Bounds on Optimal Values in Stochastic Scheduling

John R. Birge
Department of Industrial and Operations Engineering
The University of Michigan
Ann Arbor MI 48109-2117, USA

K.D. Glazebrook
Department of Statistics
University of Newcastle upon Tyne
Newcastle NE1 7RU UK

February 5, 1996

Technical Report 96-2

Bounds on Optimal Values in Stochastic Scheduling

J.R. Birge *

Department of Industrial and Operations Engineering

The University of Michigan

Ann Arbor, MI 48109

USA

K.D. Glazebrook

Department of Statistics

University of Newcastle upon Tyne

Newcastle NE1 7RU

UK

February 5, 1996

Abstract: Consistent stochastic orders of processing times and objective functions yield optimal policies in many stochastic scheduling problems. When these orders fail to hold, however, finding optimal values may be difficult. In this paper, we show how to bound these values in general situations including problems with unreliable machines and tardiness-based objectives.

1. Introduction.

Most practical scheduling problems involve some uncertainty about the lengths of time to process jobs and the availability of machines and other resources. Recent results (see, e.g., Chang and Yao [1993] and Righter [1994]) using stochastic order relationships have characterized optimal policies when these orders are consistent. While these results may be quite useful in certain situations, the conditions for optimality are most likely rare in practice. We show how the results may be extended to broad classes of problems by constructing bounds on optimal values.

While our basic approach may apply in multiple machine problems, to keep the analysis simple, we consider a single-machine and fixed set of N jobs. Costs are assumed to be the expectation of

* Based in part on work supported by the National Science Foundation under Grant DDM-9215921.

functions of job completion times, including tardiness objectives. The processing times may include breakdown processes.

Our results represent extensions of results in weighted flow time models as well as previous models with due dates. In the weighted flow time case, the application of Gittins indices (Gittins [1979], Glazebrook and Gittins [1981], Glazebrook [1976]) has led to characterizations of optimal strategies in many contexts. Special cases with machine breakdowns also appear in Glazebrook [1984], Birge et al. [1990], and Birge and Glazebrook [1988]. Birge et al [1990] also gives limited results for linear tardiness penalties. Weighted fixed cost penalties for each tardy job are considered in Glazebrook [1983].

Optimal policy results for general functions of completion times appear in Righter [1994]. We use the optimality results in Chang and Yao [1993] extensively below. Shaked and Shantikumar [1994] give general properties of stochastic orderings that lead to many of the optimality results.

In Section 2, we describe our model and previous results. Section 3 gives bounds for deviations when jobs have order specific costs. Section 4 provides corresponding results when costs are job specific. Section 5 gives bounds in the case of nonagreeable costs while Section 6 discusses optimality conditions for nonpreemptive policies in the presence of breakdowns. Section 7 presents conclusions.

2. Model and Background.

We suppose a set $J = \{1, 2, \ldots, N\}$ of jobs is to be processed on a single machine. Each job j has a random processing requirement X_j. The processing requirements are independent of each other. At $t = 0, 1, \ldots$, the state of the system consists of the set of completed jobs and the cumulative processing time x_j for all unfinished jobs in J. Decisions are made at t on which job to process during the interval $[t, t+1]$. A policy π is any nonanticipative rule dependent on the history of the process that determines which jobs to process at t until all jobs in J are completed. We assume the cost of a policy is the expectation of a function of the completion times,

$$V(\pi) = E_\pi[f(c_1, \ldots, c_N)], \tag{1}$$

where c_i denotes the completion time of job i and E_π denotes the expectation under policy π. The objective is to minimize $V(\pi)$. The objective in (1) is *job specific* as in the case where each job may correspond to a different product. When each job is substitutable, the cost may be *order specific*, only depending on the order of finish, with objective f written as $f(c_{(1)}, \ldots, c(c_{(N)})$.

A common objective in (1) is weighted flow time $f(c_1, \ldots, c_N) = \sum_{i=1}^{N} w_i c_i$. In this case, assuming no breakdowns, if job j has received x_j units of processing, then Gittins [1979] and Glazebrook [1984] show that an optimal policy processes the uncompleted job with highest Gittins index $\gamma_j(x)$ where

$$\gamma_j(x) = \sup_{r \geq 1} \left\{ \frac{w_j \sum_{t=1}^{r} \rho_j(x+t-1) \left[\prod_{s=0}^{t-2} (1 - \rho_j(x+s)) \right]}{\sum_{t=0}^{r-1} \left[\prod_{s=0}^{t-1} (1 - \rho_j(x+s)) \right]} \right\} \tag{2}$$

and $\rho_j(t) = P(X_j = t + 1 \mid X_j > t)$, $t = 0, 1, \ldots$, is the **completion rate function**. It is not hard to see that if ρ_j is nondecreasing, then the supremum in (2) is attained at $r = +\infty$ and so $\gamma_j(x) = w_j \mu_j(x)$, where $\mu_j(x) \equiv \frac{1}{E[X_j - x | X_j > x]}$. When all jobs have nondecreasing completion rates, then ordering jobs by the weighted expected remaining processing time is optimal.

