ASSESSING THE EFFECTS OF MACHINE BREAKDOWNS IN STOCHASTIC SCHEDULING

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ABSTRACT

In most scheduling problems discussed in the literature it is assumed
that the machine (i.e. key resource) is continuously available. Plainly, this
is often unrealistic. Here we suggest assessing the effects of machine
breakdowns by evaluating the strategy which is optimal when the machine is
always available as a strategy for the breakdowns case. The results extend
earlier ones of the authors and co-workers.

ALTERNATING RENEWAL PROCESS, GITTINS INDEX, RENEWAL FUNCTION, STOCHASTIC
SCHEDULING.

1. Introduction

Let $J = \{1,2,\ldots,N\}$ be a set of jobs to be processed on a single machine which is subject to breakdown. For $k = 1,2,\ldots$ the kth breakdown of the machine is associated with two random variables U_k and D_k, both taking values in the positive integers. U_k is the kth uptime for the machine, i.e. the length of the period between the $(k - 1)^{st}$ and the kth breakdown. D_k is the kth machine downtime, i.e. the length of the kth breakdown. All of these random variables are independent of each other, and further the uptimes and downtimes are (separately) identically distributed. This process of breakdowns constitutes an alternating renewal process (see Ross (1970)). We further assume that the uptimes have finite first and second moments, μ_1 and μ_2, that the downtimes have finite first moment μ_D and that the machine is up from time 0 until U_1.

The processing requirement of job j is a random variable X_j, taking values in the positive integers, $1 \leq j \leq N$. The X_j's are independent of each other and of the uptimes and downtimes. For each non-negative integer point in time t at which the machine is functioning, one of the jobs in J which has not yet been completed must be chosen for processing during $[t, t + 1)$. Processing stops when all of the jobs have been finished. A policy π is any rule for deciding how to process the jobs in J. Under π, job j is completed at time $F_j(\pi)$. The objective is to choose a policy which minimizes expected weighted flow time

$$C(\pi) = E\left\{ \sum_{j=1}^{N} w_j F_j(\pi) \right\} \tag{1}$$

where the w_j's are given positive constants.

For a machine which is continuously available (i.e. downtimes are zero), any optimal policy is known to be determined by a collection of Gittins' indices (see Gittins (1979) and Glazebrook (1984)). If job j has received x

units of processing and has still to complete, its Gittins index is defined to be

$$\gamma_j(x) = \sup_{r \geq 1} \left[\frac{w_j \sum\limits_{t=1}^{r} \rho_j(x+t-1)[\prod\limits_{s=0}^{t-2} \{1-\rho_j(x+s)\}]}{\sum\limits_{t=0}^{r-1} [\prod\limits_{s=0}^{t-1} \{1-\rho_j(x+s)\}]} \right], \quad x = 0,1,2,\ldots \quad (2)$$

where

$$\rho(t) = p(X_j = t + 1 | X_j > t), \quad t = 0,1,2,\ldots$$

is the <u>completion rate function</u> for job j. Hence in the absence of breakdowns, an optimal policy always allocates the machine to any uncompleted job with maximal Gittins index. Glazebrook (1987) pointed out that this remains optimal for a problem with breakdowns when the uptimes are geometrically distributed.

Our objective is to evaluate the simple "no breakdowns" optimal policy determined by the Gittins' indices in (2) as a policy for our general problem with breakdowns. For convenience denote by π_γ a policy which, whenever the machine is up, processes the uncompleted job with the largest Gittins index. Denote by π^* an optimal policy. We shall seek to evaluate π_γ by putting upper bounds on the quantities

$$C(\pi_\gamma) - C(\pi^*) \quad (3)$$

or

$$\{C(\pi_\gamma) - C(\pi^*)\}\{C(\pi^*)\}^{-1}. \quad (4)$$

Following on from the last remark in the previous paragraph, a natural approach is to bound the quantities in (3) or (4) by a quantity which measures the extent to which the uptimes fail to be geometric. This work is reported in Section 2 and extends results due to Glazebrook (1987). A more general result in Section 3 extends previous work by Birge, Frenk, Mittenthal and Rinnooy Kan (1987) which was restricted to the case of deterministic processing times.

