9. Gottschlich, Chad F., *ibid.*, **9**, 89 (1963).
10. Klinkenberg, A., and F. Sjenitzer, *Chem. Eng. Sci.*, **5**, 258 (1956).
11. Kasten, P. R., Leon Lapidus, and N. R. Amundson, *J. Phys. Chem.*, **56**, 683 (1952).
12. Lapidus, Leon, and N. R. Amundson, *ibid.*, 984.
13. Liles, A. W., and C. J. Geankoplis, *A.I.Ch.E. J.*, **6**, 591 (1960).
14. McHenry, K. W., Jr., and R. H. Wilhelm, *ibid.*, **3**, 83 (1958).
15. Rosen, J. B., *Ind. Eng. Chem.*, **46**, 1590 (1954).
16. ———, *J. Chem. Phys.*, **20**, 387 (1952).
17. Van Deemter, J. J., F. J. Zuiderweg, and A. Klinkenberg, *Chem. Eng. Sci.*, **5**, 271 (1956).

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Theoretical Study of Bubble Dynamics in Purely Viscous Fluids

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This paper analyzes the growth or collapse of a spherical bubble in an incompressible, viscous fluid. Theoretical results include the timewise variations in the bubble size and its growth or collapse rate, the fluid pressure, and the rate of energy dissipation. The analysis is general and may be applied to both Newtonian and non-Newtonian fluids. A comparison is given for the collapse of the bubble in several viscous fluids.

In connection with studies on cavitation and cavitation damage, it is desirable to have mathematical expressions for the pressure and velocity fields in the neighborhood of a growing or collapsing gas- or vapor-filled cavity in a liquid. The problem for a spherical bubble in an incompressible nonviscous liquid has been solved (*1*, *2*). Effects of viscosity and compressibility on the bubble dynamics have been investigated (*3*, *4*). Barlow and Langlois (*5*) considered the diffusion-fed growth of a spherical gas bubble into a Newtonian viscous liquid under isothermal conditions. Later, the problem of isothermal bubble growth, dominated by viscosity and diffusion, was studied (*6*).

In problems of heat transfer with boiling, the time history of bubble formation and growth in a superheated liquid is of great importance. For nucleate boiling in a subcooled liquid, the collapse of bubbles must be considered in addition to the formation and growth. The so-called *extended Rayleigh's equation* (*7*) has been solved for the growth of a vapor bubble in a superheated liquid (*8*, *9*). The dynamics of bubble in binary mixtures are treated in references *10* and *11*. Recently, the mechanics of vapor bubble collapse under spherically symmetrical conditions were examined to ascertain the relative importance of the liquid inertia and heat transfer rate (*12*).

In this work, the dynamic equations governing the pressure distribution and growth or collapse of a spherical gas or vapor bubble in incompressible fluids are formulated from the conservation laws of mass and momentum. The expression for the rate of energy dissipation, or the rate of irreversible conversion of mechanical energy into internal energy, in the liquid region due to the bubble motion is obtained. The analysis is general and may be applied to both Newtonian and non-Newtonian fluids. Numerical results are obtained for the collapse of the bubble.

ANALYSIS

Consider a spherical bubble growing or collapsing in an infinite mass of homogeneous incompressible liquid. The model of a spherical bubble is justified when its radius is less than 0.5 mm. (*13*). For bubbles of radius between 0.5 and 4.0 mm., which are oblate spheroid approximately, the bubble size may be specified by its equivalent spherical radius. The spherical shape of the bubble remains stable under the action of surface tension in a spherically symmetric external pressure field. The assumptions of the symmetric pressure field and the absence of wall effects and external temperature gradients are reasonable for the bubble growing or collapsing outside a thin thermal boundary layer of the wall. In case of Newtonian fluids, however, the theoretical prediction by the present model is reported in good agreement with experimental results obtained for bubbles growing on a heated surface (*8* to *10*). In the absence of body force and in laminar flow regime (including creeping flow), the equations of continuity and motion in the liquid with constant density may be expressed in spherical coordinates as (*14*)

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0 \tag{1}$$

and

$$\rho_l\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial r} - |\nabla \cdot \bar{\bar{\tau}}|_r \tag{2}$$

respectively, where

$$|\nabla \cdot \bar{\bar{\tau}}|_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{rr}) - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \quad (3)$$