An optimal nonpreemptive policy is also given in Glazebrook [1983] for an objective that includes a reward for completion before a due date, d_i for each i, and no reward beyond the due date. In this model, if weights are decreasing, $w_1 \geq \ldots \geq w_N$, due dates are increasing $d_1 \leq \ldots \leq d_N$ and processing times are stochastically increasing, $P[X_1 \leq x] \leq P[X_2 \leq x] \ldots \leq P[X_N \leq x]$ for all x, with the order unchanged when conditioning on past processing, then processing in the order $1, 2, \ldots, N$ until each job is completed is an optimal policy. For a breakdown model with deterministic processing times and with linear tardiness penalties, Birge et al. [1990] show this order is again optimal.

Chang and Yao [1993] provide results for more general objectives with various conditions. Righter and Shantikumar [1992] also provide generalizations in other ways. The variety of proof techniques and other extensions appear in Righter [1994].

We use these results in the following. Before summarizing them, we define some notation. We assume that the order, $1, 2, \ldots, N$, is consistent with shortest expected processing time (SEPT). Following this policy to the completion of each job has cost, f_{SEPT}. Following the order, $N, N-1, \ldots, 1$, until each job is completed is longest expected processing time (LEPT) order with cost, f_{LEPT}. Following the $w\mu$ order to the completion of each job is weighted shortest expected processing time (WSEPT) with cost, f_{WSEPT}.

We denote the ordinary stochastic order for two positive random variables, X and Y, by $X \leq_{st} Y$, meaning that $P\{X > t\} \leq P\{Y > t\}$ for all $t \in [0, \infty)$. We also use that X is less than Y in *hazard rate order*, $X \leq_{hr} Y$, to mean that $\frac{P\{X>t\}}{P\{Y>t\}}$ is decreasing in t for all t. When X and Y have (here discrete) densities, g_X and g_Y, respectively such that $g_X(t)/g_Y(t)$ is decreasing over the union of their support, then X is less than Y in *likelihood ratio order*, $X \leq_{lr} Y$. We note that likelihood ratio order implies hazard ratio order which, in turn, implies stochastic order.

It is often useful to have the orders consistent regardless of the amount of processing that has been completed. Here, X has increasing likelihood ratio (ILR) if $[X-t|X \geq t] \leq_{lr} [X-s|X \geq s]$ for all $0 \leq s \leq t$, which implies that X has an increasing hazard rate (IHR). If likelihood ratios of two ILR random variables, X and Y, are ordered so that $g_X(s+\delta)g_Y(t) \leq g_X(s)g_Y(t+\delta)$ for all $\delta > 0$ and any s and t, then X is less than Y in *increasing likelihood ratio order*, $X \leq_{ilr} Y$. Similarly, if $\frac{P\{X>s+\delta\}}{P\{Y>t+\delta\}}$ is decreasing in δ for all s and t with IHR X and Y , then X is less than Y in *increasing hazard rate order*, $X \leq_{ihr} Y$.

The results also generally make use of restrictions on the objective functions that are consistent with processing time orders. We say that the costs are *aggreeable* if $f(c_1, \ldots, c_i, \ldots, c_j, \ldots, c_N) \geq f(c_1, \ldots, c_j, \ldots, c_i, \ldots, c_N)$ whenever $c_i \geq c_j$. In particular, this implies, when f is *separable* $(f(c_1, \ldots, c_N) = \sum_{i=1}^{N} f_i(c_i))$, that $f_i - f_j$ is increasing for $i < j$.

From these relationships, we have the following results in Table 1 where we use $A = ST, (I)LR$ or $(I)HR$ to indicate $X_i \leq_A X_{i+1}$ for all i under ordering A. A function f is *supermodular* if $f(x+e_i\epsilon_i+e_j\epsilon_j) \geq f(x+e_i\epsilon_i)+f(x+e_j\epsilon_j)$ for any $\epsilon_i, \epsilon_j \geq 0$, where e_i is the ith unit vector. We use CY for Change and Yao [1993], RS for Righter and Shantikumar [1992] and R for Righter [1994].

