OPTIMALITY CONDITIONS FOR MATCH-UP STRATEGIES IN STOCHASTIC SCHEDULING

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Technical Report 92-58

November 1992

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Abstract : Practical scheduling problems require periodic decisions that consider future random phenomena. They can be modeled as dynamic stochastic optimization problems. In this paper, we present conditions for optimal solutions of a general class of these problems. The optimality conditions are in turn used to obtain the asymptotic optimality of strategies that match up with a turnpike strategy. We present examples of these models and discuss their implications.

Keywords: stochastic programming, optimality conditions, turnpike solutions, stochastic scheduling, match-up.

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* This author's work was supported in part by Office of Naval Research Grant N00014-86-K-0628, by Dalhousie University and the Natural Sciences and Engineering Research Council of Canada during a visit in the Department of Mathematics, Statistics, and Computing Science, by the National Research Council under a Research Associateship at the Naval Postgraduate School, Monterey, California, and by the National Science Foundation under Grant ECS 88-15101.

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1. Introduction

Practical scheduling problems require periodic decisions that consider future random phenomena. Much of the scheduling literature, however, considers problems that are deterministic and static (see, e.g., Graves [1981], Rinnooy Kan [1976] and Dempster, *et al.* [1982]). Discussions of stochastic scheduling problems with machine disruptions have generally considered single machines and limited models (Glazebrook [1984], Pinedo and Ross [1980], Mittenthal [1986]). An approach for dealing with multiple machines was introduced in Bean and Birge [1986] for single disruptions or disruptions that are well-spaced apart. In this paper, we provide a justification for this match-up scheduling approach, elaborated in Bean *et al.* [1991], through a general stochastic model. This model includes a convex objective function and decisions at discrete time intervals. It is similar to the continuous time model in Solel [1987] (see also Dempster and Solel [1987]) but allows us to treat a wider class of problems. Our main results are optimality conditions that are used to obtain conditions for optimal cyclic policies and the asymptotic optimality of match-up policies. We motivate these results through examples from scheduling.

Our results in fact apply to more general stochastic optimization models and extend the developments discussed in Dempster [1988]. They relate most closely to Flåm [1983,1985]. We extend Flåm's results by relaxing his conditions of interior optimal solutions and uniform convexity of the objective function. Our work is also related to the deterministic work of McKenzie [1976], the control results of Kushner [1972] and Arkin and Evstigneev [1979], and the finite horizon results in Rockafellar and Wets [1983] and Hiriart-Urruty [1982].

In our model, we assume a data process, $\omega := \{\omega_t : t = 0, \ldots\}$ in a (canonical) probability space (Ω, Σ, μ). We also assume a decision process $x := \{x_t : t = 0, \ldots\}$ such that x is a measurable function $x : \omega \mapsto x(\omega)$. The space of the decision processes is the space of *essentially bounded functions*, $L^n_\infty := L_\infty(\Omega \times \mathrm{N}, \Sigma \times \mathcal{P}(\mathrm{N}), \mu \times \#; \Re^n)$, where $\mathcal{P}$ is the power set and $\#$ is counting measure. Associated with the data process is a filtration

$\mathbb{F} := \{\Sigma_t\}_{t=1}^{\infty}$, where $\Sigma_t := \sigma(\omega^t)$ is the σ–field of the *history process* $\omega^t := \{\omega_0, \ldots, \omega_t\}$ and the Σ_t satisfy $\{0, \Omega\} \subset \Sigma_0 \subset \cdots \subset \Sigma$.

A fundamental property of the decision process at time t is that it must only depend on the data up to time t, i.e. $\mathbf{x}_t$ must be Σ_t–measurable. An alternative characterization of this *nonanticipative* property is that $\mathbf{x}_t = \mathbb{E}\{\mathbf{x}_t|\Sigma_t\}$ a.s., $t = 0, \ldots$, where $\mathbb{E}\{\cdot|\Sigma_t\}$ is conditional expectation with respect to the σ–field Σ_t. Using the projection operator $\Pi_t : z \mapsto \pi_t z := \mathbb{E}\{z|\Sigma_t\}, t = 0, \ldots$, on L_∞^n, this is equivalent to

$$(I - \Pi_t)x_t = 0, \ t = 0, \ldots \ . \tag{1}$$

We then let $\mathcal{N}$ denote the closed linear subspace of nonanticipative processes in L_∞^n and denote by $\Pi := (\Pi_0, \Pi_1, \ldots)$ the projection operator from L_∞^n onto $\mathcal{N}$.

Our general optimization model is to find

$$\inf_{\mathbf{x} \in \mathcal{N}} \mathbb{E} \sum_{t=0}^{\infty} f_t(\omega, x_t(\omega), x_{t+1}(\omega)), \tag{2}$$

where "$\mathbb{E}$" denotes expectation with respect to Σ. We use the notation $\mathbf{x}_t$ and $\mathbf{f}_t$ to denote respectively x_t and f_t as functions of ω, i.e. as random entities. Expression (2) then becomes

$$\inf_{\mathbf{x} \in \mathcal{N}} \mathbb{E} \sum_{t=0}^{\infty} \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}), \tag{3}$$

with objective $F(\mathbf{x}) := \mathbb{E} \sum_{t=0}^{\infty} \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1})$.

We assume in (3) that the objective components $\mathbf{f}_t$ are proper convex normal integrands (see Rockafellar [1976]) with the following additional property:

Assumption 1: For any $\mathbf{y} = (\mathbf{x}_t, \mathbf{x}_{t+1})$, there exists $\gamma > 0$ (independent of t) such that for $\pi \in \partial \mathbf{f}_t(\mathbf{y}) \subset (L_\infty^n)^*$, the Banach dual space of L_∞^n, and for all $\mathbf{w}$, either

(a) $$\pi \in \partial \mathbf{f}_t(\mathbf{w}),$$

or

(b) there exists $\mathbf{z}$ such that $\pi \in \partial \mathbf{f}_t(\mathbf{z})$ and

$$\mathbf{f}_t(\mathbf{z}) + \pi(\mathbf{w} - \mathbf{z}) \leq \mathbf{f}_t(\mathbf{w}) - \gamma||\mathbf{z} - \mathbf{y}|| \text{ a.s.} \tag{4}$$

for all $t \geq 0$. ∎

Note that uniform convexity implies the above assumption, which allows nonstrict convexity involving a von Neumann facet (see Figure 1). We use this more general assumption here because it allows us to use the common scheduling objectives which involve linear tardiness and earliness penalties. These objective functions are discussed in more detail in Section 4.

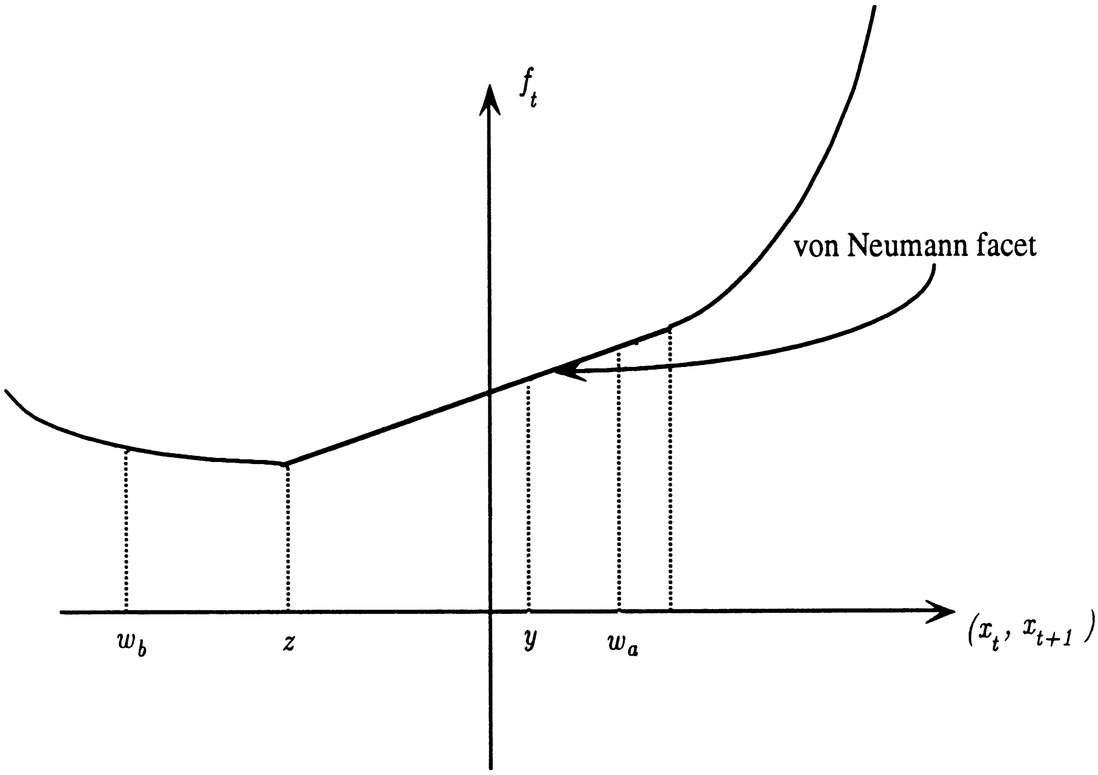

Figure 1. Illustration of the convexity of a realization of the objective function $\mathbf{f}_t$ implied by Assumption 1, Cases a) and b).