If R and $\dot{R}$ represent the instantaneous bubble radius and its time derivative, and the density ratio of gas inside the bubble to liquid is small, then the integration of Equation (1) yields

$$u = R^2 \dot{R}/r^2 \quad (4)$$

The substitution of Equation (4) into Equation (2), followed by an integration with respect to r from r to r_o at a particular time, gives

$$\ddot{R} R^2 \left(\frac{1}{r} - \frac{1}{r_o}\right) + \dot{R}^2 \left(\frac{R^4}{2r_o^4} - \frac{R^4}{2R^4} + \frac{2R}{r} - \frac{2R}{r_o}\right) = \frac{p_l(r) - p_l(r_o)}{\rho_l} - \frac{1}{\rho_l}\int_r^{r_o} |\nabla \cdot \bar{\bar{\tau}}|_r \, dr \quad (5)$$

where r_o is a reference radius in the liquid where its pressure is $p_l(r_o)$. Equation (5) represents the pressure distribution in the liquid. The dynamic equation governing the growth or collapse of the bubble may be obtained from Equation (5) by replacing r with R as

$$\ddot{R} R^2 \left(\frac{1}{R} - \frac{1}{r_o}\right) + \dot{R}^2 \left(\frac{R^4}{2r_o^4} - \frac{2R}{r_o} + \frac{3}{2}\right) = \frac{p_l(R) - p_l(r_o)}{\rho_l} - \frac{1}{\rho_l}\int_R^{r_o} |\nabla \cdot \bar{\bar{\tau}}|_r \, dr \quad (6)$$

The balance of forces at the bubble wall requires that

$$p_l(R) + \frac{2\sigma}{R} + \tau_{rr,l}(R) = p_g(R) - \tau_{rr,g}(R) \quad (7)$$

where $\tau_{rr,g}$ is the radial normal stress acting on the bubble surface due to the gas phase viscosity and may be expressed by

$$\tau_{rr,g}(R) = -2\mu_g \, (\partial u/\partial r)_{r=R} = 4\mu_g \dot{R}/R \quad (8)$$

on the assumption that the gas inside the bubble is a Newtonian fluid. The combination of Equations (6), (7), and (8) yields

$$\ddot{R} R^2 \left(\frac{1}{R} - \frac{1}{r_o}\right) + \dot{R}^2 \left(\frac{R^4}{2r_o^4} - \frac{2R}{r_o} + \frac{3}{2}\right) = p_g(R) - p_l(r_o) - \frac{2\sigma}{R} - 4\mu_g \frac{\dot{R}}{R} + \tau_{rr,l}(R) - \int_R^{r_o} |\nabla \cdot \bar{\bar{\tau}}|_r \, dr \quad (9)$$

in which $4\mu_g \dot{R}/R$ is generally very small compared with $\tau_{rr,l}(R)$ and may therefore be neglected from the expression. $\tau_{rr,l}(R)$ cannot be generally written in the form $-2\mu_l \left(\frac{\partial u}{\partial r}\right)_{r=R} = 4\mu_l \frac{\dot{R}}{R}$, similar to that of $\tau_{rr,g}(R)$, unless the liquid is a Newtonian one. The term $4\mu_g \dot{R}/R$ for $\tau_{rr,g}$ is retained in Equation (9) for generality. However, it is pointed out in the line right below Equation (9) that $\tau_{rr,g}$ is generally small compared with $\tau_{rr,l}$. So it is appropriate to leave the $\tau_{rr,g}$ or $4\mu_g \dot{R}/R$ as it is in the expressions. The instantaneous bubble size and its time derivative may be obtained by solving Equation (9) with the initial conditions

$$R(0) = R_o \quad (10a)$$

and

$$\dot{R}(0) = 0 \quad (10b)$$

which describes that initially the bubble with radius R_o is in equilibrium with the surrounding liquid.

The rate of energy dissipation (or the rate of irreversible conversion of mechanical energy into internal energy) due to the bubble motion, defined as $-(\bar{\bar{\tau}}:\nabla\bar{v})$ per unit volume, is

$$\Phi = -\int_R^{r_o} (\bar{\bar{\tau}}:\nabla\bar{v}) \, 4\pi r^2 dr \quad (11)$$

for the liquid domain between radii R and r_o.