Proc. Times	Preempt?	Conditions on f	Result	Source
LR	*No*	*Order specific, increasing*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*CY*
HR	*No*	*Order specific, increasing, supermodular*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*CY*
ST	*No*	*Order specific, increasing, separable*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*CY*
$IHR(DHR)$	*Yes*	*Order specific, increasing, supermodular*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*CY*
$ILR(DLR)$	*Yes*	*Order specific, increasing*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*RS*
LR	*No*	*Job specific, increasing, aggreeable*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*RS(R)*
HR	*No*	*Job specific, increasing, supermodular, aggreeable*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*RS(R)*
ST	*No*	*Job specific, increasing, separable, aggreeable*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*RS*
$IHR(DHR)$	*Yes*	*Job specific, increasing, supermodular, aggreeable*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*R(comment)*
$ILR(DLR)$	*Yes*	*Job specific, increasing, aggreeable*	$E[f_{SEPT}] \leq V(\pi) \leq E[f_{LEPT}]$	*R(comment)*

Table 1. Order conditions on optimal policies.

As noted, these results generally extend to models with breakdowns (see also Hirayama and Kijima [1992]) when the breakdown process does not depend on the job in process. It also applies to arrivals with two classes of jobs (assuming nonidling policies, see Righter [1994] and Chang and Yao [1993]). Our interest here, however is when the conditions in Table 1 are not satisfied. In this way, we derive results that extend the weighted flow time bounds on optimal policies in Birge and Glazebrook [1988]. We begin by considering the order specific objective case.

3. Order Specific Costs.

In general, our approach is to relax the processing time order conditions by constructing an artificial processing time $\bar{X}_{i+1}$ distribution for $i+1$ such that $\bar{X}_{i+1} \leq_{hr} X_{i+1}$ and $\bar{X}_i \leq_{hr} \bar{X}_{i+1}$. If this substitution is consistent with a given order, then we can apply the results in Table 1 and construct bounds on optimality.

One case in which ordered processing times can be constructed is when each processing time has an upper and lower geometric rate. Such conditions were also used for breakdown rates and bounds in weighted completion time problems in Birge and Glazebrook [1988]. For these rates, we assume for processing time X_i, that $\bar{\mu}_i$ is the *upper geometric rate* of X_i with $\bar{\mu}_i = \inf\{\mu > 0 \mid X_\mu \leq_{ST} X_\mu$ where X_μ is geometric with rate μ (i.e., follows a geometric distribution with expectation $1/\mu$) $\}$. The *lower geometric rate* $\underline{\mu}_i$ for X_i is defined such that $\underline{\mu}_i = \sup\{\mu > 0 \mid X_i \leq_{ST} X_\mu$ where X_μ is geometric with rate $\mu\}$.

Assuming that each processing time has an upper and lower geometric rate, we can define two permutations on $1, \ldots, N$, $\bar{\pi}$ and $\underline{\pi}$, that are optimal for the problems with geometric random variables at upper and lower rates, respectively, replacing the original processing times. Let the value with upper rates be $\bar{V}$ and the value with lower rates be $\underline{V}$. An immediate consequence is the following.

Theorem 1. *Suppose each processing time X_i has upper and lower geometric rates as defined above and suppose π^* is an optimal policy for objective, $V(\pi)$, with order specific costs and f increasing, then, for the policies following permutations $\bar{\pi}$ and $\underline{\pi}$ as defined above,*

$$\bar{V}(\bar{\pi}) \leq V(\pi^*) \leq \underline{V}(\underline{\pi}). \tag{3}$$

Proof: To see this result, note that $\bar{V}(\pi) \leq V(\pi)$ and $V(\pi) \leq \underline{V}(\pi)$ for any π. From Table 1, the optimal permutation policies $\bar{\pi}$ and $\underline{\pi}$ exist for $\bar{V}$ and $\underline{V}$ by using orders consistent with the upper and lower rates that then satisy ILR order. Note that $\bar{X}_{\bar{\mu}_i} \leq_{st} X_i$. With standard coupling results, we can define $\tilde{X}_i =_{distribution} \bar{X}_{\bar{\mu}_i}$ and $\tilde{X}_i \leq X_i$ almost surely. Then, it follows that substituting $\tilde{X}_i$ for $\bar{X}_{\bar{\mu}_i}$, $\bar{V}(\bar{\pi}) \leq \bar{V}(\pi^*) \leq V(\pi^*)$. Similarly, $V(\pi^*) \leq V(\underline{\pi}) \leq \underline{V}(\underline{\pi})$. The result follows. ∎

We can in fact strengthen the result in (3) to measure the maximum deviation possible from an optimal nonpreemptive policy to an optimal policy allowing preemption.

