Bounded and Patched Solutions for Boundary Value Problems

STUART W. CHURCHILL

The University of Michigan, Ann Arbor, Michigan

The technique of bounding, that is of constructing general solutions for all ranges of the variables and parameters from solutions for a few limiting and special cases, is outlined and illustrated for problems in conduction and convection. The technique of patching, that is of constructing approximate solutions for multiphase media by the combination of pseudo solutions for the different regions, is also outlined and illustrated for problems in conduction and freezing. The shortcomings and possible inaccuracies of these two techniques are indicated as well.

The engineer is increasingly able to describe a physical process in terms of a purely mathematical problem. This mathematical problem may then be solved exactly by analytical and numerical methods or by approximated methods. The validity of a numerical method must be established, and extensive computation may be necessary for many combinations of the parameters and variables. An analytical solution may also require the extensive use of a computing machine to obtain numerical values. The need for approximate solutions which substitute for, extend, or guide exact solutions has thus not been eliminated by the increasing analytical and computational ability of the engineer. The choice of methods is often a matter of economics, convenience, and utility rather than of mathematical capacity.

The objective of this paper is to illustrate the development and value of approximate solutions. Two general techniques are utilized: bounding, the development of complete solutions from the solutions for limiting and special cases, and patching, the development of solutions for multiphase media by combining the pseudo solutions for the different regions. Problems for which an exact solution is available have been chosen for several illustrations in order to provide a measure of the accuracy of the approximations.

The techniques described herein provide an alternative and supplement to other recognized methods of approximation (*1, 2*). For some problems they may provide a simpler, more accurate, and/or more general solution.

CONSTRUCTION OF GENERAL SOLUTIONS BY BOUNDING

Solutions for limiting conditions have of course been used extensively for convenience or in lieu of a complete solution. The construction of a complete solution from limiting solutions alone has apparently not been fully exploited. The methods, possibilities, advantages, and limitations of this procedure are illustrated in the following examples.

Example No. 1

The analytical solution for the heat flux density from a semi-infinite region to an insulated surface with the surface at constant temperature and the insulation and region at uniform initial temperature is

$$\frac{q\delta}{k'(T_o - T_s)} = \frac{\delta^2}{\pi K't}\left[1 + 2\sum_{n=1}^{\infty} \alpha^n e^{-n^2\delta^2/K't}\right] \quad (1)$$

where

$$\alpha = \frac{\sigma - 1}{\sigma + 1} \quad \text{and} \quad \sigma = \sqrt{\frac{k\rho C}{k'\rho' C'}}$$

For large (practical) values of α the series converges very slowly.

Simpler solutions can be derived for several limiting conditions: infinitely thick insulation, infinite conductivity in the semi-infinite region, insulation with negligible heat capacity, and no insulation. These solutions may be summarized as follows:

Condition	Solution for $q\delta/k'(T_o - T_s)$	
$\delta \to \infty$	$\sqrt{\delta^2/\pi K't}$	(2)
$k \to \infty$	$\sqrt{\delta^2/\pi K't}\left[1 + 2\sum_{n=1}^{\infty} e^{-n^2\delta^2/K't}\right]$	(3*a*)
$k \to \infty$	$1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 K't/\delta^2}$	(3*b*)
$C' \to 0$	$e^{K't/\sigma^2\delta^2}\,\mathrm{erfc}\sqrt{K't/\delta^2\sigma^2}$	(4)
$\delta \to 0$	$\sqrt{\sigma^2\delta^2/\pi K't}$	(5)

Two expressions are given for infinite conductivity, since Equation (3*a*) converges rapidly only for $K't/\delta^2 << 1$ and (3*b*) only for $K't/\delta^2 >> 1$.

From both mathematical analysis and physical reasoning it is apparent that the heat flux density must be greater than that given by Equations (2) and (4) and less than that given by Equations (3) and (5). Thus the exact solution must lie in the shaded region in Figure 1. For clarity the shaded region was greatly exaggerated in Figure 1 by slightly misplotting the curves of Equations (3) and (4). The curves are correctly plotted in Figure 2. The location of the solution is further defined within the shaded region by noting that the deviation of Equation (4) from the exact solution must decrease with increasing time. Since the deviation is negligible for intermediate times ($\pi Kt/\delta^2 = 10$), the deviation will be completely negligible for longer times. Similarly the deviation of

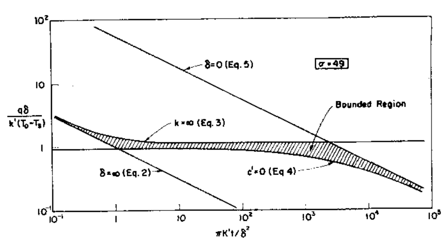

Fig. 1. Bounding solutions for an insulated slab. (The shaded region is exaggerated.)

