Freezing Front Motion and Heat Transfer Outside an Infinite, Isothermal Cylinder

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A numerical method is developed for calculation of the temperature field in a material that freezes outside an isothermal cylinder. Numerical values are presented for the freezing front location and for the rate of heat transfer to the cylinder. The computed values are compared with the analytical solutions for the limiting cases of zero latent heat, negligible heat capacity, and freezing adjacent to a flat plate. The results have many applications, including the freezing of wet soil outside underground pipes or storage tanks for cryogenic fluids.

This paper is concerned with the transient location of a freezing front in the region outside an infinitely long circular cylinder. The surroundings are assumed to be initially at a uniform temperature T_0 above the freezing temperature T_F. The surface of the cylinder is suddenly reduced to, and maintained at, a constant temperature T_w below T_F. The frozen and unfrozen regions are assumed to have constant, but different, heat capacities and thermal conductivities. The density is assumed to be the same in both phases. Heat transfer is by conduction only, and the latent heat is all absorbed at T_F.

This problem has many practical applications, including the freezing of wet soil outside underground pipes or storage tanks for cryogenic fluids. In many of these applications the length-to-diameter ratio is large enough and the surroundings are of sufficient extent so that the above idealized problem is a useful approximation. Water migration to the freezing front occurs to some extent in wet soils, but the limited available data for freezing in other geometries do not indicate that this effect is ordinarily significant. Natural convection due to density gradients behavior is not apparent from these data. Unpublished calculations indicate that a freezing range (due to capillaries in the soil or change of composition on freezing) has little effect on the rate of freezing or heat transfer for most practical conditions. These three complexities are accordingly neglected.

Analytical solutions have been derived for the cylindrical problem without freezing and for freezing adjacent to an isothermal flat plate (1). An analytical solution has apparently not been derived for the problem considered herein. Such a solution would probably be so complex as to be of little use.

Murray and Landis (2) developed a successful method for the numerical solution of freezing adjacent to a flat plate. Springer and Olson (3, 4), Teller and Churchill (5), and Seider and Churchill (6) have used modifications of this method for other freezing problems. This same general method is used herein.

Although the entire transient temperature field was computed, only the freezing front location and the heat flux density at the isothermal surface are presented herein.

MATHEMATICAL FORMULATION

The problem can be stated as follows:

$$\frac{\partial u}{\partial t} = \frac{K_S}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right), \quad a \leqq r \leqq r_F \tag{1}$$

$$\frac{\partial v}{\partial t} = \frac{K_L}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r} \right), \quad r \geqq r_F \tag{2}$$

$$\frac{dr_F}{dt} = \frac{1}{\rho L}\left[k_S \frac{\partial u}{\partial r} - k_L \frac{\partial v}{\partial r} \right]_{r_F} \tag{3}$$

$$v = T_0 \text{ at } t = 0 \text{ and } r > a \tag{4}$$

$$u = v = T_F \text{ at } r = r_F \tag{5}$$

$$u = T_w \text{ at } r = a \text{ and } t > 0 \tag{6}$$

It is convenient to reexpress the problem as follows

$$\frac{\partial U}{\partial \tau} = \frac{1}{e^{2R}} \frac{\partial^2 U}{\partial R^2}, \quad 0 \leqq R \leqq R_F \tag{7}$$

$$\frac{\partial V}{\partial \tau} = \frac{K_L}{K_S e^{2R}} \frac{\partial^2 V}{\partial R^2}, \quad R \geqq R_F \tag{8}$$

$$\frac{dR_F}{d\tau} = \frac{C_S(T_F - T_w)}{Le^{2R_F}}\left[\frac{\partial U}{\partial R} - \frac{\partial V}{\partial R} \right]_{R_F} \tag{9}$$

$$V = V_0 = k_L (T_0 - T_F)/k_S (T_F - T_w) \text{ at } \tau = 0 \text{ and } R > 0 \tag{10}$$

$$U = V = 0 \text{ at } R = R_F \tag{11}$$

$$U = -1 \text{ at } R = 0 \text{ and } \tau > 0 \tag{12}$$

where

$$U = (u - T_F)/(T_F - T_w) \tag{13}$$

$$V = k_L (v - T_F)/k_S (T_F - T_w) \tag{14}$$

$$R = \ln r/a \tag{15}$$

$$\tau = K_S t/a^2 \tag{16}$$

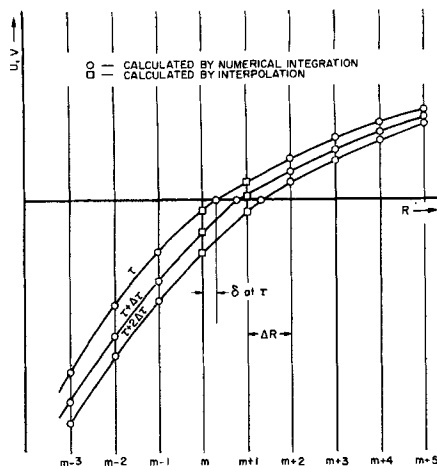The inclusion of the thermal conductivity ratio in the definition of V results in the elimination of one parameter

Fig. 1. Interpolation of temperatures near freezing front.

