AXIOMATIC AND PROBABILISTIC CHARACTERIZATIONS OF MINIMAL WINNING COALITION BASED POWER INDICES

John R. Birge

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Department of Industrial and Operations Engineering
The University of Michigan
Ann Arbor, Michigan 48109

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Abstract

Power in voting situations has been described as the probability of a player to cast the deciding vote. The Banzhaf-Coleman and Shapley-Shubik indices have been shown to have this characterization. In this paper, a general probabilistic characterization is shown to lead to a minimal winning coalition based power index similar to the Deegan-Packel index. An axiomatization is also given for a family of indices including this index.

1. Introduction.

An individual's power in a voting game can be characterized by his or her probability of being decisive in the outcome. Straffin [1976] has shown that the Shapley-Shubik (Shapley [1953]) and Banzhaf-Coleman (Banzhaf [1965], Coleman [1971]) indices fit this characterization according to different assumptions about the behavior of players in joining coalitions. The Deegan-Packel index (Deegan and Packel [1978]) differs from these indices in the assumption that only minimal winning coalitions form.

In this paper, we present a general probabilistic characterization of power indices that may be viewed as a member of Bolger's [1980] class of power indices. We show that this class includes the Shapley-Shubik and Banzhof indices and a variant of the Deegan-Packel index that does not include the "equal division of spoils" present in the Deegan-Packel characterization. We also present a set of axioms which characterizes the family of indices including this minimal winning coalition based power index.

We first define some terms that will be used throughout the discussion.

Definition: A game (N, v) is a simple game if $v(S) = 0$ or 1 for all $S \subset N = \{1, 2, \ldots, n\}$.

Definition: S is a winning coalition in a simple game (N, v) if $v(S) = 1$.

The collection of winning coalitions is $\mathcal{W}_v$.

Definition: S is a minimal winning coalition in a simple game (N, v) if $v(S) = 1$ and $v(S - \{i\}) = 0$ for all $i \in S$.

The collection of minimal winning coalitions is $\mathcal{M}_v$.

Definition: A player i is a dummy player in a simple game (N, v) if $S \cup \{i\} \in \mathcal{W}$ if and only if $S \in \mathcal{W}$ for all $S \subset N$.

Definition: A power index for a simple game (N, v) is a function ϕ_v that assigns a real number, $\phi_v(i) \geq 0$, to each player $i \in N$.

Definition: A player i is critical in a simple game (N, v) to a set S if $v(S \cup \{i\}) - v(S) = 1$.

Definition: A permutation, P_v, of the game (N, v) defines a game such that, for a permutation P of the players $i \in N$, $P_v(P(T)) = v(T)$ for all $T \subset N$.

The number of sets for which i is critical is written

$$\delta_v(i) = \sum_{\substack{S \subset N \\ S \not\ni i}} [v(S \cup \{i\}) - v(S)]. \tag{1}$$

The Banzhaf index β_v is a normalized measure of the critical sets for a player i, where

$$\beta_v(i) = \frac{\delta_v(i)}{\sum_{j \in N} \delta_v(j)} . \tag{2}$$

A non-normalized version, which we call Banzhaf-Coleman index γ_v, measures the fraction of all subsets not containing i for which i is critical, where

$$\gamma_v(i) = \frac{\delta_v(i)}{2^{n-1}} . \tag{3}$$

The Shapley-Shubik index ψ_v includes all permutations of subsets $S \subset N$ for which i may be critical. In this case,

$$\psi_v(i) = \sum_{\substack{S \subset N \\ S \not\ni i}} \frac{s!\ (n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)] \tag{4}$$

where $s = |S|$.

The Deegan-Packel index, ρ_v, is defined for i as

$$\rho_v(i) = \frac{1}{|M_v|} \sum_{S \subset M_v(i)} \frac{1}{s}, \tag{5}$$

where $M_v(i) = \{S \in M_v: i \in S\}$.

