

**A RUNOUT-TIME-BASED (Q,R) MODEL
WITH LEAD-TIME REDUCTION OPTIONS**

**Karla E. Bourland
The Amos Tuck School of Business Administration
Dartmouth College
Hanover, New Hampshire 03755**

**Candace A. Yano
Department of
Industrial & Operations Engineering
University of Michigan
Ann Arbor, MI 48109-2117**

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We model a continuous-review inventory system where demands are random, and where the lead time can be reduced at a cost. Shortage costs are proportional to the duration of the stockout, and the random variable of consequence is the time required to deplete a given amount of inventory -- a runout time. Brownian motion, which is based on a normal demand distribution for any specified time interval, has been used to model continuous demand processes and the corresponding runout time distribution is available. However, there are many difficulties with using Brownian motion to model demand processes. We define a beta demand process, and show that it has many desirable properties. We show that the partial linear loss function (that is, the shortage penalty) associated with the runout time distribution for a beta demand process is jointly convex. We also show that this loss function is jointly convex when the runout time distribution itself is a beta distribution. With this, we are able to optimally set the production quantity, reorder point, and lead time in the continuous-review problem.

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1. Introduction

In this paper we consider a single-item, continuous-review inventory system in which an order is placed for Q units whenever the inventory position reaches r . Demand is random, but the lead time is fixed and known. All excess demand is backordered. In contrast to earlier (Q,r) models (e.g., Hadley and Whitin 1961) where shortage penalties are charged on a per unit, per unit per unit time, or a stockout occasion basis, we investigate systems where the shortage penalty is proportional to the length of time the item is out of stock (stockout duration). More specifically, if the inventory level drops to zero at time t and replenishment occurs at time t' , then the shortage penalty is proportional to $t' - t$. If demand were continuous and deterministic at a constant rate, this would be equivalent to a penalty that is proportional to the number of units short. However, when demand is random, these two shortage penalties are not equivalent.

Shortage penalties are often determined, in whole or in part, by the stockout duration. For example, the shortage may result in idling or rescheduling a downstream facility. As an illustration of this, consider an item used in the assembly of an end item. The actual usage of the item varies randomly, and failure to provide the item to the end-item assembly process shuts down that facility. When the item becomes available, the assembly process can begin producing again. The penalty is proportional to the duration of the forced idle time, and may be a surrogate charge for the opportunity costs of idling the assembly facility. If it is possible to reschedule the downstream operation, then the duration of the stockout determines the amount of disruption. This type of shortage penalty may be part of a contractual agreement between the facility providing the item and the facility consuming the item, especially in cases where there is a drive to achieve tight linkage

between facilities in order to achieve "just-in-time." If the r units of inventory run out u time units before the arrival of the order, there is a shortage penalty proportional to u .

We also model a generalization of this problem where it is possible to reduce the lead time, at a cost. Although there are some situations where the lead times are fixed, it is often possible to reduce the lead time for an item, at a cost. For example, there may be a premium for faster modes of transportation, or the vendor may charge extra for faster response to an order. Shorter lead times lessen the amount of lead-time safety stock needed to achieve a target service level, and may also reduce shortage costs. There are other benefits from lead-time reduction, including improved responsiveness to customers and better quality due to faster feedback. We solve for the optimal lead time, order quantity, and reorder point simultaneously. Much of the work in this area has concentrated on lead-time reduction in a manufacturing setting, and several authors have made suggestions for reducing lead times. (See for example: Karmarkar 1987; Karmarkar, Kekre, Kekre, and Freeman 1985; and Hopp, Spearman, and Woodruff 1990)[6, 7, 4]. These papers focus on methods of improving operating policies and eliminating unnecessary waiting time in a manufacturing setting.

The fixed-lead-time problem is formulated in the next section, and this is followed by a section on the modeling of runout time distributions. In Section 4 we discuss solution procedures for the fixed-lead-time problem, and in Section 5 we examine the problem where the lead time is a decision variable. The paper concludes with a summary and directions for future research.

2. Problem Formulation

Notation

- D = average annual demand,
- t = actual lead time
- F = cumulative probability distribution function of demand during the lead time,
- μ = expected demand during the lead time,
- A = setup cost per order,

- h = holding cost per unit per unit time, and
 π = shortage cost per unit time the system is in a backordered state

We assume the reorder point r is greater than zero. We will refer to the time to deplete a given amount of inventory as the *runout time*, and to the cumulative distribution function of the runout time as a *runout time distribution*. (In the literature on stochastic processes, this is called a *first passage* or *hitting time* distribution.) The expected total cost per unit time, as a function of the reorder point, r , the order quantity, Q , and the lead time, t , can be expressed as

$$h \left(\frac{Q}{2} + r - \mu \right) + \frac{AD}{Q} + \frac{\pi D}{Q} \int_0^t (t-u) dG(u;r) \quad (1)$$

where $G(u;x) = P(T_x \leq u)$ for T_x = first time at which x units of inventory are depleted. We approximate the holding cost as in Johnson and Montgomery (1974)[5]. The second term is the standard expression for setup cost per unit time. The third term is the expected shortage cost per unit time, which is a (partial) linear loss function of the runout time distribution. Defining and manipulating runout time distributions pose many difficulties. This is one reason why they are seldom used in modeling production and inventory control systems. Note that the decision variable r appears as a parameter of the runout time distribution. This can make optimization with respect to r difficult.

3. Modeling Runout Time Distributions

The runout time distribution can be modeled in two different ways. The first way is to model the cumulative demand process as a function of time. If the distribution of cumulative demand can be specified for every time duration, it is theoretically possible to express the runout time distribution algebraically. Alternatively, it may be possible to fit the runout time distribution directly, using historical data. We consider both approaches in this section. Before doing so, we present Table 1, which contains definitions of additional variables that will be used throughout the

paper. We assume, without loss of generality, that the cumulative demand at time zero is zero. That is, $X(0) = 0$.

Table 1
Variable Definitions

$d(\tau)$	demand during time interval τ
$X(t)$	cumulative demand through time t
$F(x;t)$	cumulative distribution of $x(t)$
$f(x;t)$	$\partial F(x;t)/\partial x$
$g(u;x)$	$\partial G(u;x)/\partial u$
$\sigma^2_{x;t}$	variance of $X(t)$
T_x	time to deplete x
$\sigma^2_{T;x}$	variance of T_x

In the following subsection we discuss a Brownian motion demand process and the corresponding runout time distribution. Subsequently, we define and discuss a "beta demand process," and contrast this with the Brownian motion process.