Section 2 presents the main results on optimality conditions by showing that under the assumptions a price system exists which provides necessary and sufficient conditions for optimality. Section 3 uses these results to obtain turnpike results concerning the optimality of cyclic policies and the asymptotic optimality of match–up strategies. Scheduling examples are discussed in Section 4. Section 5 presents our conclusions.

2. Optimality Conditions

In general, the objective in (3) is infinite. We can avoid this difficulty by defining a policy $\mathbf{x}^* := \{\mathbf{x}_0, \mathbf{x}_1^*, \ldots\}$ as *(weakly) optimal*, as in McKenzie [1976], if it is not *overtaken* by any other policy, i.e. if there does not exist $\mathbf{x}'$ such that

$$\limsup_{T\to\infty} \mathbb{E} \sum_{t=0}^{T} [\mathbf{f}_t(\mathbf{x}'_t, \mathbf{x}'_{t+1}) - \mathbf{f}_t(\mathbf{x}^*_t, \mathbf{x}^*_{t+1})] \leq -\epsilon, \tag{5}$$

where $\epsilon > 0$.

We also assume that the objective functions satisfy a condition ensuring that no infinite terms are present in the sum in (5).

Assumption 2: For any t and $\delta < \infty$, there exists $\epsilon < \infty$ such that $\|\mathbf{x}_t\| < \delta$ a.s. implies $\mathbb{E}\ \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) > -\epsilon$ and $\|\mathbf{x}_{t+1}\| < \epsilon$ a.s. for $\mathbf{x}_{t+1}$ feasible. ∎

Given Assumption 2, we can subtract a constant from each $\mathbf{f}_t$ and not change the weak optimality of $\mathbf{x}^*$. By setting this constant equal to the expected objective value in each period, we obtain an infimum of 0 in (3). We therefore impose Assumption 2 so that without loss of generality we can assume a finite infimum in (3).

Our goal is to construct a sequence of prices supporting the objective terms in (3). These price supports provide the optimality conditions in the following theorem.

Theorem 1. *Suppose Assumption 2 holds and that* $\mathbf{x}^*$ *is optimal in (3) with finite infimum, and*

(a) (nonanticipative feasibility) For any $\mathbf{x} \in$ dom F *(i.e., such that* $\mathbb{E}\sum_{t=0}^{\infty} \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) < \infty$*), the projection of* $\mathbf{x}$ *onto* $\mathcal{N}$, $\Pi\mathbf{x}$, *is such that* $\mathbb{E}\sum_{t=0}^{\infty} \mathbf{f}_t(\Pi_t \mathbf{x}_t, \Pi_{t+1}\mathbf{x}_{t+1}) < \infty$,

(b) (strict feasibility) For some $\mathbf{x} \in \mathcal{N}$, *such that* $\mathbb{E}\sum_{t=0}^{\infty} \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) < \infty$, *there exists* $\delta > 0$ *such that for all* $\|\mathbf{y} - \mathbf{x}\| < \delta, \mathbf{y} \in L^n_\infty$, $\mathbb{E}\sum_{t=0}^{\infty} \mathbf{f}_t(\mathbf{y}_t, \mathbf{y}_{t+1}) < \infty$.

(c) (finite horizon continuation approximation) There exists $\mathbf{x}'$ *such that for all* T_k *in some sequence* $\{T_1, T_2, ...\}$, *and, for any* $\mathbf{x} \in \operatorname{dom} F$, $(\mathbf{x}^{T_k}, \mathbf{x}'_{T_k+1}, \mathbf{x}'_{T_k+2}, \ldots)$ *is also feasible, and the transition cost to* $\mathbf{x}'$ *is such that* $|\mathbb{E}[\mathbf{f}_{T_k-1}(\mathbf{x}_{T_k-1}, \mathbf{x}_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k}, \mathbf{x}'_{T_k+1})]| \to 0$ *as* $k \to \infty$ *and* $|\mathbb{E}[\mathbf{f}_{T_k-1}(\mathbf{x}_{T_k-1}, \mathbf{x}_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k}, \mathbf{x}'_{T_k+1})]| \geq |\mathbb{E}[\mathbf{f}_{T_k-1}(\mathbf{x}_{T_k-1}, \mathbf{x}_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k}, \mathbf{x}_{T_k+1})]|$ *for* $k = 1, \ldots$.

Then, $\mathbf{x}^*$ *is optimal with given initial conditions* $\mathbf{x}_0$ *if and only if there exist* $\mathbf{p}_t \in L_1^n(\Sigma), t = 0, \ldots,$ *such that*

(i) $\mathbf{p}_t$ *is nonanticipative, i.e.* $\mathbf{p}_t = \mathbb{E}\{\mathbf{p}_t | \Sigma_t\}$ *a.s. for* $t = 0, \ldots,$

(ii) $\mathbb{E}_0(\mathbf{f}_0(\mathbf{x}_0, \mathbf{x}_1) - \mathbf{p}_0\mathbf{x}_0 + \mathbf{p}_1\mathbf{x}_1)$ *is a.s. minimized by* $\mathbf{x}_1 := \mathbf{x}_1^*$ *over* $\mathbf{x}_1 = \mathbb{E}\{\mathbf{x}_1 \mid \Sigma_1\}$, *and, for* $t > 0$, $\mathbb{E}_t(\mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) - \mathbf{p}_t\mathbf{x}_t + \mathbf{p}_{t+1}\mathbf{x}_{t+1})$ *is a.s. minimized by* $(\mathbf{x}_t, \mathbf{x}_{t+1}) := (\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)$ *over* $\mathbf{x}_t = \mathbb{E}\{\mathbf{x}_t \mid \Sigma_t\}$ *and* $\mathbf{x}_{t+1} = \mathbb{E}\{\mathbf{x}_{t+1} \mid \Sigma_{t+1}\}$, *and*

(iii) $\mathbb{E}\ \mathbf{p}_{t_k}(\mathbf{x}_{t_k} - \mathbf{x}_{t_k}^*) \to 0$ *as* $t_k \to \infty$, *for all* $\mathbf{x} \in \operatorname{dom} F$.

Proof: The proof follows from the theory of convex normal integrands (Rockafellar [1971]) as in Flåm [1983] and [1985].

To prove sufficiency, given (i) – (iii) and denoting $\mathbb{E}\{\cdot|\Sigma_t\}$ by $\mathbb{E}_T[\cdot]$, for any $\mathbf{x} \in \mathcal{N}$ with $\mathbf{x}_0 := \mathbf{x}_0^*$, we have

$$\begin{aligned} \mathbb{E}\sum_{t=0}^{t_k-1}[\mathbf{f}_t(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*) - \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1})] &= \mathbb{E}\sum_{t=0}^{t_k-1}\mathbb{E}_t[\mathbf{f}_t(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*) - \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1})] \\ &\leq \mathbb{E}\sum_{t=0}^{t_k}\mathbb{E}_t[\mathbf{p}_t(\mathbf{x}_t^* - \mathbf{x}_t) - \mathbf{p}_{t+1}(\mathbf{x}_{t+1}^* - \mathbf{x}_{t+1})] \\ &= \mathbb{E}\mathbf{p}_{t_k}(\mathbf{x}_{t_k} - \mathbf{x}_{t_k}^*). \end{aligned} \tag{6}$$

Note that the last quantity in (6) approaches 0 as $t_k \to \infty$ by (iii) . Hence, $\mathbf{x}^*$ is weakly optimal, and optimal in (3) by the finiteness assumption.

Now, to prove necessity of (i)-(iii), suppose $\mathbf{x}^*$ is optimal in (3) and (a) – (c) hold. Define

$$F^{T_k}(\mathbf{x}) := \mathbb{E}[\sum_{t=0}^{T_k-1}\mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k}, \mathbf{x}'_{T_k+1}) + \sum_{t=T_k+1}^{\infty}\mathbf{f}_t(\mathbf{x}'_t, \mathbf{x}'_{t+1})]$$

for $\mathbf{x}'$ as in (c). Note that the objective in (3) is $F^{\infty}(\mathbf{x})$ and that it follows from (c) that $F^{\infty}(\mathbf{x}) < \infty$ implies $F^{T_k}(\mathbf{x}) < \infty$ for all $k = 1, \ldots$.