2. Probabilistic Characterizations.

Straffin [1976] characterized power in terms of the probability that a player will be critical in a voting game according to the "homogeneity" of the player set N. The player set is homogeneous if each player has the same probability p(r) of a joining a coalition for a given proposal r or the rth play of the game. The players are independent if every player i has an independent probability $p_i(r)$ of joining a coalition for any proposal. The set of proposals including r is R.

We can generalize this characterization by partitioning N into ℓ homogeneous groups where $N = N_1 \cup N_2 \cup \dots \cup N_\ell$ and $N_i \cap N_j \neq \phi$ for all $i \neq j$. We let $|N_i| = n_i$ and assume that each player $j \in N_i$ has the same probability $p_i(r)$ of joining a coalition S in (N, v). We also assume that $p_i(r)$ is independent of $p_k(r)$ for all $i \neq k$.

For a given proposal r, the probability $\pi_v^r(i)$ that i is critical is

$$\pi_v^r(i) = \sum_{S \subset C_v(i)} \{ \prod_{j=1}^{\ell} p_j^{k_j}(r) \cdot \prod_{j=2}^{\ell} p_j^{n_j - k_j}(r) \cdot p_1^{n_1-k_1-1}(r) \}, \tag{6}$$

where $C_v(i) = \{S \subset N \mid v(S \cup \{i\} - v(S) = 1\}$ and we assume that $S \subset C_v(i)$ includes k_j players from N_j for all j and that $i \in N_1$ without loss of generality.

The expected value of $\pi_v^r(i)$ over all possible proposals r is

$$\bar{\pi}_v(i) = E_r[\pi_r(i)], \tag{7}$$

$$= \sum_{S \subset C_v(i)} \int_0^1 \cdots \int_0^1 f(p_1, p_2, \ldots p_\ell) \prod_{j=1}^{\ell} p_j^{k_j} \cdot \prod_{j=2}^{\ell} p_j^{n_j - k_j} \cdot p_1^{n_1 - k_1 - 1} \, dp_1 \ldots dp_\ell,$$

where $f(p_1, p_2, \ldots, p_\ell)$ is the probability density function imputed on $p_1, p_2, \ldots, p_\ell$ from R.

Various distributions can be used in evaluating (7) based perhaps on prior data for outcomes of the game. A simple assumption is that each probability is uniformly distributed, i.e., $f(p_1, p_2, \ldots, p_\ell) = 1$. In this case,

$$\bar{\pi}_v^u(i) = \sum_{S \subset C(i)} \frac{k_1!\; k_2! \ldots k_\ell! (n_1 - k_1 - 1)! (n_2 - k_2)! \ldots (n_\ell - k_\ell)!}{n_1!\; (n_2 + 1)! \ldots (n_\ell + 1)!} . \tag{8}$$

This value can be used to obtain both the Shapley-Shubik and Banzhaf-Coleman indices.

<u>Proposition 1</u>. If $N_1 = N$, then

$$\bar{\pi}_v^u(i) = \psi_v(i)$$

for a simple game (N, v).

<u>Proof</u>: This follows immediately from (8) and (4).

<u>Proposition 2</u>. If $|N_i| = 1$ for all $i = 1, 2, \ldots, \ell$, then

$$\bar{\pi}_v^u(i) = \gamma_v(i).$$

<u>Proof</u>: This follows from (8) and (3).

If we assume that only minimal winning coalitions are possible, then the density function $f(p_1, p_2, \ldots, p_\ell)$ in (7) also depends on the coalition S. In this case $f(p_1, p_2, \ldots, p_\ell) = 0$, if S is not a minimal winning coalition. If we additionally assume that the minimal winning coalitions are equally likely, we obtain

$$\bar{\pi}_v^m(i) = \sum_{S \subset M_v(i)} \int_0^1 \cdots \int_0^1 f(p_1, p_2, \ldots p_\ell) \prod_{j=1}^{\ell} p_j^{k_j} \cdot \prod_{j=2}^{\ell} (1-p_j)^{n_j - k_j} \cdot p_1^{n_1 - k_1 - 1} \, dp_1 \ldots dp_\ell. \tag{9}$$