Theorem 2. *Suppose each processing time X_i has upper and lower geometric rates as defined above and suppose π^* is an optimal policy for objective, $V(\pi)$, with order specific, separable, and increasing costs , and $\bar{\pi}$ is an optimal policy for $\bar{V}$ as defined above, then*

$$\frac{V(\hat{\pi}) - V(\pi^*)}{V(\pi^*)} \leq \frac{\epsilon}{1-\epsilon}, \tag{4}$$

where $\epsilon = \min[1, \sum_{j=1}^{N} \frac{\bar{\mu}_j - \underline{\mu}_j}{1-\underline{\mu}_j}]$.

Proof. From Theorem 1, assuming the $\bar{\pi}$ order is $1, \ldots, N$, we have:

$$V(\bar{\pi}) - V(\pi^*) \leq V(\bar{\pi}) - \bar{V}(\pi^*) \leq \underline{V}(\bar{\pi}) - \bar{V}(\bar{\pi}) = \sum_{j=1}^{N} E[f_j(\underline{X}_1 + \cdots + \underline{X}_j) - f_j(\bar{X}_1 + \cdots + \bar{X}_j)]. \tag{5}$$

We begin with the first term on the left-hand side in (5). Here we have:

$$\begin{aligned}
E[f_1(\underline{X}_1)] - E[f_1(\bar{X}_1)] &= \sum_{x=1}^{\infty} f_1(x)\{\underline{\mu}_1(1-\underline{\mu}_1)^{x-1} - \bar{\mu}_1(1-\bar{\mu})^{x-1}\} \\
&= \sum_{x=1}^{\infty}[\sum_{y=1}^{x}\{f_1(y) - f_1(y-1)\}]\{\underline{\mu}_1(1-\underline{\mu}_1)^{x-1} - \bar{\mu}_1(1-\bar{\mu})^{x-1}\} \\
&= \sum_{y=1}^{\infty}\{f_1(y) - f_1(y-1)\}\{(1-\underline{\mu}_1)^{y-1} - (1-\bar{\mu})^{y-1}\} \\
&\leq \sum_{y=1}^{\infty}\{f_1(y) - f_1(y-1)\}(1-\underline{\mu}_1)^{y-1}[\frac{\bar{\mu}_1 - \underline{\mu}_1}{1-\underline{\mu}_1}] = [\frac{\bar{\mu}_1 - \underline{\mu}_1}{1-\underline{\mu}_1}]E[f_1(\underline{X}_1)],
\end{aligned} \tag{6}$$

where the last inequality follows since $(1-\underline{\mu}_1)^{y-1} - (1-\bar{\mu})^{y-1} \leq (\bar{\mu}_1 - \underline{\mu}_1)(1-\underline{\mu}_1)^{y-2} = (1-\underline{\mu}_1)^{y-1}[\frac{\bar{\mu}_1-\underline{\mu}_1}{1-\underline{\mu}_1}]$.

Now, we consider the second term where

$$E[f_2(\underline{X}_1+\underline{X}_2)]-E[f_2(\bar{X}_1+\bar{X}_2)]=E[f_2(\underline{X}_1+\underline{X}_2)]-E[f_2(\underline{X}_1+\bar{X}_2)] \\ +E[f_2(\underline{X}_1+\bar{X}_2)]-E[f_2(\bar{X}_1+\bar{X}_2)]. \tag{7}$$

Conditioning on $\underline{X}_1$ and $\bar{X}_2$ respectively, we can proceed as in (6) to show that

$$E[f_2(\underline{X}_1+\underline{X}_2)]-E[f_2(\underline{X}_1+\bar{X}_2)]\leq [\frac{\bar{\mu}_2-\underline{\mu}_2}{1-\underline{\mu}_2}]E[f_2(\underline{X}_1+\underline{X}_2)], \tag{8}$$

and

$$E[f_2(\underline{X}_1+\bar{X}_2)]-E[f_2(\bar{X}_1+\bar{X}_2)]\leq [\frac{\bar{\mu}_1-\underline{\mu}_1}{1-\underline{\mu}_1}]E[f_2(\underline{X}_1+\bar{X}_2)]. \tag{9}$$