Suppose the functional $\pi^{T_k} \in \partial F^{T_k}(\mathbf{x}^{*T_k}) \subset (L^n_{\infty})^*$, the Banach dual space of L^n_{∞}, where $F^{T_k}(\mathbf{x}^{T_k}) := \mathbb{E}[\sum_{t=0}^{T_k-1} \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k}, \mathbf{x}'_{T_k+1})]$, which is the expectation of a convex normal integrand by assumption. By Rockafellar [1971, Corollary 1B], there exists a representation of π^{T_k} as $\pi^{T_k} = \eta^{T_k} + \sigma^{T_k}$, where

$$\eta^{T_k} \in \partial F^{T_k}(\mathbf{x}^*) \cap L^n_1 \tag{7}$$

(i.e., there exists $\mathbf{p}^{T_k}$ such that $\eta^{T_k}(\mathbf{x}^{T_k}) = \mathbb{E}\mathbf{p}^{T_k}\mathbf{x}^{T_k}$ for all $\mathbf{x}^{T_k} \in L^n_{\infty}$) and σ^{T_k} is a *singular functional* corresponding to a *purely finitely additive measure* (see Yosida and Hewett [1952]) such that

$$\sigma^{T_k}(\mathbf{x}^{*T_k}) = \sup\{\sigma^{T_k}(\mathbf{x}^{T_k}) \mid F^{T_k}(\mathbf{x}^{T_k}) < \infty\}. \tag{8}$$

We now construct an appropriate π^{T_k}. For $t = 1, \ldots, T_k - 1$, let $\phi^*_{t-1}(\mathbf{x}_t) := \mathbb{E}[\mathbf{f}_{t-1}(\mathbf{x}^*_{t-1}, \mathbf{x}_t) + \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}^*_{t+1})]$ and $\pi_t \in \partial \phi^*_{t-1}(x^*_t)$. We can again apply Rockafellar's result as in (7), so that $\pi_t = \eta_t + \sigma_t$, for $\eta_t \in L^n_1$ and singular σ_t. By the definition of subgradient,

$$\pi_t(\mathbf{x}_t - \mathbf{x}^*_t) + \mathbb{E}[\mathbf{f}_{t-1}(\mathbf{x}^*_{t-1}, \mathbf{x}^*_t) + \mathbf{f}_t(\mathbf{x}^*_t, \mathbf{x}^*_{t+1})] \leq \mathbb{E}[\mathbf{f}_{t-1}(\mathbf{x}^*_{t-1}, \mathbf{x}_t) + \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}^*_{t+1})], \tag{9}$$

for all essentially bounded $\mathbf{x}_t$. We also define $\pi^{T_k}_{T_k}$ as a subgradient of $\mathbb{E}[\mathbf{f}_{T_k-1}(\mathbf{x}^*_{T_k-1}, \mathbf{x}_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k}, \mathbf{x}'_{T_k+1})]$, defined as a function of $\mathbf{x}_{T_k}$ and evaluated at $\mathbf{x}^*_{T_k}$, involving a trajectory $\mathbf{x}'$ whose existence is guaranteed by (c), so that we also have

$$\pi^{T_k}_{T_k}(\mathbf{x}_{T_k} - \mathbf{x}^*_{T_k}) + \mathbb{E}[\mathbf{f}_{T_k-1}(\mathbf{x}^*_{T_k-1}, \mathbf{x}^*_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}^*_{T_k}, \mathbf{x}'_{T_k+1})] \leq \mathbb{E}[\mathbf{f}_{T_k-1}(\mathbf{x}^*_{T_k-1}, \mathbf{x}_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k}, \mathbf{x}'_{T_k+1})], \tag{10}$$

for all essentially bounded $\mathbf{x}_{T_k}$. We set $\pi^{T_k} := (\pi_1, \ldots, \pi_{T_k-1}, \pi^{T_k}_{T_k})$. Combining (9) and (10), we have $\pi^{T_k} \in \partial F^{T_k}(\mathbf{x}^{*T_k})$. Using (c), $F(\mathbf{x}) = \lim_{k\to\infty} F^{T_k}(\mathbf{x})$. Thus,

$$\sum_{t=0}^{\infty} \pi_t(\mathbf{x}_t - \mathbf{x}^*_t) + \lim_{k\to\infty}(\pi^{T_k}_{T_k}(\mathbf{x}_{T_k} - \mathbf{x}^*_{T_k})) + F(\mathbf{x}^*) \leq F(\mathbf{x}), \tag{11}$$

for all $\mathbf{x} \in L^n_\infty$. From (10) and (c), the measure $\pi^{T_k}_{T_k}$ must also satisfy

$$\lim_{k\to\infty} |\pi^{T_k}_{T_k}(\mathbf{x}_{T_k} - \mathbf{x}^*_{T_k})| \le \lim_{k\to\infty} |\mathbb{E}\{[\mathbf{f}_{T_k-1}(\mathbf{x}^*_{T_k-1}, \mathbf{x}_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k}, \mathbf{x}'_{T_k+1})] - [\mathbf{f}_{T_k-1}(\mathbf{x}^*_{T_k-1}, \mathbf{x}^*_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}^*_{T_k}, \mathbf{x}'_{T_k+1})]\}| = 0, \tag{12}$$

for all $\mathbf{x} \in \text{dom } F$. (To see this, add and subtract $\mathbf{f}_{T_k-1}(\mathbf{x}^*_{T_k-1}, \mathbf{x}_{T_k}) + \mathbf{f}_{T_k}(\mathbf{x}_{T_k-1}, \mathbf{x}^*_{T_k})$ within the expectation on the right hand side.)

From (11), (12) and, since $F(\mathbf{x}) = +\infty$ if $x \notin \text{dom } F$,

$$\sum_{t=0}^{\infty} \pi_t(\mathbf{x}_t - \mathbf{x}^*_t) + F(\mathbf{x}^*) \le F(\mathbf{x}) \tag{13}$$

for all $\mathbf{x} \in L^n_\infty$. Therefore, $\pi = (\pi_1, \pi_2, \ldots) \in \partial F(\mathbf{x}^*)$.

Now, define

$$\delta(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x}_t = \mathbb{E}\{\mathbf{x}_t | \Sigma_t\} \text{ a.s.} \\ \infty & \text{otherwise.} \end{cases}$$

Let the projection from L^n_∞ onto $\mathcal{N}$ be Π, where

$$\Pi_t(\mathbf{x}_t) = \mathbb{E}[\mathbf{x}_t \mid \Sigma_t], t = 0, 1, \ldots .$$

Then, δ is the indicator function of the linear subspace $\mathcal{N} = \ker(I - \Pi)$ of L^n_∞.

Solving (3) is then equivalent to finding

$$\inf_{\mathbf{x}\in L^n_\infty}(F + \delta)(\mathbf{x}).$$

Since $\mathbf{x}^*$ is optimal in (3), $0 \in \partial(F + \delta)(\mathbf{x}^*)$. Hence, $-\pi \in \partial\delta(\mathbf{x}^*)$, i.e. $\pi \in [R(I - \Pi)]^o$, the subspace of $(L^n_\infty)^*$ of annihilators of the range of the complementary projection $(I - \Pi)$ of Π which (in an abuse of notation) we shall denote by $\mathcal{N}^\perp$. By (8) and (9),

$$\sigma^{T_k}(\mathbf{x}^{*T_k}) = \sum_{t=1}^{T_k-1} [\sigma_t(\mathbf{x}^*_t)] + \sigma^{T_k}_{T_k}(\mathbf{x}^*_{T_k}) \ge \sum_{t=1}^{T_k-1} [\sigma_t(\mathbf{x}_t)] + \sigma^{T_k}_{T_k}(\mathbf{x}_{T_k}), \tag{14}$$

for all $\mathbf{x} \in \text{dom } F^{T_k} \supset \text{dom } F$. From (12) and (b), $|\sigma^{T_k}_{T_k}(\mathbf{x}^*_{T_k} - \mathbf{x}_{T_k})| \to 0$. From (14), then,

$$\sum_{t=0}^{\infty}[\sigma_t(\mathbf{x}^*_t)] \ge \sum_{t=0}^{\infty}[\sigma_t(\mathbf{x}_t)] \quad \forall \mathbf{x} \in \text{dom } F. \tag{15}$$

We can now follow the proof technique in Flåm [1985] to obtain an equivalent subgradient without singular components.