If the growth or collapse of the bubble is affected by heat diffusion across the bubble wall, then the equation of bubble motion [(6) or (9)] has to be coupled with the equation for the temperature field in the liquid:

$$\frac{\partial T}{\partial t} + \frac{R^2\dot{R}}{r^2}\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) \quad (12)$$

The appropriate initial and boundary conditions for the energy equation are

$$T(r, 0) = T_\infty \quad (13a)$$

$$T(\infty, t) = T_\infty \quad (13b)$$

and

$$k\frac{\partial T}{\partial r} = \rho_g h_{fg} \dot{R} \quad \text{at} \quad r = R \quad (13c)$$

Equation (13a) shows that the liquid is initially at a uniform temperature T_∞. Equation (13b) indicates that the liquid temperature at a distance from the bubble remains unchanged at T_∞. Equation (13c) is obtained from the consideration of the conservation of energy at the bubble wall by neglecting the gas velocity relative to $\dot{R}$ and temperature gradient in gas phase.

The analysis is general and may be applied to both Newtonian and non-Newtonian fluids. For the latter, two common two-parameter models are investigated: power law (or Oswald-de Waele) and Bingham plastic models. Their rheological character is listed in Table 1. For special cases both models become a Newtonian fluid: zero yield stress in the Bingham plastic model and $n = 1$ in the power law model. The deviations of the yield stress from zero in the former and of n from unity in the latter indicate the degree of deviation from Newtonian behavior. For values of n in the power law model less than unity, the behavior is pseudoplastic, whereas for n greater than unity, the behavior is dilatant.

Table 2 presents the equations in the dimensionless form for the bubble motion, the pressure distribution. and the rate of energy dissipation in two non-Newtonian models and the Newtonian fluid. They are obtained by the combination of Table 1 and Equations (5), (9), and (11).

NUMERICAL RESULTS AND DISCUSSION

Only the collapse of the bubble caused by a step increase in the system pressure, that is, $p_l(\infty) = 1$ atm. (or $p_\infty^* = 1$) for the Newtonian and power law model and $p(r_o^*) = 1$ for the Bingham plastic model, is investigated numerically. It is assumed that the system is under the isothermal condition and that effects of surface tension are negligible. The gas pressure inside the bubble is considered vacuum, that is, $p_g^* = 0$. The equations of bubble motion presented in Table 2 are numerically integrated by means of a 7090-IBM digital computer for the collapse of the bubble from an initial size $R_o = 0.5$ cm. or $R^*_{(0)} = 1$ with zero initial collapsing rate $\dot{R}^*_{(0)} = 0$.

