\left(\frac{S}{\sqrt{\lambda}}\right) + \Phi\left(-\frac{S}{\sqrt{\lambda+2}}\right)\Phi\left(-\frac{S}{\sqrt{\lambda}}\right). \tag{29}$$

On close inspection, one observes that Eq. (29) is the exact equation for the cross-correlator for this case. Hence, when the signal has the same variance as the stored signal and 2WT = 1, the ideal receiver does only slightly better than the cross-correlator. Thus it can be concluded that the improvement of the ideal over the cross-correlator is much more important when the signal is actually stable but memory is poor, than when the signal is distributed and the lack of specificity is not due to poor memory.

A final calculation is for a receiver which has noise-free storage but λN_o joules per cycle added to the input. The detection index for such a receiver is

$$(d')^2 = \frac{2E}{N_o + \lambda N_o} = \frac{1}{1+\lambda}\,\frac{2E}{N_o} \tag{30}$$

so that the efficiency is

$$\eta = \frac{1}{1+\lambda} \; . \tag{31}$$

This receiver does not behave at all like those in the poor memory situation, and hence it can be concluded that, at least for this and similar cases, noisy memory does not act like "additional noise that can be reflected (invariantly) into the input."

6. SUMMARY OF EQUATIONS FOR 2WT = 1

In this section the final performance equations are repeated, together with definitions of parameters. Since one man's normalization is another man's poison, two alternative notations are used.

First Normalization:

E - signal energy at receiver input and at receiver memory input.

N_o - noise power per cycle of white noise added to signals at receiver input.

λ - ratio of memory-noise power per cycle to N_o.

Signal Specified Exactly, Ideal Receiver (SSE, Ideal):

$$P(c) = \Phi\left(\sqrt{\frac{E}{N_o}}\right)\Phi\left(\sqrt{\frac{E}{N_o}}\,\sqrt{\frac{\lambda+2}{\lambda}}\right)+\Phi\left(-\sqrt{\frac{E}{N_o}}\right)\Phi\left(-\frac{E}{N_o}\,\sqrt{\frac{\lambda+2}{\lambda}}\right) .^{*} \tag{20.1}$$

---

* Φ is the normal or Gaussian distribution function $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt.$

Signal Specified Exactly, Cross-Correlation Receiver (SSE, x-cor):

$$P(c) = \Phi\left(\sqrt{\frac{E}{N_o}}\right)\Phi\left(\sqrt{\frac{2E}{N_o\lambda}}\right) + \Phi\left(-\sqrt{\frac{E}{N_o}}\right)\Phi\left(-\sqrt{\frac{2E}{N_o\lambda}}\right) \qquad (26.1)$$

Signal Known and Specified Statistically, Ideal Receiver (SKS, Ideal):

$$P(c) = \Phi\left(\sqrt{\frac{E}{N_o}}\sqrt{\frac{2}{\lambda+2}}\right)\Phi\left(\sqrt{\frac{2E}{N_o\lambda}}\sqrt{\frac{\lambda^2+4\lambda+4}{\lambda^2+2\lambda+4}}\right)$$

$$+\Phi\left(-\sqrt{\frac{E}{N_o}}\sqrt{\frac{2}{\lambda+2}}\right)\Phi\left(-\sqrt{\frac{2E}{N_o\lambda}}\sqrt{\frac{\lambda^2+4\lambda+4}{\lambda^2+2\lambda+4}}\right). \qquad (28.1)$$

Signal Known and Specified Statistically, Cross-Correlation Receiver (SKS, x-cor):

$$P(c) = \Phi\left(\sqrt{\frac{E}{N_o}}\sqrt{\frac{2}{\lambda+2}}\right)\Phi\left(\sqrt{\frac{2E}{N_o\lambda}}\right) + \Phi\left(-\sqrt{\frac{E}{N_o}}\sqrt{\frac{2}{\lambda+2}}\right)\Phi\left(-\sqrt{\frac{2E}{N_o\lambda}}\right). \qquad (29.1)$$

Second Normalization:

E - signal energy at receiver input.