The (Banach) conjugate of the operator Π is denoted by Π^*. Define $\overline{\pi} := \Pi^*\sigma$. We will show that $\overline{\pi} \in L_1^n$. Since $\pi \in (\mathcal{N}^\perp)^o$, we have $\sigma(\mathbf{x}) = \eta(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{N}^\perp$. Hence, for any $\mathbf{x} \in \mathcal{N}^\perp$, we can write $\sigma(\mathbf{x})$ as $\sum_{t=0}^\infty \int_\Sigma p(\omega,t)x(\omega,t)P(d\omega)$ where $p \in L_1^n$. Let $\overline{p}(\cdot,t) := \mathbb{E}\{p(\cdot,t) \mid \Sigma_t\}$ for all t, so that $\overline{p} \in L_1^n$. Then, for all $\mathbf{x} \in L_\infty^n$,

$$\begin{aligned}\overline{\pi}(\mathbf{x}) = \sigma(\Pi\mathbf{x}) &= \mathbb{E}_\Sigma \sum_{t=0}^\infty p(\cdot,t)(\Pi\mathbf{x}_t) \\ &= \mathbb{E}_\Sigma \sum_{t=0}^\infty \mathbb{E}[p(\cdot,t)\mathbb{E}[x(\cdot,t) \mid \Sigma_t] \mid \Sigma_t] \\ &= \mathbb{E}_\Sigma \sum_{t=0}^\infty \bar{p}(\cdot,t)x(\cdot,t) = \mathbb{E}\bar{\mathbf{p}}\mathbf{x}.\end{aligned}$$

Hence, $\overline{\pi} \in L_1^n$. Let $\pi^* := \eta + \overline{\pi} \in L_1^n$ and note that, since $\overline{\pi} = \sigma$ on $\mathcal{N}^\perp$, $\pi^* \in (\mathcal{N}^\perp)^o$. It also follows as in Flåm [1985], since $[(I - \Pi^*)\pi^*]\mathbf{x} = \pi^*[(I - \Pi)\mathbf{x}] = 0$ for all $\mathbf{x} \in L_\infty^n$ and $(\Pi^*\pi)_t = \mathbb{E}(\pi_t|\Sigma_t)$, that

$$\pi_t^* = \mathbb{E}(\pi_t^* \mid \Sigma_t) \text{ a.s. } \forall t. \tag{16}$$

From (15) and nonanticipative feasibility,

$$\overline{\pi}(\mathbf{x}^*) = \sigma(\mathbf{x}^*) \geq \sigma(\Pi\mathbf{x}) = \overline{\pi}(\mathbf{x}), \tag{17}$$

for all $\mathbf{x} \in$ dom F. We can again apply Rockafellar's result on convex normal integrands as in (7) so that $\eta \in \partial F \cap L_1^n$. It follows that

$$\pi^* = \eta + \overline{\pi} \in \partial F \cap L_1^n. \tag{18}$$

Since π^* now excludes the singular multipliers, by (b), we can decompose π^* into multipliers for each component $\mathbf{f}_t$ of the objective (by applying Rockafellar [1971, p. 442] to the finite sum 0 to T to decompose into a finite sum for almost all ω, then decomposing this sum using Rockafellar [1969, Theorem 23.8], and noting the vanishing T terms as in (11), or, more directly, by using the results in Ioffe and Levin [1972]). Thus, $\pi^*(\mathbf{x}) = \mathbb{E}\sum_{t=0}^\infty \mathbf{p}_t'(\mathbf{x}_t, \mathbf{x}_{t+1})$, where $\mathbf{p}'_t$ be written with two components as $\mathbf{p}'_t = (\mathbf{p}^{*1}_t, -\mathbf{p}^{*2}_{t+1}) \in \partial \mathbf{f}_t(\mathbf{x}^*_t, \mathbf{x}^*_{t+1})$ a.s., where $\mathbf{p}'_t(\mathbf{x}_t, \mathbf{x}_{t+1}) = \mathbf{p}^{*1}_t \mathbf{x}_t - \mathbf{p}^{*2}_{t+1}\mathbf{x}_{t+1}$, and

$$\mathbf{p}^{*1}_t(\mathbf{x}_t - \mathbf{x}^*_t) - \mathbf{p}^{*2}_{t+1}(\mathbf{x}_{t+1} - \mathbf{x}^*_{t+1}) + \mathbf{f}_t(\mathbf{x}^*_t, \mathbf{x}^*_{t+1}) \leq \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}), \tag{19}$$

a.s. for all $\mathbf{x}_t = \mathbb{E}\{\mathbf{x}_t \mid \Sigma_t\}$ and $\mathbf{x}_{t+1} = \mathbb{E}\{\mathbf{x}_{t+1} \mid \Sigma_{t+1}\}$, $t = 0, 1, \ldots$. Since $\mathbf{x}^*_t$ minimizes $\mathbf{f}_{t-1}(\mathbf{x}^*_{t-1}, \cdot) + \mathbf{f}_t(\cdot, \mathbf{x}^*_{t+1})$ a.s., $0 \in \partial[\mathbf{f}_{t-1}(\mathbf{x}^*_{t-1}, \mathbf{x}^*_t) + \mathbf{f}_t(\mathbf{x}^*_t, \mathbf{x}^*_{t+1}]$ a.s., and we may choose $-\mathbf{p}^{*2}_t + \mathbf{p}^{*1}_t = 0$ a.s. By (16), we can define $\mathbf{p}^*$ such that

$$\mathbf{p}^*_t := \mathbb{E}(\mathbf{p}^{*2}_t \mid \Sigma_t) := \mathbb{E}(\mathbf{p}^{*1}_t \mid \Sigma_t), \quad t = 0, 1, \ldots, \tag{20}$$

proving (i). For $\mathbf{x}_t = \mathbb{E}\{\mathbf{x}_t \mid \Sigma_t\}$, $\mathbb{E}_t[\mathbf{p}^{*1}_t(\mathbf{x}_t)] = \mathbb{E}_t[\mathbf{p}^{*1}_t]\mathbf{x}_t = \mathbf{p}^*_t \mathbf{x}_t$. Also, for $\mathbf{x}_{t+1} = \mathbb{E}\{\mathbf{x}_{t+1} \mid \Sigma_{t+1}\}$, $\mathbb{E}_t[\mathbf{p}^{*2}_{t+1}(\mathbf{x}_{t+1})] = \mathbb{E}_t[\mathbb{E}_{t+1}[\mathbf{p}^{*2}_{t+1}(\mathbf{x}_{t+1})]] = \mathbb{E}_t[\mathbf{p}^*_{t+1}\mathbf{x}_{t+1}]$. Now, taking conditional expectations with respect to Σ_t in (19), we obtain $\mathbb{E}_t[\mathbf{p}^*_t(\mathbf{x}_t - \mathbf{x}^*_t) - \mathbf{p}^*_{t+1}(\mathbf{x}_{t+1} - \mathbf{x}^*_{t+1}) + \mathbf{f}_t(\mathbf{x}^*_t, \mathbf{x}^*_{t+1})] \leq \mathbb{E}_t[\mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1})]$, a.s. for all $\mathbf{x}_t = \mathbb{E}\{\mathbf{x}_t \mid \Sigma_t\}$ and $\mathbf{x}_{t+1} = \mathbb{E}\{\mathbf{x}_{t+1} \mid \Sigma_{t+1}\}$, implying condition (ii). Condition (iii) follows from (c) as in (12). ∎

Our results extend the price support results given in Flåm [1983,1985]. In Flåm [1983], strict feasibility of the optimal policy is required, while the infinite horizon results in Flåm [1985] require the objective to be completely separable in t. We use the finite continuation property to obtain these more general results which provide a more realistic framework for infinite horizon problems. The finite continuation property states that one can pay finite and absolutely decreasing amounts to reach a fixed finite cost path at given periods that extend to the horizon. This payment may, for example, represent the cost of joining a cyclic feasible policy. Examples of nonremovable singular multipliers *without* similar assumptions to ours are given in Prescott and Lucas [1972]. Nonanticipative feasibility is related to other conditions, such as *relatively complete recourse* as used in Rockafellar and Wets [1976], see e.g. Dempster [1988] for more details.

3. Turnpike Results

Our goal in this section is to show that optimal solutions approach a common facet – the *von Neumann facet* – from any given starting condition $\mathbf{x}_0$. The main implication of this result is that it is asymptotically optimal to match up with a decision process that is optimal for a *specific* initial condition even if that initial condition changes. This result is further enhanced by our showing that if the data process is cyclic then it is asymptotically optimal to return to an optimal cyclic policy even if other conditions temporarily obtain. These results lend justification to the match–up scheduling policy in Bean, et al. [1991] and extend the deterministic results in Bean and Birge [1986]. Discussion of specific scheduling policies is however postponed to the next section. In this section, we again use a general stochastic optimization model that may be applied in a variety of contexts.

Proposition 1. *Given Assumptions 1 and 2 and Conditions (a) - (c) in Theorem 1, let* $\mathbf{X}^*$ *be the set of solutions* $(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)$ *that are minimal in (ii) of Theorem 1 for* $\mathbf{p}_t^*$ *a set of optimal supporting prices given the initial condition* $\mathbf{x}_0$ *and let* $\mathbf{x}'$ *be an optimal decision process given the initial condition* $\mathbf{x}_0'$*. Then, for any* $\epsilon > 0$ *and* $\delta > 0$*, there exists* $T < \infty$*, such that, for all* $t \geq T$*,*

$$P\{\omega : \inf_{(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*) \in \mathbf{X}_t^*} \|(\mathbf{x}_t', \mathbf{x}_{t+1}') - (\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)\| > \epsilon\} < \delta. \tag{21}$$