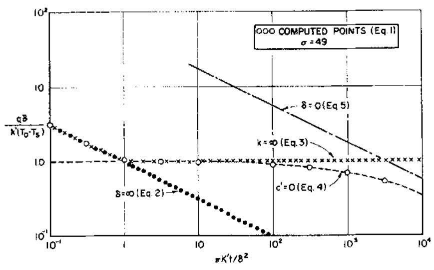

Fig. 2. Comparison of computed and bounded solution for an insulated slab.

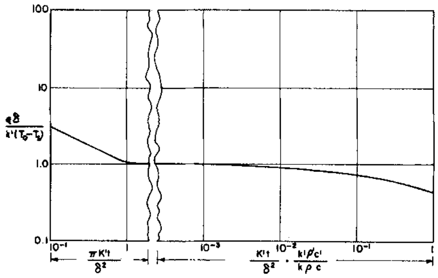

Fig. 3. Generalized solution for an insulated slab.

Equations (2) and (3) from the exact solution must decrease with decreasing time. Since the deviation is small for $\pi K't/\delta^2 = 1.0$, the deviation will be negligible for shorter times. Thus Equations (3*a*) and (3b) [or the simpler Equation (2)] constitute a satisfactory approximation for $\pi K't/\delta^2 < 1$ and Equation (4) for $\pi K't/\delta^2 > 1$. Equation (5) becomes a useful approximation only for very large values of $\pi K't/\delta^2$.

The validity of the approximations has thus been established without recourse to the exact solution. This analysis is, however, confirmed by comparing in Figure 2 the values computed from Equation (1) for $\sigma = 49$ with the approximations.

Whereas extensive calculations must be repeated for each value of σ when Equation (1) is used, satisfactory values for all values of σ can be obtained directly from Equation (2) [or (3*a*) and (3*b*)] and Equation (4). These approximate solutions are more useful and under some conditions may even yield more accurate values for a given amount of computation than the exact solution.

It is further apparent from the approximate solutions that $q\delta/k'(T_o - T_s)$ is a function only of $K't/\delta^2$ for $K't/\delta^2$ less than unity and only of $K't/\delta^2\sigma^2$ for $K't/\delta^2$ greater than unity. Hence a completely general plot for all times and parametric values can be constructed with a split abscissa as illustrated in Figure 3.

Approximate solutions can similarly be constructed for the transient temperature field in the insulation and semi-infinite region.

Example No. 2

An approximate solution can similarly be constructed for the heat flux density from the infinite region outside an insulated isothermal sphere. A general analytical solution has been derived (3) but does not appear to have any practicality. Again simple solutions can be derived for infinitely thick insulation, infinite conductivity in the infinite region, insulation with negligible heat capacity, and no insulation. The solutions may be summarized as follows:

Condition

Solution for $\dfrac{\delta q}{k'(T_o - T_s)} - \dfrac{\delta}{a}$

$\delta \to \infty$ $\quad \sqrt{\delta^2/\pi K't}$ (6)

$k \to \infty$ $\quad \sqrt{\dfrac{\delta^2}{\pi K't}}\left[1 + 2\sum_{n=1}^{\infty} e^{-n^2\delta^2/K't}\right]$ (7*a*)

$k \to \infty$ $\quad 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 K't/\delta^2}$ (7*b*)

$C' \to 0$ $\quad 1 - \dfrac{b}{a(B+1)}$

$$\times \left[1 - \Phi\left(\frac{a(B+1)}{b}\sqrt{\frac{K't}{\delta^2\sigma^2}}\right)\right] \quad (8)$$

$\delta \to 0$ $\quad B - \dfrac{\delta}{a} + \sqrt{\dfrac{\delta^2\sigma^2}{\pi K't}}$ (9)

where $B = \dfrac{k\delta}{k'a}$ and $\Phi(z) = e^{z^2}\,\mathrm{erfc}\,z$.

