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# NETWORK FUNCTION DETERMINATION FROM PARTIAL SPECIFICATIONS

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ABSTRACT

In certain problems of Network Theory the real and imaginary parts of a driving-point impedance function may be independently given in a band of interest ($\omega_1 \leq \omega \leq \omega_2$), leaving its continuations onto ($0 \leq \omega \leq \omega_1$) and ($\omega_2 \leq \omega \leq \infty$) completely unspecified.

By manipulating the Hilbert Transforms, relating the real and imaginary parts of a function of a complex variable ($p = \sigma + j\omega$) having no poles on the $j\omega$ axis or in the right-half plane, this report shows that if the continuations exist they are unique and readily obtained.

Three necessary conditions for the existence of the continuations to the given parts (of the driving-point impedance function) are obtained. Further, if these three conditions are satisfied the continuations may be obtained as a Fourier series.

Four known impedances were used as examples and their continuations determined. The results obtained were excellent. The agreement between the Fourier series solution (using the first six terms at most) and the exact expression was good up to the second significant figure.

In the appendix a computer program which obtains the Fourier coefficients of the unknown continuations, from the given real and imaginary parts, is furnished.

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## PRINCIPAL SYMBOLS

| Symbol | Description | First Reference |
|---|---|---|
| $Z(p)$ | Driving-point impedance function | Eq. (1.1) |
| $R(\omega)$ | Real part of driving point impedance function for $p = j\omega$ | Eq. (1.1) |
| $X(\omega)$ | Imaginary part of driving-point impedance function for $p = j\omega$ . | Eq. (1.2) |
| $R^*(\phi)$ | Real part of driving-point impedance function in the $\phi$ domain; i.e., $\omega = \sec \phi/2$ . | Eq. (2.13) |
| $k_n$ | Coefficients of power series for $f(\omega)$ | Eq. (2.19) |
| $B_{nm}$ | $\begin{cases} = \dfrac{1}{2^{2n}} \dbinom{2n}{n} & ; n = m \\ = \dfrac{1}{2^{2n-1}} \dbinom{2n}{m} & ; n \neq m \end{cases}$ | Eq. (2.25) |
| $\dbinom{A}{B}$ | $= \dfrac{A!}{(A-B)!\,B!}$ | Eq. (2.25) |
| $A_j$ | Fourier coefficients for unknown $R^*(\phi)$ | Eq. (2.28) |
| $H_n$ | Coefficients of power series for $g(\omega)$ | Eq. (2.36) |
| $E^*(\phi)$ | Imaginary part of driving-point impedance function multiplied by $\cos \phi/2$ in the $\phi$ domain | Eq. (2.37) |
| $T_m$ | Fourier coefficients for unknown function $E^*(\phi)$ | Eq. (2.37) |
| $H_{nm}$ | $= \begin{cases} 2^{2n-1} & ; n = m \\ (-1)^{n-m} \dfrac{2n}{n-m} 2^{2m-1} \dbinom{n+m-1}{n-m-1} & ; n \neq m \end{cases}$ | Eq. (2.43) |

List of Principal Symbols (Cont'd)

| Symbol | Description | First Reference |
|---|---|---|
| $R^*(\alpha)$ | Real part of driving-point impedance function for $0 \le \omega \le 1$ and $\omega = \cos \alpha/2$ | Eq. (2.41) |
| $R_m$ | Coefficients of power series about the origin for $R(\omega)$ | Eq. (2.45) |
| $B_n$ | Fourier coefficients for $R^*(\alpha)$ | Eq. (2.41) |
| $\frac{(2j-1)!!}{(2j)!!}$ | $= \frac{1 \cdot 3 \cdot 5 \cdots (2j-1)}{2 \cdot 4 \cdot 6 \ldots (2j)}$ | Eq. (2.48) |
| $C_n$ | Fourier coefficients for $\omega X(\omega)$ in $(0 \le \omega \le 1)$ | Eq. (2.54) |
| $D_n$ | Fourier coefficients for $R(\omega)$ in $(\omega_1 \le \omega \le \omega_2)$ when $\omega^2 = (\omega_2^2 - \omega_1^2) \cos^2 \frac{\beta}{2} + \omega_1^2$ | Eq. (2.74) |

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## Chapter One

## INTRODUCTION

It has been known for some time that the back scattering radar cross section of a metallic object may be controlled by the method of impedance loading[1]. That is, the return from an object may be increased or decreased, depending on the loading impedance.

By properly loading an object its return can, in some instances, be reduced considerably. For example, in the case of a dipole antenna, Chen and Liepa[2] have shown that the return can be reduced by 33 db. at broadside incidence. For the case of a sphere, Liepa and Senior[3] have shown that the return can be reduced by 22 db. Using a different approach, Chang and Senior[4] have obtained a 15 db. reduction in the case of a sphere.

The loading impedance function necessary to achieve this reduction (for the sphere case) is shown in Fig.1-1. As can be seen, the impedance function is not positive real (the real part is not positive for all frequencies) and cannot be realized with passive elements. However, considering that the real part of this function remains positive in the band $0 \leq ka^{**} \leq 1.5$, we might hope to change its behavior outside the band such that the function obtained is positive real. However, once the behavior of an impedance function is given in a band of interest, how much freedom is there as to the choice of its continuations outside the band? Is the behavior of an impedance function completely determined for all frequencies by specifying its behavior in a band of interest? In other words:

> Given independently (in the form of a graph) the real and imaginary parts of a passive driving-point impedance function in a band (0, 1)*
>
> (a) What can be said about the compatibility of the given parts; i.e. are they the real and imaginary

---

* For simplicity, the band of interest has been taken as (0, 1). However, in a later section, the solution to the problem will be extended to the band $(\omega_1, \omega_2)$.

** For this case a: radius of sphere, k: $\frac{2\pi}{30}$ f (f in MHz).

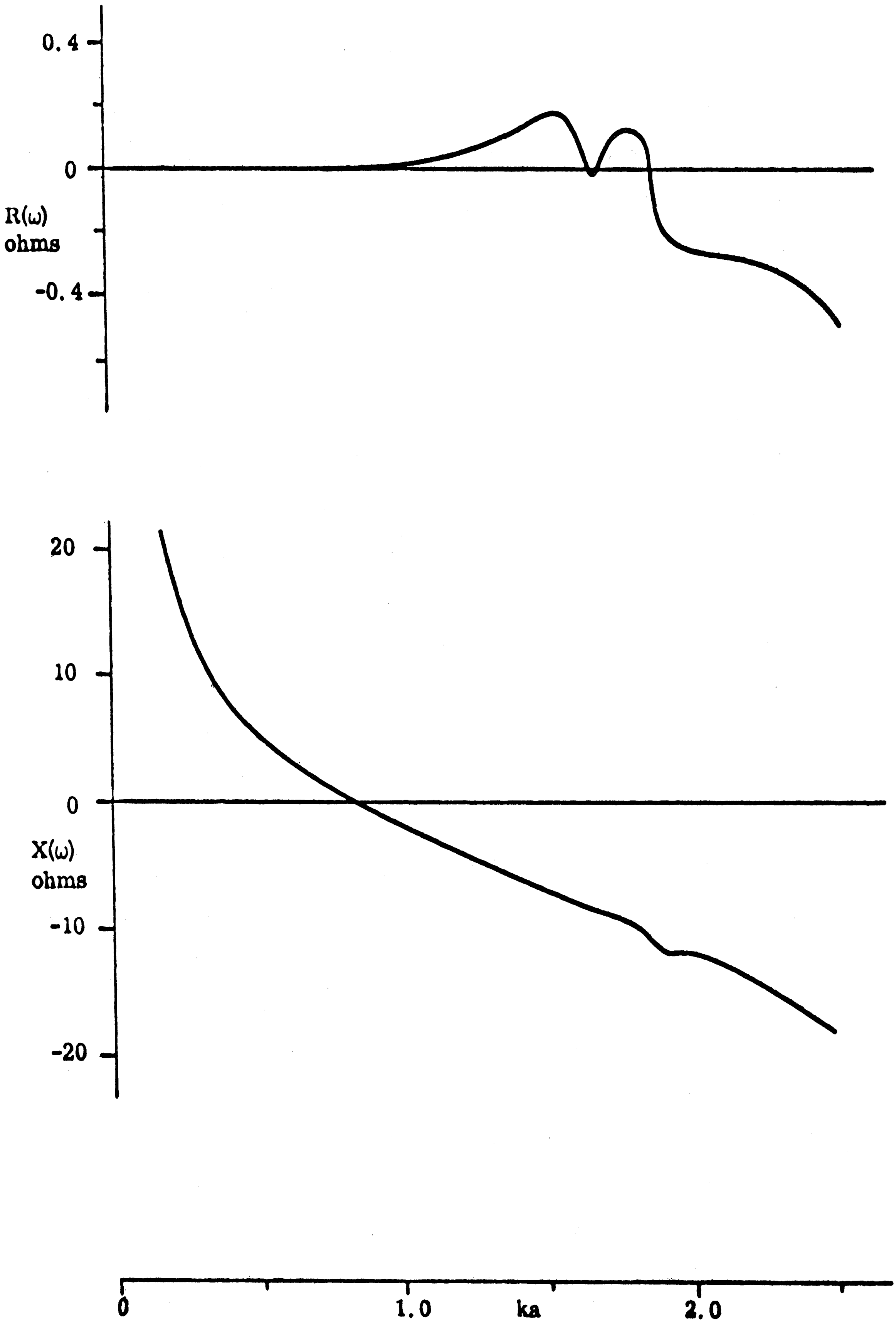


**FIG. 1-1: IMPEDANCE NEEDED TO REDUCE THE RETURN FROM A SPHERE BY 15 db. (after Chang and Senior).**

parts of a positive real function in the band (0, 1)?

(b) What can be said about the continuations onto $(1, \infty)$? Are they also the real and imaginary parts of the same positive real function in $(1, \infty)$?

(c) What conditions must be imposed on the given parts, in the band (0, 1), such that (a) and (b) are satisfied?

(d) Are the continuations unique?

(e) What are the continuations onto $(1, \infty)$?

Assuming that the impedance function in question:

(i) has no poles on the $j\omega$ axis or in the right-half plane

this report gives an adequate answer to each of the above questions.

This assumption is necessary for the sole purpose of uniqueness; i.e. a pole on the $j\omega$ axis may be partially or completely removed without changing the behavior of the real part (thereby making the impedance function not unique).

In answering question (a), (b) and (c) a set of three necessary (though not sufficient) conditions are obtained which test whether the given real and imaginary parts are compatible in the band of interest.

Unfortunately, the solution for the continuations of the given parts can not be obtained in a closed form due to the nature of the integral equations relating the given parts and their continuations.[13] However, a Fourier series solution can be obtained. This series, as the example solved in a later chapter will show, converges very rapidly.

Before going into the solution of the problem, let us review some of the work that has been done previously.

Bode[5] pointed out that having the real or imaginary part of a passive driving-point impedance function as a rational function of frequency would completely specify the impedance function for all frequencies. He noted that in going from the real to the complex frequency domain ($\omega \to p/j$) one would obtain:

$$R(\omega \to p/j) = \mathrm{Ev}\, Z(p) = 1/2 \left[ Z(p) + Z(-p) \right] \tag{1.1}$$

and

$$X(\omega \to p/j) = \mathrm{Od}\, Z(p) = 1/2 \left[ Z(p) - Z(-p) \right] \quad . \tag{1.2}$$

Hence, expanding the even or odd part of the impedance function (whichever is known and is rational) in a partial fraction expansion, one can identify those terms having left-half plane poles with $Z(p)$ and those with right-half plane poles $Z(-p)$.