FINITE DIFFERENCE FORMULATION

from the dimensionless representation of the temperature field, and hence in the representation of the freezing front location and the heat flux density at the isothermal surface.

The region outside the cylinder was divided into equal increments of R and the problem rewritten in the corresponding finite difference form. Equations (7) and (8) were expressed in the following explicit forms:

$$U'_i = U_i + (U_{i-1} - 2U_i + U_{i+1}) \frac{\Delta \tau}{e^{2R}(\Delta R)^2} \quad (17)$$

and

$$V'_i = V_i + (V_{i-1} - 2V_i + V_{i+1}) \frac{K_L \Delta \tau}{K_S e^{2R}(\Delta R)^2} \quad (18)$$

The new temperature at the two grid lines immediately adjacent to the freezing front cannot be computed with stability from Equations (17) and (18) because of the discontinuity in the temperature gradient at the freezing front. Murray and Landis (2) suggested calculation of these two temperatures by a three-point interpolation. The following similar expressions based on a four-point Lagrangian interpolation were utilized herein.

$$U'_m = \frac{\delta}{3\Delta R + \delta} U'_{m-3} - \frac{3\delta}{2\Delta R + \delta} U'_{m-2} +$$
$$\frac{3\delta}{\Delta R + \delta} U'_{m-1} \quad (19)$$

$$V'_{m+1} = \frac{3(\Delta R - \delta)}{2\Delta R - \delta} V'_{m+2} - \frac{3(\Delta R - \delta)}{3\Delta R - \delta} V'_{m+3} +$$
$$\frac{\Delta R - \delta}{4\Delta R - \delta} V'_{m+4} \quad (20)$$

where m indicates grid number immediately inside the freezing front and

$$\delta = R_F - R_m \quad (21)$$

as illustrated in Figure 1. The temperature at the freezing front was utilized in the formulation of Equations (19) and (20) but does not appear explicitly because of its zero value in terms of V and U. When the freezing front crosses a grid line during a time step, Equation (19) is used with m replaced by $m + 1$ to find the new temperature just inside the frozen region.

The motion of the freezing front was calculated from the following finite difference form of Equation (9).

$$\Delta R_F = \frac{C_S (T_F - T_W) \Delta \tau}{L e^{3R_{F_\tau} - R_{F_{\tau-\Delta\tau}}}} \left[\frac{\partial U}{\partial R} - \frac{\partial V}{\partial R} \right]_{R_F} \quad (22)$$

$(\partial U/\partial R)_{R_F}$ and $(\partial V/\partial R)_{R_F}$ were evaluated from the following expressions obtained by differentiation of Lagrange's interpolation formulas using the same four points as in Equations (19) and (20).

$$\left(\frac{\partial U}{\partial R}\right)_{R_F} = -\frac{(2\Delta R + \delta)(\Delta R + \delta)}{2(3\Delta R + \delta)(\Delta R)^2} U_{m-3} +$$
$$\frac{(3\Delta R + \delta)(\Delta R + \delta)}{(2\Delta R + \delta)(\Delta R)^2} U_{m-2} - \frac{(3\Delta R + \delta)(2\Delta R + \delta)}{2(\Delta R + \delta)(\Delta R)^2} U_{m-1}$$
$$(23)$$

and

$$\left(\frac{\partial V}{\partial R}\right)_{R_F} = \frac{(3\Delta R - \delta)(4\Delta R - \delta)}{2(2\Delta R - \delta)(\Delta R)^2} V_{m+2} -$$
$$\frac{(2\Delta R - \delta)(4\Delta R - \delta)}{(3\Delta R - \delta)(\Delta R)^2} V_{m+3} + \frac{(2\Delta R - \delta)(3\Delta R - \delta)}{2(4\Delta R - \delta)(\Delta R)^2} V_{m+4}$$
$$(24)$$

The exact evaluation of R'_F would require the use of the correct mean value for the right side of Equation (9). As indicated R_F at τ plus half the previous increment in R_F was used as a mean value for R_F in the exponential. The reiterative use of average values of $(\partial U/\partial R)_{R_F}$ and $(\partial V/\partial R)_{R_F}$ over the incremental time step did not result in a significant correction, and the values at τ were actually employed.