These solutions are plotted in Figure 4. The asymptotic solutions for long times are also indicated. Analysis indicates that the exact heat flux must be greater than that given by Equations (6) and (8) and less than that given by Equations (7*a*), (7*b*), and (9). Furthermore, Equations (6) [or (7*a*) and (7*b*)] for short times and Equation (8) for intermediate and long times provide a complete and sufficiently accurate approximation. Equation (9) does not contribute to the construction of the approximate solution. A generalized plot for all times and parametric values is shown in Figure 5.

A similar approximation has been developed for conduction in the infinite region outside an insulated cylinder, although the limiting solutions are more complicated (4).

Example No. 3

Transient free convection to a vertical plate provides an example of the limitations of the method of bounding. The steady state solution for the heat transfer coefficient with $\nu/K = 0.733$ is

$$\frac{h}{k}\left[\frac{\nu^2 x}{g\beta\Delta T}\right]^{1/4} = 0.37 \quad (10)$$

For short times, before the motion develops, the solution for conduction to a semi-infinite region should hold. This solution can be rearranged in the form of Equation (10) as follows:

$$\frac{h}{k}\left[\frac{\nu^2 x}{g\beta\Delta T}\right]^{1/4} = \left[t\left(\frac{g\beta\Delta T}{x}\right)^{1/2}\right]^{-1/2}\left[\frac{\nu/K}{\pi}\right]^{1/2} \quad (11)$$

An approximate solution might be sketched between these two limiting solutions as indicated in Figure 6.

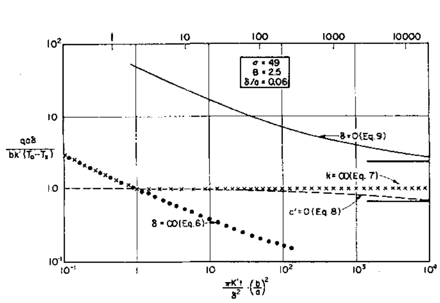

Fig. 4. Bounding solutions for an insulated sphere.

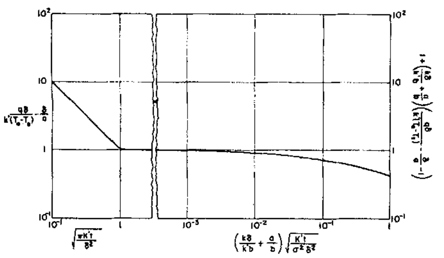

Fig. 5. Generalized solution for an insulated sphere.

However, the measurements of Gebhart (5) and Klei [as discussed by Siegel (6)] suggest that the heat transfer coefficient for free convection may go through a minimum before approaching the steady state. The analysis of Bosworth (7) for electrically heated wires indicated that this minimum is only the first swing of a dampened oscillation. The complete transient solution computed by Hellums and Churchill (8) is included in Figure 6 for comparison with the interpolated solution.

Thus the limiting solutions for short and long times may be an insufficient guide to the fine structure of the solution even though they provide a reasonable guide to the overall behavior.

CONSTRUCTION OF APPROXIMATE SOLUTIONS BY PATCHING

Approximate solutions may be constructed for problems involving multiple regions by combining pseudo solutions for the individual regions which would exist in the absence of coupling between the regions. This procedure has also been utilized previously but does not appear to have received much formal attention. The possibilities, accuracy, and limitations of the method are illustrated in the following examples.

Example No. 4

The problem of Example No. 1 will again be considered. The solution for the semi-infinite region will be approximated by the transient solution for a semi-infinite region with pseudo constant interface temperature T_i, and the solution for the insulation will be approximated by the transient solution for a slab with a pseudo constant interface temperature T_i. The two corresponding expressions for the heat flux density at the interface are then equated:

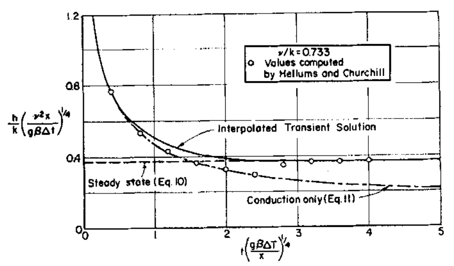

Fig. 6. Comparison of computed and limiting solutions for free convection to a vertical plate.

$$q_i = \frac{k[T_o - T_i]}{\sqrt{\pi K t}} = \frac{k'[T_i - T_s]}{\delta} f_i\left\{\frac{\delta^2}{K't}\right\} \quad (12)$$

where

$$f_i\{z\} = 1 - 2\sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2/z} \quad \text{(for small } z) \quad (13a)$$

$$= 2\sqrt{\frac{z}{\pi}}\sum_{n=1}^{\infty} e^{-n^2 z/4} \quad \text{(for large } z) \quad (13b)$$

