Aggregation Bounds in Stochastic Production Problems

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Abstract:

Capacitated stochastic production problems can be modelled as multi-stage stochastic programs. Simple solution procedures for these problems do not exist. Some simplification procedure is generally applied to allow their solution. In this paper, we apply the principle of aggregation to stochastic production problems by aggregating over random variables and time periods. The solution of the aggregate problem is then used to obtain bounds on the value of the full stochastic program.

I. Introduction

The optimization of multi-stage production-inventory control systems with uncertain demands and capacitated production represents a challenging problem that most production planning methods have not considered. This problem is an example of a multi-stage stochastic program for which few computational procedures exist. When only two periods are present, the methods of El Agizy [6], Everett and Ziemba [7] and Wets [9] may be applied. For three or four period examples, Birge's [3] general stochastic programming code may be applied for random variables with discrete realizations. The only approaches to specifically consider the production problem are Beale, Forrest, and Taylor's [2] and Ashford's [1] approximations for normally distributed demands. Large general problems are still, however, not readily solved.

To simplify these models to some solvable form, the general approach of aggregating variables and constraints (Zipken [10, 11]) may be applied. When the weights of these aggregations coincide with the distributions of the random variables, it has been shown (Birge [4]) that the resulting aggregate problem is the stochastic problem with expected values replacing random variables. This expected value problem has been analyzed in Bitran and Yanasse [5] under different assumptions about the distributions of the random variables.

Other types of aggregation are also possible. In this paper, we show that bounds on the value of the full stochastic problem can be found from solutions of aggregate problems that combine both random variables and time periods. The assumptions usually necessary for these bounds in general aggregation are shown to be true by virtue of the problem structure in Section 2. In Section 3, a specific aggregation for combining random variables and time periods is given and the bounds resulting from this aggregate problem are presented. Section 4 presents an example and other potential aggregations.

2. Problem Definition and Variable Bounds

The formulation we consider is similar to that in Beale, Forrest and, Taylor [2] and Bitran and Yanasse [5]. We write the single product multi-stage stochastic production problem as

$$\max \qquad z = E\left[\sum_{t=1}^{T} \rho^{t-1}(p_t x_t - q_t o_t - h_t i_t^{+})\right] \tag{1}$$

subject to

$$x_t - o_t \leq k_t \tag{1.1}$$

$$y_{t-1} + x_t - y_t = 0, \tag{1.2}$$

$$y_t \geq b_t, \tag{1.3}$$

$$+ i_{t-1}^{+} - i_{t-1}^{-} + x_t - i_t^{+} + i_t^{-} = d_t, \tag{1.4}$$

$$x_t, o_t, y_t\ i_t^{+}, i_t^{-} \geq 0, \qquad t = 1,\ldots,T;$$

where the decision variables are x_t, production in time period t, o_t, overtime used in period t, i_t^{+}, inventory after period t, i_t^{-} , back orders after period t, and y_t, total production through period t. p_t is net production revenue, q_t is overtime cost and h_t is inventory costs.

These costs and the capacity k_t and minimum total production b_t are assumed known. The demand d_t is a random variable defined on an interval $[d_t^{min}, d_t^{max}]$ with distribution function $F_t(d_t)$. The random variables $d_1, \ldots d_t$ have a joint distribution function $F(d_1, \ldots, d_t)$. E[ ] signifies mathematical expectation with respect to these random variables. The decision variables depend on past outcomes so x_t, for example, is really $x_t(d_1, \ldots, d_t)$. The The expected value can then be written as

$$E\left(\sum_{t=1}^{T} \rho^{t-1} (px_t - qo_t - h_t \, i_t^+) \right.$$

$$= \sum_{t=1}^{T} \rho^{t-1} \left\{ \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (px_t(d_1, \ldots, d_t) - qo_t(d_1, \ldots, d_t) \right.$$

$$- h \; i_t^+(d_1, \ldots, d_t)) dF(d_1, \ldots, d_t).$$

The constraint (1.3) has been added to (1) as in Bitran and Yanasse [5] as an alternative formulation of a constraint for demand satisfaction with some confidence. Constraint (1.2) is used to keep the staircase structure of the problem so that period t is only linked directly to periods t - 1 and t + 1 through the constraints.