Guillemin[6] pointed out that by making use of the Hilbert Transforms the real (or imaginary) part of a driving-point impedance function may be obtained from the imaginary (or real) part. The Hilbert Transforms assume that the impedance function in question has no poles in the right-half plane or on the $j\omega$ axis. These transforms are:

$$R(\omega) = -\frac{2}{\pi} \int_0^\infty \frac{\lambda X(\lambda)\, d\lambda}{\lambda^2 - \omega^2} + R(\infty) \tag{1.3}$$

and

$$X(\omega) = \frac{2\omega}{\pi} \int_0^\infty \frac{R(\lambda)\, d\lambda}{\lambda^2 - \omega^2} \quad . \tag{1.4}$$

Both of these methods require some information for all frequencies; i.e. the real or imaginary part of the impedance function must be known throughout the whole spectrum.

However, for the problem of impedance loading the real and imaginary parts of the impedance function are given in a band of frequencies $\omega_1 \leq \omega \leq \omega_2$

leaving its continuations onto $0 \leq \omega \leq \omega_1$ and $\omega_2 \leq \omega \leq \infty$ completely unspecified.

For this case Calahan[7] has suggested matching the known impedance function behavior with an expansion of positive-real, rational functions in the band $\omega_1 \leq \omega \leq \omega_2$; the coefficients of these rational functions being determined by computer optimization techniques.

Another method (due to Redheffer[8]) of solving this problem makes use of the change of variable $\omega = \tan \frac{\phi}{2}$. Then, the band of interest $\omega_1 \leq \omega \leq \omega_2$ becomes $\phi_1 = 2 \tan^{-1} \omega_1 \leq \phi \leq \phi_2 = 2 \tan^{-1} \omega_2$. In the $\phi$ domain Fourier series which give the best fit (in the least square sense) to the real and imaginary parts in $\phi_1 \leq \phi \leq \phi_2$ can be found (subject to the Weiner-Lee criterion for physical realizability*) . Unfortunately, the equations for the Fourier coefficients are rather difficult to generate and, as pointed out by Redheffer, it may be shown that the Fourier series will not in general be bounded for $\phi_2 \leq \phi \leq \pi$ .

Still another method is that suggested by Zeheb and Lempel[9]. If the behavior of an impedance function is known for a finite number of frequencies, a passive network can be synthesized such that its impedance takes on the prescribed values at the given frequencies. However, there is no restriction as to the behavior of the impedance other than at the given frequencies, thereby allowing more than one solution.

This report is concerned with the problem of determining the behavior of an impedance function outside of the band where its behavior is completely specified. That is, given an impedance function with a prescribed behavior in a band $\omega_1 \leq \omega \leq \omega_2$ it is desired to find the solution for the function's continuation onto $0 \leq \omega \leq \omega_1$ and $\omega_2 \leq \omega \leq \infty$.

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* If $R(\phi) = \sum_{n=0}^{\infty} A_n \cos n\phi$ and $X(\phi) = \sum_{n=1}^{\infty} B_n \sin n\phi$, then:

$A_n + B_n = 0$, for all $n = 1, 2, \ldots$ .

In contrast to the methods outlined above, this report does not use any approximation techniques (other than the truncation of an infinite series) and if certain necessary conditions are satisfied, it gives bounded and unique solutions for the continuations.

# Chapter Two

# THEORETICAL FORMULATION

## 2.1 Hilbert Transforms

If $Z(p)$ (where $p$ is the complex variable $p = \sigma + j\omega$) is a driving-point impedance function, then it is positive-real and by definition:

a) $Z(p)$ is analytic in the right half plane

b) Re $\left[Z(j\omega)\right] \geq 0$ for all $\omega$

c) $Z(p)$ is real for $p$ real.

Given a function $G(p)$, having no poles on the $j\omega$ axis or the right half plane, we define the function $F(p)$ by:

$$F(p) \triangleq \frac{G(p)}{p - j\omega_o} \tag{2.1}$$

Then from the Cauchy-Goursat integral theorem[10] we have:

$$\oint_c F(p)\, dp = 0 \tag{2.2}$$

where $c$ is the contour shown in Fig. 2-1. Writing $G(p)$ in the form:

$$G(p) = H_1(\sigma, \omega) + j\, H_2(\sigma, \omega) \tag{2.3}$$

we have from (2.2)

$$\oint_c \frac{H_1(\sigma, \omega) + j\, H_2(\sigma, \omega)}{p - j\omega_o}\, dp = 0 \ . \tag{2.4}$$

Carrying out the contour integration, letting $r \to 0$ and $R \to \infty$ and separating real and imaginary parts, we obtain:

$$\left[H_1(0,\omega) + jH_2(0,\omega)\right] - \left[H_1(0,\infty) + jH_2(0,\infty)\right] = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H_2(0,\lambda) - jH_1(0,\lambda)}{\lambda - \omega}\, d\lambda \ . \tag{2.5}$$

Let:
$$G(p) = \frac{Z(p)}{\sqrt{1+p^2}} = \frac{R+jX}{\sqrt{1+p^2}} \tag{2.6}$$

so that:

$$H_1(0,\omega) = H_1(\omega) = \begin{cases} \dfrac{R(\omega)}{\sqrt{1-\omega^2}} & 0 \leq \omega \leq 1 \\ \dfrac{X(\omega)}{\sqrt{\omega^2-1}} & 1 \leq \omega \leq \infty \end{cases} \tag{2.7}$$

$$H_2(0,\omega) = H_2(\omega) = \begin{cases} \dfrac{X(\omega)}{\sqrt{1-\omega^2}} & 0 \leq \omega \leq 1 \\ -\dfrac{R(\omega)}{\sqrt{\omega^2-1}} & 1 \leq \omega \leq \infty \end{cases} \tag{2.8}$$

It should be noted that since $Z(p)$ has no poles on the $j\omega$ axis or in the right-half plane, then:

$$\lim_{p \to \infty} G(p) = 0 \quad \text{for } \sigma \geq 0 \ .$$

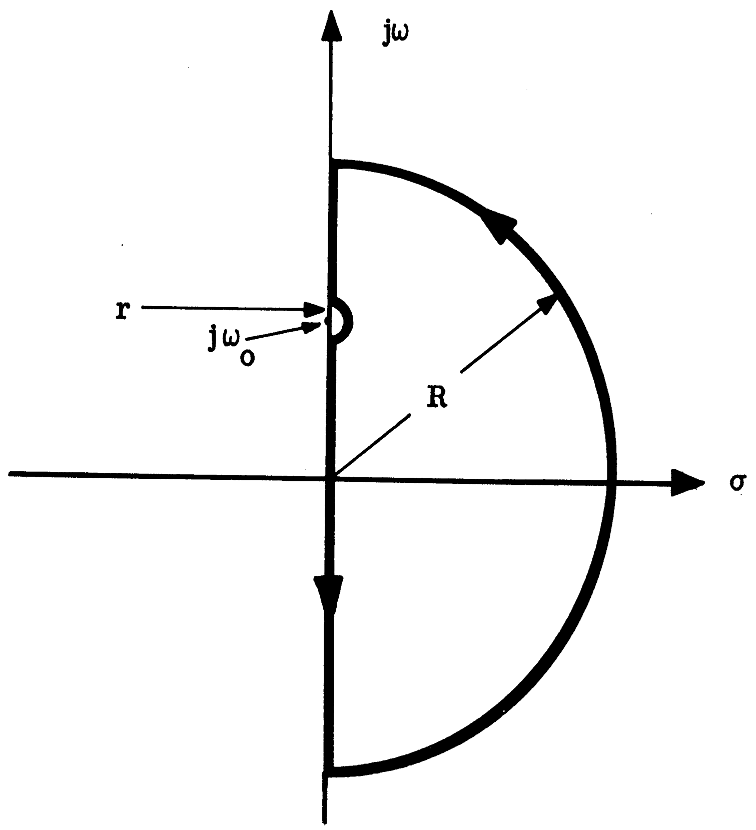


FIG. 2-1: CONTOUR OF INTEGRATION FOR EQ. (2.2).

Since we have assumed that Z(p) is real for p real, it follows that $R(\omega)$ and $X(\omega)$ are even and odd functions of frequency respectively.

Using Eqs. (2.7), (2.8), (2.5) and (2.6) we obtain:

$$\frac{\pi R(\omega)}{2\sqrt{1-\omega^2}} + \int_0^1 \frac{\lambda X(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}} = \int_1^\infty \frac{\lambda R(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{\lambda^2-1}} \triangleq f(\omega), \quad 0 \le \omega \le 1 \tag{2.9}$$

$$\frac{\pi X(\omega)}{2\omega\sqrt{1-\omega^2}} - \int_0^1 \frac{R(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}} = \int_1^\infty \frac{X(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{\lambda^2-1}} \triangleq g(\omega), \quad 0 \le \omega \le 1 . \tag{2.10}$$

These integrals are to be evaluated in the Cauchy principal value sense; i.e.

$$\int_0^1 \frac{R(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}} = \lim_{\epsilon \to 0} \left[ \int_0^{\omega-\epsilon} \frac{R(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}} + \int_{\omega+\epsilon}^1 \frac{R(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}} \right]; \quad 0 \le \omega \le 1 \tag{2.11}$$

and so on.

Eqs. (2.9) and (2.10) are Hilbert Transforms relating the real and imaginary parts of a function of the complex variable $p = \sigma + j\omega$ for $\sigma = 0$ and $0 \le \omega \le 1$ .

Note the following:

1) Since $R(\omega)$ and $X(\omega)$ are given in the band (0, 1) then the functions $f(\omega)$ and $g(\omega)$ are known.
2) We are seeking a solution for $R(\lambda)$ and $X(\lambda)$ in $(1, \infty)$.
3) Since $X(\omega)/\omega$ is even, $\omega X(\omega)$ is even and $\omega R(\omega)$ is odd, Eqs. (2.9) and (2.10) have the same functional representation. Hence, we need solve only one of these equations. For our purposes, we will solve equation (2.9) which is reproduced here for convenience.

$$f(\omega) = \int_1^{\infty} \frac{\lambda R(\lambda)\, d\lambda}{(\lambda^2 - \omega^2)\sqrt{\lambda^2 - 1}} , \quad 0 \leq \omega \leq 1 . \tag{2.9}$$

## 2.2 Necessary Conditions for Positive Real Continuations.

Making the change of variable

$$\lambda = \sec \frac{\phi}{2} \tag{2.12}$$

in Eq. (2.9) we obtain:

$$f(\omega) = \frac{1}{2} \int_0^{\pi} \frac{R^*(\phi)\, d\phi}{1 - \omega^2 \cos^2 \frac{\phi}{2}} , \quad \begin{cases} 0 \leq \phi \leq \pi \\ 0 \leq \omega \leq 1 \end{cases} \tag{2.13}$$

where:

$$R^*(\phi) = R\left(\lambda = \sec\frac{\phi}{2}\right) \quad . \tag{2.14}$$

If $R^*(\phi)$ is positive for all $\phi$, then we must have that $f(\omega)$ is non-negative for all $0 \leq \omega \leq 1$. Hence, in order for $R(\omega)$ and $X(\omega)$ to be the real and imaginary parts of a passive driving-point impedance for $0 \leq \omega \leq 1$, Eq. (2.9) implies that the following inequality must be satisfied.

$$R(\omega) \geq - \frac{2\sqrt{1 - \omega^2}}{\pi} \int_0^1 \frac{\lambda\, X(\lambda)\, d\lambda}{(\lambda^2 - \omega^2)\sqrt{1 - \lambda^2}} \ ; \quad 0 \leq \omega \leq 1 . \tag{2.15}$$