Eliminating T_i and rearranging Equation (13) one gets

$$\frac{q_i\,\delta}{k'(T_o - T_s)} = \frac{1}{\dfrac{1}{f_i\left\{\dfrac{\delta^2}{K't}\right\}} + \sqrt{\dfrac{\pi K't}{\delta^2\sigma^2}}} \quad (14)$$

The heat flux at the outer surface of the insulation is then equal to the flux at the inner surface [Equation (14)] times the ratio of the flux at the outer and inner surface:

$$\frac{q\delta}{k'(T_o - T_s)} = \frac{f_s\left\{\dfrac{\delta^2}{K't}\right\}}{f_i\left\{\dfrac{\delta^2}{K't}\right\}}\left[\frac{1}{\dfrac{1}{f_i\left\{\dfrac{\delta^2}{K't}\right\}} + \sqrt{\dfrac{\pi K't}{\delta^2\sigma^2}}}\right] \quad (15)$$

where $f_s\,\{z\}$ is the expression given by Equations (3*a*) and (3*b*).

If the heat capacity of the insulation is neglected, the functions in Equation (15) both become unity, and the following simpler approximation is obtained:

$$\frac{q\delta}{k'(T_o - T_s)} = \frac{1}{1 + \dfrac{\pi K't}{\delta^2\sigma^2}} \quad (16)$$

Equations (1) and (14) can be seen to approach Equation (16) as $t \to \infty$. Examination also indicates that Equations (1) and (15) approach each other as $t \to 0$. Equations (15), (16), and (1) are compared over the full range of time in Figure 7. (In this plot σ is a significant parameter only for $\pi K't/\delta^2\sigma^2 < 10^{-3}$.) The agreement between (15) and (1) is excellent for $\pi K't/\delta^2\sigma^2 < 10^{-3}$, fair for intermediate values, and good again for large values. Equation (16) differs negligibly from Equation (15) for $\pi K't/\delta^2\sigma^2 > 10^{-3}$.

Patched solutions can be developed in the same way for insulated spheres and cylinders and for other boundary or initial conditions.

Example No. 5

The motion of a freezing front in a semi-infinite region with a uniform initial temperature above the freezing point and a constant surface temperature below the freez-

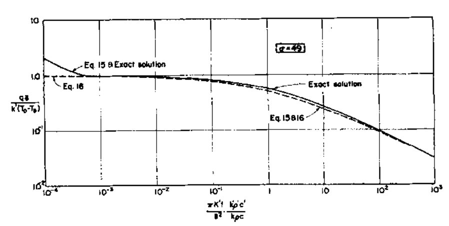

Fig. 7. Comparison of exact and patched solutions for an insulated slab.

ing point provides a nonlinear problem for which an analytical solution exists (9). Patched solutions will be derived on two different bases and examined.

An energy balance over the freezing front can be written formally as

$$\rho L\left(\frac{dx_f}{dt}\right) = q_f - q_u \tag{17}$$

If q_f is approximated by the transient solution for a slab with uniform initial temperature, constant surface temperatures, and a pseudo constant thickness, and q_u by the transient solution for a semi-infinite region with a uniform initial temperature and a constant surface temperature at a pseudo-fixed position, the following differential equation is obtained:

$$\rho L\left(\frac{dx_f}{dt}\right) \cong \frac{k_f(T_f - T_s)}{x_f} f_i\left\{\frac{x_f^2}{K't}\right\} - \frac{k(T_o - T_f)}{\sqrt{\pi K t}} \tag{18}$$

The solution of Equation (18) is

$$x_f = 2\lambda\sqrt{K_f t} \tag{19}$$

where λ is the solution of the algebraic equation

$$\lambda^2 + VR\,\lambda - \frac{R}{2} f_i\{4\lambda^2\} = 0 \tag{20}$$

with

$$V = \sqrt{\frac{K_f}{\pi K}}\left(\frac{k}{k_f}\right)\left(\frac{T_o - T_f}{T_f - T_s}\right) \text{ and } R = \frac{C_f(T_f - T_s)}{L}$$

If $\lambda << \frac{1}{2}$, $f_i\{4\lambda^2\}$ approaches unity, and

$$\frac{x_f}{2\sqrt{K_f t}} = \lambda = \frac{VR}{2}\left(\sqrt{1 + \frac{2}{RV^2}} - 1\right) \tag{21}$$