In order to obtain bounds on the optimal value, $z^* = E[\sum_{t=1}^{T} \rho^{t-1} (p_t \, x_t^* - qo_t - h_t \, i_t^{+*})]$, bounds on optimal primal variable levels, x_t^*, o_t^*, y_t^* , $i_t^{+,*}$, $i_t^{-,*}$, must be found. The general conditions in Zipken [11] and Birge [4] may be met by assuming the variables are bounded. The structure of problem 1, however, provides bounds on the variables without extra assumptions on the variable values. For dual variables, $(\pi_t, \nu_t, \sigma_t, \mu_t)$ associated with (1.1), (1.2), and (1.4) respectively, optimal dual variable levels π_t^* ν_t^*, σ_t^*, and μ_t^* can also be found.

First, note that total production will never exceed total demand over the planning horizon for 0 to T. Hence,

$$\sum_{t=1}^{T} x_t^* \leq \sum_{t=1}^{T} d_t^{max}. \tag{2}$$

From (2), we also obtain

$$y_t \leq \sum_{t=1}^{T} d_t^{max}; \; t = 1,\ldots,T, \tag{3}$$

$$\sum_{t=1}^{T} o_t^* \leq \sum_{t=1}^{T} d_t^{max} - \min_{1 \leq t \leq T} \ell_t, \tag{4}$$

where (4) follows because x_t^* must be nonzero in at least one period so that at least one period's capacity (without overtime) was used in producing the total production.

Constraint (1.3) forces back orders to be bounded above by

$$i_t^- \leq \sum_{\tau=1}^{t} d_\tau^{max} - \ell_t. \tag{5}$$

Inventory is also constrained in the last period by

$$i_T^+ \leq d_T^{max} - d_T^{min} , \tag{6}$$

since having any inventory in the event of d_T^{min} would be sub-optimal. For periods t < T, the inventory is at most the remaining demand,

$$i_t^+ \leq (d_t^{max} - d_t^{min}) + \sum_{\tau=t+1}^{T} d_\tau^{max} . \tag{7}$$

The dual variables can be bounded in a similar manner. First, observe that

$$- \rho^{t-1} q_t \leq - \pi_t, \quad \text{hence}$$

$$0 \leq \pi_t \leq \rho^{t-1} q_t, \tag{8}$$

for all t. Dual feasibility in the inventory variables in period T implies

$$\mu_T \leq \rho^{T-1} h_T, \tag{9}$$

$$\mu_T \geq 0. \tag{10}$$

Iterating backwards, we obtain

$$\mu_t \leq \sum_{\tau=t}^{T} \rho^{\tau-1} h_\tau, \tag{11}$$

$$\mu_t \geq 0. \tag{12}$$

Constraints on ν_t and σ_t are obtained through dual feasibility corresponding to the x and y variables. Note that

$$\sigma_t \leq 0, \tag{13}$$

$$\nu_T \leq \sigma_T \leq 0, \tag{14}$$

$$\nu_{t-1} - \nu_t \leq \sigma_{t-1}, \ t = 2,\ldots,T. \tag{15}$$

(13), (14) and (15) imply

$$\nu_t \leq 0, \quad t = 1, \ldots, T. \tag{16}$$

For the dual constraint corresponding to x_t,

$$\rho^{t-1} p_t \leq \pi_t + \nu_t + \mu_t,$$

so

$$\nu_t \geq \rho^{t-1} p_t - \rho^{t-1} q_t - \sum_{\tau=t}^{t} \rho^{\tau-1} h_\tau. \tag{17}$$

From (14), (15), and (17),

$$\sigma_t \geq \rho^{t-1} p_t - \rho^{t-1} q_t - \sum_{\tau=t}^{T} \rho^{\tau-1} h_\tau. \tag{18}$$

Equation (8) - (18) represent upper and lower bounds on the dual variables that will be used in obtaining bounds on z^*.

3. Stochastic Production Aggregation

Problem (1) will be simplified by aggregating both random variables and time periods. The resulting problem will be a single period deterministic approximation of the original multi-stage stochastic problem. Our procedure is similar to those in Zipken [10, 11] and Birge [4]. This will represent the most extreme aggregation possible, although less extreme aggregations involving conditional means and the aggregation of a subset of the periods may be possible.

For constraints (1.1), (1.3) and (1.4), our aggregation procedure is similar to the method in Birge [4]. We define

R_i = {Rows corresponding to constraint (1.i) in periods 1,...,T for i = 1, 2, 3, 4}.

The rows in R_i are summed together using a weighting function $f^{\beta}(i)$ where i_t is some constraint (1.i) in period t and

$$f^{\beta}(i_t) = \rho^{t-1}\, dF(d_2,\ldots,d_t), \tag{19}$$

for i = 1, 3, 4.

