Using Sequential Approximations in the L-Shaped and Generalized Programming Algorithms for Stochastic Linear Programs

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Abstract

Stochastic programs with continuous random variables are usually approximated by programs with discrete valued random variables approximating the continuous distribution. Algorithms can be applied to these problems and bounds can be found on the value of the solution of the original problem. In order to obtain a more precise bound on the solution value, the problem is generally re-solved with a finer discrete approximation. In this paper, we present a method for achieving optimality within any pre-defined error bound that can be applied directly to the L-shaped and generalized programming algorithms for stochastic linear programs.

1. Introduction

A variety of algorithms can be applied to stochastic linear programs with recourse and with discretely distributed random variables. When these variables are allowed continuous distributions, however, the methods are generally unable to solve the problem efficiently in a direct implementation. The standard procedure as in Kall [4] is to approximate the continuous random variables using a discrete distribution and solve the problem iteratively taking finer and finer partitions of the random variable in the approximation. We give below a method that solves the continuous problem in a single application of two algorithms.

We consider the problem:

$$
\begin{aligned}
&\text{minimize} && z = cx + \mathcal{Q}(x) \\
(1)\quad &\text{subject to} && Ax = b \\
& && x \geqq 0,
\end{aligned}
$$

where $\mathcal{Q}(x) = E_{\xi}[Q(x\ ,\ \xi)]$ and

$$
\begin{aligned}
Q(x, \xi) = \min\quad & qy \\
\text{subj. to}\quad & Wy = \xi - Tx \\
& y \geqq 0.
\end{aligned}
$$

This problem represents the standard stochastic linear program with recourse in which random variables are restricted to the right-hand-side coefficients $\xi \in \Xi$.

The difficulty in finding $\mathcal{Q}(x)$ is that the solutions of an infinite number of subproblems (2) may be required in order to calculate the expectation. Several methods have been suggested for approximating $\mathcal{Q}(x)$ (viz. [1], [2], [3], [5]). These methods may be incorporated into our modification to solve (1) in a single pass procedure.

We assume that we can obtain successively better approximations of $\mathcal{Q}(x)$ by upper and lower bounding. We assume there exist sequences $\{\mathcal{Q}_k^L(x)\}$ and $\{\mathcal{Q}_k^u(x)\}$ such that

$$\mathcal{Q}_{k+1}^L(x) \geq \mathcal{Q}_k^L(x),$$

$$\mathcal{Q}_{k+1}^u(x) \leq \mathcal{Q}_k^u(x),$$

$$\lim_{k\to\infty} \mathcal{Q}_k^L(x) = \mathcal{Q}(x),$$

and

$$\lim_{k\to\infty} \mathcal{Q}_k^u(x) = \mathcal{Q}(x),$$

for all x such that $Ax = b$, $x \geqq 0$. These approximations may be found by discretizations of Ξ.

2. The L-Shaped Algorithm

We first consider the L-shaped algorithm of Van Slyke and Wets [7]. The kth master problem in this algorithm is:

(3) Minimize $z^k = cx + \theta$

(3.1) Subject to: $Ax = b$

$D_\ell x \geq d_\ell; \ell = 1,\ldots,s,$

(3.2) $E_\ell x + \theta \geq e_\ell ; \ell = 1,\ldots,t,$

$x \geqq 0,$

where the constraints in (3.2) induce feasibility and constraints (3.3) form an outer envelope of $\mathcal{Q}(x)$. We adapt the construction of these constraints to the successive approximation.

We let (x^k, θ^k) be optimal in (3). We use x^k as an input to solve (2) for all ξ used in the lower approximation of $\mathcal{Q}(x)$. If any subproblem (2) is not feasible then we add a feasiblity constraint,

$$D_{s+1}\, x \geq d_{s+1},$$

let s = s+ 1, k = k + 1, and resolve (3).

(This process does not require independent solutions of all subproblems (2) [8].)

We let the lower approximation found be $\mathcal{Q}_k^L(x)$. If $\theta^k < \mathcal{Q}_k^L(x) \leq \mathcal{Q}(x)$, then x is not optimal in (1). We let $F_k^L(\xi)$ be the distribution function used in this lower approximation of $\mathcal{Q}(x)$ and let $\pi_k(\xi)$ be the optimal dual multipliers for subproblem (2).

$$E_{t+1} = \int_{\Xi} \pi_k(\xi)\, T\, dF_k^L(\xi),$$

and

$$e_{t+1} = \int_{\Xi} \pi_k(\xi)\, \xi\, dF_k^L(\xi).$$

$E_{t+1}x + \theta \geq e_{t+1}$ is added as a constraint to (3.3), t is set to t+ 1, and k is set to k + 1. (3) is again solved. The constraints in (3.3) then represent an outer linearization or lower approximation of $\mathcal{Q}(x)$.