TABLE 1. RHEOLOGICAL CHARACTER OF NEWTONIAN AND NON-NEWTONIAN FLUIDS

	Non-Newtonian fluids		
	Power law model	Bingham model	Newtonian fluid
Stress tensor $\overline{\overline{\tau}}$	$\overline{\overline{\tau}} = -m \left\| \left[\frac{1}{2}(\overline{\overline{\Delta}} : \overline{\overline{\Delta}}) \right]^{1/2} \right\|^{n-1} \overline{\overline{\Delta}}$	if $\left\| \frac{1}{2}(\overline{\overline{\tau}} : \overline{\overline{\tau}}) \right\| \geqq \tau_o$ $\overline{\overline{\Delta}} = -[\eta \pm \tau_o/(\overline{\overline{\Delta}} : \overline{\overline{\Delta}})^{1/2}]\, \overline{\overline{\Delta}}$ if $\left\| \frac{1}{2}(\overline{\overline{\tau}} : \overline{\overline{\tau}}) \right\| \leqq \tau_o$ $\overline{\overline{\Delta}} = 0$	$\overline{\overline{\tau}} = -\mu \overline{\overline{\Delta}}$
Stress component $\tau_{rr} = -2\tau_{\theta\theta}$ $= -2\tau_{\phi\phi}$	$\tau_{rr} = 4m(2\sqrt{3})^{n-1} \left\| \frac{R\dot{R}}{r^2} \right\|^{n-1} \left(\frac{R^2\dot{R}}{r^3} \right)$	$\tau_{rr} = 4\eta \frac{\partial u}{\partial r} \pm \frac{2}{\sqrt{3}} \tau_o = 4\eta \frac{R^2\dot{R}}{r^3} \pm \frac{2}{\sqrt{3}} \tau_o$ for $\left\| \frac{\partial u}{\partial r} \right\| = \left\| 2\frac{R^2\dot{R}}{r^3} \right\| > 0$	$\tau_{rr} = 4\mu \frac{\partial u}{\partial r}$ $= 4\mu \frac{R^2\dot{R}}{r^3}$
Reference radius r_o	∞	r_o	∞
$\int_r^{r_o} \left\| \nabla \cdot \overline{\overline{\tau}} \right\|_r dr$	$\pm \frac{4m(1-n)}{n} (2\sqrt{3})^{n-1} \left(\pm \frac{R\dot{R}}{r^3} \right)^n$ for $n \neq 0$	$\pm 2\sqrt{3}\, \tau_o \ln(r_o/r)$	0
$\int_R^{r_o} \left\| \nabla \cdot \overline{\overline{\tau}} \right\|_r dr$	$\pm \frac{4m(1-n)}{n} (2\sqrt{3})^{n-1} \left(\pm \frac{\dot{R}}{R} \right)^n$ for $n \neq 0$	$\pm 2\sqrt{3}\, \tau_o \ln(r_o/R)$	0
$\Phi = -\int_R^{r_o} (\overline{\overline{\tau}} : \nabla \overline{v}) 4\pi r^2 dr$	$\pm 16\pi \frac{m}{n} (2\sqrt{3})^{n-1} R^2\dot{R} \left(\pm \frac{\dot{R}}{R} \right)^n$	$16\pi\eta \left(R\dot{R}^2 - \frac{R^4\dot{R}^2}{r_o^3} \right)$ $\pm 8\sqrt{3}\pi \tau_o R^2\dot{R} \ln(r_o/R)$	$16\pi \mu_{2l} R\dot{R}^2$

Note: For terms with plural signs, the upper sign for $\dot{R}^* \geqq 0$, the lower sign for $\dot{R}^* \leqq 0$.

R^*, $\dot{R}^*$, and $\ddot{R}^*$ are obtained by steps of $t^* = 0.001$. The method of Runge and Kutta was employed for this purpose. The reference radius r_o is selected at 20 corresponding to $r_o = 10$ cm. for the Bingham plastic model and at infinity for the other fluids. Now with the numerical values of R^*, $\dot{R}^*$, and $\ddot{R}^*$ available, the liquid pressure p_l^* and the energy dissipation rate Φ^* are calculated from the equations listed in Table 2. The pressure-time history is evaluated at $r = 0.75$ cm. or $r^* = 1.5$ for all models.

The time variations in the bubble size and collapsing rate, the fluid pressure, and the rate of energy dissipation are presented elsewhere† for four different viscous fluids at 71°F.: 10% volume fraction of alumina particles (average 3 mu in diameter) suspended in 25% sulfuric acid solution (*15*) $\eta = 6.81 \times 10^{-3}$ (lb.$_f$)(sec.)/sq. ft., $\tau_o = 1.254 \times 10^{-3}$ lb.$_f$/sq. ft., $\eta^* = 6.48 \times 10^{-3}$, $\tau_o^* = 5.90 \times 10^{-7}$; 5% polyisobutylene in water, $n = 0.34$, $m = 1.7$ (lb.$_f$) (sec.^{n})/sq. ft., $m^* = 1.09 \times 10^{-2}$ (*16*); 0.83% solution of ammonium alginate in water, $n = 0.78$, $m = 1.865 \times 10^{-3}$ (lb.$_f$)(sec.^{n})/sq. ft., $m^* = 3.95 \times 10^{-4}$ (*17*); and pure water, $\mu = 0.658 \times 10^{-3}$ (lb.$_f$)(sec.)/sq. ft., $\mu_l^* = 6.26 \times 10^{-4}$. The first fluid may be considered as a Bingham plastic, the second and third fluids as power law models, and the fourth fluid as a Newtonian model. The effects of the parameters n, m^* for power law model and τ_o^* for Bingham plastic model on the bubble size and velocity, the fluid pressure, and the rate of energy dissipation are illustrated in Figures 1, 2, and 3, respectively. Since the Newtonian fluid may be regarded as a special case of the Bingham plastic model for $\tau_o^* = 0$ and $r_o^* = \infty$, one observes from the figures that a decrease in τ_o^* or an increase in r_o^* result in increases in