Columns are grouped together as

$$S_i = \{\text{Columns corresponding to variables } i = 1,\ldots,5 \text{ in periods } t = 1,\ldots,T\},$$

where i = 1, 2, 3, 4, 5 corresponds to x_t, o_t, y_t, i_t^+, and i_t^- respectively. These columns are summed directly with a weight $g^{\alpha} \equiv 1$ for each column.

The rows corresponding to constraints (1.2) are treated differently because of the problem structure. The y variables represent total production and, therefore, should increase each period. If a given x is used throughout the horizon then the y variables increase exactly in multiplies of x. This observation leads to a definition of weights $\alpha(t)$ in (1.2) where

$$f^{\beta}(i_t) = \alpha(t)dF(d_2,\ldots,d_t)$$

for i = 2.

To see this, we first observe that constraint (1.3) in the aggregate problem becomes

$$\left(\sum_{t=1}^{T} \rho^{t-1}\right) y \geq \sum_{t=1}^{T} b_t, \tag{20}$$

where y is the aggregate variable. The left hand side considering disaggregated variables is

$$\left(\sum_{t=1}^{T} \rho^{t-1}\right) y = \sum_{t=1}^{T} \rho^{t-1} y_t. \tag{21}$$

Now, in constraint (1.2) in the aggregate problem, we have

$$\left(\sum_{t=1}^{T} \alpha(t)\right) X - Y = 0. \tag{22}$$

Now, we would like $X = y_1$ and $y_t = t X$. From (21), this would lead to

$$Y = \left(\frac{\sum_{t=1}^{T} \rho^{t-1} t}{\sum_{t=1}^{T} \rho^{t-1}} \right) X. \tag{23}$$

By defining,

$$\alpha(t) = \left(\frac{\sum_{\tau=t}^{T} \rho^{t-1}}{\sum_{t=1}^{T} \rho^{t-1}} \right), \tag{24}$$

(23) is obtained and (20) and (22) are, therefore, consistent with the problem structure.

Given these weighting functions, the aggregate problem is:

$$\max \quad z = \left(\sum_{t=1}^{T} \rho^{t-1} p_t\right) X - \left(\sum_{t=1}^{T} \rho^{t-1} q\right) 0 - \left(\sum_{t=1}^{T} \rho^{t-1} h_t\right) I^+ \qquad (24)$$

subject to

$$\left(\sum_{t=1}^{T} \rho^{t-1}\right) X - \left(\sum_{t=1}^{T} \rho^{t-1}\right) 0 \leq \sum_{t=1}^{T} \rho^{t-1} k_t, \qquad (24.1)$$

$$\left(\sum_{t=1}^{T} \alpha(t)\right) X - Y = 0, \qquad (24.2)$$

$$\left(\sum_{t=1}^{T} \rho^{t-1}\right) Y \geq \sum_{t=1}^{T} \rho^{t-1} b_t, \qquad (24.3)$$

$$\left(\sum_{t=1}^{T} \rho^{t-1}\right) X - I^+ + I^- = \sum_{t=1}^{T} \rho^{t-1} \bar{d}_t, \qquad (24.4)$$

$$X,\ 0,\ Y,\ I^+,\ I^- \geq 0,$$

where $\bar{d}_t = E[d_t]$. Optimal primal variables in (24) are X^*, 0^*, Y^*, $I^{+,*}$, $I^{-,*}$ and the optimal dual variables are Π^*, N^*, Σ^*, M^*. The disaggregated solution obtained from (24) is $(\hat{x}_t, \hat{o}_t, \hat{i}_t^+, \hat{i}_t^-) = (X^*, 0^*, I^{+,*}, I^{-,*})$ and $\hat{y}_t = tX^*$ for all $d_1,\ldots,d_t$. For dual variable disaggregation, we have

$$(\hat{\pi}_t, \hat{\gamma}_t, \hat{\sigma}_t, \hat{\mu}_t) = (\rho^{t-1} \pi^*, \alpha(t) \quad N^* \quad , \rho^{t-1}\Sigma^*, \rho^{t-1} \quad M^*)dF(d_2,\ldots,d_t). \quad (25)$$

Note that dissagregating is not performed exactly as in aggregation due to the particular problem structure of (1).