Hence, from (7-9),

$$\begin{aligned} E[f_2(\underline{X}_1+\underline{X}_2)]-E[f_2(\bar{X}_1+\bar{X}_2)] &\leq [\frac{\bar{\mu}_2-\underline{\mu}_2}{1-\underline{\mu}_2}]E[f_2(\underline{X}_1+\underline{X}_2)] \\ &+[\frac{\bar{\mu}_1-\underline{\mu}_1}{1-\underline{\mu}_1}]E[f_2(\underline{X}_1+\bar{X}_2)] \\ &\leq \{[\frac{\bar{\mu}_1-\underline{\mu}_1}{1-\underline{\mu}_1}]+[\frac{\bar{\mu}_2-\underline{\mu}_2}{1-\underline{\mu}_2}]\}E[f_2(\underline{X}_1+\underline{X}_2)]. \end{aligned} \tag{10}$$

Repeating the argument for (10) up to job j, we obtain

$$E[f_j(\underline{X}_1+\cdots+\underline{X}_j)]-f_j(\bar{X}_1+\cdots+\bar{X}_j)]\leq \{\sum_{i=1}^{j}[\frac{\bar{\mu}_j-\underline{\mu}_j}{1-\underline{\mu}_j}]\}E[f_j(\underline{X}_1+\cdots+\underline{X}_j)]. \tag{11}$$

Now, we have from (5)

$$V(\bar{\pi})-V(\pi^*)\leq \{\sum_{i=1}^{N}[\frac{\bar{\mu}_j-\underline{\mu}_j}{1-\underline{\mu}_j}]\}\underline{V}(\bar{\pi}). \tag{12}$$

We can then let $\epsilon=\min\{1,\sum_{i=1}^{N}[\frac{\bar{\mu}_j-\underline{\mu}_j}{1-\underline{\mu}_j}]\}$ to obtain

$$V(\bar{\pi})-V(\pi^*)\leq \underline{V}(\bar{\pi})-\bar{V}(\bar{\pi})\leq \epsilon\underline{V}(\bar{\pi}). \tag{13}$$

Noting that we have $\underline{V}(\bar{\pi})\leq(1-\epsilon)^{-1}\bar{V}(\bar{\pi})\leq(1-\epsilon)^{-1}V(\pi^*)$ from (13), we obtain the result. ∎

4. Bounds with Job Specific Costs.

The results of the previous section assume that costs are order specific. We now wish to extend the results to job specific cases. For these results, we must define rates to be consistent with an agreeable objective function order. We assume that the objective function has an agreeable order, $1, \ldots, N$. Then, we define $\tilde{\mu}_i = \max_{j \geq i} \bar{\mu}_j$ for $i = 1, \ldots$. We then have $\tilde{X}_i$ which is geometric with rate $\tilde{\mu}_i$ in IHR order. We suppose $\tilde{V}$ is the objective for these processing distributions and that $\tilde{\pi}$ is a corresponding optimal policy. We then obtain the following.

Theorem 3. *Suppose each processing time X_i has upper and lower geometric rates as defined above and suppose π^* is an optimal policy for objective, $V(\pi)$, with job specific and agreeable costs, then, for the policies following permutations $\tilde{\pi}$ and $\underline{\pi}$ as defined above,*

$$\tilde{V}(\tilde{\pi}) \leq V(\pi^*) \leq \underline{V}(\underline{\pi}). \tag{14}$$

Moreover,

$$\frac{V(\tilde{\pi}) - V(\pi^*)}{V(\pi^*)} \leq \frac{\tilde{\epsilon}}{1 - \tilde{\epsilon}}, \tag{15}$$

where $\tilde{\epsilon} = \min[1, \sum_{j=1}^{N} [\frac{\tilde{\mu}_j - \underline{\mu}_j}{1 - \underline{\mu}_j}]] = \min[1, \sum_{j=1}^{N} [\frac{\tilde{\mu}_j - \bar{\mu}_j}{1 - \underline{\mu}_j} + \frac{\bar{\mu}_j - \underline{\mu}_j}{1 - \underline{\mu}_j}]$.

Proof. This follows exactly as in the proof of Theorem 2 where we note that the $\tilde{\mu}$ values provide the necessary ordering. ∎

Note in (15) that $\tilde{\epsilon}$ has two components, the first for the extent of incorrect ordering and the second for the deviation from geometric times.