The dimensionless heat flux density was calculated from the following derivative of a Lagrangian (or Newtonian forward) interpolation through the first four grid points

$$\frac{qa}{k_S(T_F - T_W)} = \left(\frac{\partial U}{\partial R}\right)_{R=0} =$$
$$\frac{-11U_0 + 18U_1 - 9U_2 + 2U_3}{6\Delta R} \quad (25)$$

The four-point interpolation used in Equations (19), (20), (23), and (24) resulted in a significant improvement in accuracy over the three-point formulas suggested by Murray and Landis (2).

NUMERICAL CALCULATIONS

Analysis of Equations (7) through (12) reveals that the dimensionless freezing front location r_F/a and the
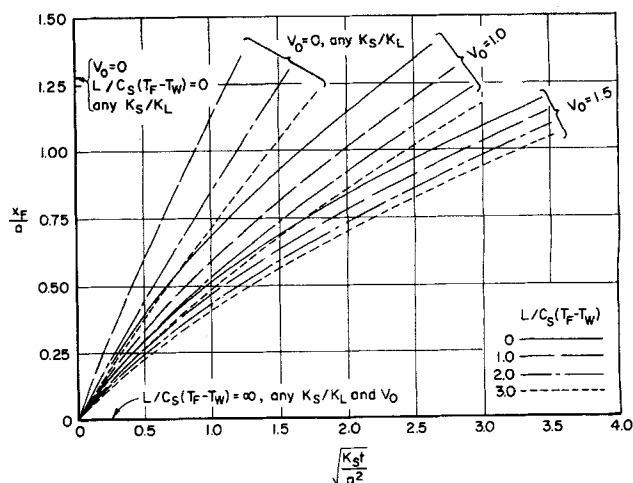

Fig. 2. Location of freezing front for $K_S/K_L = 1.0$.

dimensionless heat flux density $qa/k_s (T_F - T_W)$ are functions of only the three dimensionless parameters $k_L (T_0 - T_F)/k_s(T_F - T_W)$, $L/C_s(T_F - T_W)$, and K_s/K_L, and not additionally of k_L/k_s. This simplification produces a significant economy in computation and presentation of the results.

The conventional criterion for the stability of the explicit formulation is still valid for the case of latent heat (7). For Equation (17) this criterion can be written as

$$\Delta\tau \leq (\Delta R)^2 e^{2R}/2 \qquad (26)$$

Accordingly $\Delta\tau$ was set equal to 0.4 $(\Delta R)^2$ in all but one set of computations. A test calculation with $\Delta\tau = 0.2$ $(\Delta R)^2$ did not produce a significant change in the solution.

A grid size of $\Delta R = 0.025$ for $R_F = 0.1$ to 0.2, of $\Delta R = 0.050$ for $R_F = 0.2$ to 0.4 and of $\Delta R = 0.10$ for $R_F > 0.4$ was used. In all cases the computations were cut off at the grid where $(V_0 - V) < 10^{-6}$.

To initiate the computations, R_F must have a finite value. For very short times the cylindrical solution must approach the flat plate solution. Accordingly the flat plate solution was used to provide a starting temperature distribution, including R_F, at a small value of time. The small error from this approximation dies out rapidly with time as was confirmed by calculations begun with different initial temperature distributions.

In the computer program, ΔR_F is first computed from Equations (22), (23), and (24), then R_F and δ at $\tau + \Delta\tau$. This value of δ is used in Equations (19) and (20) to compute U'_m and V'_{m+1} [or in Equation (19) to compute U'_m and U'_{m+1} if the freezing front crosses a grid line]. Equations (17) and (18) are then used to compute the other temperatures at $\tau + \Delta\tau$.

The computer program was used with appropriate modifications to make test calculations for the flat plate. The results up to $\tau = 9$ agreed within 0.5% with the analytical solution. The numerical results for the two limiting cases of $L = 0$ and $C = 0$ also agreed very well with the corresponding analytical solutions.

All in all, the above numerical method appears to be stable and accurate. The computed values of x_F/a deviate positively and the computed values of $qa/k_s(T_F - T_W)$ deviate negatively. Both are believed to be reliable within 1%.

The calculations were carried out on an IBM 7090. The computations were stopped when $qa/k_s(T_F - T_W)$ fell to unity or τ exceeded 16. The computing time ranged from 0.02 to 0.05 sec. per time increment and averaged about 2 min. per case.