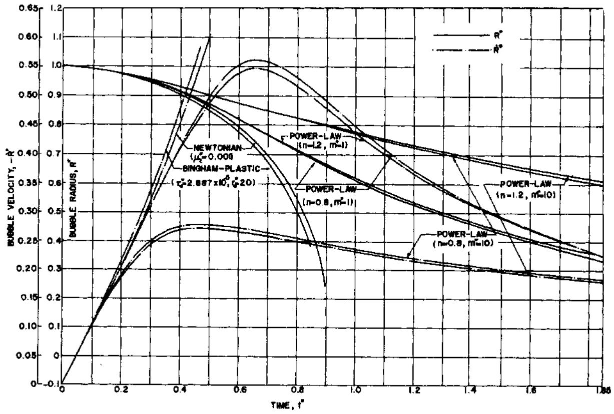

Fig. 1. Radius and collapsing rate time history of collapsing bubbles in Newtonian and non-Newtonian fluids.

† Deposited as document 8874 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington 25, D. C., and may be obtained for $1.25 for photoprints or 35-mm. microfilm.

Table 2. Equations for the Bubble Motion, Pressure Distribution, and Rate of Energy Dissipation in Newtonian and Non-Newtonian Fluids

		Non-Newtonian fluids: Power law model	Non-Newtonian fluids: Bingham plastic model	Newtonian fluid
Equation of bubble motion		$R^*\ddot{R}^* + \frac{3}{2}(\dot{R}^*)^3 = p_g^* - p_\infty^* - \frac{2\sigma}{R^*}$ $-4\mu_g^* \dot{R}^*/R^*$ $\mp 4(2\sqrt{3})^{n-1}(m^*/n)(\pm \dot{R}^*/R^*)^n$	$\ddot{R}^*\left[-\frac{(R^*)^2}{r_o^*} + R^*\right] + (\dot{R}^*)^2\left[-\frac{2R^*}{r_o^*} + \frac{(R^*)^4}{2(R_o^*)^4} + \frac{3}{2}\right]$ $= p_g^* - p_l^*(r_o^*) - \frac{2\sigma^*}{R^*} - (\eta^* + \mu_g^*)\frac{\dot{R}^*}{R^*}$ $\mp 2\sqrt{3}\,\tau_o^*\left[\ln(r_o^*/R^*) + \frac{1}{3}\right]$	$R^*\ddot{R}^* + \frac{3}{2}(\dot{R}^*)^2$ $= p_g^* - p_\infty^* - \frac{2\sigma^*}{R^*}$ $-4(\mu_l^* + \mu_g^*)\dot{R}^*/R^*$
Pressure distribution	At time >0	$p_l^* - p_\infty^* = \mp\frac{4m^*}{n}(1-n)(2\sqrt{3})^{n-1}\left[\pm\frac{(R^*)^2\dot{R}^*}{(r^*)^3}\right]^n$ $+ \ddot{R}^*(R^*)^2/r^* + (\dot{R}^*)^2[-(R^*)^4/2(r^*)^4$ $+ 2(R^*/r^*)]$	$p_l^* - p_l^*(r_o^*) = \ddot{R}^*(R^*)^2\left(\frac{1}{r^*} - \frac{1}{r_o^*}\right)$ $+ (\dot{R}^*)^2\left[\frac{(R^*)^4}{2(r_o^*)^4} - \frac{(R^*)^4}{2(r^*)^4} + \frac{2R^*}{r^*} - \frac{2R^*}{r_o^*}\right]$ $\pm 2\sqrt{3}\,\tau_o^* \ln(r_o^*/r^*)$	$p_l^* - p_\infty^* = \frac{\ddot{R}^*(R^*)^2}{r^*}$ $+ (R^*)^2\left[-\frac{(R^*)^4}{2(r^*)^4} + \frac{2R^*}{r^*}\right]$
	At zero time	$p_{lo}^* - p_\infty^* = \frac{R_o^*}{r^*}\left(p_g^* - p_\infty^* - \frac{2\sigma}{R_o^*}\right)$	$p_{lo}^* - p_l^*(r_o^*) = \frac{R_o^*}{r^*}\left(p_g^* - p_\infty^* - \frac{2\sigma^*}{R_o^*}\right)$ $- 2\sqrt{3}\,\tau_o^* \ln(r_o^*/r^*)$	$p_{lo}^* - p_\infty^* = \frac{R_o^*}{r^*}\left(p_g^* - p_\infty^* - \frac{2\sigma^*}{R_o^*}\right)$
Rate of energy dissipation		$\Phi^* = \pm 16\pi\frac{m^*}{n}(2\sqrt{3})^{n-1}(R^*)^2\dot{R}^*\left(\pm\frac{\dot{R}^*}{R^*}\right)^n$	$\Phi^* = -16\pi\,\eta^*[R^*(\dot{R}^*)^2 - (R^*)^4(\dot{R}^*)^2/(r_o^*)^3$ $\pm 8\sqrt{3}\,\pi\,\tau_o^*(R^*)^2\dot{R}^* \ln(r_o^*/R^*)$	$\Phi^* = -16\pi\,\mu_l^* R^*(R^*)^2$