Expanding the kernel of Eq. (2.13) in a power series:

$$\frac{1}{1 - \omega^2 \cos^2 \frac{\phi}{2}} = \sum_{n=0}^{\infty} \omega^{2n} \cos^{2n} \frac{\phi}{2} ; \quad \left| \omega^2 \cos^2 \frac{\phi}{2} \right| < 1 \tag{2.16}$$

and substituting this expansion in Eq. (2.13) we obtain

$$f(\omega) = \frac{1}{2}\int_0^{\pi} R^*(\phi) \sum_{n=0}^{\infty} \omega^{2n} \cos^{2n} \frac{\phi}{2}\, d\phi . \qquad (2.17)$$

However, since the series in Eq. (2.17) converges within the limits of integration, we may integrate term by term and obtain:

$$f(\omega) = \frac{1}{2} \sum_{n=0}^{\infty} \omega^{2n} \int_0^{\pi} R^*(\phi) \cos^{2n} \frac{\phi}{2}\, d\phi . \qquad (2.18)$$

Let us now expand $f(\omega)$ in a power series about the origin

$$f(\omega) = \sum_{n=0}^{\infty} K_n \omega^{2n} = \frac{1}{2} \sum_{n=0}^{\infty} \omega^{2n} \int_0^{\pi} R^*(\phi) \cos^{2n} \frac{\phi}{2}\, d\phi . \qquad (2.19)$$

Equating coefficients of like powers in Eq. (2.19) we obtain:

$$K_n = \frac{1}{2}\int_0^{\pi} R^*(\phi) \cos^{2n} \frac{\phi}{2}\, d\phi . \qquad (2.20)$$

From Eq. (2.20) we note that if $R^*(\phi)$ is positive for all $\phi$, as it must be for a passive impedance, then:

$$K_n > 0 \qquad \text{for all } n . \qquad (2.21)$$

Also note that since $\cos^{2n} \frac{\phi}{2} \geq \cos^{2n+2} \frac{\phi}{2} \geq \ldots$ it follows that:

$$\ldots > K_{n-1} > K_n > K_{n+1} > \ldots \qquad (2.22)$$

Finally then, we conclude that for the real and imaginary parts of an impedance given in a band $[0, 1]$ to be the real and imaginary parts of a positive real function in that band we must require that:

$$\begin{cases} f(\omega) > 0 \quad \text{or} \quad R(\omega) \geq -\dfrac{2\sqrt{1-\omega^2}}{\pi}\displaystyle\int_0^1 \frac{\lambda\, X(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}}\ ; \ 0 \leq \omega \leq 1 \\ K_n > 0 \qquad \text{for all } n \\ \ldots > K_{n-1} > K_n > K_{n+1} > \ldots \quad \text{for all } n\,. \end{cases} \tag{2.23}$$

Clearly conditions (2.23) are only <u>necessary</u> and not sufficient conditions.

At this point we have made use of the fact that:

$R(\omega) \geq 0, \qquad 0 \leq \omega < \infty$

$Z(p)$ is real for $p$ real

$Z(p)$ is analytic on the $j\omega$ axis and in the right half plane.

## 2.3 Uniqueness of Continuations.

If there exists another solution $R_1^*(\phi)$ to Eq. (2.20) then:

$$0 = \frac{1}{2}\int_0^{\pi} \left[R_1^*(\phi) - R^*(\phi)\right] \cos^{2n}\frac{\phi}{2}\, d\phi \qquad \text{for all } n \geq 0\,. \tag{2.24}$$

However, replacing $R_1^*(\phi)$ and $R^*(\phi)$ by their respective Fourier series it can be shown that both series have the same coefficients, implying that there is a unique solution to (2.20).

Making in (2.24) the transformation

$$\cos^{2n}\frac{\phi}{2} = \sum_{m=0}^{n} B_{nm} \cos(n-m)\,\phi \tag{2.25}$$

where:

$$B_{nm} = \begin{cases} \dfrac{1}{2^{2n}}\dbinom{2n}{n} & n = m \\ \dfrac{1}{2^{2n-1}}\dbinom{2n}{m} & n \neq m \end{cases}$$

and replacing $R_1^*(\phi)$ and $R^*(\phi)$ by their Fourier series one obtains:

$$0 = \sum_{m=0}^{n} B_{nm} \int_0^{\pi} \sum_{j=0}^{\infty} (A_j - B_j) \cos j\phi \cos (n-m)\phi \; d\phi \; . \qquad (2.26)$$

Integrating (2.26), noting that the integral vanishes for $j \neq (n-m)$

$$0 = \sum_{m=0}^{n} \frac{1}{2^{2n}} (A_{n-m} - B_{n-m}) \binom{2n}{m} \quad \text{for all } n \geq 0$$

which implies that $A_{n-m} = B_{n-m}$. Q.E.D.

## 2.4 Solution for Continuations

Making use of (2.25) Eq. (2.20) becomes:

$$K_n = \frac{1}{2} \sum_{m=0}^{n} B_{nm} \int_0^{\pi} R^*(\phi) \cos (n-m)\phi \, d\phi \; . \qquad (2.27)$$

However, since $R^*(\phi)$ is a continuous even function, we can expand it in a convergent Fourier cosine series as follows:

$$R^*(\phi) = \sum_{j=0}^{\infty} A_j \cos j\phi \; ; \qquad (2.28)$$

$$A_j = \frac{2}{\pi} \int_0^{\pi} R^*(\xi) \cos j\xi \; d\xi$$

Substituting (2.28) into (2.27) we obtain:

$$K_n = \frac{1}{2} \sum_{m=0}^{n} B_{nm} \int_0^{\pi} \sum_{j=0}^{\infty} A_j \cos j\phi \cos (n-m)\phi \, d\phi \; . \qquad (2.29)$$

Making use of the orthogonality relation:

$$\int_0^{\pi} \cos (n-m) \phi \cos j \phi \, d\phi = \begin{cases} \pi & \text{if} \quad j = n-m = 0 \\ \pi/2 & \text{if} \quad j = n-m \neq 0 \\ 0 & \text{if} \quad j \neq n-m \end{cases} \tag{2.30}$$

we obtain for (2.29):

$$K_n = \frac{1}{2} \int_0^{\pi} \sum_{m=0}^{n} B_{nm} A_{(n-m)} \cos^2 (n-m) \phi \, d\phi \tag{2.31}$$

or, replacing $B_{nm}$ from (2.25)

$$\frac{2}{\pi} K_n = \sum_{m=0}^{n} \frac{1}{2^{2n}} A_{(n-m)} \binom{2n}{m} . \tag{2.32}$$

Equation (2.32) is then the recursion relation for the coefficients of the Fourier series (2.29) for $R^*(\phi)$ . Solving Eq. (2.32) for $A_n$ we obtain

$$A_n = 2^{2n} \left[ \frac{2}{\pi} K_n - \sum_{m=1}^{n} \frac{A_{(n-m)}}{2^{2n}} \binom{2n}{m} \right] \tag{2.33}$$

or

$$A_n = \frac{2}{\pi} 2^{2n} K_n - \sum_{m=1}^{n} A_{(n-m)} \binom{2n}{m} . \tag{2.34}$$

In order to solve for the continuation of the imaginary part onto $(1, \infty)$, we make use of Eq. (2.10) which is reproduced here for convenience:

$$g(\omega) = \int_1^{\infty} \frac{X(\lambda) \, d\lambda}{(\lambda^2 - \omega^2) \sqrt{\lambda^2 - 1}} \; ; \quad 0 \leq \omega \leq 1 \; . \tag{2.10}$$

Letting $X(\lambda) = \lambda\, E(\lambda)$ ; where $E(\lambda)$ is an even function, Eq. (2.10) becomes:

$$g(\omega) = \int_1^\infty \frac{\lambda\, E(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{\lambda^2-1}} \qquad 0 \leq \omega \leq 1 \ . \tag{2.35}$$

Note that Eq. (2.35) has the same functional representation as does Eq. (2.9). Hence, we should expect to obtain a similar solution; i.e.

$$g(\omega) = \sum_{n=0}^{\infty} H_n\, \omega^{2n} \tag{2.36}$$

$$E(\lambda) = E^*(\phi) = \sum_{m=0}^{\infty} T_m \cos m\phi \tag{2.37}$$

and from (2.34):

$$T_n = \frac{2}{\pi}\, 2^{2n} H_n - \sum_{m=1}^{n} T_{(n-m)} \binom{2n}{m} \tag{2.38}$$

where:

$$X^*(\phi) = \cos\frac{\phi}{2} \sum_{m=0}^{\infty} T_m \cos m\phi \ . \tag{2.39}$$

So far a functional representation for the real and imaginary parts of the driving point impedance in $[0, 1]$ has been used. However, our goal is to be able to find the continuations from a graphical representation of the real and imaginary parts in $[0, 1]$ . To this we now turn our attention.

## 2.5 Solution From a Graphical Representation.

From a graphical representation of the real and imaginary parts of a driving point impedance in $[0, 1]$, we can always find a Fourier series for these parts by making use of the following change of variable:

$$\omega = \cos\frac{\alpha}{2} \tag{2.40}$$

obtaining

$$R(\omega) = R(\cos\frac{\alpha}{2}) = R^*(\alpha) = \sum_{n=0}^{\infty} B_n \cos n\alpha \qquad \text{for } -\pi \leq \alpha \leq 0 \quad . \tag{2.41}$$

Similarly:

$$\omega X(\omega) = E(\omega) = E^*(\alpha) = \sum_{n=0}^{\infty} C_n \cos \alpha \quad . \tag{2.42}$$

Using the trigonometric identity:

$$\cos n\alpha = \sum_{m=0}^{n} b_{nm} \cos^{2m} \frac{\alpha}{2} \tag{2.43}$$

where:

$$b_{nn} = 2^{2n-1} \qquad b_{oo} \triangleq 1$$

$$b_{nm} = (-1)^{n-m} \frac{2n}{n-m} \binom{n+m-1}{n-m-1} 2^{2m-1} \quad ; \quad m \neq n$$

we obtain for Eq. (2.41)

$$R^*(\alpha) = \sum_{n=0}^{\infty} B_n \sum_{m=0}^{n} b_{nm} \cos^{2m} \frac{\alpha}{2} \quad . \tag{2.44}$$

If the summation signs in (2.44) are inverted with the proper change of subscripts, we obtain:

$$R^*(\alpha) = \sum_{m=0}^{\infty} \cos^{2m} \frac{\alpha}{2} \sum_{n=m}^{\infty} B_n b_{nm} = \sum_{m=0}^{\infty} R_m \cos^{2m} \frac{\alpha}{2} \tag{2.45}$$

where

$$R_m = \sum_{n=m}^{\infty} B_n b_{nm} \quad . \tag{2.46}$$

Using (2.40) in (2.45) we obtain:

$$R^*(\alpha) = \sum_{m=0}^{\infty} R_m \omega^{2m} = R(\omega) \quad . \tag{2.47}$$

This is a power series for $R(\omega)$ about the origin.