Brownian Motion. The normal distribution is often used to model demand during an interval of time. That is, the demand, $d(\tau)$, during an interval of duration τ is assumed to be normal with mean $\tau\mu$ and variance $\tau\sigma^2$ for constants μ and σ . When t is a continuous variable, the demand process $X(t)$ is *Brownian motion* with drift μ and variance σ^2 . For convenience, we will refer to this simply as Brownian motion. The discrete-time version is referred to as a *random walk* with drift μ and variance σ^2 . We shall restrict our discussion to the Brownian motion process, but the concepts are easily applied to the discrete-time case, as well.

Although there are many situations where the normal distribution would not be an accurate representation of demand, it is often chosen because (1) forecast errors are often assumed to be normally distributed, (2) the two parameters of the normal distribution can be chosen to obtain different coefficients of variation, (3) the normal distribution can be standardized so that analysis

can be performed without specifying μ and σ , and (4) the normal distribution is an adequate approximation for many applications.

The normal distribution has another attractive property: its generating function is infinitely divisible. That is, the n th root of the generating function of the normal probability distribution is again a normal probability generating function. Thus, if for *any* interval of *arbitrary* duration τ , $d(\tau)$ is normal, then $X(t)$ is also normal.

For a Brownian motion demand process, the density of $X(t)$ is given by:

$$f(x;t) = (2\pi\sigma^2t)^{-1/2} \exp \left[-\frac{(x - \mu t)^2}{2\sigma^2t} \right].$$

The corresponding runout time distribution, $G(t;x)$, cannot be expressed in closed form, but the runout time density is given by:

$$g(t;x) = \frac{x}{\sigma\sqrt{2\pi t^3}} \exp \left[-\frac{(x - \mu t)^2}{2\sigma^2t} \right]. \quad (2)$$

It is interesting to note that $g(t;x) = \frac{x}{t}f(x;t)$. Figure 1 shows $g(t;x)$ for $\mu = 100$, $\sigma = 30$ and $x \in \{60, 80, 100\}$. Notice that the densities are positively skewed.

Although the normal distribution is often used to model probabilistic demand, there may be difficulties in doing so. The normal distribution cannot accurately represent, or even approximate, distributions that are skewed. In addition to being symmetric, the normal density has infinite tails. That is, $f(x;t) > 0$ for arbitrarily small and large values of x . This also means that all runout times greater than zero have positive probability. This is unrealistic in most production settings. In many manufacturing environments, natural limitations, such as limits on capacity or contractual agreements between the producer and customer, place bounds on the amount that the manufacturer is required to supply in any period of time. In other cases, due dates for large orders may be negotiated.

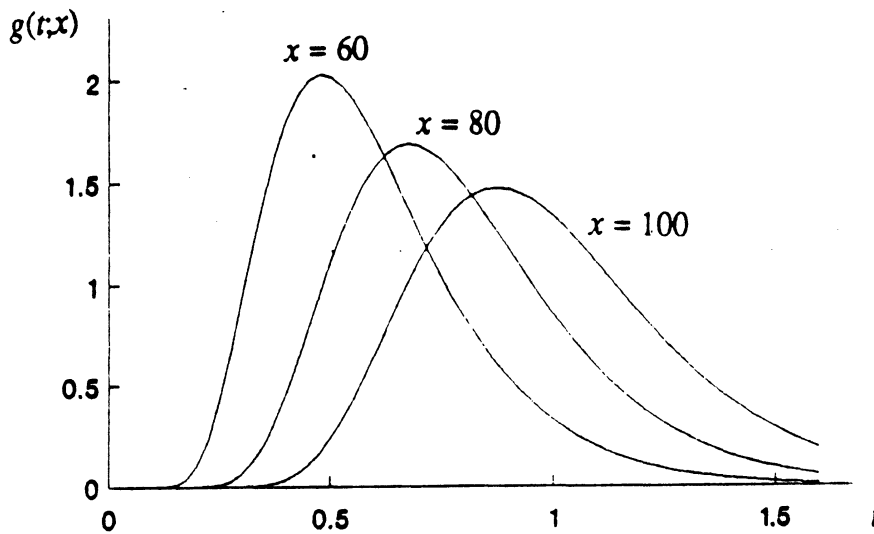


Figure 1. Runout time density functions corresponding to a Brownian motion process

Brownian motion also presents difficulties from an analytical standpoint. $G(t;x)$ can neither be expressed in closed form, nor standardized, nor computed easily and stored in a table. Hence, the runout time distribution given in (2) is difficult to use in an optimization routine. Modeling demand as a Brownian motion results in a problem that is numerically and analytically intractable.

As we mentioned earlier, the normal distribution is often chosen because it is common to assume forecast errors are normal. In practice, *cumulative* forecast errors usually are not normal (Brown 1963)^[1]. Even if the true demand process is stationary, the parameters of the process may be unknown. The forecast of the cumulative demand through time t is usually computed as the sum of single-period forecasts. In other words, for $t = n\tau$ where n is an integer and τ is the duration of a single forecasting period, the forecast for $X(t)$ is computed as the sum of the forecasts for the next n periods. As a result, the variance of the cumulative forecast errors through time t is determined by the variance of the underlying process, which is unknown, *and* the variances of the forecasts. Determining the variance of cumulative forecast errors can be difficult, especially if little is known about the underlying demand process.

There is empirical evidence that the variance of the forecast error for cumulative demand through time t (for $t = n\tau$) can be represented fairly accurately as $n^c \tau^c \sigma^2$, where σ^2 is the variance of the single period forecast and c is a constant that typically lies between 1.0 and 2.0 (Silver and Peterson 1979)^[10]. Notice that $c = 1.0$ implies that the forecast errors for each of the n periods are independent random variables. In other words, when the underlying process is a Brownian motion, the variance of the forecast errors equals the variance of the process. This implies that the parameters of the process are known exactly. When the parameters are *not* known, then the forecast errors for the n successive periods are serially correlated and the variance of the cumulative forecast errors is larger than the variance of the cumulative demand process. Other work suggests that $c = 3$ when exponential smoothing is used to forecast demand (Wecker 1979)^[11]. In these cases, the use of Brownian motion to model a cumulative demand process will significantly understate the true uncertainty about $X(t)$.

An Alternative to Brownian Motion. Now we define and discuss a *beta demand process* model which has many advantages over Brownian motion. This process has both range and shape parameters so it is possible to model demand distributions with a variety of shapes, and the values of $X(t)$ are bounded. For integral shape parameters, the cumulative demand distributions *and* the runout time distributions can be expressed as polynomials. Hence, analytic treatment becomes a realistic option. In addition, in many situations a beta process may provide a more accurate representation of the distribution of $X(t)$.