ANALYTICAL SOLUTIONS FOR LIMITING CASES

The analytical solutions for the three limiting cases of no latent heat, no heat capacity, and the flat plate provide bounding values and useful tests for the numerical calculations as indicated above.

For no latent heat the analytical solution is

$$\frac{T_0 - T}{T_0 - T_W} = 1 + \frac{2}{\pi} \int_0^\infty e^{-\tau\xi^2}$$

$$\frac{J_0\left(\frac{r}{a}\xi\right)Y_0(\xi) - Y_0\left(\frac{r}{a}\xi\right)J_0(\xi)}{J_0^2(\xi) + Y_0^2(\xi)} \frac{d\xi}{\xi} \qquad (27)$$

and

$$\frac{qa}{k(T_0 - T_W)} = \frac{4}{\pi^2} \int_0^\infty e^{-\tau\xi^2} \frac{d\xi}{\xi[J_0^2(\xi) + Y_0^2(\xi)]} \qquad (28)$$
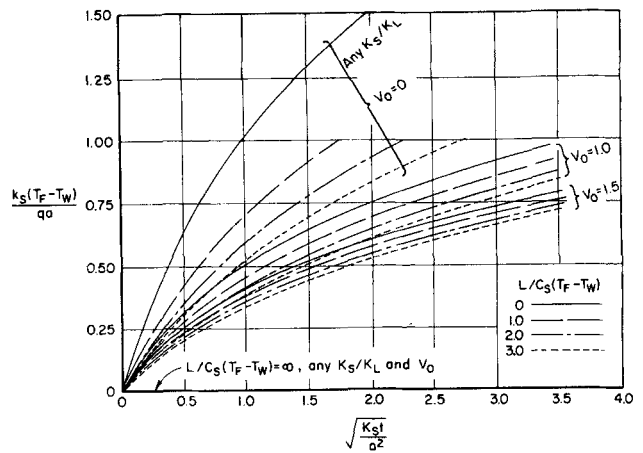

Fig. 3. Heat flux density for $K_S/K_L = 1.0$.

The location of the pseudo freezing front can be found by equating T to T_F in Equation (1) and solving for the corresponding r/a. Plots of the solution of Equations (27) and (28) are included in Carslaw and Jaeger (1). However, because of the uncertainty in reading values from these plots computer programs were written and utilized to compute the freezing front location and heat flux density. A series solution was used for short times and numerical integration of the differential equation for long times rather than Equations (27) and (28).

The following analytical solution can be derived for zero heat capacity.

$$\frac{4 k_s(T_F - T_W)t}{\rho L a^2} = 1 + \left(\frac{r_F}{a}\right)^2 \left[\ln\left(\frac{r_F}{a}\right)^2 - 1\right] \qquad (29)$$

and

$$\frac{qa}{k_s(T_F - T_W)} = 1/\ln\left(\frac{r_F}{a}\right) \qquad (30)$$

For the flat plate

$$x_F/a = 2\lambda\sqrt{K_s t/a^2} \qquad (31)$$

and

$$\frac{qa}{k_s(T_F - T_W)} = \frac{1}{(\text{erf }\lambda)} \sqrt{\frac{a^2}{\pi K_s t}} \qquad (32)$$