5. Bounds for Nonagreeable Costs.

The conditions of the optimal permutations in Theorems 1-3 are fairly restrictive, but they may also be used to obtain bounds on optimal objective values whenever processing times follow a consistent stochastic order or have upper and lower bounding geometric (or, in the continuous case, exponential) distributions. For the following, we first assume an ordering on the processing times and then define an alternate objective that is agreeable.

To define the new objective, suppose f is separable and increasing. We define f^+ by

$$f^+(c_1, \ldots, c_N) = \sum_{i=1}^{N} f_i^+(c_i), \tag{16}$$

where $f^+(c_i) = f_i(c_i) + \delta_i^+(c_i)$ and δ_i^+ is such that f^+ is agreeable. For example, we may define δ_i^+ by

$$\delta_i^+(t) = \sum_1^t \dot{\delta}_i^+(t), \tag{17}$$

and

$$\dot{\delta}_i^+(t) = \sup_{k \geq i}\{\dot{f}_k(t) - \dot{f}_i(t), 0\}, \tag{18}$$

where $\dot{f}(t) = f(t) - f(t-1)$ (or df/dt in the continuous case with an integral replacing the sum in (17)). Note that $\dot{\delta}_i^+$ is nonnegative so that δ_i^+ is increasing for all i and that δ_i^+ is identically zero whenever costs are agreeable.

Similarly, we can define f^- as in (16) with $f^-(c_i) = f_i(c_i) + \delta_i^-(c_i)$, $\delta_i^-(t) = \sum_1^t \dot{\delta}_i^-(t)$ and $\dot{\delta}_i^-(t) = -\sup_{k \leq i}\{\dot{f}_i(t) - \dot{f}_k(t), 0\}$. In this case, we again have δ_i^- identically zero whenever costs are agreeable.

As an example, for the case of expected weighted tardiness objectives, $f_i(t) = w_i(t-d_i)^+$, where d_i is a due date for job i. Suppose that the processing times are ILR, we then have

$$\begin{aligned}\dot{\delta}_i^-(t) &= -\sup_{k \leq i}\{\dot{f}_i(t) - \dot{f}_k(t), 0\} \\ &= -sup\{w_i 1_{t \geq d_i, k<i:t<d_k}, sup_{k \leq i, t \geq d_k; t \geq d_i}\{w_i - w_k\}, 0 1_{t<d_i}\}. \\ &= -w_i 1_{t \geq d_i, k<i:t<d_k} + sup_{k \leq i, t \geq d_k; t \geq d_i}\{w_i - w_k\} 1_{t \geq d_k, k \leq i}\end{aligned} \tag{19}$$

We, therefore, have that $\delta_i^- = -w_i(t-d_i)1_{t \geq d_i, k<i:t<d_k} - sup_{k \leq i, t \geq d_k; t \geq d_i}\{w_i - w_k\}(t - sup_{k \leq i} d_k)1_{t \geq d_k, k \geq i}$. If we define a new set of weights, w_i^-, $i = 1, \ldots, N$, recursively by $w_1^- = w_1$, $w_{i+1}^- = \min\{w_i^-, w_{i+1}\}$, $i = 1, \ldots, N-1$ and a new set of due dates, d_i^-, $i = 1, \ldots, N$, by $d_1^- = d_1$, $d_{i+1}^- = \max\{d_i^-, d_{i+1}\}$, $i = 1, \ldots, N-1$, then $f_i^-(t) = w_i^-(t - d_i^-)^+$. Here, if, for example, $w_1 = 3, w_2 = 2, w_1 = 1$, and $d_1 = 3, d_2 = 2, d_1 = 1$, then, for f^-, we have $w_1^- = 3, w_2^- = 2, w_1^- = 1$ and $d_1^- = 3, d_2^- = 3, d_1^- = 3$. On the other hand, if the weights are, $w_1 = 1, w_2 = 2, w_1 = 3$, then $w_1^- = w_2^- = w_1^- = 3$.

Suppose the objective value of any policy π with $f^{+(-)}$ replacing f is $V^{+(-)}(\pi)$. This provides bounds on the optimal value of $V(\pi)$. Combined with the result in Theorem 3, we can then define $\tilde{V}^-(\pi)$ as the objective value under policy π for using distributions with the upper geometric rates, $\tilde{\mu}$, and the objective f^-. We can also define $\underline{V}^+(\pi)$ as the objective value under policy π for using distributions with the lower geometric rates, $\underline{\mu}$, and the objective f^+. We then have a general bound whenever processing times about upper and lower geometric rates.