The exact solution of the freezing problem has the form of Equation (19), but the dependence of the values of λ on the parameters is given by the transcendental equation

$$\frac{R\,e^{-\lambda^2}}{\sqrt{\pi}\,\lambda\,\mathrm{erfc}\,\lambda} - \frac{VR\,e^{-\lambda^2}K_f/K}{\lambda\,\mathrm{erfc}\,(\lambda\sqrt{K_f/K})} - 1 = 0 \tag{22}$$

The values of λ, and hence of $x_f/2\sqrt{K_f t}$, obtained from Equations (20) and (21) are compared with the exact solution in Figures 8 and 9 for the case of $K = K_f$. The accuracy of the approximations is strongly dependent on the values of the parameters $k(T_o - T_f)/k_f(T_f - T_s)$ and $L/C_f(T_f - T_s)$. Surprisingly, Equation (21) is a better approximation than (20). The error is somewhat exaggerated by plotting x_f^2 rather than x_f.

An alternative and somewhat better patched solution is obtained by assuming a pseudo constant velocity for the freezing front. The steady state temperature distribution is a slab of frozen material moving at uniform velocity u with the plane $x = 0$ maintained at T_f, and the plane $x = -x_f$ at T_s is

$$\frac{T_f - T}{T_f - T_s} = \frac{1 - e^{-xu/K_f}}{1 - e^{-x_f u/K_f}} \tag{23}$$

The steady state temperature distribution in a semi-infinite (unfrozen) region moving at velocity u with the plane $x = 0$ maintained at T_f and with $T \to T_o$ as $x \to \infty$ is

$$\frac{T_o - T}{T_o - T_f} = e^{-xu/K} \tag{24}$$

Substituting Equations (23) and (24) in the following energy balance at the freezing plane

$$\rho L u = k_f\left(\frac{dt}{dx}\right)_{0^-} - k\left(\frac{dt}{dx}\right)_{0^+} \tag{25}$$

and rearranging one obtains

$$\frac{ux_f}{K_f} = \ln\left[1 + \frac{R}{1 + \sqrt{\pi K_f/K}\,VR}\right] \tag{26}$$

Substituting dx_f/dt for u and rearranging one gets

$$\frac{x_f^2}{2K_f t} = \ln\left[1 + \frac{R}{1 + \sqrt{\pi K_f/K}\,VR}\right] \tag{27}$$

Equation (27) has the same form as the exact solution and thus indicates the following approximate expression for the constant:

$$\lambda^2 = \ln\left[1 + \frac{R}{1 + \sqrt{\pi K_f/K}\,VR}\right]^{1/2} \tag{28}$$

Equation (28) is compared with the exact solution for $K = K_f$ in Figures 8 and 9. Equation (28) is symmetrical for $L/C_f(T_f - T_s)$ and $k(T_o - T_f)/k_f(T_f - T_s)$ in this case. Equation (28) appears to be a better approximation than Equation (20) for small values of $k(T_o - T_f)/k_f(T_f - T_s)$ and $L/C_f(T_f - T_s)$.

The above approximations are limited in direct value because of the availability of an exact solution. However, the same techniques are readily utilized to develop corresponding approximations for problems for which a complete analytical solution has not yet been obtained. Examples are freezing in cylindrical and spherical geometries and freezing over a temperature range. The accuracy and

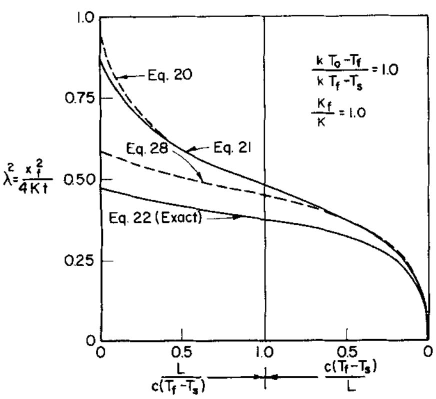

Fig. 8. Effect of $L/C_f(T_f - T_s)$ on patched solutions.