Expanding in a power series about the origin the term

$$\frac{1}{\sqrt{1-\omega^2}} = \sum_{j=0}^{\infty} \omega^{2j} \frac{(2j-1)!!}{(2j)!!} \tag{2.48}$$

we obtain for the first term in the right hand side of equation (2.9)

$$\frac{\pi R(\omega)}{2\sqrt{1-\omega^2}} = \frac{\pi}{2} \sum_{m=0}^{\infty} R_m \omega^{2m} \sum_{j=0}^{\infty} \omega^{2j} \frac{(2j-1)!!}{(2j)!!} \tag{2.49}$$

or:

$$\frac{\pi R(\omega)}{2\sqrt{1-\omega^2}} = \frac{\pi}{2} \sum_{m=0}^{\infty} \omega^{2m} \sum_{j=0}^{m} R_{(m-j)} \frac{(2j-1)!!}{(2j)!!} \quad . \tag{2.50}$$

We are now left with the task of finding a power series for the second term in the right hand side of Eq. (2.9) so that by equating coefficients of like powers, the needed $K_n$'s of equation (2.34) can be found; i.e.

$$I \triangleq \int_0^1 \frac{\lambda X(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}} = \sum_{n=0}^{\infty} G_n \omega^{2n} ; \; 0 \leq \omega < 1 \quad . \tag{2.51}$$

However, Eq. (2.51) as it stands has a singular integrand. In order to remove this singularity, we make use of the fact that adding or subtracting a constant (any function not dependent on $\lambda$) does not change the value of the integral* Since

$$\int_0^1 \frac{d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}} = 0 ; \qquad 0 \leq \omega \leq 1 \tag{2.52}$$

we can write

$$I = \int_0^1 \frac{\lambda\, X(\lambda) - \omega\, X(\omega)}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}}\, d\lambda \; ; \qquad 0 \leq \omega \leq 1 \; . \tag{2.53}$$

Equation (2.53) has no singularities** within the limits of integration. We may now use the substitution (2.42) as follows:

$$\omega\, X(\omega) = \sum_{n=0}^{\infty} C_n \cos n\,\alpha$$

$$\lambda\, X(\lambda) = \sum_{n=0}^{\infty} C_n \cos n\,\xi \tag{2.54}$$

with the following change of variable:

$$\lambda = \cos \frac{\xi}{2} \; . \tag{2.55}$$

Then (2.53) becomes:

$$I = \frac{1}{2}\int_0^{\pi} \frac{\sum_{n=0}^{\infty} C_n (\cos n\,\xi - \cos n\,\alpha)}{\cos^2 \frac{\xi}{2} - \cos^2 \frac{\alpha}{2}}\, d\xi \; . \tag{2.56}$$

Again, since the series of (2.56) converges within the limits of integration, we may integrate term by term and obtain

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* For a detailed proof of this, see appendix E.

** For a detailed proof of this, see appendix F.

$$I = \frac{1}{2} \sum_{n=0}^{\infty} C_n \int_0^{\pi} \frac{\cos n\xi - \cos n\alpha}{\cos^2 \frac{\xi}{2} - \cos^2 \frac{\alpha}{2}} \, d\alpha \quad . \tag{2.57}$$

However, using expression (2.43) we have:

$$\frac{\cos n\xi - \cos n\alpha}{\cos^2 \frac{\xi}{2} - \cos^2 \frac{\alpha}{2}} = \sum_{m=1}^{n} b_{nm} \frac{\left(\cos^{2m} \frac{\xi}{2} - \cos^{2m} \frac{\alpha}{2}\right)}{\left(\cos^2 \frac{\xi}{2} - \cos^2 \frac{\alpha}{2}\right)} \tag{2.58}$$

and:

$$\frac{\cos^{2m} \frac{\xi}{2} - \cos^{2m} \frac{\alpha}{2}}{\cos^2 \frac{\xi}{2} - \cos^2 \frac{\alpha}{2}} = \sum_{i=1}^{m} \cos^{2(i-1)} \frac{\alpha}{2} \cos^{2(m-i)} \frac{\xi}{2} \, . \tag{2.59}$$

Using (2.59) in (2.58) we obtain:

$$\frac{\cos n\xi - \cos n\alpha}{\cos^2 \frac{\xi}{2} - \cos^2 \frac{\alpha}{2}} = \sum_{m=1}^{n} b_{nm} \sum_{i=1}^{m} \cos^{2(i-1)} \frac{\alpha}{2} \cos^{2(m-i)} \frac{\xi}{2} \, . \tag{2.60}$$

Substituting (2.60) in (2.57) we obtain:

$$I = \frac{1}{2} \sum_{n=1}^{\infty} C_n \sum_{i=1}^{n} \cos^{2(i-1)} \frac{\alpha}{2} \sum_{m=i}^{n} b_{nm} \int_0^{\pi} \cos^{2(m-i)} \frac{\xi}{2} \, d\xi \tag{2.61}$$

$$= \sum_{n=1}^{\infty} C_n \sum_{i=1}^{n} \cos^{2(i-1)} \frac{\alpha}{2} \sum_{m=i}^{n} b_{nm} \frac{\pi}{2} \frac{[2(m-i)-1]!!}{[2(m-i)]!!}$$

$$I = \frac{\pi}{2} \sum_{i=0}^{\infty} \omega^{2i} \sum_{n=i+1}^{\infty} C_n \sum_{m=i+1}^{n} b_{nm} \frac{[2(m-i)-3]!!}{[2(m-i)-2]!!} \tag{2.62}$$

where we have gone back to the variable $\omega$ instead of $\cos \frac{\alpha}{2}$ .

Comparing Eq. (2.62) with (2.51), we see that:

$$G_i = \sum_{n=i+1}^{\infty} C_n \sum_{m=i+1}^{n} b_{nm} \frac{[2(m-i)-3]!!}{[2(m-i)-2]!!} \tag{2.63}$$

and we can write:

$$\frac{\pi}{2} \sum_{m=0}^{\infty} \omega^{2m} \sum_{j=0}^{m} R_{(m-j)} \frac{(2j-1)!!}{(2j)!!} + \frac{\pi}{2} \sum_{i=0}^{\infty} \omega^{2i} \sum_{n=i+1}^{\infty} C_n \times$$

$$\times \sum_{m=i+1}^{n} b_{nm} \frac{[(2(m-i)-3]!!}{[(2(m-i)-2]!!} = \sum_{n=0}^{\infty} K_n \omega^{2n} . \tag{2.64}$$

Equating coefficients of like powers in (2.64) we obtain:

$$\frac{2}{\pi} K_p = \sum_{j=0}^{p} R_{(p-j)} \frac{[2(j)-1]!!}{[2j]!!} + \sum_{n=p+1}^{\infty} C_n \sum_{m=p+1}^{n} b_{nm} \frac{[2(m-p)-3]!!}{[2(m-p)-2]!!} . \tag{2.65}$$

If we wish to obtain a similar expression for the imaginary part, we proceed as follows:

From Eq. (2.10) we have:

$$\frac{\pi X(\omega)}{2\omega\sqrt{1-\omega^2}} = \frac{\pi \omega X(\omega)}{2\omega^2\sqrt{1-\omega^2}} = \frac{\pi E(\omega)}{2\omega^2\sqrt{1-\omega^2}} \tag{2.66}$$

From (2.42) we have:

$$E(\omega) = E^*(\alpha) = \sum_{n=0}^{\infty} C_n \cos n\alpha \tag{2.67}$$

so that (2.66) becomes:

$$\frac{\pi X(\omega)}{2\omega\sqrt{1-\omega^2}} = \frac{\pi}{2} \sum_{m=0}^{\infty} \omega^{2m} \sum_{j=0}^{m} P_{(m-j)} \frac{(2j-1)!!}{(2j)!!} \tag{2.68}$$

where:

$$P_m = \sum_{n=m}^{\infty} C_n b_{nm} . \tag{2.69}$$

Making use of Eq. (2.52) we note that:

$$\int_0^1 \frac{R(\lambda) - R(\omega)}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}}\, d\lambda = \int_0^1 \frac{R(\lambda)\, d\lambda}{(\lambda^2-\omega^2)\sqrt{1-\lambda^2}} \quad ; \; 0 \leq \omega \leq 1 \tag{2.70}$$

and, carrying out a derivation similar to that in Eq. (2.57) to (2.65) we obtain:

$$\frac{2}{\pi} H_p = \sum_{j=0}^{p} P_{(p-j)} \frac{(2j-1)\,!!}{(2j)\,!!} + \sum_{n=p+1}^{\infty} B_n \sum_{m=p+1}^{n} b_{nm} \frac{[2(m-p)-3]\,!!}{[2(m-p)-2]\,!!} \; . \tag{2.71}$$

With the aid of Eq. (2.71) we can use Eq. (2.38) to find the Fourier coefficients of the continuation to the imaginary part onto $(1, \infty)$.

Let us at this point rewrite expressions (2.33) and (2.38) for convenience:

$$A_n = \frac{2}{\pi}\, 2^{2n} K_n - \sum_{m=1}^{n} A_{(n-m)} \binom{2n}{m} \tag{2.33}$$

$$T_n = \frac{2}{\pi}\, 2^{2n} H_n - \sum_{m=1}^{n} T_{(n-m)} \binom{2n}{m} \quad . \tag{2.38}$$

Note that from the graphical representation of the real and imaginary parts of a passive driving-point impedance function in (0, 1) we can always find $R_{(p-j)}$, $P_{(p-j)}$, $B_n$ and $C_n$ for any $p, j$ and $n$. Hence, knowing these constants, we can always find the constants $A_n$ and $T_n$ which give the desired solution.

## 2.6 Extension to the Band Case.

So far we have been using (0, 1) as the band of interest. If we wish to extend this method to the case where the band of interest is $(\omega_1, \omega_2)$ this can be done by first finding the continuations onto $0 \leq \omega \leq \omega_1$ and then knowing the real and imaginary parts for $0 \leq \omega \leq \omega_1$ use the method described in section (2.4) to find the continuations onto $(\omega_2 \leq \omega \leq \infty)$ .

Let:

$$\cos^2 \frac{\beta}{2} = \frac{\omega^2 - \omega_1^2}{\omega_2^2 - \omega_1^2} \tag{2.72}$$

so that:

$$\omega^2 = \left(\omega_2^2 - \omega_1^2\right) \cos^2 \frac{\beta}{2} + \omega_1^2 \tag{2.73}$$

and:

$$R(\omega) = R\left[\sqrt{\left(\omega_2^2 - \omega_1^2\right) \cos^2 \frac{\beta}{2} + \omega_1^2}\right] = \sum_{n=0}^{\infty} D_n \cos n\beta \tag{2.74}$$

i.e. we can always find a Fourier series which describes the behavior of the given real part in the band $\omega_1 \leq \omega \leq \omega_2$ .

However, we know from (2.43) that:

$$\cos n\,\beta = \sum_{m=0}^{n} b_{nm} \cos^{2m} \frac{\beta}{2} \tag{2.75}$$

so that (2.74) becomes:

$$\begin{aligned} R(\omega) &= \sum_{m=0}^{\infty} \cos^{2m} \frac{\beta}{2} \sum_{n=m}^{\infty} b_{nm} D_n \\ &= \sum_{m=0}^{\infty} \left(\frac{\omega^2 - \omega_1^2}{\omega_2^2 - \omega_1^2}\right)^m \sum_{n=m}^{\infty} b_{nm} D_n \ . \end{aligned} \tag{2.76}$$

Note that:

$$\lim_{m \to \infty} \left| \frac{\sum_{n=m+1}^{\infty} b_{nm+1} D_n}{\sum_{n=m}^{\infty} b_{nm} D_n} \right| \to 1 \ , \tag{2.77}$$

Thus, for Eq. (2.76) to converge, we require that:

$$\left| \frac{\omega^2 - \omega_1^2}{\omega_2^2 - \omega_1^2} \right| < 1 \quad . \tag{2.78}$$

However, we are interested in representing R(ω) down to $\omega = 0$, so that we can apply the method outlined in the preceding pages.