Theorem 4. *Suppose each processing time X_i has upper and lower geometric rates, f is job specific, separable and increasing, π^* is an optimal policy for objective, $V(\pi)$, and $\pi^{+(-)}$ is an optimal policy for $V^{+(-)}$ with objective $f^{+(-)}$ as defined above, then*

$$\tilde{V}^-(\pi^{+(-)}) \le V(\pi^*) \le V(\pi^{+(-)}) \le \underline{V}^+(\pi^+). \tag{20}$$

Proof. By construction, π^- is optimal for the objective $\tilde{V}^{+(-)}$. This policy in fact correspond to SEPT (under the $\tilde{\mu}$ order) according to the result in Table 1 and our assumptions. For any policy π, we have

$$\begin{aligned} V(\pi^*) = E_{\pi^*}[f(c_1,\ldots,c_N)] &\le E_\pi[f(c_1,\ldots,c_N)] \\ &\le E_\pi[f^+(c_1,\ldots,c_N)] \\ &= E_\pi[f(c_1,\ldots,c_N)] + E_\pi[\sum_{i=1}^N \delta_i^+(c_i)]. \end{aligned} \tag{21}$$

For $\pi = \pi^{+/-}$, we have the middle inequality of (20). We also have

$$\begin{aligned} V(\pi^*) &\ge E_{\pi^*}[f(c_1,\ldots,c_N)] + E_{\pi^*}[\sum_{i=1}^N \delta_i^-(c_i)] \\ &\ge \tilde{V}^-(\pi^*) \\ &\ge \tilde{V}^-(\pi^-). \end{aligned} \tag{22}$$

∎

Since the differences in (20) depend on all the costs, we do not have as readily available comparisons as in Theorems 2 and 3. We have an additional bound possible using the increasing property of the δ_i^+ functions. If we apply the same process as in the construction of f^+ to obtain δ^{++} from δ^+ which is agreeable. Then, for any policy π under the conditions of Theorem 4, we have $E_\pi[\sum_{i=1}^N \delta_i^{++}(c_i)] \le E_\pi[\sum_{i=1}^N \delta_i^{++}(c_i(\pi))] \le E_{LEPT}[\sum_{i=1}^N \delta_i^{++}(c_i)]$. In this case,

$$V(\pi^*) = V^+(\pi^*) - E_{\pi^*}[\sum_{i=1}^N \delta_i^+(c_i)] \ge \underline{V}^+(\pi^\pm) - E_{LEPT}[\sum_{i=1}^N \delta_i^{++}(c_i)], \tag{23}$$

where, again the last term vanishes when costs are agreeable.

6. Optimality of Nonpreemptive Policies with Breakdowns.

Other results may also be possible with specific distributions when costs are not agreeable. It may be advantageous, for example, to identify when nonpreemptive policies are actually optimal within the class of preemptive policies. We have seen this result for ILR and IHR families. The next result shows that it is extendable to arbitrary deterministic processing times with increasing separable convex costs and a breakdown process. In the following, we assume integral processing times, p_i, $i = 1, \dots, N$, and suppose that the class of preemptive policies is *stationary*, meaning that decisions may depend on the completed processing of each job but cannot depend explicitly on the time or breakdown process.

Theorem 5. *Suppose that $X_i = p_i$ (deterministic), completion times are affected by a random breakdown process which is independent of the job order and has finite expected number and downtimes in any fixed interval, and f is separable, increasing and convex, then there exists an optimal policy within the class of stationary preemptive policies which is nonpreemptive.*

Proof: Suppose, without loss of generality, that an optimal preemptive policy $\hat{\pi}$ completes jobs in the order, $1, 2, \dots, N$. By our stationarity assumption and deterministic processing times, this order does not depend on the breakdown process. Thus, the completion order remains deterministic.