Proof: By summing terms in (ii) of Theorem 1 for $\mathbf{x}^*$ and $\mathbf{x}'$ for each $\mathbf{f}_t$, $t \geq 1$, we obtain

$$\mathbb{E}_t\{(\mathbf{p}_{t+1}^* - \mathbf{p}_{t+1}')(\mathbf{x}_{t+1}' - \mathbf{x}_{t+1}^*) - (\mathbf{p}_t^* - \mathbf{p}_t')(\mathbf{x}_t' - \mathbf{x}_t^*)\} \leq 0, \tag{22}$$

where $\mathbf{p}^*$ is the price support process corresponding to $\mathbf{x}^*$, an optimal solution given $\mathbf{x}_0$, and $\mathbf{p}'$ is the sequence of prices supporting $\mathbf{x}'$. Next let $\mathbf{v}_t(\mathbf{x}_t^*) := (\mathbf{p}_t^* - \mathbf{p}_t')(\mathbf{x}_t' - \mathbf{x}_t^*)$, so that (22) becomes

$$\mathbb{E}_t \mathbf{v}_{t+1}(\mathbf{x}_{t+1}^*) \leq \mathbb{E}_t \mathbf{v}_t(\mathbf{x}_t^*) \quad \text{for all } t \geq 1, \tag{23}$$

for any $(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*) \in \mathbf{X}_t^*$. By (iii) of Theorem 1, $\lim_{t\to\infty} \mathbb{E}\mathbf{v}_t(\mathbf{x}_t^*) = 0$ for any $\mathbf{x}_t^*$. Note however that if (21) does not hold, then, for some $\epsilon > 0$, $\delta > 0$, for $t = T_{k_1}, \dots,$

$$P\{\omega : \inf_{(\mathbf{x}_t^*,\mathbf{x}_{t+1}^*)\in\mathbf{X}_t^*} \|(\mathbf{x}_t', \mathbf{x}_{t+1}') - (\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)\| > \epsilon\} \geq \delta. \tag{24}$$

From Assumption 2, since $(\mathbf{p}_t^*, -\mathbf{p}_{t+1}^*) \in \partial f_t((\mathbf{x}_t^*, \mathbf{x}_{t+1}^*))$, either $(\mathbf{p}_t^*, -\mathbf{p}_{t+1}^*) \in \partial f_t((\mathbf{x}_t', \mathbf{x}_{t+1}'))$ (in which case $(\mathbf{x}_t', \mathbf{x}_{t+1}') \in \mathbf{X}_t^*$) or there exists $(\mathbf{z}_t, \mathbf{z}_{t+1}) \in \mathbf{X}_t^*$ such that

$$\begin{aligned} \mathbf{f}_t(\mathbf{z}_t, \mathbf{z}_{t+1}) - \mathbf{p}_t^*\mathbf{z}_t + \mathbf{p}_{t+1}^*\mathbf{z}_{t+1} \leq \mathbf{f}(\mathbf{x}_t', \mathbf{x}_{t+1}') - \mathbf{p}_t^*\mathbf{x}_t' + \mathbf{p}_{t+1}^*\mathbf{x}_{t+1}' \\ - \gamma\|(\mathbf{z}_t, \mathbf{z}_{t+1}) - (\mathbf{x}_t', \mathbf{x}_{t+1}')\| \quad \text{a.s.} \end{aligned} \tag{25}$$

By the definition of $\mathbf{X}_t^*$ for given $\mathbf{x}_0$, for any $(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)$ we have that

$$\mathbb{E}\{\mathbf{f}(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*) - \mathbf{p}_t^*\mathbf{x}_t^* + \mathbf{p}_{t+1}^*\mathbf{x}_{t+1}^*\} = \mathbb{E}\{\mathbf{f}_t(\mathbf{z}_t, \mathbf{z}_{t+1}) - \mathbf{p}_t^*\mathbf{z}_t + \mathbf{p}_{t+1}^*\mathbf{z}_{t+1}\}.$$

Hence, for any T, summing (25) and taking expectations yields

$$\begin{aligned} \mathbb{E}[\sum_{t=0}^{T} \mathbf{f}_t(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)] - \mathbb{E}[\sum_{t=0}^{T} \mathbf{f}_t(\mathbf{x}_t', \mathbf{x}_{t+1}')] &\leq \mathbb{E}((\mathbf{p}_0^* - \mathbf{p}_0')(\mathbf{x}_0' - \mathbf{x}_0^*)) \\ &\quad - \mathbb{E}((\mathbf{p}_T^* - \mathbf{p}_T')(\mathbf{x}_T' - \mathbf{x}_T^*)) \\ &\quad - \sum_{t=0}^{T} \gamma\mathbb{E}\|(\mathbf{z}_t, \mathbf{z}_{t+1}) - (\mathbf{x}_t', \mathbf{x}_{t+1}')\|. \end{aligned} \tag{26}$$

For $t = T_{k_i}, i = 1, \dots,$ using the Markov inequality and (24), we have that

$$\begin{aligned} \mathbb{E}\|(\mathbf{z}_t, \mathbf{z}_{t+1}) - (\mathbf{x}_t', \mathbf{x}_{t+1}')\| &\geq \epsilon P\{\inf_{(\mathbf{x}_t^*,\mathbf{x}_{t+1}^*)\in\mathbf{X}_t^*} \|(\mathbf{x}_t', \mathbf{x}_{t+1}') - (\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)\| > \epsilon\} \\ &\geq \epsilon\delta. \end{aligned}$$

Since $\mathbb{E}\mathbf{v}_t(\mathbf{x}_t^*) \to 0$, the right–hand side in (26) therefore approaches $-\infty$, contradicting the finiteness of $F^\infty(\mathbf{x}')$. ∎

Theorem 2. *Under the conditions of Proposition 1, we may conclude that as* $t \to \infty$,

$$\inf_{(\mathbf{x}_t^*,\mathbf{x}_{t+1}^*)\in\mathbf{X}_t^*} \|(\mathbf{x}_t', \mathbf{x}_{t+1}') - (\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)\| \to 0 \quad \text{a.s.} \tag{27}$$

Proof: Proposition 1 states that

$$\inf_{(\mathbf{x}_t^*,\mathbf{x}_{t+1}^*)\in\mathbf{X}_t^*} \|(\mathbf{x}_t', \mathbf{x}_{t+1}') - (\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)\| \to 0$$

in probability. Since the $\mathbf{f}_t, t = 0, 1, 2, \ldots$, are proper convex normal integrands, the infima over the $\mathbf{X}_t^*$ are actually achieved. Hence, we may use the vector form of Skorohod's Theorem (see, e.g., Billingsley [1979], Theorem 29.6, pp. 337-340) to adjust the paths of the data process so that (27) holds. ∎

For Theorem 2 to be fully applicable, we would like to have a method for determining an optimal policy for some initial state so that the policy of matching up to that strategy can be implemented. This determination is simpler if we can show that cyclic policies are optimal. In this case, only a single cycle needs to be analyzed to determine the turnpike optimal policy. An example of such a policy is given in the next section.

In this development, we follow a similar approach to Arkin and Evstigneev [1979]. We first assume that the data process has a left tail, i.e. that ω_0 can be interpreted as $\ldots, \omega'_{-1}, \omega'_0$. An alternative is to assume some type of Markovian property of the data process (see Arkin and Evstigneev). The data process is assumed to be *cyclic with cycle k* if the measure μ is invariant with respect to the *k-period forward shift operator* T_k where $T_k\omega = \omega'$ such that $\omega'_t := \omega_{t+k}$, i.e., $\omega_t = T_k\omega_t = \omega_{t+k}$ a.s. It follows that we may define $T_k\Sigma_t := \Sigma_t$ for $t = 0, \ldots, k-1$. We also assume that the objective is invariant with respect to T_k so that $\mathbf{f}_{t+k}(T_k\mathbf{x}_t, T_k\mathbf{x}_{t+1}) = \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1})$ a.s., where $T_k x_t(\omega) := x_t(T_k\omega)$. In this context, $\mathbf{x}$ is a *cyclic policy* if $\mathbf{x}_{t+k} = T_k\mathbf{x}_t$ a.s. .

Corollary 1. *Given conditions (a) - (c) of Theorem 1 and a cyclic data process with cycle k, then there exists a weakly optimal policy with cycle k for any initial condition* $\mathbf{x}_0$.

Proof: First consider the problem :

$$\inf \mathbb{E}(\sum_{t=0}^{k-2} \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) + \mathbf{f}_{k-1}(\mathbf{x}_{k-1}, T_k\mathbf{x}_0)) \tag{28}$$

subject to $\mathbf{x}_t$ nonanticipative in L^n_∞ for $t = 0, \ldots, k-1$ and where $T_k x_0(\omega) := x_0(T_k\omega)$.