In order for (2.76) to extend its radius of convergence all the way down to $\omega = 0$, we require that:

$$\left| \omega^2 - \omega_1^2 \right| < \left| \omega_2^2 - \omega_1^2 \right| \tag{2.79}$$

or:

$$\omega^2 > 2\omega_1^2 - \omega_2^2 \quad . \tag{2.80}$$

Hence, for $\omega = 0$ we must have that:

$$\frac{\omega_2^2}{\omega_1^2} > 2 \quad . \tag{2.81}$$

If the band of interest is such that (2.81) holds then we are assured that we can always find the behavior of the function R(ω) all the way down to the origin. However, if this should not be the case, then we can find the behavior of R(ω) down to some new frequency, say $\omega_3$, such that $\omega_3 < \omega_1$, and then use Eq. (2.74) with $\omega_1$ replaced by $\omega_3$, and so on until we have the representation of the function R(ω) down to the point $\omega = 0$.

In order to find the behavior of the imaginary part for $0 \leq \omega \leq \omega_1$, we use a technique similar to that used for finding R(ω). However, in this case we let:

$$\frac{X(\omega)}{\omega} = E(\omega) \tag{2.82}$$

where E(ω) is an even function of ω .

Then using the change of variables (2.72) we obtain:

$$E(\omega) = E\left[\sqrt{\left(\omega_2^2 - \omega_1^2\right)\cos^2\frac{\beta}{2} + \omega_1^2}\right] = \sum_{n=0}^{\infty} C_n \cos n\beta \ . \qquad (2.83)$$

Using (2.75) in (2.83) and inverting the summation signs we obtain:

$$E(\omega) = \sum_{m=0}^{\infty}\left[\frac{\omega^2 - \omega_1^2}{\omega_2^2 - \omega_1^2}\right]\sum_{n=m}^{\infty} b_{nm}\, C_n \qquad (2.84)$$

or:

$$X(\omega) = \omega \sum_{m=0}^{\infty}\left[\frac{\omega^2 - \omega_1^2}{\omega_2^2 - \omega_1^2}\right]\sum_{n=m}^{\infty} C_n\, b_{nm} \ . \qquad (2.85)$$

Note that (2.85) is similar to (2.76). Hence, we can conclude that in order for (2.85) to extend its radius of convergence all the way down to $\omega = 0$, we require that:

$$\frac{\omega_2^2}{\omega_1^2} > 2 \ . \qquad (2.81)$$

The statements made in regards to $R(\omega)$, following Eq. (2.81) also hold for $X(\omega)$. Hence, in practice, we only need to know the behavior of the real and imaginary parts of the driving point impedance to a passive network in a finite range of frequencies in order to find its behavior outside this band.

Up to now, we have been using infinite series without being concerned with the actual solution to a practical problem. However, if we wish to solve a problem using the method outlined in the preceding pages, we must truncate the infinite series. This truncation introduces an error, for which we can obtain a measure as follows:

## 2.7 Measure of Truncation Error.

From Eq. (2.70) we can see that truncating the power series for $f(\omega)$ is equivalent to truncating the infinite series in the right hand side of (2.70). Hence, we can define a measure of the error as:

$$f(\omega) - \sum_{n=0}^{N} K_n \omega^{2n} \tag{2.86}$$

this presents some difficulties since $f(\omega)$ is singular at $\omega = 1$. In order to remove this singularity, we can multiply $f(\omega)$ by the factor $\sqrt{1-\omega^2}$ and obtain:

$$F(\omega) = \sqrt{1-\omega^2}\; f(\omega) \;, \tag{2.87}$$

or:

$$E(\omega) \triangleq F(\omega) - \sqrt{1-\omega^2} \sum_{n=0}^{N} K_n \omega^{2n} \;. \tag{2.88}$$

We have shown, in preceding pages, that:

$$f(\omega) > 0 ;\; 0 \leq \omega \leq 1 \tag{2.89}$$

and

$$K_n > 0 \;;\; \text{for all } n \;. \tag{2.90}$$

Then we note that:

$$F(\omega) = \sqrt{1-\omega^2}\; f(\omega) > 0 \quad \text{for} \quad 0 \leq \omega \leq 1 \;. \tag{2.91}$$

Since in (2.88) we are only taking the first N terms of the power series for $f(\omega)$ (all the $K_n$'s are positive) we can conclude that:

$$E(\omega) \geq 0 \quad \text{for} \quad 0 \leq \omega \leq 1 \;.$$

And then a measure of the normalized error becomes:

$$\epsilon = \frac{\int_0^1 E(\omega)\,d\omega}{\int_0^1 F(\omega)\,d\omega} = 1 - \frac{\frac{\pi}{2}\sum_{n=0}^{N} K_n \frac{(2n-1)(2n-3)\ldots(3)(1)}{(2n+2)(2n)\ldots(4)(2)}}{\int_0^1 F(\omega)\,d\omega} \tag{2.92}$$

This is not an accurate measure of the error. However, it gives some measure of the error incurred in the truncation of the infinite series. An exact expression for the truncation error can not be obtained since we do not know anything about $R^*(\omega)$; in addition it appears only under an integral sign. Therefore, the usual least square error technique cannot be applied here.

## 2.8 Extension to Transfer Impedances.

The method described in section 2.4 readily extends to the case of a transfer impedance function. We note that if the function in question has no poles in the right half plane or the $j\omega$ axis of the complex plane, then we may use the Hilbert Transforms of Eqs. (2.9) and (2.10). These equations require no restrictions on the location of the zeros of the transfer impedance function. Thus, it follows that the method derived in section 2.4 is indeed applicable to the case of a transfer impedance function if and only if this function has no poles on the $j\omega$ axis or the right half plane. Uniqueness will be achieved only up to an additive constant, since the concept of a minimum resistance function is not meaningful for a transfer impedance.

Since there are no restrictions on the behavior of the real part of a transfer impedance function, we can not state any conditions, necessary and/or sufficient, for the compatibility of the given parts. Hence, we must obtain the continuations of the given parts in order to find out whether they are compatible or not; i.e. whether the Fourier series obtained converge or diverge.

## Chapter Three

## NUMERICAL COMPUTATIONS

### 3.1 Transformation of Equations

Several examples have been used to verify the method outlined in the preceeding pages. Functions whose behavior was known outside the band of interest were used as the impedance functions and the results obtained were excellent.

A computer program was written to solve for the unknown parts using $4N + 5$ points as data; i.e. $R(\omega)$ and $X(\omega)$ were given at $4N + 5$ different frequencies equally distributed along the band of interest. The program gives as results the first $N + 1$ Fourier coefficients for $R^*(\phi)$ and $X^*(\phi)$.

The reason why we only obtain $N + 1$ coefficients from $4N + 5$ points relates to the nature of the subroutine used to obtain the Fourier coefficients of $R(\omega)$ and $\omega X(\omega)$ in the band $(0, 1)$. From $4N + 5$ points we can obtain only the first $2N + 3$ Fourier coefficients for $R(\omega)$ and $\omega X(\omega)$ in the band $(0, 1)$. It will become apparent that this leads to $N + 1$ coefficients for $R^*(\phi)$ and $\cos \frac{\phi}{2} X^*(\phi)$.

For convenience we reproduce here the expression for the coefficients of the power series for $f(\omega)$ about the origin.

$$\frac{2}{\pi} K_k = \sum_{n=0}^{k} R_{(k-n)} \frac{(2n-1)!!}{(2n)!!} + \sum_{n=k+1}^{\infty} C_n \sum_{j=k+1}^{n} b_{nj} \frac{[2(j-k)-3]!!}{[2(j-k)-2]!!} \tag{3.1}$$

where

$$R_j = \sum_{i=j}^{\infty} B_i b_{ij} \tag{3.2}$$

Looking at the second term in the right hand side we note that $b_{nj}$ varies as $2^{2j}$, whereas the double factorial varies as $1/2^{2j}$. Hence, we would like to remove this term from both $b_{nj}$ and the double factorial.

In order to do that we note that:

$$\left.\begin{aligned} b_{nj} &= 2^{2n-1} \quad ; \quad n = j \neq 0 \\ b_{nj} &= (-1)^{n-j} \frac{2n}{n-j} \binom{n+j-1}{n-j-1} 2^{2j-1} ; n \neq j \neq 0 \\ b_{nj} &\triangleq 1 \; ; \; n = j = 0 \end{aligned}\right\} \tag{3.3}$$

also,

$$\frac{[2(j-k)-3]!!}{[2(j-k)-2]!!} = \frac{1}{2^{2(j-k)-2}} \binom{2(j-k-1)}{j-k-1} \tag{3.4}$$

Thus we have:

$$\sum_{j=k+1}^{n} b_{nj} \frac{[2(j-k)-3]!!}{[2(j-k)-2]!!} = 2^{2n-1} \frac{1}{2^{2(n-k-1)}} \binom{2(n-k-1)}{n-k-1} + \tag{3.5}$$

$$+ \sum_{j=k+1}^{n-1} (-1)^{n-j} \frac{2n}{n-j} \binom{n+j-1}{n-j-1} 2^{2j-1} \frac{1}{2^{2(j-k-1)}} \binom{2(j-k-1)}{j-k-1}$$

$$= 2^{2k+1} \binom{2(n-k-1)}{n-k-1} + \sum_{j=k+1}^{n-1} (-1)^{n-j} \frac{2n}{n-j} \binom{n+j-1}{n-j-1} \binom{2(j-k-1)}{j-k-1} 2^{2k+1}$$

$$= 2^{2k+1} \left[ \binom{2n-2k-2}{n-k-1} + \sum_{j=k+1}^{n-1} (-1)^{n-j} \frac{2n}{n-j} \binom{n+j-1}{n-j-1} \binom{2j-2k-2}{j-k-1} \right] \tag{3.6}$$

Referring to Eq. (2.1) we note that it contains two infinite series. For computational purposes we must truncate these series after the first N terms. In order to do this and maintain a negligible truncation error we must use the first N terms of each of the infinite series. However, note that when $j \equiv N$

$$R_N = \sum_{i=N}^{N} B_i b_{iN} = B_N b_{NN} \qquad (3.7)$$

Hence, we are only using one term to compute $R_N$, thereby making the error not negligible.

To be able to truncate the infinite series and maintain the truncation error negligible we make; in (3.2) the following change of subscripts

$$R_j = \sum_{i=0}^{\infty} B_{i+j} b_{(i+j)j} \qquad (3.8)$$

and using (3.6) in (3.1):

$$\frac{2}{\pi} K_k = \sum_{n=0}^{k} R_{(k-n)} \frac{(2n-1)!!}{(2n)!!} + T_k \qquad (3.9)$$

$$T_k = \sum_{n=0}^{\infty} C_{n+k+1} 2^{2k+1} \left[ \binom{2n}{n} + \sum_{j=k+1}^{n+k} (-1)^{n+k+1-j} \frac{2(n+k+1)}{n+k+1-j} \binom{n+k+j}{n+k-j} \binom{2j+2k}{j-k-1} \right] \qquad (3.10)$$

where:

$$T_k = \sum_{n=0}^{\infty} C_{n+k+1} 2^{2k+1} \left[ \binom{2n}{n} + \sum_{j=k+1}^{n+k} (-1)^{n+k+1-j} \frac{2(n+k+1)}{n+k+1-j} \binom{n+k+j}{n+k-j} \binom{2(j-k-1)}{j-k-1} \right] \qquad (3.11)$$

Making another change of subscripts in (3.11)

$$T_k = \sum_{n=0}^{\infty} C_{n+k+1} \left[ 2^{2k+1} \right] \left[ \binom{2n}{n} + \sum_{j=0}^{n-1} (-1)^{n-j} \frac{2(n+k+1)}{n-j} \binom{n+j+2k+1}{n-j-1} \binom{2j}{j} \right] \qquad (3.12)$$

Now we can truncate Eq. (3.10), using (3.8) and (3.12) to obtain:

$$\frac{2}{\pi} K_k = \sum_{n=0}^{k} R_{(k-n)} \frac{(2n-1)!!}{(2n)!!} + \sum_{n=0}^{N} C_{n+k+1}\, 2^{2k+1} \left[ \binom{2n}{n} + \right.$$

$$\left. + \sum_{j=0}^{n-1} (-1)^{n-j} \frac{2(n+k+1)}{n-j} \binom{n+j+2k+1}{n-j-1} \binom{2j}{j} \right] \qquad (3.13)$$

where as before:

$$R_j = \sum_{i=0}^{N} B_{(i+j)} b_{(i+j)j} \qquad (3.14)$$

## 3.2 First Example

For the first example the pair of functions used were:

$$R(\omega) \quad \frac{1}{1+\omega^2} \quad ; \quad X(\omega) \quad \frac{-\omega}{1+\omega^2} \qquad (3.15)$$

For this case $N = 4$ was used. The coefficients for the power series of the function $f(\omega)$ are then:

| n | $K_n$ |
|---|---|
| 0 | 0.292 893 2 |
| 1 | 0.207 106 5 |
| 2 | 0.167 894 5 |
| 3 | 0.144 602 9 |
| 4 | 0.128 839 3 |

(3.16)

Note that as predicted earlier, the Kn's are all positive and monotonically decreasing, and the necessary conditions for compatibility are satisfied.