We let $\hat{z} = \sum_{t=1}^{T} \rho^{t-1} (c_t \hat{x}_t - q_t \hat{o}_t - h_t \hat{i}_t^+)$

and wish to find ε^+ and ε^- such that

$$\hat{z} - \varepsilon^- \leq z^* \leq \hat{z} + \varepsilon^+ . \tag{26}$$

To simplify the exposition of these bounds, we assume first that the problem data is stationary, except for demand. That is, $p_t = p$, $q_t = q$, $h_t = h$, $k_t = k$, $b_t = tb$ for all t. Without this assumption, bounds are still attainable, but they will involve more complicated formulas for ε^- and ε^+. We, therefore, assume stationarity as a simplifying device.

In obtaining values for ε^+ and ε^-, we first check primal and dual feasibility. By stationarity and the definition of $\hat{x}_t$ and $\hat{o}_t$, (1.1) is always satisfied. Also by the definition of $\hat{x}_t$ and $\hat{y}_t$, (1.2) is satisfied. For constraint (1.3), note that (24.2) and the definition of $\alpha(t)$ imply

$$Y^* = \frac{\sum_{t=1}^{T} \rho^{t-1} t\, x^*}{\sum_{t=1}^{T} \rho^{t-1}} ,$$

so (24.3) can be re-written as

$$(\sum_{t=1}^{T} \rho^{t-1} t)\, X^* \geq (\sum_{t=1}^{T} \rho^{t-1} t)\, b.$$

Hence, $y_t = tx^* \geq tb$, and constraint (1.3) is satisfied. The only remaining infeasibilities may occur in (1.4) as demand varies.

For dual feasibility, stationarity implies that

$$\hat{\pi}_t = \rho^{t-1} \Pi^* \leq \rho^{t-1} q, \tag{27}$$

and $\hat{\pi}_t \geq 0$. For constraints associated with the variables y_t, for feasibility, we need

$$0 \leq -\hat{\nu}_c + \hat{\sigma}_t + \hat{\nu}_{t+1},$$

or

$$\rho^t N^*(1-\rho) \leq \rho^t \Sigma^*,$$

or

$$N^*(1-\rho) \leq \Sigma^*. \tag{28}$$

However, dual feasibility in (24) implies

$$N^* \left(\frac{1}{\sum_{t=1}^{T} \rho^{t-1}} \right) \leq \Sigma^*$$

and $1/\sum_{t=1}^{T} \rho^{t-1} \leq (1-\rho)$ implies (28). We also have $\hat{\sigma}_t = \rho^{t-1} \Sigma^* \leq 0$.

A similar argument applies for the i_t^+ variables where we want

$$-\rho^t h \leq -\hat{\mu}_t + \hat{\mu}_{t+1},$$

or

$$M^*(1-\rho) \leq h. \tag{29}$$

From (24), we have

$$M^* \leq \sum_{t=1}^{T} \rho^{t-1} h,$$

which implies (29).

For i_t^-, we require

$$0 \leq \hat{\mu}_t - \hat{\mu}_{t+1}$$

or $$0 \leq M^*(1-\rho), \qquad (30)$$

which, for $\rho < 1$, and dual feasibility in (24) is true.

The only dual infeasibilities may then incur in the constraint associated with the x_t variables. The possibility for this infeasibility must be considered in calculating bounds on z^*. We are now able to state bounds in the optimal value of the solution in (1). We first assume that no duality gap exists in (1) by assuminy that $d_t^{max} \leq +\infty$ for all t. In this case, for any realization of $d_1,\ldots,d_t$, the function

$$\sum_{t=1}^{T} \rho^{t-1}(p\ x_t^*(d_1,\ldots,d_t) - qo_t^*\ (d_1,\ldots,d_t)$$
$$- h\ i_t^*(d_1,\ldots,d_t))$$

is bounded. This implies that no duality gap exists in (1) (Rockafellar and Wets [8]) and that

$$z^* = \sum_{t=1}^{T} (\int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (\pi_t^*(d_1,\ldots,d_t)k$$
$$+ \sigma_t^*\ (d_1,\ldots,d_t)\ b_t + \mu_t^*(d_1,\ldots,d_t)\,d_t)\},$$

for $(\pi_t^*, \nu_t^*, \sigma_t^*, \mu_t^*)$ optimal in the dual of (1).