N_o - noise power per cycle at receiver input.

e - signal energy at undistorted memory.

n_o - noise power per cycle of memory noise.

(SSE, Ideal):

$$P(c) = \Phi\left(\sqrt{\frac{E}{N_o}}\right)\Phi\left(\sqrt{\frac{E}{N_o}+\frac{2e}{n_o}}\right) + \Phi\left(-\sqrt{\frac{E}{N_o}}\right)\Phi\left(-\sqrt{\frac{E}{N_o}+\frac{2e}{n_o}}\right). \qquad (20.2)$$

(SSE, x-cor):

$$P(c) = \Phi\left(\sqrt{\frac{E}{N_o}}\right)\Phi\left(\sqrt{\frac{2e}{n_o}}\right) + \Phi\left(-\sqrt{\frac{E}{N_o}}\right)\Phi\left(-\sqrt{\frac{2e}{n_o}}\right). \qquad (26.2)$$

(SKS, Ideal):

$$P(c) = \Phi\left(\sqrt{\frac{2eE}{En_o+2eN_o}}\right)\Phi\left(\sqrt{\frac{2e}{n_o}}\sqrt{\frac{E^2n_o^2+4EeN_on_o+4e^2N_o^2}{E^2n_o^2+2EeN_on_o+4e^2N_o^2}}\right)$$

$$+\ \Phi\left(-\sqrt{\frac{2eE}{En_o+2eN_o}}\right)\Phi\left(-\sqrt{\frac{2e}{n_o}}\sqrt{\frac{E^2n_o^2+4EeN_on_o+4e^2N_o^2}{E^2n_o^2+2EeN_on_o+4e^2N_o^2}}\right) \qquad (28.2)$$

(SKS, x-cor):

$$P(c) = \Phi\left(\sqrt{\frac{2eE}{En_o+2eN_o}}\right)\Phi\left(\sqrt{\frac{2e}{n_o}}\right) + \Phi\left(-\sqrt{\frac{2eE}{En_o+2eN_o}}\right)\Phi\left(-\sqrt{\frac{2e}{n_o}}\right). \qquad (29.2)$$

In all cases, with no internal noise, both receivers are ideal and

(SKE, Ideal):

$$P(c) = \Phi\left(\sqrt{\frac{E}{N_o}}\right) . \qquad (32)$$

7. CONCLUSIONS

Several receivers have been discussed and evaluated. The one of primary concern in this report is the receiver that is the optimum receiver when restricted to receivers with noisy memory and detecting a signal specified exactly in white Guassian noise. The second receiver is the crosscorrelator, which would be the optimum receiver if the memory were perfect. The evaluations of Sections 3 and 4 are for the noisy-memory, signal-specified-exactly situation, for which the receiver under study is optimum and the crosscorrelator is not. In Section 5 the signal was actually distributed and both receivers evaluated;

neither is the optimum receiver for the condition. In Section 5 a receiver with additional noise at the input but no memory noise was treated. It can be concluded that the performance of the "noisy-memory, signal-specified-exactly" optimum receiver detecting a signal under the conditions for which it is optimum is not equivalent to the performance in any of the other receiver-signal combinations.

Two normalizations have been presented in Section 6, and the interpretation inherent with these normalizations deserves discussion. The first normalization, used in the analysis, considered the internal noise proportional to the external noise. Of course, this is possible for any fixed situation. However, when the efficiency η is plotted against the input signal quality $\frac{2E}{N_o}$ the rise in efficiency embodies the fact that the memory-signal quality is correspondingly increasing. The result is that for very small signals the efficiency is very low and rises rapidly as the signal level increases. In contrast, if the model being studied has a fixed-quality memory signal $\frac{2e}{n_o}$ the increase of efficiency with signal level is quite different. Figure 8 shows this effect. The efficiency is constant for low level signals and then rises to 1 when the external quality becomes much larger than the internal quality.