For this case the $A_n$'s (Fourier coefficients for the unknown $R^*(\phi)$) are found to be:

| n | $A_n$ | |
|---|---|---|
| 0 | 0.292 893 2 | |
| 1 | 0.242 639 6 | |
| 2 | -0.041 639 0 | (3.17) |
| 3 | 0.006 761 5 | |
| 4 | 0.003 385 6 | |

Figure (3-1) shows a plot of $R^*(\phi)$ as compared with $R(\omega)$ given in Eq. (3.15) (where $\omega = \sec \frac{\phi}{2}$ ). For this case it is not easy to distinguish the points of the exact expression (3.15) and those from the Fourier series. Therefore a comparison of both points is given in Table 3.1 below. The points obtained from the exact expression are labeled $R(\phi)$ and those from the Fourier series $R^*(\phi)$. (only for $0 \leq \phi \leq \pi$).

| $\phi$ | $R(\phi)$ | $R^*(\phi)$ |
|---|---|---|
| 0 | 0.5 | 0.504 073 9 |
| $\pi/4$ | 0.460 49 | 0.459 658 9 |
| $\pi/2$ | 0.333 333 | 0.337 884 8 |
| $3\pi/4$ | 0.122 | 0.122 749 9 |
| $\pi$ | 0.000 | 0.005 271 7 |

Table 3.1: Comparison Between Points from the Exact Expression and those from the Fourier Series Solution for Example One.

From this table we see that there is good agreement between the exact expression for $R(\phi)$ and its Fourier series, obtained by the method described in the preceeding pages. As a matter of fact the agreement is good up to the third significant figure.

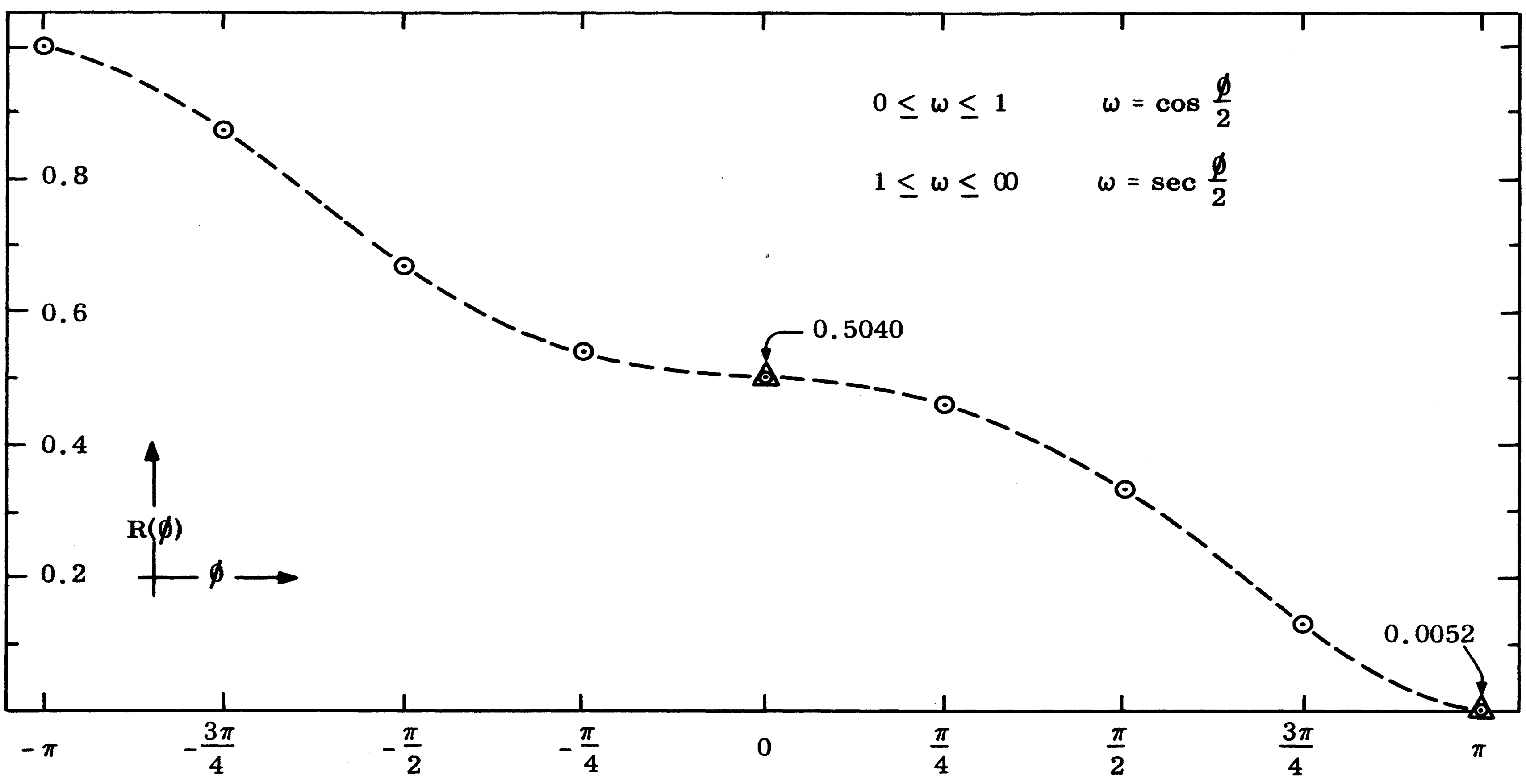


FIG. 3-1: PLOT OF R(ω) FOR EXAMPLE NUMBER ONE. ⊙ POINTS FROM EXACT EXPRESSION, △ POINTS FROM FOURIER SERIES SOLUTION.

## 3.3 Second Example

As a second example the following functions were used:

$$R(\omega) = \frac{(\omega^2 - 4)^2}{(1 + \omega^2)(4 + \omega^2)} \quad ; \quad X(\omega) = \frac{7\omega^3 - 68\omega}{3(1 + \omega^2)(4 + \omega^2)} \tag{3.18}$$

$N = 4$ was used for this case and the $K_n$'s obtained were:

| n | $K_n$ |
|---|---|
| 0 | 0.492 582 8 |
| 1 | 0.296 269 6 |
| 2 | 0.256 525 3 |
| 3 | 0.230 256 2 |
| 4 | 0.210 581 7 |

(3.19)

Once again we see that the $K_n$'s are all positive and monotonically decreasing. As expected, the necessary conditions are satisfied.

For this example the Fourier coefficients $(A_n)$ corresponding to the real part $R^*(\phi)$ are:

| n | $A_n$ |
|---|---|
| 0 | 0.492 582 8 |
| 1 | 0.199 912 6 |
| 2 | 0.349 257 5 |
| 3 | 0.209 492 8 |
| 4 | 0.129 740 8 |

(3.20)

Figure (3-2) shows a plot of $R^*(\phi)$ as compared with the exact expression. Once again we observe that, the points obtained using the Fourier series technique are almost indistinguishable from those of the exact expression. Therefore, a comparison of both points is given in Table 3.2 below.

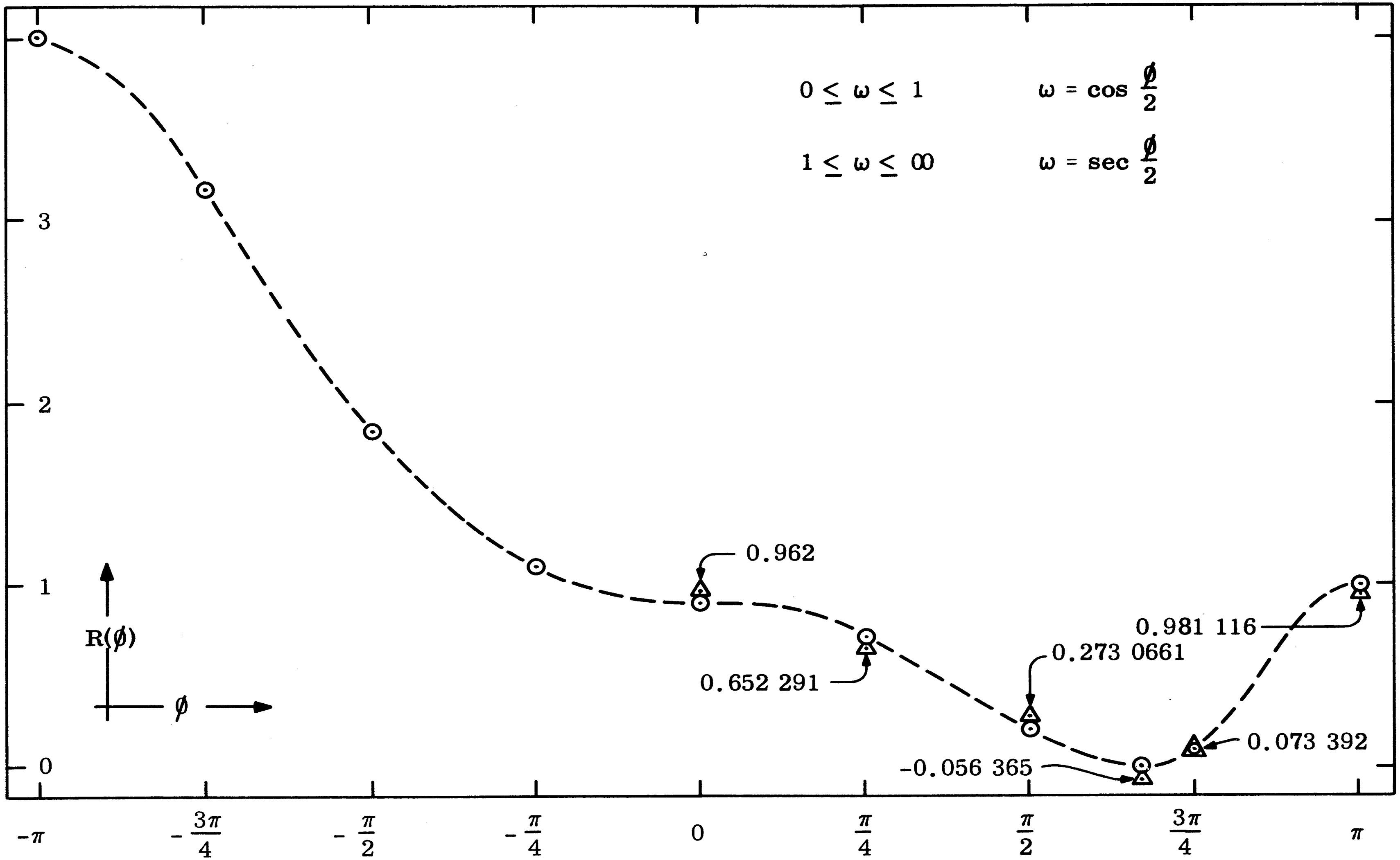


FIG. 3-2: PLOT OF R(ω) FOR EXAMPLE NUMBER TWO. ⊙ POINTS FROM EXACT EXPRESSION, △ POINTS FROM FOURIER SERIES SOLUTION.

| $\phi$ | $R(\phi)$ | $R^*(\phi)$ |
|---|---|---|
| 0 | 0.900 | 0.962 001 0 |
| $\pi/4$ | 0.716 | 0.652 241 6 |
| $\pi/2$ | 0.222 222 2 | 0.273 066 1 |
| $3\pi/4$ | 0.094 379 | 0.073 347 3 |
| $\pi$ | 1.000 000 | 0.981 116 1 |

Table 3.2: Comparison between Points from the Exact Expression and those from the Fourier Series Solution for Example two.