In general, the beta cumulative probability distribution function can be expressed in closed form. The *normalized* beta probability density function is given below for shape parameters (p, q) .

$$f_{\beta_{p,q}}(x) = \frac{1}{\beta_{p,q}} x^{p-1} (1-x)^{q-1}, \quad 0 < x < 1$$

where

$$\beta_{p,q} = \int_0^1 t^{p-1} (1-t)^{q-1} dt.$$

For integral p and q ,

$$\beta_{p,q} = \frac{(q-1)! (p-1)!}{(p+q-1)!}$$

and $F_{\beta_{p,q}}(x)$, which equals

$$\int_0^x f_{\beta_{p,q}}(x) dx,$$

is given by

$$\sum_{j=p}^{p+q-1} \frac{(p+q-1)!}{j! (p+q-1-j)!} x^j (1-x)^{p+q-1-j}.$$

To simplify the notation, we shall drop the shape parameter subscripts when they are clear from the context.

Recall that $X(t)$ is the total demand through time t where $X(0) = 0$. We shall consider such a process where $X(t)$ has a beta distribution with range parameters (at, bt) , that is, $at < X(t) < bt$.

The density and distribution functions are given below:

$$f(x;t) = \frac{1}{\beta t (b-a)} \left(\frac{x-at}{t(b-a)} \right)^{p-1} \left(\frac{bt-x}{t(b-a)} \right)^{q-1}, \quad at < x < bt; \quad (3)$$

and

$$F(x;t) = \int_{at}^x f(u;t) du = \frac{1}{\beta} \int_0^{\frac{x-at}{t(b-a)}} u^{p-1} (1-u)^{q-1} du = F_{\beta} \left(\frac{x-at}{t(b-a)} \right).$$

We shall refer to this process as a "beta process." The mean and variance are given below for fixed t :

$$E(X;t) = at + t(b-a) \frac{p}{p+q},$$

$$\sigma_{X;t}^2 = \frac{t^2(b-a)^2 pq}{(p+q)^2 (p+q+1)}. \quad (4)$$

(See Rohatgi (1976)[9] for a detailed discussion of the beta distribution.)

There are several differences between the beta and Brownian motion processes. As mentioned above, the possible values of $X(t)$ are bounded above and below in the beta process,

and the parameters (p, q, a, b) can be chosen to obtain a variety of shapes, means, and variances. Another difference between the beta and the Brownian motion demand processes is the relationship between the standard deviation of $X(t)$ and t . In a Brownian motion demand process, $\sigma_{X;t}$ increases as the square root of t . From (4) we see that $\sigma_{X;t}$ increases linearly with t when $X(t)$ is a beta process. In other words, if we use $X(t)$ as a representation of the forecast of cumulative demand through time t (for $t = n\tau$), then the variance of $X(t)$ is equal to $n^c \tau^c \sigma^2$ where σ^2 is the variance of the single-period forecast and $c = 2$. In some settings, this may overestimate the uncertainty about $X(t)$.

The linear increase in the variance of $X(t)$ is directly related to the fact that, unlike the normal distribution, the beta is *not* infinitely divisible. Therefore, although $X(t)$ has a beta distribution, $d(\tau)$ for *arbitrary* τ does not. In fact, there is little we can say analytically about the distribution of $d(\tau)$ for *arbitrary* τ . The beta process may be a poor approximation for Brownian motion, unless the range of t that is of interest is relatively small. However, we are not suggesting the beta process be used only as an approximation. Instead, we suggest that it may be *more appropriate* in many production settings.

In the remainder of this section, we assume $X(t) \geq 0 \forall t$. That is, $a \geq 0$, and hence, $d(\tau) \geq 0, \forall \tau$. In many situations demand is never negative, because any customer returns are more than offset by orders. When the lower bound on demand is greater than or equal to zero, then the event $X(t) \leq x$ is equivalent to $T_x \geq t$, and we have

$$G(t;x) = 1 - F(x;t).$$

From this it is a simple matter to express the general form of $g(t;x)$:

$$\begin{aligned} g(t;x) &= \frac{x}{\beta t^2 (b-a)} \left(\frac{x-at}{t(b-a)} \right)^{p-1} \left(\frac{bt-x}{t(b-a)} \right)^{q-1} \\ &= \frac{x}{t} f(x;t). \end{aligned} \tag{5}$$

In circumstances where the beta demand process is not appropriate, it is possible to model the runout time distribution itself as a beta distribution. In other words, we assume that

$$g(t;x) = \frac{1}{\beta x (b-a)} \left(\frac{t-ax}{x(b-a)} \right)^{p-1} \left(\frac{bx-t}{x(b-a)} \right)^{q-1}, \quad ax < t < bx \quad (6)$$

and then fit $g(t;x)$ to observations of T_x . However, recall that we are minimizing a function of $g(t;r)$ with respect to r . Hence, it would be necessary to fit $g(t;r)$ to observations of T_x for many (if not all) values of x . Continuing with our assumption that $X(t) \geq 0 \forall t$, $f(x;t)$ is given by

$$f(x;t) = \frac{t}{\beta x^2 (b-a)} \left(\frac{t-ax}{x(b-a)} \right)^{p-1} \left(\frac{bx-t}{x(b-a)} \right)^{q-1}$$

In this section we have given two possible methods of modeling runout time distributions for production and inventory control problems that provide alternatives to Brownian motion runout time. We believe the first--that is, modeling the demand process as a beta process--is the more practical alternative. Our primary interest in this paper is to find a tractable method of finding near optimal order quantities and reorder points in continuous review inventory systems. However, the beta demand process may be useful in many other problems.

4. Finding Optimal Solutions

In this section we discuss properties of the objective function when demand is modeled as a beta demand process and when the runout time distribution is itself modeled as a beta distribution. Detailed proofs of results are given in the Appendix. We also discuss solution procedures and incorporation of service level constraints.

Recall that our objective is to minimize the expected costs in a (Q,r) inventory system, and that the objective function (1) is:

$$h \left(\frac{Q}{2} + r - \mu \right) + \frac{AD}{Q} + \frac{\pi D}{Q} \int_0^t (t-u) dG(u;r) .$$

We will show that this is strictly quasiconvex jointly in Q and r for a beta demand process and for a beta runout time distribution.

First, we define the linear loss function, Z

$$Z(x,t) = \int_{-\infty}^t (t-u) dG(u;x).$$

Let $f(x;t)$ and $F(x;t)$ be the beta demand process density and cumulative distribution functions, respectively; and let $g(t;x)$ and $G(t;x)$ be the beta process runout density and distribution functions, respectively. Unless otherwise noted, it is assumed that (p,q) are the shape parameters, and we assume that they are integers. Also, let

$$\mathcal{F}_t = \{x : 0 < at < x < bt\}.$$

PROPOSITION 1. The objective function (1) is jointly quasiconvex in t and Q for $g(t;x)$ given by (5) or (6).