If $\theta^k \geq \mathcal{Q}_k^L(x^k)$, then we solve (2) for any additional values of ξ necessary to find $\mathcal{Q}_k^u(x^k)$. Again, if an infeasibility is found, a constraint is added to (3.2) and we repeat the process. After finding $\mathcal{Q}_k^u(x^k)$, we check whether $\theta^k \geq \mathcal{Q}_k^u(x^k)$. If so, (x^k, θ^k) are an optimal pair, and we are done. If not, we have optimized $cx + \mathcal{Q}^L_k(x)$, but are not sure if x^k optimizes $cx + \mathcal{Q}(x)$. In this case, we refine the approximations $\mathcal{Q}^u_k(x^k)$ and $\mathcal{Q}^L_k(x^k)$ and update k, until either $\theta^k \geq \mathcal{Q}^u_k(x^1)$ or $\theta^k < \mathcal{Q}^L_k(x)$. Since $\lim_{k\to\infty} \mathcal{Q}^L_k(x) = \lim_{k\to\infty} \mathcal{Q}^u_k(x)$, one of these conditions must be obtained. In the former case, (x^k, θ^k) is optimal and in the latter case, another constraint is added to (3.3).

The algorithm converges because of the convergence of the approximations. We, of course, have bounds for x* optimal in (1) of

$$c\, x^k + \theta^k \leq cx^* + \mathcal{Q}(x^*) \leq cx^k + \mathcal{Q}^u_k(x^k), \tag{4}$$

at every iteration. The algorithm may be terminated whenever the interval in (4) is sufficiently small.

3. The Generalized Programming Method

A new algorithm for (1) based on generalized linear programming has been proposed by Nazareth and Wets [6]. For this algorithm, it is more convenient to write (1) as

$$\begin{aligned}
\text{minimize}\quad & z = cx + \Psi(\chi) \\
\text{subject to}\quad & Ax = b \\
& Tx - \chi = 0 \\
& x \geq 0,
\end{aligned} \tag{5}$$

where $\Psi(\chi) = E_\xi[\psi(\chi, \xi)]$ and

$$\psi(\chi, \xi) = \min \quad qy$$

$$\text{subj. to} \quad Wy = \xi - \chi$$

$$y \geqq 0.$$

We assume again that converging approximations $\Psi_k^u(\chi)$ and $\Psi_k^L(\chi)$ are available.

The generalized linear programming algorithm proceeds by noting that χ can be found as a convex combinations of extreme points in its domain which we assume is bounded. In our modification of this algorithm, the restricted master problem of (5) becomes

(6) $\quad$ minimize $\quad z^k = c\,x + \sum_{i=1}^{k} \lambda_i \Psi_i^u(\chi_i)$

subj. to

(6.1) $\quad Ax = b,$

(6.2) $\quad Tx - \sum_{i=1}^{k} \lambda_i \chi_i = 0,$

(6.3) $\quad \sum_{i=1}^{k} \lambda_i = 1,$

$$\lambda_i \geq 0, \quad i = 1,\ldots,k,$$

$$x \geq 0.$$

Let the optimal solution of (6) be (x^k, λ^k), and let the optimal dual multipliers be $(\pi^k, \sigma^k, \rho^k)$ corresponding to (6.1), (6.2) and (6.3) respectively. The subproblem associated with (6) is then

(7) minimize $\Psi_k^u(\chi) + \sigma^k\chi$.

Let χ^k be optimal in (7). If $\Psi_k^u(\chi^k) + \sigma^k \chi^k < \rho^k$, then x^k is not optimal in (5). χ^k and $\Psi(\chi^k)$ are added to (6), k is updated, and (6) is solved again.

If $\Psi_k^u(\chi^k) + \sigma^k \chi^k \geq \rho^k$, then x^k is optimal relative to Ψ_k^u but is not necessarily optimal relative to Ψ. $\Psi_k^L(\chi^k)$ is then computed. If $\Psi_k^L(\chi^k) + \sigma^k \chi^k \geq \rho^k$, then x^k is optimal in (5). If, however, $\Psi_k^L(\chi^k) + \sigma^k \chi^k < \rho^k$, then the approximations Ψ_k^u and Ψ_k^L are refined and (7) is resolved if we still have $\Psi_k^u(x^k) + \sigma^k x^k \geq \rho^k$ and $\Psi_k^L(\chi^L) + \sigma \chi^k < \rho^k$.

This refinement and re-optimization is repeated until either $\Psi_k^u(\chi^k) + \sigma^k \chi^k < \rho^k$ or $\Psi_k^L(\chi^k) + \sigma^k \chi^k \geq \rho^k$. In either case, we again update (6) and repeat the procedure. These conditions must be met because of the convergence of the approximations to the true distribution. The optimality condition must also be reached eventually because of our assumptions.

Bounds may also be obtained in this problem. Clearly, for (x^*, χ^*) optimal,

(8) $$cx^* + \Psi(\chi^*) \leq cx^k + \sum_{i=1}^{k} \lambda_i^k \Psi_i^u(\chi_i) = z^k.$$

By duality, we also have

(9) $$cx^* + \Psi(\chi^*) \geq z^k - \rho^k + \Psi_k^L(\chi^k) + \sigma^k \chi^k.$$

(8) and (9) may then be used to obtain sufficient bounds on z^*.

5. Conclusion

Procedures incorporated into two algorithms for stochastic linear programming have been presented that solve the stochastic l.p. with continuous random variables in a single pass. The procedures take advantage of the outer linearization of the L-shaped method and the inner linearization of the generalized programming method. They extend previous results on approximations for stochastic linear programs that converge monotonically to the true value.

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