Once more we see that the Fourier series technique does indeed give a solution which agrees with the exact solution. Here however, we note that the discrepancy occurs in the second significant figure. This is due to the fact that for this example $N = 4$ is not enough; i.e. more terms of the Fourier series are required.

3.4 Third Example:

As a third example we take the transfer impedance function whose real and imaginary parts are given by:

$$R(\omega) = \frac{6 - \omega^2}{\omega^4 + 13\omega^2 + 36} \quad ; \quad X(\omega) = \frac{-5\omega}{\omega^4 + 13\omega^2 + 36} \tag{3.21}$$

For this case $N = 5$ was used. The $K_n$'s obtained were:

$$\begin{array}{cc} n & K_n \\ 0 & 0.048\ 469\ 12 \\ 1 & 0.039\ 559\ 93 \\ 2 & 0.032\ 979\ 22 \\ 3 & 0.028\ 658\ 62 \\ 4 & 0.025\ 627\ 36 \\ 5 & 0.023\ 370\ 66 \end{array} \tag{3.22}$$

and the Fourier coefficients corresponding to the unknown real part $R^*(\phi)$ are:

| n | $A_n$ | |
|---|---|---|
| 0 | 0.048 469 1 | |
| 1 | 0.061 301 47 | |
| 2 | -0.008 353 18 | (3.23) |
| 3 | -0.004 633 85 | |
| 4 | 0.005 843 25 | |
| 5 | -0.003 502 56 | |

Figure (3-3) shows a plot of the real part $R(\phi)$ for the range $0 \leq \phi \leq \pi$ $(1 \leq \omega \leq \infty)$. Note that the points from the exact expression and those from the Fourier series (using six terms) are almost indistinguishable. Again a comparison of both sets of points is given below in table 3-3.

| $\phi$ | $R(\phi)$ | $R^*(\phi)$ |
|---|---|---|
| 0 | 0.100 | 0.099 124 25 |
| $\pi/4$ | 0.091 816 | 0.091 726 10 |
| $\pi/2$ | 0.0606 | 0.062 665 55 |
| $3\pi/4$ | -0.001 102 | -0.001 514 09 |
| $\pi$ | 0.000 | -0.007 205 87 |

Table 3-3: Comparison Between Points from the Exact Expression and those from the Fourier Series Solution for Example Three.

From this table we see that the Fourier series technique gives a solution which indeed agrees with the exact expression, for the continuation. Once again we observe that the discrepancy occurs in the third significant figure.

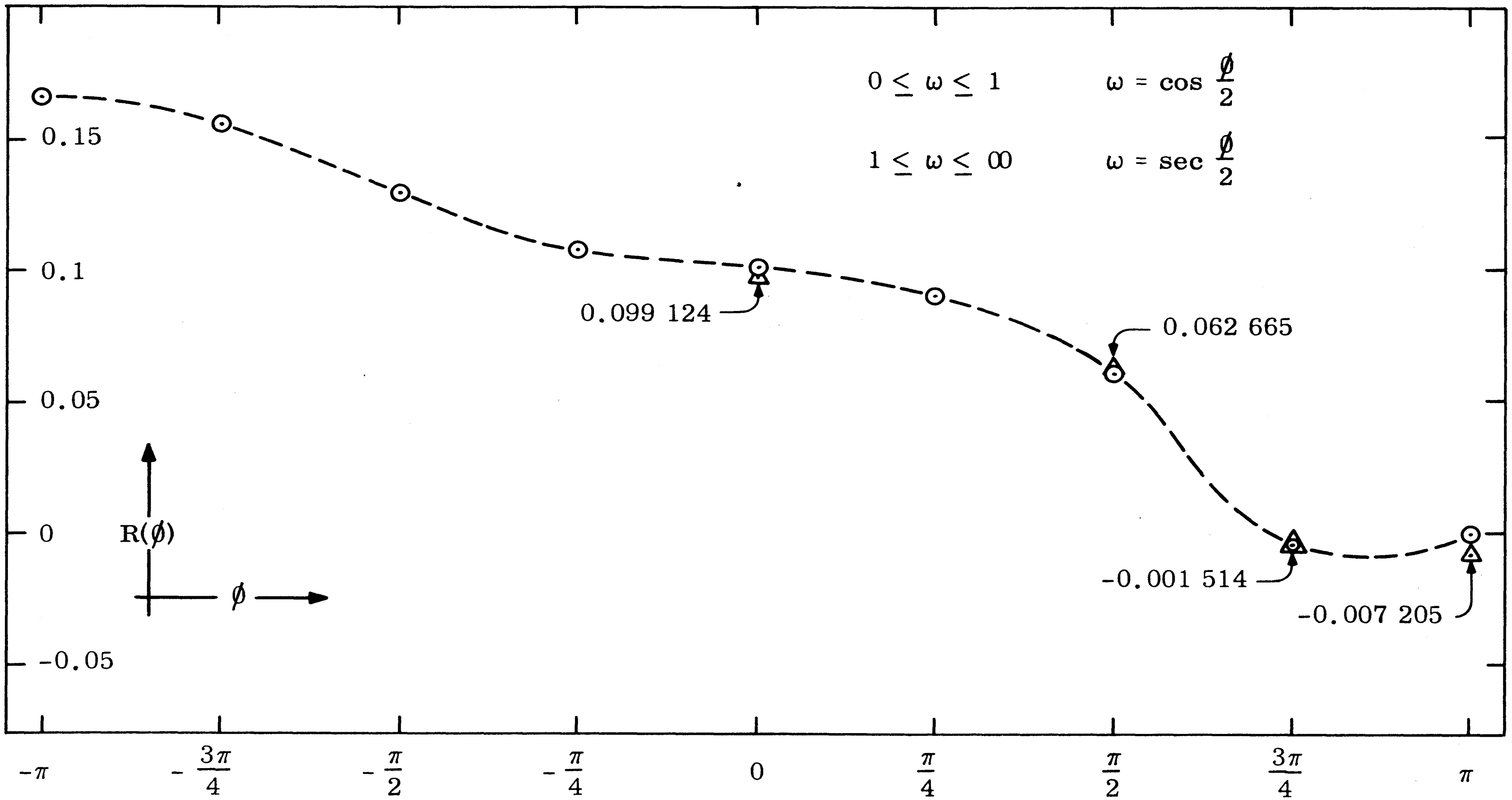


FIG. 3-3: PLOT OF R(ω) FOR EXAMPLE NUMBER THREE. ⊙ POINTS FROM EXACT EXPRESSION, △ POINTS FROM FOURIER SERIES SOLUTION.

### 3.5 Fourth Example

As a fourth example the following transfer impedance function was used.

$$R(\omega) = \frac{24 - 9\omega^2}{576 + 244\omega^2 + 29\omega^4 + \omega^6}$$

and (3.24)

$$X(\omega) = \frac{\omega^3 - 26\omega}{576 + 244\omega^2 + 29\omega^4 + \omega^6}$$

For this case N = 4 was used. The $K_n$'s obtained were

| n | $K_n$ |
|---|---|
| 0 | 0.004 955 57 |
| 1 | 0.005 691 39 |
| 2 | 0.005 180 53 |
| 3. | 0.004 671 78 |
| 4 | 0.004 261 05 |

(3.25)

and the Fourier coefficients corresponding to the unknown real part $R^*(\phi)$ were:

| n | $A_n$ |
|---|---|
| 0 | 0.004 955 57 |
| 1 | 0.012 854 42 |
| 2 | 0.001 737 44 |
| 3 | -0.003 358 14 |
| 4 | 0.002 308 59 |

(3.26)

Figure (3-4) shows a plot of the real part $R(\phi)$ for the range $0 \leq \phi \leq \pi$ $(1 \leq \omega \leq \infty)$. Note that the points from the exact expression and those from the Fourier series (using the first five terms) are almost

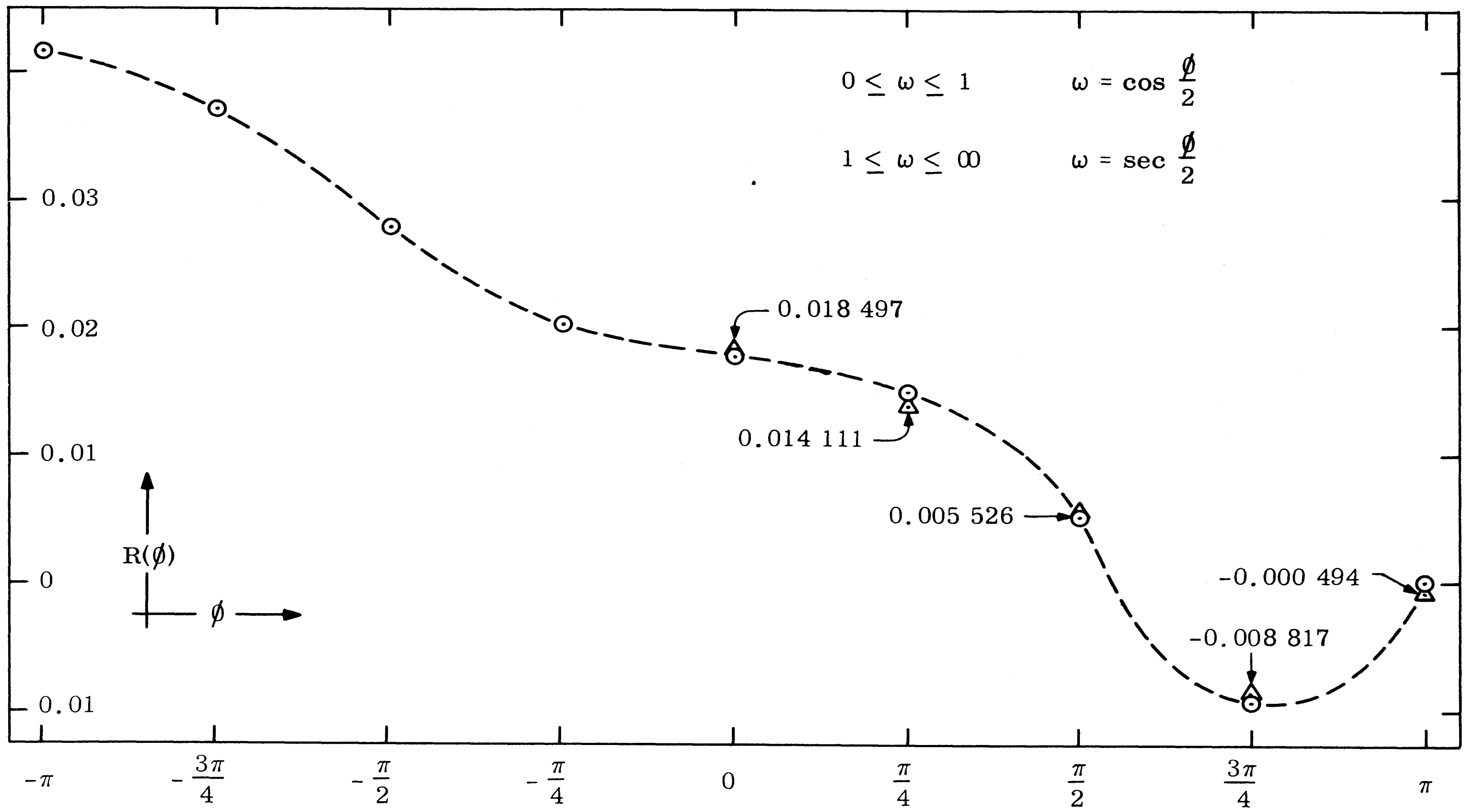


FIG. 3-4: PLOT OF R(ω) FOR EXAMPLE NUMBER FOUR. ⊙ POINTS FROM THE EXACT EXPRESSION, △ POINTS FROM FOURIER SERIES SOLUTION.