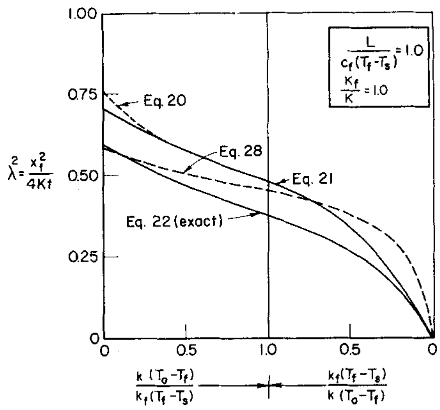

Fig. 9. Effect of $k(T_o - T_f)/k_f(T_f - T_s)$.

range of applicability of these approximations can be estimated from the results illustrated in Figures 8 and 9.

NOTATION

a = radius of sphere
B = $k\delta/k'a$
C = heat capacity
$f_i\{z\}$ = function of z defined by Equations (13a) and (13b)
$f_s\{z\}$ = function of z defined by Equations (3a) and (3b)
g = acceleration due to gravity
h = heat transfer coefficient
k = thermal conductivity
K = thermal diffusivity = $k/\rho C$
L = latent heat of solidification
n = any integer
R = $C_f(T_f - T_s)/L$
q = heat flux density
t = time
T = temperature
u = velocity
V = $k\sqrt{K_f}(T_o - T_f)/k_f\sqrt{\pi K}(T_f - T_s)$
x = distance
ΔT = $T_s - T_o$

Greek Letters

α = $(\sigma - 1)/(\sigma + 1)$
β = volumetric coefficient of expansion
ν = kinematic viscosity
δ = thickness of insulation
λ = coefficient defined by Equation (22)
σ = $\sqrt{k\rho C/k'\rho' C'}$
$\Phi(x)$ = e^{z^2} erfc z
ρ = density

Subscripts

i = interface
f = of frozen materials, or freezing front
o = initial
s = surface
u = of unfrozen material

Superscripts

$(')$ = insulation

LITERATURE CITED

1. Costello, A. F., *Am. Soc. Mech. Engrs. Paper No. 63HT-44* (1963).
2. Churchill, S. W., *A.I.Ch.E. Journal*, **3**, 289 (1957).
3. Chen, J.-M., and S. W. Churchill, "Transient Heat Transfer from a Composite Sphere," unpublished report, Univ. of Michigan, Ann Arbor, Michigan.
4. Churchill, S. W., "Conduction to an Insulated Cylinder from a Semi-Infinite Region," under review.
5. Gebhart, Benjamin, *J. Heat Transfer*, **83**, Series C, No. 1, 61 (1961).
6. Siegel, Robert, *Trans. Am. Soc. Mech. Engrs.*, **80**, 347 (1958).
7. Bosworth, R. C. L., *Australian J. Phys.*, **13**, 84 (1960).
8. Hellums, J. D., and S. W. Churchill, *A.I.Ch.E. Journal*, **8**, 690 (1962).
9. Carslaw, H. S., and J. C. Jaeger, "Conduction of Heat in Solids," Ind Ed., Oxford at the Clarendon Press, England (1959).

Manuscript received August 28, 1964; revision received November 25, 1964; Paper accepted November 30, 1964. Paper presented at A.I.Ch.E. Buffalo meeting.

Significance of Pressure Gradients in Porous Materials:

Part I. Diffusion and Flow in Fine Capillaries

NORIAKI WAKAO, SEIYA OTANI, and J. M. SMITH
University of California, Davis, California

Equations are derived for the effect of a total pressure gradient on flow and diffusion of a binary gas system in a fine capillary. The results apply over the pressure and pore radius regimes from Knudsen flow to Poiseuille flow.

Application of the equations to a pure component leads to an expression for the permeability which predicts the observed minimum flow in the slip-flow region. This expression agrees well with Knudsen's data for carbon dioxide and for new measurements for nitrogen in a glass capillary (radius = 0.01244 cm.).

Gaseous diffusion in porous media at constant pressure has been explained effectively in recent years (*2, 7, 8*). The development applies when both Knudsen and bulk diffusion are significant, a situation of common occurrence in porous catalysts. In some applications diffusion with a pressure gradient is also of interest. For example, in a gas-solid catalytic reaction for which there is a change in number of moles, a pressure difference will exist along the catalyst pores. Evans, Watson, and Mason (*3*) have used a dusty-gas model successfully to treat the diffusion problem with variable pressure. These authors also proposed two methods of relating the flow to the pressure gradient: an extension of the dusty-gas model, and application of the equation of motion.

The purpose of Part I of this paper is to show that an equation with no disposable constants can be derived for

Noriaki Wakao is with the University of Tokyo, Tokyo, Japan.