If we further assume uniformity in forming these coalitions, we obtain

$$\pi_v^{m,u}(i) = \sum_{S \subset M(i)} \frac{k_1!\, k_2! \ldots k_\ell!\, (n_1-k_1-1)(n_2-k_2)! \ldots (n_\ell-k_\ell)!}{n_1!\,(n_2+1)! \ldots (n_\ell+1)!}. \tag{10}$$

We can again obtain indices similar to the Shapley-Shubik of Banzhaf-Coleman indices by assuming $N_1 = N$ or $|N_i| = 1$ for all i. In the latter case, we obtain

$$\eta_v(i) \equiv \pi_v^{m,u}(i) = \frac{|M_v(i)|}{2^{n-1}}. \tag{11}$$

The index η_v is similar to the Deegan-Packel index in its counting over minimal winning coalitions, but it does not depend on the size of these coalitions. This aspect may be interpreted as saying that a player receives equal amounts of power from groups for which he is critical regardless of the size of that group.

η_v is also not normalized in terms of dividing one unit of power among all players as the Deegan-Packel index does. This may be an advantage in characterizing power among different games. A normalized version would

$$\bar{\eta}_v(i) = \frac{|M_v(i)|}{\sum_{i \in N} |M_v(i)|}. \tag{12}$$

We note that $\bar{\eta}_v$ is actually a member of the family of indices in Packel and Deegan [1980] and, therefore, fits their set of axioms. This family consists of indices ρ_v such that

$$\rho_v^f(i) = \frac{1}{\sum_{S\in M_v} P_v^f(S)} \sum_{S\in M_v(i)} \frac{P_v^f(S)}{s}, \tag{13}$$

where

$$P_v^f(S) = \frac{1}{\alpha_v^f} \cdot \begin{cases} 0 & \text{if } S \notin M_v \\ f(S) & \text{if } S \in M_v \end{cases}, \tag{14}$$

and

$$\alpha_v^f = \sum_{S\in M_v} f(S). \tag{15}$$

If we assume $f(S) = s$, $\rho_v^f(i)$ and the axioms for ρ_v^f with this definition of $f(S)$ lead to $\bar{\eta}_v(i)$. In the next section, we develop a set of axioms for η_v.

3. Axiomatization

For simple games, (N, v) and (N, w), we define the compositions, $v \vee w$ and $v \wedge w$ by

$$v \vee w(S) = \begin{cases} 1 & \text{if } v(S) = 1 \text{ or } w(S) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$v \wedge w(S) = \begin{cases} 1 & \text{if } v(S) = 1 \text{ and } w(S) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We also introduce the elementary game, (N, E_i) as in Owen [1978], where

$$E_i(S) = \begin{cases} 1 & \text{if } i \in S, \\ 0, & \text{otherwise,} \end{cases}$$

and the game (N, v_S) such that

$$v_S(T) = \begin{cases} 1 & \text{if } T \supseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

The index $\eta_v(i)$ will be shown to be a member of the family of indices satisfying a set of axioms based on the compositions $v \vee w$ and $v \wedge w$. This family η_v^f is defined in the following theorem.

<u>Theorem.</u> There exists a one parameter family of power indices ϕ_v that satisfy the following axioms:

a) If i is a dummy in (N, v), then $\phi_v(i) = 0$.

b) If ϕ_v is a permutation of (N, v), then $\phi_{P_v}(P(i)) = \phi_v(i)$.

c) If $S \nsubseteq T$ and $T \nsubseteq S$ for all $S \in M_v$ and $T \in M_w$ for simple games (N, v) and (N, w), then

$$\phi_{v \vee w}(i) = \phi_v(i) + \phi_w(i). \tag{16}$$

d) If $R \in M_{v \wedge w}$ if and only if $R = S \cup T$ for $S \in M_v$ and $T \in M_w$ for simple games (N, v) and (N, w), then

$$\phi_{v \wedge w}(i) = \phi_v(i)\ |M_w| + \phi_w(i)\ |M_v| - \phi_v(i) \cdot \phi_w(i) \cdot q, \tag{17}$$

where q is a nonnegative parameter.