In any application of a noisy-memory model the experimenter has one more variable to contend with. On the one hand, it is conceivable that this type of model could unify previously conflicting data. On the other hand, one would not expect the relation of internal quality to external quality of signal to be the proportional relation of Fig. 5 or completely independent as in Fig. 8.

The basic analysis of this problem has been only partially completed in this report. The determination of the ideal receiver is

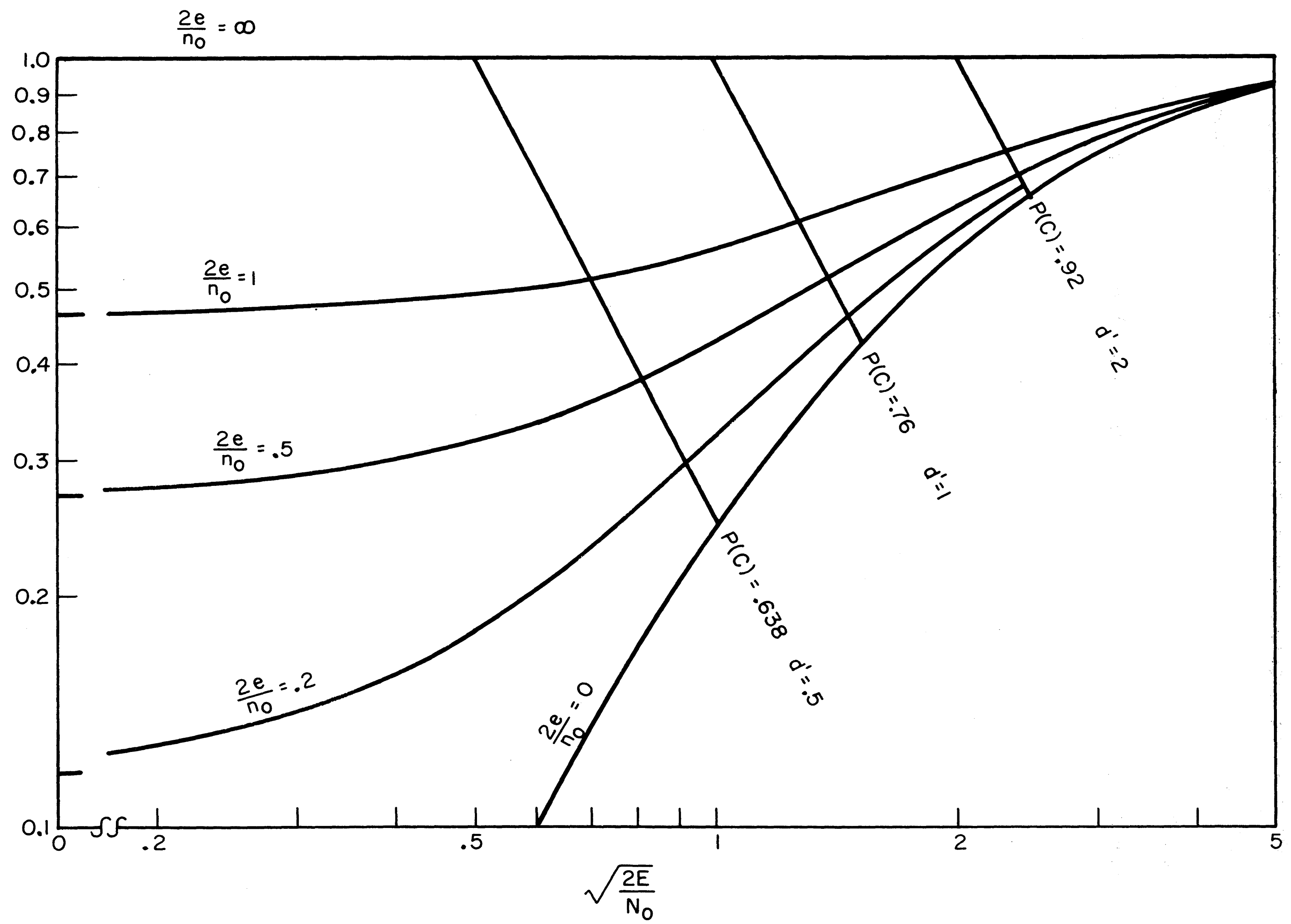

FIG. 8 RECEIVER EFFICIENCY, η, NOISY MEMORY LIKELIHOOD RATIO RECEIVER

completed in Section 2. However, the numerical evaluation has been completed only for the special case of 2WT = 1 and a symmetric forced choice in time where no memory degradation took place between presentations. Although the author feels this special case indicates the effect of "noisy-memory", a more complete evaluation might shed more light on this model.