Proof: In the Appendix we show that for $g(t;x)$ given by (5) or (6), $Z(x,t)$ is convex in x for all x and t such that $x \in \mathcal{F}_t$. (We also note that Z is *strictly convex* $\forall x \in \mathcal{F}_t$ whenever $t > 0$.)

Thus, (1) can be expressed as a convex function divided by a nonnegative linear function, and is strictly quasiconvex. ■

In addition, a per-unit shortage cost can also be included in the analysis without sacrificing the strict quasiconvexity.

With the strict quasiconvexity of the objective function established, it is easy to find an optimal order quantity and reorder point for a beta demand process or for a beta runout time distribution. One approach would be to use standard nonlinear programming techniques. However, we observe that for a beta demand process, the optimal control parameters, Q^* and r^* can be found by simultaneously solving

$$Q = \sqrt{\frac{2D[A + \pi \tau(r)]}{h}} \quad (7)$$

and

$$\bar{\tau}(r) = \frac{hQ}{\pi D} \quad (8)$$

where

$$\bar{\tau} = \frac{1}{r} \int_{r/b}^t u g(u;r) du$$

and

$$\bar{\tau}(r) = \int_0^t (t-u) dG(u;r).$$

This can be accomplished by initializing r , then iteratively solving for Q and r until the solution converges. Solving (7) and (8) requires finding the roots of polynomials that may be of order $p + q - 1$. However, since $Z(x,t)$ is strictly convex for $t > 0$, (7) and (8) have a single solution for $r > 0$

For a beta runout time distribution, we can again solve for Q^* and r^* using (7) and (8) where

$$\bar{\tau}(r) = a(1 - F(x;t)) + \frac{p(b-a)}{p+q} (1 - F_{p+1,q}(x;t)).$$

This also requires finding the root of a polynomial of order $p + q - 1$ that has a single root for $r > 0$

In the absence of information about shortage penalties, service level constraints often are used to specify a reorder point. One such service level requirement might be a maximum acceptable probability of shortage, α . Another might be a maximum acceptable fraction of time the system is in a backorder position, α' . Without shortage costs, it is easy to show that these service levels should be satisfied at equality. If the probability of shortage is fixed, then r satisfies $F(r;t) = 1 - \alpha$. Fixing the percentage of time the item is out of stock equal requires an iterative

solution to (7) and $\bar{\tau}(r) = \frac{\alpha'Q}{D}$.

5. Lead-time Reductions

In this section we solve for the optimal lead time, order quantity, and reorder point where we assume that the lead time can be *reduced* to t at a cost of $k(t)$ dollars per order. The function $k(t)$ is convex decreasing in t for $0 > t \geq t'$ where t' is the original lead time and t is the improved lead time (cf. Porteus 1985)^[8]. We now have the following minimization problem:

$$\min_{Q,r,t} = h \left(\frac{Q}{2} + r - \mu \right) + \frac{(k(t) + AD)}{Q} + \frac{\pi D}{Q} \int_0^t (t-u) dG(u;r) \quad (9)$$

Q, r, t

subject to

$$0 \leq t \leq t'$$

This is a more complex problem. In the Appendix we prove the following result:

PROPOSITION 2. The objective function (9) is strictly quasiconvex in Q , r and t for $g(t;x)$ given by (5) or (6).

Proof: We continue with Z defined as

$$Z(x,t) = \int_0^t (t-u) dG(u;x).$$

In the Appendix we show that $Z(x,t)$ is jointly convex in x and t for $g(t;x)$ given by (5) or (6).

(We also note that $\frac{\partial^2 Z}{\partial t^2} = g(t;x) > 0$, $\forall (x,t)$ for which $g(t;x)$ is defined. Hence Z is strictly convex in t for any runout time distribution.) Therefore, (9) can be expressed as a convex function divided by a nonnegative linear function, and is therefore strictly quasiconvex. ■

Having proved the strict quasiconvexity of the objective function, we can solve this problem using standard nonlinear optimization techniques.

For example, consider an item with the characteristics given in Table 2 where the demand during lead time follows a beta demand process with $p = 5$ and $q = 5$, $a = 0.1$, and $b = 1.9$. If there is no opportunity for lead-time reduction, the optimal order quantity is 15 units, the optimal reorder point is 14.6 units, and the expected cost is \$19.56 per week.

Table 2
Base Case Parameter Values

Parameter	Units	Value
Mean demand rate	units / week	50
Holding cost rate ^a	\$/ unit-year	1
Fixed ordering cost	\$/ order	100
Late charge rate ^b	\$/ week	500
Current lead time	weeks	1

^a This assumes a 50 week year.

^b This assumes 5 days per week.

Now assume the cost to reduce the lead time to t is given by $-C \ln(t/t')$ where the C is the *lead-time reduction coefficient* and has units of dollars per order. This function has many intuitively appealing properties. The cost to reduce the current lead time by a given *fraction* is constant for any current lead time. It is also (i) convex decreasing in t ; (ii) zero at $t = t'$; and (iii) asymptotic to infinity as $t \rightarrow 0$. Examples of this function are given in Figure 2 for t/t' in the interval from 0.3 to 1.0. In our example $C = \$50$ per order.