Proposition : The optimal value z* of (1) is bounded by

$$\hat{z} - \varepsilon^- \leq z^* \quad \hat{z} + \varepsilon^+,$$

where

$$\varepsilon^- = \sum_{t=1}^{T} \int_{d_t} \max\{-d_t + X_t^* - I^{+,*} + I^{-,*}, \quad 0\} \; (\sum_{\tau=t}^{T} \rho^{\tau-1} h_\tau) dF_t(d_t) \qquad (30)$$

and

$$\varepsilon^+ = \max_{1 \leq t \leq T}\{ \rho^{t-1}(p - \Pi^* - N^* - M^*), 0\} \sum_{t=1}^{T} d_t^{max}. \qquad (31)$$

Proof: To show (30), we first note that

$$z^* \geq \sum_{t=1}^{T} [\int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (\pi_t^*(d_1,\ldots,d_t)k + \sigma_t^*(d_1,\ldots,d_t)b_t$$

$$+ \mu_t^* (d_1,\ldots,d_t)d_t)]$$

$$+ \sum_{t=1}^{T} \{ \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (\rho^{t-1} p - \pi_t^* (d_1,\ldots,d_t) - \nu_t^*(d_1,\ldots,d_t)$$

$$- \mu_t^* (d_1,\ldots,d_t)) \hat{x}_t(d_1,\ldots,d_t)\}$$

$$+ \sum_{t=1}^{T} \{ \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (- \rho^{t-1} q + \pi_t^* (d_1,\ldots,d_t)) \hat{o}_t (d_1,\ldots,d_t)\}$$

$$+ \sum_{t=1}^{T} \{ \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (- \rho^{t-1} h + \nu_t^* (d_1,\ldots,d_t)$$

$$- \nu_{t+1}^*(d_1,\ldots,d_t)) \hat{i}_t^+(d_1,\ldots,d_t)\}$$

$$+ \sum_{t=1}^{T} \{ \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (+ \nu_t^*(d_1,\ldots,d_t) - \sigma_t^*(d_1,\ldots,d_t)$$

$$- \nu_{t+1}^*(d_1,\ldots,d_t))\hat{y}_t(d_1,\ldots,d_t)\}$$

$$+ \sum_{t=1}^{T} \{ \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (- \mu_t(d_1,\ldots,d_t) + \mu_{t+1}(d_1,\ldots,d_t))\ \hat{i}_t^-(d_1,\ldots,d_t)\}$$

$$= \hat{z} + \sum_{t=1}^{T} \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (\pi_t(d_1,\ldots,d_t)(k + \hat{o}_t(d_1,\ldots,d_t) - \hat{x}_t(d_1,\ldots,d_t))$$

$$+ \sum_{t=1}^{T} \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} \nu_t^*(d_1,\ldots,d_t)(\hat{y}_t(d_1,\ldots,d_t)$$

$$- \hat{x}_t(d_1,\ldots,d_t) - \hat{y}_{t-1}(d_1,\ldots,d_t))$$

$$+ \sum_{t=1}^{T} \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} \sigma_t^*(d_1,\ldots,d_t)(b_t - y_t(d_1,\ldots,d_t))$$

$$+ \sum_{t=1}^{T} \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{min}} \mu_t^*(d_1,\ldots,d_t)(d_t - \hat{i}_t^-(d_1,\ldots,d_t)$$

$$+ \hat{i}_t(d_1,\ldots,d_t) - \hat{x}_t(d_1,\ldots,d_t) + \hat{i}_{t-1}(d_1,\ldots,d_t)$$

$$- \hat{i}_{t-1}(d_1,\ldots,d_t)) \qquad (32)$$

$$\geqq \hat{z} - \varepsilon^-,$$

since only the last term in (32) can be negative and μ_t^* is bounded as in (11).

For the upper bound, we follow a similar development.

$$z^* \leqq \sum_{t=1}^{T} \left[\int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} (\rho^{t-1} p\ x_t^*(d_1,\ldots,d_t) - q\ o_t^*(d_1,\ldots,d_t) - h_t\ i_t^*(d_1,\ldots,d_t))dF(d_1,\ldots,d_t)\right]$$

$$+ \sum_{t=1}^{T} \left(\int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} \hat{\pi}_t(d_1,\ldots,d_t)(k + o_t^*(d_1,\ldots,d_t) - x_t^*(d_1,\ldots,d_t))\right.$$

$$+ \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} \hat{\nu}(d_1,\ldots,d_t)(y_t^*(d_1,\ldots,d_t) - x_t^*(d_1,\ldots,d_t) - y_{t-1}^*(d_1,\ldots,d_t))$$

$$+ \int_{d_t^{min}}^{d_t^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} \hat{\sigma}_t(d_1,\ldots,d_t)(b_t - y_t^*(d_1,\ldots,d_t))$$

$$+ \int_{d_1^{min}}^{d_1^{max}} \cdots \int_{d_1^{min}}^{d_1^{max}} \hat{\mu}_t(d_1,\ldots,d_t)(d_t - i_t^{-,*}(d_1,\ldots,d_t) + i_t^{+,*}(d_1,\ldots,d_t) - x_t^*(d_1,\ldots,d_t) + i_{t-1}^{-,*}(d_1,\ldots,d_t) - i_t^{+,*}(d_1,\ldots,d_t))$$

$$\leq \hat{z} + \varepsilon^+,$$

by rearranging terms and noting dual feasibility in all but the constraints corresponding to x variables. ∎