Proof. We will find a set of power indices that are functions of the parameter q in (17). The proof follows as in Deegan and Packel [1978] and Owen [1978].

We can write any game as $v = \bigvee_{S \in M_v} v_S$ and v_S can be written as $v_S = \bigwedge_{i \in S} E_i$. In the game E_i, by (a), we have

$$\phi_{E_i}(j) = \begin{cases} a & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

for some a.

Now using (b) and (d), we find

$$\phi_{v_S}(i) = \begin{cases} a & \text{if } i \in S, \\ 0, & \text{otherwise,} \end{cases}$$

and from (c),

$$\phi_v(i) = \sum_{S \in M_v(i)} a. \tag{18}$$

Hence, if ϕ_v satisfies (a) - (d), then it has the form in (18). Next, for any game $z = v \bigvee w$ satisfying the conditions in (d), we have:

$$M_z(i) = (M_v(i) \cup M_w) \cup (M_w(i) \cup M_v),$$

and $$|M_z(i)| = |M_v(i)| \cdot |M_w| + |M_v| \cdot |M_w(i)| - |M_v(i)| \cdot |M_w(i)|.$$

So, from (d),

$$\phi_z(i) = a(|M_v(i)| \cdot |M_w| + |M_v| \cdot |M_w(i)| - |M_v(i)| \cdot |M_w(i)|)$$

$$= \phi_v(i)\ |M_w| + \phi_w(i)|M_v| - \phi_v(i)\ \phi_w(i)q$$

$$= a(|M_v(i)|\ |M_w| + |M_v|\ |M_w(i)|)$$

$$- a^2q\ (|M_v(i)| \cdot |M_w(i)|),$$

so we must have $a = a^2q$ if $|M_v(i)| \neq 0$ and $|M_w(i)| \neq 0$. Therefore, $a = \frac{1}{q}$ or 0. The nonzero one-parameter family is $\phi_v(i) = \frac{|M_v(i)|}{q}$. This index clearly satisfies (a) and (b) and the above shows that it satisfies (d). To show that (c) is satisfied we note that if the conditions for (c) are satisfied then $|M_{v \vee w}(i)| = |M_v(i)| + |M_w(i)|$.

η_v is the unique nonzero index satisfying the axioms when $q = 2^{n-1}$. These axioms are similar to those given by Shapley, [1953], Dubey [1975], Owen [1978], and Deegan and Packel [1978] for other power indices in their giving dummy players zero value, being symmetric, and describing relationships in composition games.

We can see how η_v may differ from the other indices in the following <u>weighted voting game</u>, $[q; w_1, w_2, w_3, w_4] = [7, 5, 3, 2, 1]$ where $v(S) = 1$ if $\sum_{i \in S} w_i \geq q$. We also look at its <u>complementary</u> game $(N, \bar{v})$ where $\bar{v}(S) = v(N-S)$.

For these games, we obtain the values in Table 1. We note that the Shapley-Shubik, Banzhaf and Banzhaf-Coleman indices have the same value in complementary games but that ρ_v, η_v and $\bar{\eta}_v$ may differ. In this example, $\rho_v = \rho_{\bar{v}}$ but $\eta_v \neq \eta_{\bar{v}}$.

4. Summary

A general probabilistic characterization of power indices was given which led to a variant of the Deegan-Packel index. In this characterization, power was defined as the expected probability of making a critical decision when only minimal winning coalitions are allowed to form and a uniform distribution is assumed for the probability of joining a coalition in a given play of the game. This index was also shown to be characterized by a set of axioms similar to the axioms for other power indices.

Table 1. Comparative Index Values.

Player	ψ_v	β_v	γ_v	ρ_v	$\rho_{\bar{v}}$	η_v	$\eta_{\bar{v}}$	$\bar{\eta}_v$	$\bar{\eta}_{\bar{v}}$
1	2/3	3/5	3/4	1/2	1/2	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{3}$
2	1/6	1/5	1/4	1/4	1/4	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{3}$
3	1/6	1/5	1/4	1/4	1/4	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{3}$
4	0	0	0	0	0	0	0	0	0

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