Note: For terms with plural signs, the upper sign for $\dot{R}^* \geq 0$, the lower sign for $\dot{R}^* \leq 0$.

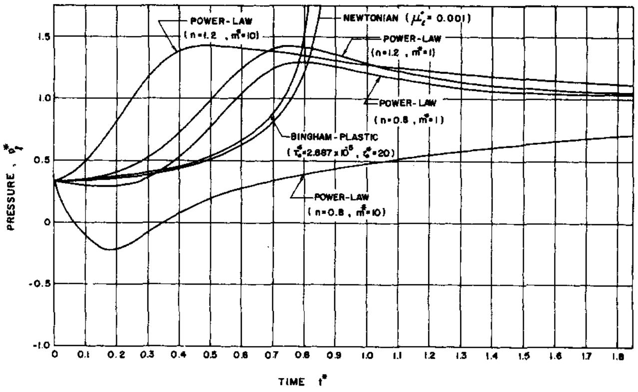

Fig. 2. Pressure-time history at $r^* = 1.5$ in Newtonian and non-Newtonian fluids.

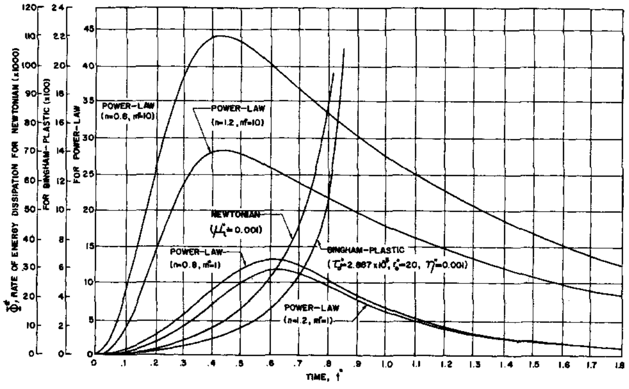

Fig. 3. The time history of the rate of energy dissipation in Newtonian and non-Newtonian fluids.

the collapsing rate and fluid pressure and in a decrease in the rate of energy dissipation. For the power law fluids it is disclosed that increases in the collapsing rate with a decrease in n or n^* increases the pressure with an increase in n, and the rate of energy dissipation decreases with an increase in n or a decrease in m^*.