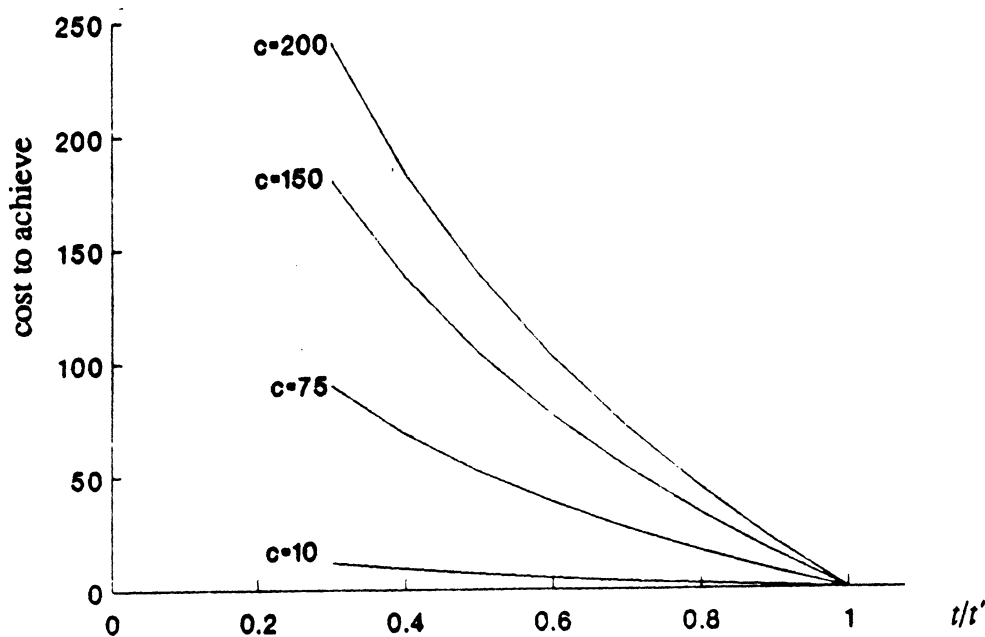


Figure 2. Cost to reduce lead time from t' to t

It is now optimal to reduce the lead time to 6.3 weeks at a cost of \$23.42 per order. The optimal reorder point drops to 9.1 units, and the order quantity is reduced slightly to 14.7 units. The expected cost is \$17.54 per week, or 11% less than the cost without lead-time reduction. We will refer to this problem (with the problem parameters given in Table 2 and $C = \$50$ per order) as the *base case*.

We solved problems for every combination of the parameters given in Table 3 with the other parameters fixed at the values from the base case. For each problem, we found the optimal order quantity, reorder point and lead time. Each problem was solved again with the lead time fixed at 10 weeks. Solutions were found using the General Algebraic Modeling System (GAMS)[2].

Table 3
Other Parameter Values

Lead-time Reduction Coefficient (\$/order)	Holding Cost Rate (\$/unit/week)	Late Charge Rate (\$/week)
25	2.0	1,000
50	1.0	750
785	0.75	500
100	0.5	250
150		100

It was optimal to reduce the lead time in only 53 of the 144 problems. In most of these problems, the optimal order quantity was not significantly different from the optimal order quantity with the lead time fixed. In addition, an average 85 % of the cost reduction came from lower holding costs.

As one might expect, the lead-time reduction coefficient, C , had a significant effect on the optimal lead time. With C at \$150 or \$200 per order, there was no benefit in reducing the lead time. These coefficients are not unusually high. For example, with C at \$200 per order, a 10% reduction in the lead time would cost \$21 per order, or a 21% increase in the fixed ordering cost.

With C set at \$100 per order, there were only three problems where a lead-time-reduction reduced costs. In these problems, the holding cost rate was at its highest level (\$2 per unit year), the late charges were at the three highest levels (\$1,000, \$750, and \$500 per week), and yet the reduction in total cost was less than 7%. To illustrate the effects of problem parameters, we give the ratio of the expected cost *without* lead-time reduction to the expected cost *with* the optimal lead-time reduction in Tables 4, 5, and 6, for several combinations of the problem parameters.

Table 4
Varying the Lead-time Reduction Coefficient

Lead-time Reduction Coefficient (\$/order)	Holding Cost Rate (\$/unit/week)	Late Charge Rate (\$/week)	Cost Ratio
25	1.0	500	1.27
50	1.0	500	1.12 ^a
75	1.0	500	1.02
100	1.0	500	1.0
150	1.0	500	1.0

^aThis row corresponds to the base case.

Table 5
Varying The Holding Cost Rate

Lead-time Reduction Coefficient (\$/order)	Holding Cost Rate (\$/unit/week)	Late Charge Rate (\$/week)	Cost Ratio
50	2.0	500	1.23
50	1.0	500	1.12 ^a
50	0.75	500	1.08
50	0.5	500	1.03

^aThis row corresponds to the base case.

Table 6
Varying The Late Charge Rate

Lead-time Reduction Coefficient (\$/order)	Holding Cost Rate (\$/unit/week)	Late Charge Rate (\$/week)	Cost Ratio
50	1.0	1,000	1.14
50	1.0	750	1.13
50	1.0	500	1.12 ^a
50	1.0	250	1.08
50	1.0	100	1.02
50	1.0	50	1.00

^aThis row corresponds to the base case.

Table 4 confirms the importance of the lead-time reduction coefficient. With $C = \$25$ per order, it is possible to reduce the costs by 27%. However, this means that a 50% reduction in the lead time increases the fixed ordering charge by only 17%. This may be unrealistic in many situations. From Table 5 we note that the holding cost rate is also a major contributor to the cost effectiveness of lead-time reduction. The largest holding cost rate (\$2 per unit per year) is not unusually high. In fact, we would expect to see much higher holding cost rates in many applications.

6. Summary and Directions for Further Research

In this paper we have modeled a continuous review inventory system where the shortage penalty is proportional to the length of time the item is out of stock. This requires the definition and manipulation of a runout time distribution. Few production and inventory control models use runout time distributions. In the runout-time-based (Q,r) model, the decision variable r appears as a parameter of the runout time distribution. This makes optimization with respect to r difficult.

Although Brownian motion is often used to model demand processes, there are difficulties with doing so. Brownian motion cannot adequately approximate demand distributions that are skewed, there is a positive probability of arbitrarily small (negative) and arbitrary large demands,

and the runout time distribution associated with Brownian motion is difficult to use in an optimization routine. In addition, modeling demand as a Brownian motion process may understate the uncertainty of demand when the parameters of the process are not known. We have defined a beta demand process and suggest this as an alternative that overcomes these problems.

We have shown that the objective function of the runout-time-based (Q,r) model is strictly quasiconvex in Q and r when the demand process is modeled as a beta process. This result also holds when the runout time has a beta distribution. Therefore, optimal solutions can be found with standard nonlinear programming techniques.

We have also modeled a more general problem, where lead-time reductions are available at a cost. We show that the objective function in this case is also strictly quasiconvex when the demand process is modeled as a beta process and when the runout time is beta-distributed. We give numerical examples that demonstrate the potential savings from reducing the lead time. In these examples the cost to reduce the lead time from t' to t is $-C \ln(t/t')$. As expected, the lead-time reduction coefficient, C , has a strong impact on the optimal lead times. The holding cost rate is also important. In many cases it is *not* optimal to reduce the lead time. When it *is* optimal to reduce the lead time, the majority of the savings come from reduced lead-time safety stock.

The beta demand process given here can be included in other interesting production and inventory control models. For example, the beta process may be useful in modeling the distribution of the inventory position of an item in a periodic-review inventory system. In these systems, the distribution of the inventory position can be determined by conditioning on a runout time. Runout times are also important when items share a common production resource.