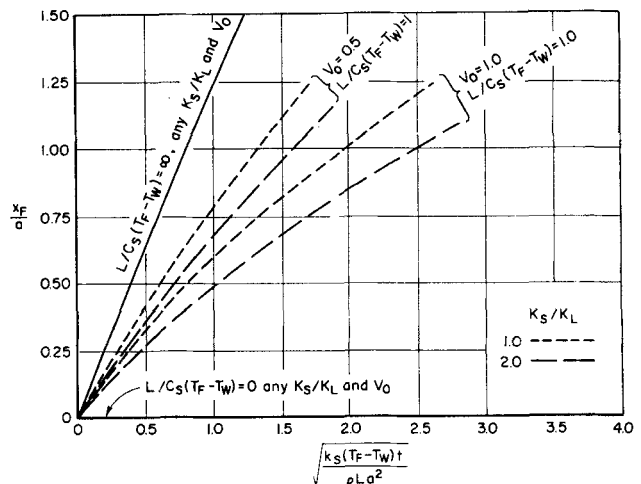where λ is the root of

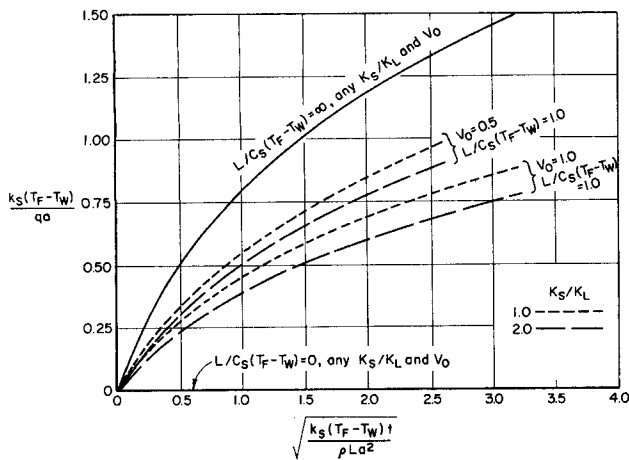Fig. 4. Effect of thermal diffusivity ratio on the location of freezing front.

Fig. 5. Effect of thermal diffusivity ratio on the heat flux density.

$$\frac{e^{-\lambda^2}}{\mathrm{erf}\,\lambda} = \frac{L\sqrt{\pi}\,\lambda}{C_s(T_F - T_W)} +$$

$$\frac{k_L}{k_S}\sqrt{\frac{K_S}{K_L}}\left(\frac{T_0 - T_F}{T_F - T_W}\right)\frac{e^{-\lambda^2 K_S/K_L}}{\mathrm{erfc}\,\lambda\sqrt{K_S/K_L}} \qquad (33)$$

NUMERICAL RESULTS

Because of the three parameters and two independent variables, only illustrative values can be plotted. In Figure 2 the freezing front location is plotted vs. $\sqrt{K_s t/a^2}$ for $K_s/K_L = 1.0$ and a number of values $V_0 = k_L(T_0 - T_W)/k_s(T_F - T_W)$ and $L/C_s(T_F - T_W)$. The corresponding solutions for a flat plate are straight lines with the same gradient (equal to 2λ) at time zero. These straight lines are omitted to avoid confusion. The limiting solutions for zero latent heat (except for $V_0 = 0$) are shown but the limiting solutions for negligible heat capacity fall along the abscissa in this plot.

The corresponding plot for the reciprocal of the dimensionless heat flux density is shown in Figure 3. Again the solutions for a flat plate would be straight lines with the same gradient [equal to $\sqrt{\pi}\,\mathrm{erf}\,\lambda$] at the origin. The limiting solutions for $L = 0$ are included but the limiting solutions for $C = 0$ fall along the abscissa.

The effect of the property ratio K_s/K_L is illustrated in Figures 4 and 5 for $L/C_s(T_F - T_W) = 1.0$ and two values of V_0. In these plots the alternative dimensionless group $\sqrt{k_s(T_F - T_W)\,t/\rho\,La^2}$ is chosen as the abscissa, permitting the inclusion of the limiting solution for zero heat capacity. In this case the limiting solution for zero latent heat falls along the abscissa.

The computed values of x_F/a and $qa/k_s(T_F - T_W)$ for eighteen conditions have been deposited.* The computer program, which is written in MAD (Michigan Algorithm Decoder), and a simplified flow diagram for the computations are included in the deposited material. The transient temperature distribution is not tabulated but may be obtained from the computer program.

CONCLUSIONS

The method described herein and computer programs that have been made available can be used to calculate the temperature field, including the freezing front loca-

tion, and the heat flux density in the region outside an infinitely long circular cylinder.

The numerical values that are presented illustrate the effects of the several parameters and variables. The minimum number of parameters with which the problem can be described was determined. This permitted a minimum of computation and an economical representation of the results.

NOTATION

a	= radius of cylinder, length
C	= heat capacity, energy/(mass)(degree)
J_0	= Bessel function of first kind and zero order
k	= thermal conductivity, energy/(length)(time)(degree)
K	= thermal diffusivity, length^2/time
L	= latent heat, energy/mass
q	= heat flux density, energy/(time)(length)^2
r	= radial distance, length
R	= $\ln(r/a)$
t	= time
T	= temperature
u	= temperature of frozen region
U	= $(u - T_F)/(T_F - T_W)$
v	= temperature of unfrozen region
V	= $k_L(v - T_F)/k_s(T_F - T_W)$
x	= $r - a$, length
Y_0	= Bessel function of second kind and zero order

Greek Letters

δ	= $R_F - R_m$, length
λ	= root of Equation (33)
ρ	= density, mass/length^3
τ	= $K_s t/a^2$

Subscripts

F	= at freezing front
i	= grid number
L	= in unfrozen region
m	= grid number adjacent to freezing front, inside frozen region
0	= initial
S	= in frozen region
W	= at cylindrical surface

Superscript

$(')$	= at time $\tau + \Delta\tau$

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