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NOTATION

h_{fg} = latent heat of evaporation
k = thermal conductivity of liquid
n, m = parameters in power law model
m^* = $(m/R_o^n \Delta p)(\Delta p/\rho)^{n/2}$
p = pressure; $p_g(R)$, of gas at bubble wall; p_{go}, of gas inside at zero time; $p_l(r)$, of liquid, $p_l(r_o)$, of liquid at a reference radius r_o; $p_{lo}(r)$, of liquid at zero time; p_∞, of liquid at infinity or system pressure
Δp = pressure difference, $= p_\infty - p_{go}$
p^* = $\dfrac{p(r) - p_{go}}{\Delta p}$; $p_g^* = \dfrac{p_g(R) - p_{go}}{\Delta p}$; $p_l^* = \dfrac{p_l(r) - p_{go}}{\Delta p}$; $p_{lo}^* = \dfrac{p_{lo}(r) - p_{go}}{\Delta p}$
R = bubble radius
R_o = initial bubble radius
$\dot{R}$ = dR/dt
$\ddot{R}$ = d^2R/dt^2
R^* = R/R_o
$\dot{R}^*$ = $\dot{R}(\rho_l/\Delta p)^{1/2}$
$\ddot{R}^*$ = $\ddot{R} R_o \rho_l/\Delta p$
r = distance from the center of spherical bubble
r_o = reference radius
r_i^* = r/R_o
r_o^* = r_o/R_o
T = liquid temperature; T_∞, at infinity
t = time
t^* = $(t/R_o)(\Delta p/\rho_l)^{1/2}$
u = radial velocity of liquid at r

Greek Letters

α = thermal diffusivity of liquid
η = parameter of Bingham model
η^* = $(\eta/R_o)(1/\rho_l \Delta p)^{1/2}$
μ = viscosity; μ_l, of liquid; μ_g, of gas
μ^* = $(\mu/R_o)(1/\rho_l \Delta p)^{1/2}$
ρ = density; ρ_l, of liquid; ρ_g, of gas
Δ = rate of deformation tensor
σ = surface tension
σ^* = $\sigma/(R_o \Delta p)$
τ = normal stress; τ_{rr}, $\tau_{\theta\theta}$ and $\tau_{\phi\phi}$, in the direction of r, θ and ϕ, respectively
τ_o = yield stress of Bingham model
τ_o^* = $\tau_o/\Delta p$
$\bar{\bar{\tau}}$ = stress tensor
$(\bar{\bar{\tau}}:\nabla \bar{v})$ = rate of energy dissipation per unit volume
$|\nabla \cdot \bar{\bar{\tau}}|_r$ = viscous force per unit volume in radial direction
Φ = rate of energy dissipation
Φ^* = $(\Phi/R_o^2)(\rho_l/\Delta p^3)^{1/2}$

Superscripts

$*$ = dimensionless physical quantity
$\cdot$ = first derivative with respect time
$\cdot\cdot$ = second derivative with respect time

Subscripts

g = gas phase
l = liquid phase
o = at zero time, except r_o, r_o^*, τ_o, and τ_o^*

LITERATURE CITED

1. Lord Rayleigh, *Phil. Mag.*, **34**, 94 (1917); **45**, 257 (1923).
2. Lamb, H., "Hydrodynamics," pp. 122, 482, 489-491, Dover, New York (1945).
3. Gilmore, F. R., "Heat Transfer and Fluid Mechanics Institute," pp. 53-64, Stanford Univ. Press, Calif.
4. Flynn, H. G., *TM No. 33, NR-014-908, ONR* (March, 1957).
5. Barlow, E. L., and W. E. Langlois, *IBM J. Res. Develop.*, **6**, 329 (1962).
6. Langlois, W. E., *J. Fluid Mech.*, **15**, 111 (1963.)
7. Plesset, M. S., *J. Appl. Mech.*, **16**, 277 (1949).
8. ———, and S. A. Zwick, *J. Appl. Phys.*, **25**, 499-500 (1954).
9. Forster, H. K., and N. Zuber, *ibid.*, 474-478.
10. Scriven, L. E., *Chem. Eng. Sci.*, **10**, 1-12 (1959).
11. Yang, W. J., and J. A. Clark, *J. Heat Transfer*, **83**, 207-212 (1964).
12. Florschuetz, L. W., and B. T. Chao, *ibid.*, **87**, 209-220 (1965).
13. Saffman, P. G., *J. Fluid Mech.*, **1**, 249-275 (1956).
14. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," Chap. 3, Wiley, New York (1960).
15. Henning, C. D., Ph.D. thesis, Univ. Michigan, Ann Arbor (1965).
16. Foresti, R., Jr., and T. Liu, *Ind. Eng. Chem.*, **51**, 860 (1959).
17. Merrill, E. W., *ibid.*, 868.

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