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APPENDIX

A.1 Introduction

Most of this appendix deals with the derivation of partial derivatives of the following function:

$$Z(x,t) = \int_{-\infty}^t (t-u) dG(u;x) \quad (\text{A.1})$$

where $G(u;x)$ is a probability distribution function. In this work we consider two runout-time distributions. The first applies when demand is modeled as a beta process. That is,

$$f(x;t) = \frac{1}{\beta t(b-a)} \left(\frac{x-at}{t(b-a)} \right)^{p-1} \left(\frac{bt-x}{t(b-a)} \right)^{q-1}, \quad at < x < bt; \quad (\text{A.2})$$

$$F(x;t) = \int_{at}^x f(u;t) du = \frac{1}{\beta} \int_0^{\frac{x-at}{t(b-a)}} u^{p-1} (1-u)^{q-1} du \quad (\text{A.3})$$

which

$$= \sum_{j=p}^{p+q-1} \frac{p+q-1}{j!(p+q-1-j)!} \left(\frac{x-at}{t(b-a)} \right)^j \left(\frac{bt-x}{t(b-a)} \right)^{p+q-1-j}; \quad (\text{A.4})$$

and

$$g(t;x) = \frac{x}{\beta t^2(b-a)} \left(\frac{x-at}{t(b-a)} \right)^{p-1} \left(\frac{bt-x}{t(b-a)} \right)^{q-1} \quad (\text{A.5})$$

where

$$\beta = \frac{(q-1)!(p-1)!}{(p+q-1)!};$$

and p and q are integers.

We also consider the case where the runout-time distribution itself has a beta distribution.

Then we have

$$g(t;x) = \frac{1}{\beta x (b-a)} \left(\frac{t-ax}{x(b-a)} \right)^{p-1} \left(\frac{bx-t}{x(b-a)} \right)^{q-1}, \quad ax < t < bx; \quad (\text{A.6})$$

and

$$f(x;t) = \frac{t}{\beta x^2 (b-a)} \left(\frac{t-ax}{x(b-a)} \right)^{p-1} \left(\frac{bx-t}{x(b-a)} \right)^{q-1}. \quad (\text{A.7})$$

We assume that $a > 0$. Then we have:

$$G(t;x) = 1 - F(x;t), \quad (\text{A.8})$$

and

$$g(t;x) = \frac{x}{t} f(x;t). \quad (\text{A.9})$$

A.2 Preliminary Results

In this section we will derive $\frac{\partial^2 Z(x,t)}{\partial x^2}$. These results, are given in Lemmas A.1 and A.2.

For notational convenience we will use Z in place of $Z(x,t)$ whenever the meaning is clear from context.

We rewrite Z as

$$Z(x,t) = tG(t;x) - \int_{-\infty}^t u dG(u;x). \quad (\text{A.10})$$

The region over which Z is defined is given as

$$\mathcal{F} = \{ x,t : 0 < at < x < bt \}.$$

LEMMA A.1: For $g(t;x)$ given by (A.5) and $f(x;t)$ given by (A.2),

$$\frac{\partial^2 Z}{\partial x^2} \text{ exists } \forall x \in \mathcal{F} \text{ and } \frac{\partial^2 Z}{\partial x^2} = \frac{t^2}{x^2} g(t;x).$$

Proof: Z is continuous and twice differentiable for $x \in \mathcal{F}$. Hence $\frac{\partial Z}{\partial x}$ and $\frac{\partial^2 Z}{\partial x^2}$ exist for

$\forall x \in \mathcal{F}$. Substituting (A.8) and (A.9) into (A.10), we have

$$Z = t (1 - F(x;t)) - \int_{x/b}^t x f(x;u) du.$$

From this we have

$$\frac{\partial Z}{\partial x} = t f(x;t) - \int_{x/b}^t \left(x \frac{\partial}{\partial x} f(x;u) + f(x;u) \right) du + \frac{x}{b} f(x;x/b).$$

The last term is zero, and next we show that $\int_{x/b}^t x \frac{\partial}{\partial x} f(x;u) du = -t f(x;t)$. We will then

have the following simple expression

$$\frac{\partial Z}{\partial x} = - \int_{x/b}^t f(x;u) du.$$

$$\begin{aligned} \frac{\partial}{\partial x} f(x;t) &= \frac{(p+q-1)!}{t(b-a)(p-1)!(q-1)!} * \\ &\quad \left[\frac{p-1}{t(b-a)} \left(\frac{t-ax}{x(b-a)} \right)^{p-2} \left(\frac{bx-t}{x(b-a)} \right)^{q-1} \right. \\ &\quad \left. - \frac{q-1}{t(b-a)} \left(\frac{t-ax}{x(b-a)} \right)^{p-1} \left(\frac{bx-t}{x(b-a)} \right)^{q-2} \right] \\ &= \frac{(p-1)(p+q-1)(p+q-2)!}{(p-1)(p-2)!(q-1)!t^2(b-a)^2} \left(\frac{t-ax}{x(b-a)} \right)^{p-2} \left(\frac{bx-t}{x(b-a)} \right)^{q-1} \\ &\quad - \frac{(q-1)(p+q-1)(p+q-2)!}{(p-1)!(q-1)(q-2)!t^2(b-a)^2} \left(\frac{t-ax}{x(b-a)} \right)^{p-1} \left(\frac{bx-t}{x(b-a)} \right)^{q-2} \\ &= \frac{p+q-1}{t(b-a)} \left(f_{p-1,q}(x;t) - f_{p,q-1}(x;t) \right). \end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{x/b}^t x \frac{\partial}{\partial x} f(x;u) du \\
&= \frac{x(p+q-1)}{(b-a)} \int_{x/b}^t \frac{1}{u} f_{p-1,q}(x;u) du - \frac{x(p+q+1)}{(b-a)} \int_{x/b}^t \frac{1}{u} f_{p,q-1}(x;u) du. \quad (\text{A.11})
\end{aligned}$$

Now

$$\begin{aligned}
& \frac{x(p+q-1)}{(b-a)} \int_{x/b}^t \frac{1}{u} f_{p-1,q}(x;u) du \\
&= \frac{x(p+q-1)}{\beta_{p-1,q}(b-a)^2} \int_{x/b}^t \frac{1}{u^2} \left(\frac{x-au}{u(b-a)} \right)^{p-2} \left(\frac{bu-x}{u(b-a)} \right)^{q-1} du
\end{aligned}$$

which, using the substitution $v = \frac{x-au}{u(b-a)}$ gives us

$$\frac{(p+q-1)}{(b-a)} \left(1 - F_{p-1,q}(x;t) \right) \quad (\text{from (A.3)}). \quad (\text{A.12})$$