We can interpret (28) as

$$\inf \mathbb{E}(\mathbf{f}(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1})), \tag{29}$$

over the same decisions. Program (29) is a finite horizon problem as in Flåm [1985], hence there exist supporting prices $\mathbf{p}_t^*, t = 0, \ldots, k-1$, such that an optimal policy $\mathbf{x}^*$ almost surely minimizes over nonanticipative policies in L_∞^n:

$$\mathbb{E}_t(\mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) - \mathbf{p}_t^*\mathbf{x}_t + \mathbf{p}_{t+1}^*\mathbf{x}_{t+1}), \quad t = 0, \ldots, k-2, \tag{30}$$

and

$$\mathbb{E}_{k-1}(\mathbf{f}_{k-1}(\mathbf{x}_{k-1}, T_k\mathbf{x}_0) - \mathbf{p}_{k-1}^*\mathbf{x}_{k-1} + \mathbf{p}_0^*\mathbf{x}_0). \tag{31}$$

Define $x_k^*(\omega) := x_0^*(T_k\omega)$ and $p_k^*(\omega) := p_0^*(T_k^{-1}\omega)$. From the cyclic property of the data process,

$$\begin{aligned}\mathbb{E}[x_k^*(\omega) \mid \Sigma_k] &= \mathbb{E}[x_0^*(T_k\omega) \mid T_k\Sigma_0] \\ &= \mathbb{E}[x_0^*(\omega') \mid \Sigma_0] \\ &= x_0^*(\omega') = x_o^*(T_k\omega) = x_k^*(\omega) \;\text{ a.s. }\;.\end{aligned} \tag{32}$$

Similarly, $\mathbf{p}_k^*$ is nonanticipative. Continue to define $p_{t+Lk}^*(\omega) := p_t^*(T_{Lk}^{-1}\omega)$ for $0 \le t \le k-1$ and $L = 1, \ldots$, and $x_{t+Lk}^*(\omega) = x_t^*(T_{Lk}\omega)$ for $0 \le t \le k-1$ and $L = 1, \ldots$. We then have conditions (i) and (ii) in Theorem 1 for optimality by applying (30), (31), and repetitions of (32). By summing the terms in (ii) up to $\tau = Lk$ and noting that $\mathbb{E}\mathbf{x}_0 = \mathbb{E}\mathbf{x}_\tau$, we obtain

$$\mathbb{E}[\sum_{t=0}^{\tau}(\mathbf{f}_t(\mathbf{x}_t', \mathbf{x}_{t+1}') - \mathbf{f}_t(\mathbf{x}_t^*, \mathbf{x}_{t+1}^*)] \ge 0, \tag{33}$$

for all τ. Inequality (33) implies that $\mathbf{x}^*$ is weakly optimal by definition. ∎

4. Application Examples

The primary concern of our analysis is in application to stochastic scheduling problems. Our first example is a small problem that involves a general type of scheduling objective and meets our requirements for cyclic optimal policies. We show that these conditions are met and determine the optimal cyclic policy.

Assume a single machine and that units of a single item produced on the machine. The *state* x_t of the process is the amount of inventory (positive or negative) of the item at the beginning of each period. One unit is demanded in each period t. One unit may be produced in regular time in each period. An additional unit may be produced with overtime if the machine is available. The uncertainty is in the availability of the machine. We assume that the machine is *available* with probability 2/3 independent of the time period.

This model is a small stochastic version of the general deterministic example in Bean *et al.* [1991]. The objective includes a penalty (equal 10) for any backordered products, unit holding costs for any positive inventory, regular time production at cost 2 per unit, overtime production at cost 4 per unit, and a possible outside vendor purchase at cost p per unit.

The result is that, without production, the state moves from x_t to $x_{t+1} = x_t - 1$ from t to $t+1$. The cost of this transition is x_t if $x_t > 0$ or $-10x_t$ if $x_t < 0$. The value of x_{t+1} given x_t is the production/purchase decision. Production is only possible if the machine is available. The cost is $2(x_{t+1} + 1 - x_t)$ if $x_t \geq x_{t+1} \geq x_t - 1$ or $2 + 4(x_{t+1} - x_t)$ if $x_t + 1 \geq x_{t+1} \geq x_t$. If the machine is not available, then outside purchases are possible at a cost $p(x_{t+1} + 1 - x_t)$.

The objective function is then

$$f_t(\omega_{t+1}, x_t, x_{t+1}) := \begin{cases} -10x_t + p(x_{t+1} + 1 - x_t) & \text{if } x_t < 0 \text{ and } -1 \leq x_{t+1} - x_t \leq 0, \\ x_t + p(x_{t+1} + 1 - x_t) & \text{if } x_t \geq 0 \text{ and } -1 \leq x_{t+1} - x_t \leq 0, \\ \infty & \text{otherwise,} \end{cases} \tag{34}$$

if ω_{t+1} corresponds to "machine unavailable." If ω_{t+1} corresponds to an available machine, we have

$$f_t(\omega_{t+1}, x_t, x_{t+1}) := \begin{cases} -10x_t + 2 + 4(x_{t+1} - x_t) & \text{if } x_t < 0 \text{ and } 0 < x_{t+1} - x_t \leq 1, \\ -10x_t + 2(x_{t+1} + 1 - x_t) & \text{if } x_t < 0 \text{ and } -1 \leq x_{t+1} - x_t \leq 0, \\ x_t + 2 + 4(x_{t+1} - x_t) & \text{if } x_t \geq 0 \text{ and } 0 < x_{t+1} - x_t \leq 1, \\ x_t + 2(x_{t+1} + 1 - x_t) & \text{if } x_t \geq 0 \text{ and } -1 \leq x_{t+1} - x_t \leq 0, \\ \infty & \text{otherwise.} \end{cases} \tag{35}$$

Note that the functions in (34) and (35) are convex and satisfy Assumption 1. Finite values over the infinite horizon can be obtained as in Section 2 by subtracting constants in each period corresponding to the expected values of contributions from weakly optimal schedules. The data process here is Markovian, so we have the conditions for an optimal *stationary* strategy.

We seek an optimal solution to (28) for $k := 1$. The penalty term p is used as the price of obtaining one unit of the product elsewhere. For finite p, an optimal solution exists in L^n_∞. To simplify the analysis, we consider optimal solutions for $p \to \infty$. Results for finite p can be easily obtained as perturbations (depending on p) from the results below. The solution of these problems only requires the determination of additional parameters corresponding to the levels at which outside products should be purchased.

This problem can be interpreted as an abstract linear program. Suppose $\omega_{t+1} = 0$ if the machine is unavailable, and $\omega_{t+1} = 1$ if the machine is available. Problem (28) becomes:

$$\begin{aligned} \inf\ & \mathbb{E}[10\mathbf{x}_t^- + \mathbf{x}_t^+ + 2\mathbf{y}_{t+1}^1 + 4\mathbf{y}_{t+1}^2 | \Sigma^{t+1}] \\ \text{s. t. } & \mathbf{x}_t^+ - \mathbf{x}_t^- = \mathbf{x}_t \ \text{ a.s.}, \\ & E[\mathbf{x}_t|\Sigma_t] = \mathbf{x}_t \ \text{ a.s.}, \\ & \mathbf{x}_{t+1} = T\mathbf{x}_t \ \text{ a.s.}, \\ & \mathbf{x}_t + \mathbf{y}_{t+1}^1 + \mathbf{y}_{t+1}^2 = \mathbf{x}_{t+1} - 1 \ \text{ a.s.}, \\ & 0 \le \mathbf{x}_t^+, 0 \le \mathbf{x}_t^-, 0 \le \mathbf{y}_{t+1}^1 \le 1_{\omega_{t+1}=1}, 0 \le \mathbf{y}_{t+1}^2 \le 1_{\omega_{t+1}=1} \ \text{ a.s.} \end{aligned} \tag{36}$$

An optimal solution occurs at an extreme point of the feasible region of stationary distributions for $\mathbf{x}_t$. In our case, the solution occurs at a discrete distribution with atoms spaced units apart. There are two classes of these extreme solutions:-

$\mathcal{E}_1$: solutions corresponding to $x_{t+1} = x_t + 1$ if the machine is available and $x_t \le \alpha$ and $x_{t+1} = x_t - 1$ otherwise.

$\mathcal{E}_2$: solutions corresponding to $x_{t+1} = x_t + 1$ if the machine is available and $x_t \le \alpha - 1$, $x_{t+1} = x_t$ if the machine is available and $\alpha \ge x_t > \alpha - 1$, and $x_{t+1} = x_t - 1$ otherwise. Note that a stationary solution cannot have a wider range of regular time production of units since we can never achieve any higher x_{t+1} than α. The optimal value of α and the corresponding optimal strategy remain to be determined.

For $\mathcal{E}_1$, we consider $\alpha := 3$. This yields the stationary distribution :

x_t	**machine status**	probability
4	up	$\frac{1}{6}$
4	down	$\frac{1}{12}$
3	up	$\frac{1}{4}$
3	down	$\frac{1}{8}$
2	up	$\frac{1}{8}$
2	down	$\frac{1}{16}$
1	up	$\frac{1}{16}$
1	down	$\frac{1}{32}$
$-l$	up	$(\frac{1}{16})(\frac{1}{2})^{l+1}$
$-l$	down	$(\frac{1}{32})(\frac{1}{2})^{l+1}$

The expected objective value is then $6\frac{17}{32}$.

For $\mathcal{E}_2$, the stationary distribution for $\alpha := 3$ is:

x_t^*	**machine status**	probability
3	up	$\frac{1}{3}$
3	down	$\frac{1}{6}$
2	up	$\frac{1}{6}$
2	down	$\frac{1}{12}$
1	up	$\frac{1}{12}$
1	down	$\frac{1}{24}$
$-l$	up	$(\frac{1}{12})(\frac{1}{2})^{l+1}$
$-l$	down	$(\frac{1}{24})(\frac{1}{2})^{l+1}$

Here the expected value is $6\frac{1}{24}$.