The bounds given for the aggregate problem represent the simplest problem attainable from (1) with the same basic structure. Other possible aggregations are presented in Birge [4]. One of these amounts to the expected value approach in Bitran and Yanasse [5]. Another possibility is to allow the random variables to remain but to aggregate time periods. The result from this aggregation is a simple recourse problem that can be solved by the methods in Everett and Ziemba [7] or Wets [9]. The problem has the basic form:

$$\max \; z = \Big(\sum_{t=1}^{T} \rho^{t-1} p\Big) X - \Big(\sum_{t=1}^{T} \rho^{t-1} q\Big) O + E\Big[- \Big(\sum_{t=1}^{T} \rho^{t-1} h\Big) I^{+}(D)\Big]$$

$$\begin{aligned} \text{subject to} \quad & \Big(\sum_{t=1}^{T} \rho^{t-1}\Big) X - \Big(\sum_{t=1}^{T} \rho^{t-1}\Big) O \le \sum_{t=1}^{T} \rho^{t-1} k, \\ & \Big(\sum_{t=1}^{T} \alpha(t)\Big) X - Y = 0, \\ & \sum_{t=1}^{T} \rho^{t-1} Y \ge \sum_{t=1}^{T} \rho^{t-1} b_t, \\ & \Big(\sum_{t=1}^{T} \rho^{t-1}\Big) X - I^{+}(D) + I^{-}(D) = D, \\ & X,\ O,\ Y,\ I^{+}(D), I^{-}(D) \ge 0, \end{aligned} \tag{33}$$

where D is random variable equal to $\sum_{t=1}^{T} \rho^{t-1} d_t$. The value of (33) can then be used as in the development provided above to give a tighter bound on the optimal value z*.

4. Example and Extensions

In this section, we present an example to demonstrate how the bounds may apply. The example is similar to those in Bitran and Yanasse [5] where they have given bounds from the expected value problem for various distribution assumptions. We assume in this example that p represents cost of production. The parameter are

$p = -19$/unit produced

$h = .4$/unit/time period

$O = 1.9$/unit of overtime

$k = 20000$ units/month

$b = 9500$ units,

and demand is uniformly distributed on [8000, 10000].

The aggregate problem as in (24) is

$$\max \quad z = -\,51.5X - 5.15O - 1.08\,I^{+} \tag{34}$$

subject to

$$\begin{array}{lllll}
2.71X & -\,2.71O & & & \leq 54{,}200, \\
1.93X & & -\,Y & & = 0, \\
 & & 2.71\,Y & & \geq 49{,}685\,, \\
2.71X & & & -\,I^{+} + I^{-} & = 24{,}390, \\
\end{array}$$

$$X,\ O,\ Y,\ I^{+},\ I^{-} \geq 0.$$

The value obtained from (34) is $\hat{z} = -490684$. From Proposition, we obtain $\varepsilon^- = 11$ and $\varepsilon^+ = 243600$.

$$-490695 \leq z^* \leq -247084.$$

In this example, ε^+ is very large because of the loose bound in (2) on x_t^*. If some other bound (such as $x_t^* \leq d_t^{max}$) is available then this error could be reduced significantly. This shows that additional information may help bound the problem. In many problems, less extreme aggregations, such as keeping a larger number of periods or possible values for the random variables, may be used.

The general approach in bounding z* may also be used for multi-product multi-stage stochastic production problems. The basic difference in these problems would be in identifying constraints such as (1.2) which require special consideration in aggregations. Otherwise, the general procedures of Birge [4] may be used.

5. Conclusions

A method for simplifying multi-stage capacitated stochastic production problems has been presented. The method employs the principle of aggregation and combines both random variables and time period. The solution of the aggregated problem provides bounds on the optimal value of the original problem. These bounds may be improved by using the same principles on less extremely aggregated problems that may allow for more than one period and for some randomness in the demands.

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