Similarly,

$$\frac{x(p+q-1)}{(b-a)} \int_{x/b}^t \frac{1}{u} f_{p,q-1}(x;u) du = \frac{(p+q-1)}{(b-a)} \left(1 - F_{p,q-1}(x;t) \right) \quad (\text{A.13})$$

Now we have

$$\int_{x/b}^t x \frac{\partial}{\partial x} f(x;u) du = -\frac{(p+q-1)}{(b-a)} (F_{p,q-1}(x;t) - F_{p-1,q}(x;t)).$$

However,

$$\left(F_{p,q-1}(x;t) - F_{p-1,q}(x;t) \right) = -\frac{t(b-a)}{p+q-1} f(x;t) \quad (\text{from (A.4)}) \quad (\text{A.14})$$

and hence,

$$\int_{x/b}^t x \frac{\partial}{\partial x} f(x;u) du = -t f(x;t).$$

Thus,

$$\frac{\partial z}{\partial x} = - \int_{x/b}^t f(x;u) du = -\frac{1}{x} \int_{x/b}^t u g(u;x) du. \quad (\text{A.15})$$

So,

$$\frac{\partial^2 Z}{\partial x^2} = - \int_{x/b}^t \frac{\partial}{\partial x} f(x;u) du$$

which, by reasoning similar to (A.11) through (A.14), gives

$$\frac{\partial^2 Z}{\partial x^2} = \frac{t}{x} f(x;t) = \frac{t^2}{x^2} g(t;x). \quad \blacksquare$$

LEMMA A.2. For $g(t;x)$ given by (A.6) and $f(x;t)$ given by (A.7), $\frac{\partial^2 Z}{\partial x^2}$ exists for all

$$x \in \mathcal{F} \text{ and } \frac{\partial^2 Z}{\partial x^2} = \frac{t^2}{x^2} g(t;x).$$

Proof: We begin by noting that

$$\int_{-\infty}^t u dG(u;x) = \frac{(p+q-1)!}{(p-1)!(q-1)!} \int_{ax}^t \frac{u}{x(b-a)} \left(\frac{u-ax}{x(b-a)} \right)^{p-1} \left(\frac{bx-u}{x(b-a)} \right)^{q-1} du$$

which, using the substitution $v = \frac{u-ax}{t(b-a)}$, gives us

$$\frac{p}{p+q} G_{p+1,q}(t;x) + ax G(t;x).$$

Hence, using this and (A.8), the objective function (A.10) can be written as

$$Z = t(1 - F(x;t)) - ax(1 - F(x;t)) - \frac{px(b-a)}{p+q} (1 - G_{p+1,q}(t;x)).$$

Now,

$$\begin{aligned} \frac{\partial Z}{\partial x} &= (ax - t)f(x;t) - a(1 - F(t;x)) - \frac{p(b-a)}{p+q} (1 - F_{p+1,q}(t;x)) \\ &\quad + \frac{p(b-a)x}{p+q} f_{p+1,q}(t;x). \end{aligned} \quad (\text{A.16})$$

To simplify this further, we note that

$$g_{p+1,q}(t;x) = \frac{(p+q)(t-ax)}{xp(b-a)} g(t;x). \quad (\text{A.17})$$

Substituting (A.9) into (A.17) and then (A.17) into (A.16) we have:

$$\frac{\partial Z}{\partial x} = -a(1 - F(x;t)) - \frac{p(b-a)}{p+q} (1 - F_{p+1,q}(x;t)).$$

Now,

$$\begin{aligned} \frac{\partial^2 Z}{\partial x^2} &= af(x;t) + \frac{p(b-a)}{p+q} f_{p+1,q}(x;t) \\ &= \frac{t^2}{x^2} g(t;x) \quad (\text{from (A.9) and (A.17)}). \end{aligned}$$

■

A.3 Proofs of Lemmas and Propositions

In this paper we consider two optimization problems. In the first, the decision variables are r and Q . The objective function is given below.

$$h \left(\frac{Q}{2} + r - \mu \right) + \frac{AD}{Q} + \frac{\pi D}{Q} \int_0^t (t-u) dG(u;r) \quad (\text{A.18})$$

We begin this subsection with a proof of the convexity of $Z(x,t)$ with respect to the variable x .

This is then used to show that (A.18) is strictly quasiconvex in r and Q .

Next, we address the convexity of the second objective function,

$$h \left(\frac{Q}{2} + r - \mu \right) + \frac{AD + K(t)}{Q} + \frac{\pi D}{Q} \int_0^t (t-u) dG(u;r). \quad (\text{A.19})$$

This is very similar to (A.18), but the decision variables are t , Q , and r and we include the convex function $K(t)$. Lemmas A.5 and A.6 show that Z given by (A.10) is jointly convex in t and x .

These results are then used to establish the strict quasiconvexity of (A.19).

LEMMA A.3. For $g(t;x)$ given by (A.5) or (A.6), $Z(x,t)$ given by A.1 is convex in x for $\forall x \in \mathcal{F}$.

Proof: In Lemmas A.1 and A.2 we show that for $g(t;x)$ given by (A.5) or (A.6), $\frac{\partial^2 Z}{\partial x^2}$ exists

$\forall x \in \mathcal{F}$ and is given by

$$\frac{t^2}{x^2} g(t;x) \quad (\text{A.20})$$

which is nonnegative $\forall x \in \mathcal{F}$. Hence, Z is convex in X for $g(t;x)$ given by (A.5) or (A.6). ■

With the convexity of Z with respect to x established, it is easy to show that (A.19) is strictly quasiconvex in the decision variables t and Q . To do so, we give the following lemma.

LEMMA A.4. Let $f:R^n \rightarrow R^1$ be a convex, differentiable function, and let $g:R^n \rightarrow R^1$ be a positive linear function. Then $f(x) / g(x)$ is strictly quasiconvex.