2. Bounds based on stochastic orderings

We shall need some preliminary remarks and definitions relating to the renewal process $N_U(t)$ determined by the up-times, i.e.

$$N_U(t) \overset{\Delta}{=} \sup\{k \geq 0 : U_1 + U_2 + \ldots + U_k \leq t\} .$$

The renewal function $m_U(t)$ is defined as usual to be $E\{N_U(t)\}$. We shall use the notation $N_V(t)$, $N_W(t)$, $m_V(t)$, $m_W(t)$ to represent renewal processes and functions defined with respect to i.i.d. random variables $\{V_1, V_2, \ldots\}$ and $\{W_1, W_2, \ldots\}$ respectively. Throughout, all random variables are assumed to be positive integer-valued with finite mean. We make use of the following ideas from reliability theory (see Barlow and Proschan (1975)).

DEFINITION 1. Positive integer-valued random variable V is new better than used in expectation (N.B.U.E.) if

$$E(V) \geq E(V - v | V \geq v + 1) , \qquad v = 0,1,2,\ldots .$$

DEFINITION 2. Positive integer-valued random variable V is new worse than used in expectation (N.W.U.E.) if

$$E(V) \leq E(V - v | V \geq v + 1) , \qquad v = 0,1,2,\ldots .$$

The following result follows fairly simply from Theorem 3.14 in Chapter 6 of Barlow and Proschan (1975). In Lemma 1 "$\leq_{ST}$" denotes stochastic ordering.

LEMMA 1. Suppose that there exists a N.B.U.E. random variable V_1 and a N.W.U.E. random variable W_1 such that $V_1 \leq_{ST} U_1 \leq_{ST} W_1$ then

$$t/E(V_1) \geq m_V(t) \geq m_U(t) \geq m_W(t) \geq t/E(W_1), \quad t \geq 0.$$

Geometric random variables are both N.B.U.E. and N.W.U.E. and hence Lemma 1 relates to the situation where the uptimes are stochastically bounded above and below by such variates. The following definitions relate to the "closest" such bounding variables obtainable.

DEFINITION 3. The <u>upper geometric rate</u> $\bar{\eta}$ of uptime U_1 is defined as

$$\bar{\eta} = \inf_{\eta}\{\eta > 0; \ V_1 \leq_{ST} U_1 \text{ where } V_1 \text{ is geometric with probability } \eta\} \ .$$

DEFINITION 4. The <u>lower geometric rate</u> $\underline{\eta}$ of uptime U_1 is defined as

$$\underline{\eta} = \sup_{\eta}\{\eta > 0; \ U_1 \leq_{ST} W_1 \text{ where } W_1 \text{ is geometric with probability } \eta\} \ .$$

It is not difficult to show algebraically that if $\eta(\cdot)$ is the completion rate function of uptime U_1 then

$$\sup_j \eta(j) \geq \bar{\eta} = \sup_j [1 - (\prod_{k=1}^{j} \{1 - \eta(k)\})^{1/j}] \geq \sup_j \{\sum_{k=1}^{j} \eta(k)\cdot j^{-1}\}$$

$$\geq \inf_j \{\sum_{k=1}^{j} \eta(k)\cdot j^{-1}\} \geq \inf_j [1 - (\prod_{k=1}^{j} \{1 - \eta(k)\})^{1/j}] = \underline{\eta} \geq \inf_j \eta(j). \tag{5}$$

We are now in a position to use the above bounds on the uptimes and the related bounds on renewal functions to give the evaluation of the "no breakdowns" optimal strategy π_γ we seek. Let π be a strategy for our stochastic scheduling problem with breakdowns, which is stationary with regard to the values of past downtimes. Optimal π^* plainly has this property. Under π job j is completed at time $F_j(\pi)$. Denote by $F_j^0(\pi)$ the implied completion time of j under policy π when there are no breakdowns (i.e. U_1 is infinite). The assumptions concerning independence and finiteness of moments imply via a simple conditioning argument that

$$C(\pi) = E\left\{\sum_{j=1}^{N} w_j F_j(\pi)\right\}$$

$$= E\left[\sum_{j=1}^{N} w_j (F_j^0(\pi) + m_u \{F_j^0(\pi)\}\mu_D)\right] \tag{6}$$

Theorem 1 embodies an evaluation of π_γ in terms of a measure of the extent to which the uptimes fail to be geometric.