APPENDIX

EVALUATION OF EQUATION (4)

Insertion of Eq. (3) and (5) into Eq. (4) yields

$$\ell(x) = \left(\frac{1}{2\pi\lambda N_o W}\right)^{WT} \int_S \exp\left[\frac{1}{N_o} \int_0^T -S^2(t)+2x(t)S(t) - \frac{1}{\lambda}\left\{s(t)-S(t)\right\}^2 dt\right] dS.$$

In evaluating this, a simplified notation is used to recognize the completion of the square in the exponent integrals; specifically, the time argument was omitted.

The negative of the integrand of the inner integral is

$$S^2 - 2xS + \frac{1}{\lambda}(s-S)^2 = S^2 - 2xS + \frac{s^2}{\lambda} - \frac{2sS}{\lambda} + \frac{S^2}{\lambda}$$

$$= S^2\left(1 + \frac{1}{\lambda}\right) - 2S\left(x + \frac{s}{\lambda}\right) + \frac{s^2}{\lambda}$$

$$= \left(S\sqrt{1 + \frac{1}{\lambda}} - \frac{x + \frac{s}{\lambda}}{\sqrt{1 + \frac{1}{\lambda}}}\right)^2 - \frac{\left(x + \frac{s}{\lambda}\right)^2}{1 + \frac{1}{\lambda}} + \frac{s^2}{\lambda}$$

$$= \left(1 + \frac{1}{\lambda}\right)\left(S - \frac{x + \frac{s}{\lambda}}{1 + \frac{1}{\lambda}}\right)^2 - \frac{\lambda x^2 + 2sx - s^2}{1+\lambda}.$$

Hence

$$\ell(x) = \left(\frac{1}{2\pi\lambda N_o W}\right)^{WT} \exp\left[\int_0^T \frac{\lambda x^2 + 2xs - s^2}{N_o(1+\lambda)}\, dt\right] \int_S \exp\left[-\frac{\lambda+1}{N_o \lambda} \int_0^T \left(S - \frac{\lambda x + s}{\lambda + 1}\right)^2 dt\right] dS.$$

Now this should be examined in careful detail. The integral with respect to S is a Lebesgue integral in the 2WT dimensional function space. For each x(t) and s(t) the expression

$$\int_0^T \left[S(t) - \frac{\lambda x(t)+s(t)}{1+\lambda} \right]^2 dt$$

represents the square of the distance from S(t) to the fixed point $\frac{\lambda x+s}{1+\lambda}$. Hence the integral

$$\int_S \exp\left[-\frac{\lambda+1}{N_o\lambda}\int_o^T \left(S - \frac{\lambda x+s}{\lambda+1} \right)^2 dt \right] dS$$

is proportional to the probability of the region of integration because the integrand is proportional to the normal probability-density function in 2WT dimensional space. The region of integration is the whole space and hence has probability one, independent of the center $\frac{\lambda x+s}{1+\lambda}$ of the distribution. This integral has some (non-zero) value k_1 which is not a function of S(t), s(t), or x(t), although it is a function of λ and N_o. This value need not be determined since the desired result is to show that the likelihood ratio can be written as

$$\ell(x) = k_1 \exp \int_o^T \frac{\lambda x^2+2sx-s^2}{N_o(1+\lambda)} dt.$$

Thus $\ell(x)$ is strictly monotone increasing with the exponent

$$\ell(x) \text{ -smi- } \int_o^T \left(x^2+2\frac{sx}{\lambda} - \frac{s^2}{\lambda} \right) dt.$$