To show the optimality of $\mathbf{x}_t^*$, consider changes from the values of $\mathbf{y}_{t+1}^1$, which we can consider as nonbasic variables in (36). Note that, since $\mathbf{x}_t^*$ is stationary, we must have $\mathbb{E}[\mathbf{y}_{t+1}^1 + \mathbf{y}_{t+1}^2] = 1$. Since $\mathbb{E}[\mathbf{y}_{t+1}^1] = 2/3$, it cannot be increased. The only possible changes are then to decrease $\mathbf{y}_{t+1}^1$ and to increase $\mathbf{y}_{t+1}^2$. We never would reduce $\mathbf{y}_{t+1}^1$ before reducing $\mathbf{y}_{t+1}^2$ to 0, so consider first an ϵ reduction of $\mathbf{y}_{t+1}^1$ for any subset of $\Omega^t(3)$ of measure $\delta > 0$ where $x_t^*(\omega) = 3$ for all $\omega \in \Omega^t(3)$. Correspondingly, we must increase $\mathbf{y}_{t+1}^2$ on $\Omega^t(3)$. The resulting stationary distribution $\mathbf{x}_t' \leq \mathbf{x}_t^*$ a.s. with $P\{\mathbf{x}_t' = \mathbf{x}_t^* - \epsilon\} = \delta$ and $P\{\mathbf{x}_t' = \mathbf{x}_t^*\} = 1/3 - \delta$. So, the difference in expected inventory cost from $\mathbf{x}_t^*$ to $\mathbf{x}_t'$ is

$$\delta(-\epsilon P\{\mathbf{x}_t^* > 0\} + 10\epsilon P\{\mathbf{x}_t^* \leq 0\}) = \delta\epsilon(3/8).$$

Thus, both inventory and production costs would increase in this case. The only other alternative change in the strategy is to keep $\mathbf{y}_{t+1}^1 \equiv 1_{\omega_t = 1}$ and to translate the distribution. In this case, however, any ϵ increase in α yields an expected cost increase of $\frac{5}{16}\epsilon$, and, as before, an ϵ decrease in α yields an expected cost increase of $\frac{3}{8}\epsilon$. With a finite penalty p the results are similar. For example, if $p = 1000$, then the same α value is optimal and the distribution is truncated at $l = 35$.

The results of the previous sections then show that it is asymptotically optimal to match up with this strategy regardless of our initial conditions. A match–up strategy to accomplish this is simply not to produce if inventory is greater than 3, to produce one unit for inventory equal 3, to produce in overtime one plus δ for inventory of $2 + \delta$ for $\delta > 0$, and to produce two units for inventories of 2 or less.

This first example includes an objective with tardiness or shortage penalties and holding costs that are common in scheduling models. Here the randomness was confined to the machine availability. Other scheduling models with varying degrees of uncertainty can also be incorporated into our general stochastic optimization model.

For example, consider a *k-period cyclic* model in which each of several commodities $i = 1, \ldots, n$ has a random *processing time* $\mathbf{p}(i)$, a random *release date* $\mathbf{r}(i)$, a random *due* *date* $\mathbf{d}(i)$, and a *penalty weight* of w_i for every period after the due date in which processing is not completed. The evolution of the data process is such that $\mathbf{p}(i)$ is known given $d(i)$. We also assume that $\mathbf{r}(i) < \mathbf{d}(i) < k$ a.s.

Decisions are the amount of processing performed on each item i in each period t. We may consider the *state* of each item to be reduced by the processing requirement at each due date, i.e. the state is changed by the total processing $\mathbf{x}_{t+1}(i) - \mathbf{x}_t(i)$ in period t if t is not a due date ($t \neq d(i)$) or by $\mathbf{x}_{t+1}(i) + \mathbf{p}(i) - \mathbf{x}_t(i)$ if t is a due date ($t = d(i)$). The decisions are constrained so that no processing can occur if an item is not released ($t < \mathbf{r}(i)$) and processing in each period on each item is at most one (for simplicity). Other restrictions on feasible processes appear in an indicator function $\delta(\omega, \mathbf{x}_t, \mathbf{x}_{t+1})$ that considers all resource availabilities.

The only costs in this model are due to tardiness. The penalty w_i is charged in each period for every unit of item i *overdue* ($\mathbf{x}_t(i) < 0$). The *total tardiness cost* at time t given ω is then $\sum_{i=1}^{n} w_i(-x_t(\omega, i))^+$. The objective is to minimize the expected total tardiness.

The single period objective contribution is then:

$$\mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1}) := \sum_{i=1}^{n} \mathbf{f}_t^i(\mathbf{x}_t(i), \mathbf{x}_{t+1}(i)) + \delta(\omega, \mathbf{x}_t, \mathbf{x}_{t+1}), \tag{37}$$

where

$$\mathbf{f}_t^i(\mathbf{x}_t(i), \mathbf{x}_{t+1}(i)) = \begin{cases} -w_i \mathbf{x}_t(i) & \text{if } -1_{\{t=\mathbf{d}(i)\}}\mathbf{p}(i) \le \mathbf{x}_{t+1}(i) - \mathbf{x}_t(i) \\ & \le 1_{\{t \ge \mathbf{r}(i)\}} - 1_{\{t=\mathbf{d}(i)\}}\mathbf{p}(i), \mathbf{x}_t(i) < 0 \text{ a.s.}, \\ 0 & \text{if } -1_{\{t=\mathbf{d}(i)\}}\mathbf{p}(i) \le \mathbf{x}_{t+1}(i) - \mathbf{x}_t(i) \\ & \le 1_{\{t \ge \mathbf{r}(i)\}} - 1_{\{t=\mathbf{d}(i)\}}\mathbf{p}(i), \mathbf{x}_t(i) \ge 0 \text{ a.s.}, \\ \infty & \text{otherwise.} \end{cases}$$

The data process is assumed to determine the availability of the resources (such as machines, labor and tools) for processing all commodities. We allow the δ indicator term to represent feasibility generally by assuming a value of "0" if $\mathbf{x}_{t+1}$ is feasibly reached from $\mathbf{x}_t$ and "∞" otherwise. For example, suppose that the processing of each commodity i requires a *resource* $m(i)$ where $m(i) \in \{1, \ldots, M\}$, the set of resources, and each resource can process at most one unit during a time interval if available and cannot process anything if unavailable. In this case, and independent component $\bar{\omega}_t$ of ω_t can be interpreted as an M-vector of ones and zeroes corresponding to availability and unavailability. We then have

$$\delta(\omega, \mathbf{x}_t, \mathbf{x}_{t+1}) := \begin{cases} 0 & \text{if } \sum_{i=1}^{n} 1_{j=m(i)}(\mathbf{x}_{t+1}(i) - \mathbf{x}_t(i) + \mathrm{p}(i)1_{\{t=\mathrm{d}(i)\}}) \\ & \leq \bar{\omega}_t(j) \text{ for } j = 1, \ldots, M, \\ \infty & \text{otherwise.} \end{cases}$$

Other constraints can also be represented in this way. Our only requirement is that $\delta(\omega, \cdot, \cdot)$ must be convex.

This model is the basic *minimum expected weighted tardiness multiprocessor* scheduling problem. With the convexity assumption, it meets the criteria for optimality, asymptotic stability and cyclic optimality given in Theorems 1 and 2 and Corollary 1. In some cases, an optimal stationary distribution for this model follows a path between disruptions such that an optimal match point is achieved as quickly as possible (*cf.* Bean *et al.* [1991]). To see this, let $\mathbf{x}^*$ be an optimal cyclic turnpike schedule in $\mathbf{X}^*$, the set of optimal cyclic turnpike schedules. Assume that some state $\mathbf{x}'_0 \leq \mathbf{x}^*_0$ a.s. is the initial state instead of $\mathbf{x}^*_0$. Let $\mathbf{x}'$ be an optimal trajectory given $\mathbf{x}'_0$. We wish to show that a trajectory $\bar{\mathbf{x}}$ that starts at $\mathbf{x}'_0$ and matches up with $\mathbf{x}^*$ at the earliest feasible t can be constructed with the same objective value as $\mathbf{x}'$.