Proof: Let $x_1, x_2, \in R^n$ be arbitrary points such that

$$\frac{f(x_1)}{g(x_1)} \neq \frac{f(x_2)}{g(x_2)}$$

and assume without loss of generality that

$$\frac{f(x_1)}{g(x_1)} > \frac{f(x_2)}{g(x_2)}$$

For strict quasiconvexity we need to show

$$\frac{f(x_1)}{g(x_1)} > \frac{f(\lambda x_1 + (1 - \lambda)x_2)}{g(\lambda x_1 + (1 - \lambda)x_2)} \text{ for } 0 < \lambda < 1.$$

However, since f is convex, we have $\lambda f(x_1) + (1 - \lambda)f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2)$ and g is linear, we have $g(\lambda x_1 + (1 - \lambda)x_2) = \lambda g(x_1) + (1 - \lambda)g(x_2)$. Hence, it suffices to show that

$$\frac{f(x_1)}{g(x_1)} > \frac{\lambda f(x_1) + (1 - \lambda)f(x_2)}{\lambda g(x_1) + (1 - \lambda)g(x_2)}.$$

By assumption $g(x) > 0 \forall x$ and

$$\frac{f(x_1)}{g(x_1)} > \frac{f(x_2)}{g(x_2)}$$

which implies

$$(1 - \lambda)f(x_1)g(x_2) > (1 - \lambda)g(x_1)f(x_2)$$

and

$$f(x_1)[(1 - \lambda)g(x_2) + \lambda g(x_1)] > g(x_1)[(1 - \lambda)f(x_2) + \lambda f(x_1)].$$

Thus,

$$\frac{f(x_1)}{g(x_1)} > \frac{\lambda f(x_1) + (1 - \lambda)f(x_2)}{\lambda g(x_1) + (1 - \lambda)g(x_2)}$$

and $\frac{f(x_1)}{g(x_1)}$ is strictly quasiconvex. ■

We are now ready to prove the quasiconvexity of the objective function.

PROPOSITION 3.1. The objective function (A.18) is strictly quasiconvex in t and Q for $g(t;x)$ given by (A.5) or (A.6).

Proof: From Lemma A.3 we conclude that

$$AD + \pi D \int_0^t (t-u) dG(u;r)$$

is a convex function (and strictly convex for $t > 0$.) The strict quasiconvexity of (A.18) follows directly from Lemma A.4. ■

In the following two lemmas we show the joint convexity of $Z(t,x)$ for runout-time densities given by (A.5) and (A.6).

LEMMA A.5. For $g(t;x)$ given by (A.5) $Z(t,x)$ is jointly convex in t and x for all $(x,t) \in \mathcal{F}$.

Proof: Let $H = \text{Hessian}$ of Z with $g(t;x)$ given by (A.5).

$$\text{Det}(H) = \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial t^2} - \left(\frac{\partial^2 Z}{\partial t \partial x} \right)^2 \quad (\text{A.21})$$

Z is continuous and twice differentiable for all $(x,t) \in \mathcal{F}$. Hence, $\frac{\partial Z}{\partial t}$ and $\frac{\partial^2 Z}{\partial t^2}$ exist for all $(x,t) \in \mathcal{F}$, and

$$\frac{\partial Z}{\partial t} = G(t;x), \quad (\text{A.22})$$

$$\frac{\partial^2 Z}{\partial t^2} = g(t;x). \quad (\text{A.23})$$

From (A.22) and (A.8),

$$\frac{\partial^2 Z}{\partial t \partial x} = \frac{\partial}{\partial x} (1 - F(x;t)) = -f(x;t). \quad (\text{A.24})$$

Substituting (A.23), (A.20), and (A.24) into (A.21), gives $\det(H) = 0$ and, hence, there is at most one nonzero eigenvalue of H . Let λ be the nonzero eigenvalue of H , and let I be the identity matrix. Then $\det(H - \lambda I)$ is given by

$$\left(\frac{\partial^2 Z}{\partial x^2} - \lambda \right) \left(\frac{\partial^2 Z}{\partial t^2} - \lambda \right) - \left(\frac{\partial^2 Z}{\partial t \partial x} \right)^2. \quad (\text{A.25})$$

If we let

$$\lambda = \frac{x^3 + xt^2}{tx^2} f(x;t) \quad (\text{A.26})$$

then substituting this, (A.23), (A.20), and (A.24) into (A.25) gives $\det(H - \lambda I) = 0$. Hence, λ given by (A.26) is an eigenvalue of H . Now $x > 0$, $t > 0$, and $f(x;t) > 0 \forall x, t \in \mathcal{F}$. Therefore, H is positive semidefinite and hence, Z is jointly convex in t and x . ■

Note also, from (A.23) that $\frac{\partial^2 Z}{\partial t^2} = g(t;x)$ which is zero for all x and t for which $g(t;x)$ is defined. Hence, Z is strictly convex in t for any runout-time distribution.

LEMMA A.6. For $g(t;x)$ given by (A.6), Z is jointly convex in t and $x \forall (x,t) \in \mathcal{F}$.

Proof: Let $H = \text{Hessian of } Z \text{ with } g(t;x) \text{ given by (A.6)}$. In Lemma A.5 we have shown that

$$\frac{\partial Z}{\partial x} = \left(-a G(t;x) - \frac{p(b-a)}{p+q} G_{p+1,q}(t;x) \right). \text{ Therefore,}$$

$$\frac{\partial^2 Z}{\partial t \partial x} = -a g(t;x) - \frac{p(b-a)}{p+q} g_{p+1,q}(t;x).$$

However,

$$g_{p+1,q}(t;x) = \frac{(p+q)(t-ax)}{xp(b-a)} g(t;x)$$

which gives

$$\frac{\partial^2 Z}{\partial t \partial x} = -\frac{t}{x} g(t;x). \quad (\text{A.27})$$

Thus,

$$\det(H) = \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial t^2} - \left(\frac{\partial^2 Z}{\partial t \partial x} \right)^2. \quad (\text{A.28})$$

Substituting (A.20), (A.27), and (A.23) into (A.28) gives $\det(H) = 0$ and, hence, there is at most one nonnegative eigenvalue of H . Let λ be the nonzero eigenvalue of H , and let I be the identity matrix. Then $(H - \lambda I)$ is given by

$$\left(\frac{\partial^2 Z}{\partial x^2} - \lambda \right) \left(\frac{\partial^2 Z}{\partial t^2} \right) - \left(\frac{\partial^2 Z}{\partial t \partial x} \right)^2. \quad (\text{A.29})$$

Letting

$$\lambda = \frac{\partial^2 Z}{\partial x^2} + g(t;x) \quad (\text{A.30})$$

and substituting this, (A.23), (A.20), and (A.27) into (A.29) gives $(H - \lambda I) = 0$. Hence, λ

given by (A.30) is an eigenvalue of H . Now $\frac{\partial^2 Z}{\partial x^2} > 0$ and $g(t;x) > 0 \quad \forall (x,t) \in \mathbf{F}$.

Therefore, H is positive semidefinite and Z is jointly convex in t and x .

With the results of these two lemmas it is again easy to show the quasiconvexity of the objective function for either runout-time distribution.

PROPOSITION 2. The objective function (A.19) is strictly quasiconvex in Q, r , and t for $g(t;x)$ given by (A.5) or (A.6).

Proof: Follows directly from Lemmas A.4, A.5, and A.6.