THEOREM 1

(a) $C(\pi_\gamma) - C(\pi^*) \leq (\bar{\eta} - \underline{\eta})\mu_D E[\sum_{j=1}^{N} w_j F_j^0(\pi_\gamma)]$

(b) $\{C(\pi_\gamma) - C(\pi^*)\}\{C(\pi^*)\}^{-1} \leq \{(\bar{\eta} - \underline{\eta})\mu_D\}\{1 + \underline{\eta}\mu_D\}^{-1}$

$\leq \{\bar{\eta}(\underline{\eta})^{-1}\} - 1 .$

Proof

From Lemma 1, Definition 3 and (6) it follows that

$$C(\pi_\gamma) = E[\sum_{j=1}^{N} w_j (F_j^0(\pi_\gamma) + m_U\{F_j^0(\pi_\gamma)\}\mu_D)]$$

$$\leq (1 + \bar{\eta}\mu_D)E[\sum_{j=1}^{N} w_j F_j^0(\pi_\gamma)] .$$

Similarly we have that

$$C(\pi^*) \geq (1 + \underline{\eta}\mu_D)E[\sum_{j=1}^{N} w_j F_j^0(\pi^*)]$$

$$\geq (1 + \underline{\eta}\mu_D)E[\sum_{j=1}^{N} w_j F_j^0(\pi_\gamma)] ,$$

the latter inequality following from the optimality of π_γ for the

no-breakdowns case. Inequalities (a) and (b) now follow simply.

Note that since $(\bar{\eta} - \underline{\eta})$ and $\{\bar{\eta}(\underline{\eta})^{-1}\} - 1$ are both natural measures of the

extent to which uptimes fail to be geometric then in Theorem 1 we have

achieved our stated objective - namely the evaluation of π_γ in terms of such

measures. It follows fairly simply from Lemma 1 that we can do rather better

than Theorem 1 in the case of N.B.U.E. and N.W.U.E. uptime distributions.

This is of considerable practical importance since all of the standard

discrete distributions have monotonic completion rate functions and hence are

either N.B.U.E. or N.W.U.E. Theorem 2 has a proof which is a simple

elaboration of that of Theorem 1 and hence it will not be given.

THEOREM 2

(i) If uptime U_1 is N.B.U.E. then

(a) $C(\pi_\gamma) - C(\pi^*) \leq (\mu_1^{-1} - \underline{\eta})\mu_D E[\sum_{j=1}^{N} w_j F_j^0(\pi_\gamma)]$

(b) $\{C(\pi_\gamma) - C(\pi^*)\}\{C(\pi^*)\}^{-1} \leq \{(\mu_1^{-1} - \underline{\eta})\mu_D\}\{1 + \underline{\eta}\mu_D\}^{-1}$

$\leq (\underline{\eta}\mu_1)^{-1} - 1$.

(ii) If uptime U_1 is N.W.U.E. then

(c) $C(\pi_\gamma) - C(\pi^*) \leq (\bar{\eta} - \mu_1^{-1})\mu_D E[\sum_{j=1}^{N} w_j F_j^0(\pi_\gamma)]$

(d) $\{C(\pi_\gamma) - C(\pi^*)\}\{C(\pi^*)\}^{-1} \leq \{(\bar{\eta} - \mu_1^{-1})\mu_D\}\{1 + \mu_1^{-1}\mu_D\}^{-1}$

$\leq \bar{\eta}\mu_1 - 1$.

3. Bounds based on Lorden's inequality

The bounds described here extend Theorem 2.5 in Birge, Frenk, Mittenthal and Rinnooy Kan (1987) to the case with random processing times. They originate from Lorden's inequality for the renewal function which, under the conditions assumed here, states that

$$\frac{t}{\mu_1} - 1 \leq m_U(t) \leq \frac{t}{\mu_1} + \frac{\mu_2}{\mu_1^2} - 1 , \qquad t \geq 0 . \tag{7}$$

THEOREM 3

$$C(\pi_\gamma) - C(\pi^*) \leq \mu_2 \mu_D \mu_1^{-2}(\sum_{j=1}^{N} w_j) .$$

Proof

From the right-hand side of (7), together with (6), it follows that

$$C(\pi_\gamma) = E[\sum_{j=1}^{N} w_j(F_j^0(\pi_\gamma) + m_U\{F_j^0(\pi_\gamma)\}\mu_D)]$$

$$\leq E[\sum_{j=1}^{N} w_j(F_j^0(\pi_\gamma) + \mu_D\{\frac{F_j^0(\pi_\gamma)}{\mu_1} + \frac{\mu_2}{\mu_1^2} - 1\})]$$

Similarly, from the left-hand side of (7), we have that

$$C(\pi^*) \geq (1 + \mu_D\mu_1^{-1})E[\sum_{j=1}^{N} w_j F_j^0(\pi^*)] - \mu_D(\sum_{j=1}^{N} w_j)$$

$$\geq (1 + \mu_D\mu_1^{-1})E[\sum_{j=1}^{N} w_j F_j^0(\pi_\gamma)] - \mu_D(\sum_{j=1}^{N} w_j) ,$$

the latter inequality following from the optimality of π_γ for the no-breakdowns case. The result now follows.

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