Theorem 3. *Suppose $\mathbf{x}^*$ is optimal from x^*_0 in the above tardiness model, find $\inf_{\mathbf{x} \in \mathcal{N}, \mathbf{x}_0 = \mathbf{x}^*_0 \text{ a.s.}} \mathbb{E} \sum_{t=0}^{\infty} \mathbf{f}_t(\mathbf{x}_t, \mathbf{x}_{t+1})$ given $\mathbf{x}^*_0 \geq \mathbf{x}'_0$ a.s., with $\mathbf{f}_t$ defined in (37), and that there exists a feasible solution $\bar{\mathbf{x}}$ such that $\bar{\mathbf{x}}_0 = \mathbf{x}'_0$ a.s., and $\bar{\mathbf{x}}_\tau = \mathbf{x}^*_\tau$ a.s. for some $\tau < \infty$, then there exists an optimal solution $\mathbf{x}'$ given $\mathbf{x}'_0$ such that $\mathbf{x}'_t = \mathbf{x}^*_t, t \geq \tau$ a.s. .*

Proof: The existence of the feasible path $\bar{\mathbf{x}}$ that matches up to $\mathbf{x}^*$ at τ is equivalent to the availability of

$$\sum_{i=1}^{n} 1_{\{j=m(i)\}}(x_0(\omega, i) - x'_0(\omega, i)), \tag{38}$$

additional units of capacity on every machine j beyond the processing required on machine j under policy $\mathbf{x}^*$ for a.e. ω, $j = 1, \ldots, M$, before time τ. This is true because we implicitly assume that if $x_0^*(\omega, i) > x_0'(\omega, i)$, then $r(\omega, i) \leq 0$. Given this surplus capacity, consider $\mathbf{x}'$ optimal given $\mathbf{x}_0'$ and construct $\bar{\mathbf{x}}_t, 0 \leq t \leq \tau$ such that for all $i = 1, \ldots, n$ and all ω:

(i) $\bar{x}_t(\omega, i) := x_t'(\omega, i)$ if $x_t'(\omega, i) \leq x_t^*(\omega, i)$,

(ii) $x_\tau'(\omega, i) := x_\tau^*(\omega, i)$ if $x_t'(\omega, i) > x_t^*(\omega, i)$.

Note that $\bar{\mathbf{x}} := \mathbf{x}' \wedge \mathbf{x}^*$, i.e. a minimum of the two optimal policies in the sense of coordinatewise order in t, so that $\bar{\mathbf{x}}_t \leq \mathbf{x}_t^*, t = 1, \ldots, \tau - 1$, $\bar{\mathbf{x}}_\tau = \mathbf{x}_\tau^*$, and no processing takes $\bar{x}_t(i)$ beyond $\mathbf{x}_t^*(i)$ for any i. Since no more processing capacity than (38) has been used on any mahcine $j = 1, \ldots, M$, $\bar{\mathbf{x}}$ defined by (i) and (ii) can be defined as $\bar{\mathbf{x}}_t := \mathbf{x}_t^*$ for $t > \tau$. Note also that for $\mathbf{x}'$ and $\mathbf{x}^*$ nonanticipative, $\bar{\mathbf{x}}$ is nonanticipative by construction.

Suppose that $\bar{\mathbf{x}}$ is not optimal from the initial point $\mathbf{x}_0'$. Then, there exists a sequence $\delta \mathbf{x} := \{\delta \mathbf{x}_0 := 0, \delta \mathbf{x}_1, \delta \mathbf{x}_2, \ldots\}$ such that $\mathbf{x}' = \bar{\mathbf{x}} + \delta \mathbf{x}$ a.s. and $F(\bar{\mathbf{x}} + \delta \mathbf{x}) < F(\bar{\mathbf{x}})$. We will show that this implies that there exists $\delta \mathbf{x}^*$ such that $F(\mathbf{x}^* + \delta \mathbf{x}^*) < F(\mathbf{x}^*)$ in contradiction to optimality.

First, define $F(\mathbf{x}') := \sum_{i=1}^n F_i(\mathbf{x}'(i))$, where F_i is the contribution of commodity i to the overall objective, and $F_i(\mathbf{x}'(i)) := \mathbb{E}[g_i(\omega, \mathbf{x}(i))]$, where $g_i(\omega, \mathbf{x}(i)) := \sum_{t=0}^{\infty} \mathbf{f}_t^i(\mathbf{x}_t(i), \mathbf{x}_{t+1}(i))$. Now suppose

$$F_i(\mathbf{x}'(i)) < F_i(\bar{\mathbf{x}}(i)) \tag{39}$$

for some i $(i = 1, \ldots, n)$. Then, there exists $\Omega_i' \subset \Omega$, $P\{\Omega_i'\} > 0$, such that $g_i(\omega, x'(\omega, i)) < g_i(\omega, \bar{x}(\omega, i))$ for all $\omega \in \Omega_i'$. This in turn implies that there exists $\tau(\omega, i) \leq \tau$ such that for all $\omega \in \Omega_i'$,

$$x'_{\tau(\omega,i)}(\omega, i) = x^*_{\tau(\omega,i)}(\omega, i) = \bar{x}_{\tau(\omega,i)}(\omega, i),$$

and

$$\sum_{t=\tau(\omega,i)}^{\infty} f_t^i(\omega, x_t'(\omega, i), x_{t+1}'(\omega, i)) < \sum_{t=\tau(\omega,i)}^{\infty} f_t^i(\omega, \bar{x}_t(\omega, i), \bar{x}_{t+1}(\omega, i)), \tag{40}$$

since $\bar{\mathbf{x}} := \mathbf{x}' \wedge \mathbf{x}^*$ up to period τ. Note that $\delta x_{\tau(\omega,i)}(\omega, i) = 0$ and define the feasible new policy $x^*(\omega, i) + \delta x^*(\omega, i)$ for $\omega \in \Omega_i'$ by setting

$$\delta x_t^*(\omega, i) := \begin{cases} 0 & t = 0, \ldots, \tau(\omega, i), \\ \delta x_t(\omega, i) & t = \tau(\omega, i) + 1, \ldots \end{cases},$$

and setting $\delta x_t^*(\omega, i) := 0,\ t = 1, 2, \ldots$ for $\omega \in \Omega \setminus \Omega_i'$. It follows, for all $\omega \in \Omega_i$, using (40) that

$$
\begin{aligned}
g_i(\omega, x^*(\omega, i) + \delta x_t^*(\omega, i)) &= \sum_{t=0}^{\infty} f_t^i(\omega, x_t^*(\omega, i), x_{t+1}^*(\omega, i)) \\
&+ f_{\tau(\omega,i)-1}^i(\omega, x_{\tau(\omega,i)}^*(\omega, i), x_{\tau(\omega,i)}^*(\omega, i) + \delta x_{\tau(\omega,i)}^*(\omega, i)) \\
&+ \sum_{t=\tau(\omega,i)}^{\infty} f_t^i(\omega, x_t^*(\omega, i) + \delta x_t^*(\omega, i), x_{t+1}^*(\omega, i) + \delta x_{t+1}^*(\omega, i)) \\
&< \sum_{t=0}^{\infty} f_t^i(\omega, x_t^*(\omega, i), x_{t+1}^*(\omega, i)) \\
&+ f_{\tau(\omega,i)-1}^i(\omega, x_{\tau(\omega,i)}^*(\omega, i), x_{\tau(\omega,i)}^*(\omega, i) + \delta x_{\tau(\omega,i)}^*(\omega, i)) \\
&+ \sum_{t=\tau(\omega,i)}^{\infty} f_t^i(\omega, x_t^*(\omega, i), x_{t+1}^*(\omega, i)) \\
&= g_i(\omega, x^*(\omega, i)) \\
&+ f_{\tau(\omega,i)-1}^i(\omega, x_{\tau(\omega,i)-1}^*(\omega, i), x_{\tau(\omega,i)}^*(\omega, i) + \delta x_{\tau(\omega,i)}^*(\omega, i)) \\
&- f_{\tau(\omega,i)-1}^i(\omega, x_{\tau(\omega,i)-1}^*(\omega, i), x_{\tau(\omega,i)}^*(\omega, i)) \\
&= g_i(\omega, x^*(\omega, i)),
\end{aligned}
$$

since $\delta x_{\tau(\omega,i)}^*(\omega, i) = \delta x_{\tau(\omega,i)}(\omega, i) = 0$. This implies that

$$F_i(\mathbf{x}^*(i) + \delta\mathbf{x}^*(i)) := \mathbb{E}[g_i(\omega, \mathbf{x}^*(i) + \delta\mathbf{x}^*(i)] < F_i(\mathbf{x}^*(i)) := \mathbb{E}[g_i(\omega, \mathbf{x}^*(i))].$$

Making this construction for each i for which (39) holds and summing over $i = 1, \ldots, n$ gives

$$F(\mathbf{x}^* + \delta\mathbf{x}^*) < F(\mathbf{x}^*),$$

the required contradiction. ∎

This tardiness model can be extended further by including other constraints in the δ term of the objective. The only requirement is that the overall objective remain convex. Hence we need only require convex feasible sets.

5. Conclusions

We have presented a general stochastic optimization model with discrete time periods and infinite horizon. We showed that optimality conditions in terms of supporting prices can be derived for the model. This conclusion extends previous results by allowing extreme value optimal solutions and a more general convexity condition. These generalizations make the model more applicable to a variety of problems. They allow for characterizations of turnpike and cyclic optimal solutions that justify match–up strategies.

In particular, we showed how these results apply to scheduling. In these problems, a cyclic schedule can be developed "off–line" using whatever computing resources are available and the cyclic policy is implemented whenever conditions evolve as predicted by the model. When other conditions obtain (for example, a catastrophic failure of a machining center), it is optimal to match– up with the cyclic schedule sometime in the future. Determining this match–up strategy can be accomplished "on–line" while the remainder of the cyclic schedule is maintained. In practice, the match–up point can be set so that matching up is both computationally feasible and cost effective.

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