Suppose job i consists of p_i separate unit length jobs labelled, $i1, i2, \dots, ip_i$. We define a new cost function, f', on job $ij, 1 \leq j \leq p_i$, by

$$f'_{ij}(t) = \sum_{k \geq i} \{f_k(t) - f_k(t-1)\}. \tag{24}$$

Let $V'(\pi)$ be the value for the process defined with objective f' in (24) and some policy π. We then have

$$\begin{aligned} V'(\pi) &= E_\pi[\sum_{i=1}^{N} \sum_{j=1}^{p_i} f'_{ij}(c_{ij})] \\ &\geq E_\pi[\sum_{j=1}^{N} f_j(\max[c_1, \dots, c_j])] \\ &\geq E_\pi[\sum_{j=1}^{N} f_j(c_j)]. \end{aligned} \tag{25}$$

We can show that $\hat{\pi}$ is in fact optimal for V'. For the jobs in the process for V', consider the lexicographic order on $\{ij\}$. We have $f'_{jl}(t) - f'_{jk}(t) = 0$ for any k, l. Also, whenever $j < k$, we have

$f'_{jl}(t) - f'_{km}(t) = \sum_{i=j}^{k-1}(f_i(t) - f_i(t-1))$, which is increasing by the convexity of f. Therefore, by Theorem 2.2 in Birge et al. [1990], the lexicographic ordering, $11, 12, \ldots, 1p_1, 21, \ldots, 2p_2, \ldots, N1, \ldots, Np_N$, is an optimal policy for V'. This policy in fact corresponds to the nonpreemptive policy $\tilde{\pi}$ to process in the order $1, 2, \ldots, N$. From (25),

$$
\begin{aligned}
V(\hat{\pi}) &= E_{\hat{\pi}}[\sum_{j=1}^{N} f_j(c_j)] \\
&= E_{\hat{\pi}}[\sum_{j=1}^{N} f_j(\max[c_1, \ldots, c_j])] \\
&\geq E_{\tilde{\pi}}[\sum_{j=1}^{N} f_j(\max[c_1, \ldots, c_j])] \\
&\geq E_{\tilde{\pi}}[\sum_{j=1}^{N} f_j(c_j)] = V(\tilde{\pi}).
\end{aligned}
\tag{26}
$$

Thus, the nonpreemptive sequence is optimal. ∎

We can extend the result in Theorem 5 to stochastic processing times if $X_i = \sum_{j=1}^{p(i)} Y_j$, where the Y_j are independent and identically distributed. In this way, we can replace the time $t-1$ with the time of the last completion before t and apply the approach in Righter [1994] to show that interchanges cannot improve the objective for V' as in the proof of Theorem 5.

7. Conclusions.

We have presented results that provide bounds on the value of optimal policies in stochastic scheduling. The bounds measure the extent to which distributions and cost functions vary from the orderings that produce optimality for simple orders. These bounds may be used in heuristic or optimum seeking algorithms to eliminate suboptimal strategies or provide ε-optimal results.

References.

1. Birge, J.R., Frenk, J.B.G, Mittenthal, J., and Rinnooy Kan, A.H.G., "Single-Machine Scheduling Subject to Stochastic Breakdown," *Naval Research Logistics* 37 (1990) 661-677.
2. Birge, J.R., and Glazebrook, K.D., "Assessing the Effects of Machine Breakdowns in Stochastic Scheduling," *OR Letters* 6 (1988) 267-271.

3. Chang, C.-S., and Yao, D.D., "Rearrangement, Majorization, and Stochastic Scheduling," *Mathematics of Operations Research* 18 (1993) 658-684.

4. Gittens, J.C., "Bandit Processes and Dynamic Allocation Indices," *J. Royal Statistical Society, Series B* 41 (1979) 148-177.

5. Glazebrook, K.D., "Stochastic Scheduling with Order Constraints," *International Journal of Systems Science* 7 (1976) 657-666.

6. Glazebrook, K.D., "On Stochastic Scheduling Problems with Due Dates," *Int. J. Systems Sci.* 14 (1983) 1259-1271.

7. Glazebrook, K.D., "Scheduling Stochastic Jobs on a Single Machine Subject to Breakdowns," *Naval Res. Logist. Qualt.* 31 (1984) 251-264.

8. Glazebrook, K.D., and Gittins, J.C., "On Single-Machine Scheduling with Precedence Relations and Linear or Discounted Costs," *Operations Research* 29 (1981)161-173.

9. Hirayama, T., and Kijima, M., "Single machine scheduling problem when the machine capacity varies stochastically," *Operations Research* 40 (1992) 376-383.

10. Righter, R.D., "Scheduling," Chapter 13 in M. Shaked and J.G. Shantikumar, *Stochastic Orders and their Applications,* Academic Press, San Diego, 1994.

11. Righter, R.D., and Shantikumar, J.G., "Extremal properties of the FIFO discipline in queueing networks," *J. Applied Probability* 29 (1992) 967-978.

12. Shaked, M., and Shantikumar, J.G., *Stochastic Orders and their Applications,* Academic Press, San Diego, 1994.