indistinguishable. Again a comparison of both sets of points is given below in Table 3-4:

| $\phi$ | $R(\phi)$ | $R^*(\phi)$ |
|---|---|---|
| 0 | 0.017 647 06 | 0.018 497 89 |
| $\pi/4$ | 0.014 923 36 | 0.014 111 04 |
| $\pi/2$ | 0.005 050 50 | 0.005 526 72 |
| $3\pi/4$ | -0.009 458 00 | -0.008 817 08 |
| $\pi$ | 0.000 | -0.000 494 68 |

Table 3-4: Comparison Between Points from the Exact Expression and those from the Fourier Series Solution for Example Four.

From this table we see that the Fourier series technique gives a solution which indeed agrees with the exact expression, for the continuation.

# Chapter Four

## COMMENTS AND SUGGESTIONS FOR FURTHER WORK

### 4.1 Minimum Amount of Information Required to Specify an Impedance Function.

Guillemin[6] has shown that knowing the real (or imaginary) part of a positive real function, for all frequencies, the imaginary (or real) part is completely specified for all frequencies. Further, he showed that knowing the real (or imaginary) part in a band of frequencies, and the imaginary (or real) part in the rest of the spectrum, the unknown parts are completely specified. That is, it is necessary to have some information for all frequencies.

However, in this report, it has been shown that the minimum amount of information which is sufficient to completely and uniquely specify the behavior of a passive driving-point impedance function or a transfer impedance function (assuming these functions have no poles in the right-half plane or on the $j\omega$ axis) is the knowledge of the function's behavior for a finite (no matter how narrow) band of frequencies. Thus, information is only required in a band.

In most problems we have a complete description of the impedance function in a band of interest and we are interested in obtaining its behavior outside this band of interest. However, there is no assurance that there exists an impedance function whose real and imaginary parts, in the band of interest, coincide with the given ones. If such a function exists, then it is sufficient to know its behavior in the band of interest.

### 4.2 Uses of This Method.

In problems similar to that of impedance loading, the impedance is known in a band of interest, leaving its continuations completely unspecified. These continuations may be desired to either synthesize the impedance or to calculate something else about the device in question. Thus, one can use the method described in this report to obtain the desired continuations.

In other problems the behavior of the impedance function may be measured in only a band, and we may want to know its behavior outside of this band. A typical example of this type of problems may be that of trying to find the transfer characteristics of a transistor. As frequency increases it becomes more difficult to measure the transistor characteristics. Hence, this method may be used to compute the behavior of these characteristics, beyond the frequency where it becomes difficult to measure them.

In general, then whenever we know the behavior of the driving-point impedance function, or the transfer impedance function, of a stable system (no poles in the right-half plane or on the $j\omega$ axis) for a range of frequencies, its behavior for all frequencies is completely determined and may be obtained by the method described in this report.

4.3 Problems in Computation.

There are three main problems in computing the impedance continuations, using the method described in this report. These arise mainly because in Eqs. (2.76), (2.33), (2.82) and (2.38) we multiply large numbers by small numbers to obtain small numbers. These small numbers are added or subtracted to obtain other small numbers.

As can be seen from Eqs. (2.33) and (2.38) the first error in computation derives from the fact that two large numbers

$$2^{2n} K_n \qquad \text{and} \qquad \sum_{m=1}^{n} A_{(n-m)} \binom{2n}{m}$$

are subtracted to obtain a small number ($A_n$). Thus, any slight error in the given data ($K_n$ being obtained from the given data) will be enhanced by the factor $2^{2n}$.

Data with an error of 5 percent to 10 percent was used and the continuations obtained were completely erroneous. Therefore, great care should be taken in using accurate data.

The second source of error in computation arises when the radius of convergence of the power series for $R(\omega)$ about the origin is less than one. If this should be the case, the coefficients of the power series will not necessarily be monotonically decreasing; indeed, they may be monotonically increasing. Therefore, if the coefficients of this power series are $R_n$, the first term in the right-hand side of Eq. (2.76)

$$\sum_{j=0}^{p} R_{(p-j)} \frac{(2j-1)!!}{(2j)!!} \tag{4.1}$$

will increase as p increases. However, the second term in the right-hand side of Eq. (2.76) (an infinite series), which is truncated for computation purposes, will not increase at the same rate as does the first term; thereby introducing an error. Moreover, when the $K_n$ obtained from (2.76) (containing the error) is substituted into Eq. (2.33) the previously incurred error will be enhanced by the factor $2^{2n}$.

Accordingly, if the given impedance function has one or more poles within the left-hand unit semicircle, the computation error incurred may make the continuations obtained completely erroneous.

The third source of error comes from the fact that to adequately represent a function with sharp variations by its Fourier series requires more than the first six or ten terms. However, if more than say the first ten terms of the Fourier series are required to represent the function $R^*(\phi)$, the computation error existing in the $K_n$'s obtained will be enhanced by the factor $2^{2n}$ as can be seen from Eq. (2.33). (Note that for $n = 10$; $2^{2n} \sim 10^7$). For this reason the method described in this thesis is limited to functions having <u>no</u> sharp variations.

It has been brought to the author's attention that the problems in computation may have arisen from expanding both sides of Eq. (2.9) in a power series about the origin. To circumvent some or all of the problems Calahan has suggested using, for the expansion of both sides of Eq. (2.9), a series with a different (larger) radius of convergence.

Two expansions which show some promise are Chebyshev polynomials and Fourier series. It will be shown, however that both of these expansions lead to the same result.

A Chebyshev polynomial is defined* as:

$$T_n(\omega) = \cos n(\cos^{-1}\omega)\,; \quad 0 \leq \omega \leq 1\ . \tag{4.2}$$

Using the change of variable:

$$\cos\frac{\theta}{2} = \omega \qquad \text{for} \begin{cases} 0 \leq \omega \leq 1 \\ 0 \leq \theta \leq \pi \end{cases} \tag{4.3}$$

one obtains for the Chebyshev polynomials:

$$T_n(\omega = \cos\frac{\theta}{2}) = \cos n\frac{\theta}{2} \quad . \tag{4.4}$$

Thus, using the change of variable (4.3) in (2.9) :

$$f(\omega = \cos\frac{\theta}{2}) = \int_1^{\infty} \frac{\lambda R(\lambda)\, d\lambda}{\left(\lambda^2 - \cos^2\frac{\theta}{2}\right)\sqrt{\lambda^2 - 1}} \quad . \tag{4.5}$$

Making once again the change of variable

$$\lambda = \sec\frac{\phi}{2} \tag{4.6}$$

in (4.5),

$$f(\theta) = \frac{1}{2}\int_0^{\pi} \frac{R^*(\phi)\, d\phi}{1 - \cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}} \quad . \tag{4.7}$$

Note that $f(\omega)$ is an even function of frequency, and if expanded in a series of Chebyshev polynomials one obtains:

$$\sum_{n=0}^{\infty} K_n \cos n\,\theta = \frac{1}{2}\int_0^{\pi} \frac{R^*(\phi)\, d\phi}{1 - \cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}} \quad . \tag{4.8}$$

---

* Weinberg, L.,"Network Analysis and Synthesis, McGraw-Hill (1962), pp. 448.

However, note that the expansion for $f(\theta)$ in a series of Chebyshev polynomials is equivalent to the Fourier series expansion of this function. Therefore, we may conclude that expanding both sides of Eq. (2.9) in a series of Chebyshev polynomials is equivalent to a Fourier series expansion of these terms.

Expanding the kernel of Eq. (4.8) in a Fourier series of two variables:

$$\frac{1}{1-\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T_{nm}\cos n\theta\cos m\phi \qquad (4.9)$$

where:

$$T_{nm} = \frac{4}{\pi}\int_0^{\pi}\int_0^{\pi}\frac{\cos n\theta\cos m\phi\, d\theta\, d\phi}{1-\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}}\ , \qquad (4.10)$$

and substituting (4.9) in (4.8) one obtains:

$$\sum_{n=0}^{\infty} K_n\cos n\theta = \sum_{n=0}^{\infty}\cos n\theta\sum_{m=0}^{\infty} T_{nm}\frac{1}{2}\int_0^{\pi} R^*(\phi)\cos m\phi\, d\phi \qquad (4.11)$$

or:

$$K_n = \sum_{m=0}^{\infty} T_{nm}\frac{1}{2}\int_0^{\pi} R^*(\phi)\cos m\phi\, d\phi\ . \qquad (4.12)$$

Replacing $R^*(\phi)$ by its Fourier series, (4.12) becomes:

$$K_n = \sum_{m=0}^{\infty} T_{nm}\frac{1}{2}\int_0^{\pi}\sum_{j=0}^{\infty} A_j\cos_j\phi\cos m\phi\, d\phi\ . \qquad (4.13)$$

Integrating (4.13),

$$K_n = \sum_{m=0}^{\infty} T_{nm} \frac{\pi}{4} A_m + T_{no} \frac{\pi}{2} A_o \tag{4.14}$$

or:

$$\frac{4}{\pi} \begin{bmatrix} K_0 \\ K_1 \\ \vdots \\ K_N \\ \vdots \end{bmatrix} = \begin{bmatrix} T_{00} & T_{01} \cdots & T_{0N} \cdots \\ T_{10} & T_{11} \cdots & T_{1N} \cdots \\ \vdots & \vdots & \vdots \\ T_{N0} & T_{N1} \cdots & T_{NN} \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 2A_0 \\ A_1 \\ \vdots \\ A_N \\ \vdots \end{bmatrix} . \tag{4.15}$$

Consequently, we have replaced the problem of solving an integral equation by that of solving an infinite system of equations.

As far as the computation problems are concerned, we have gained very little or nothing at all since we are left with problems of different nature which may or may not be easier to circumvent.

These problems are:

(i) We don't know if there exists an inverse matrix to the system of equations.

(ii) If the inverse matrix exists, we can only truncate the system and obtain an approximate solution.

(iii) The coefficients $T_{nm}$ must be obtained numerically.

(iv) The function $f(\omega = \cos \theta/2)$ must be expanded in a Fourier series.

(v) If more terms in the Fourier series for $R^*(\phi)$ are required, an enlarged version of the truncated system of equations must be solved.

However, it should be noted that even though using a Fourier series expansion the gain may have been little or nothing, there may very well be another complete set of orthogonal functions which does indeed simplify, or at least reduce, the problems in computation. Therefore, it is suggested that further work should be in this direction, that is, finding other suitable complete sets of orthogonal functions which would reduce the degree of the computational problems.

Another approach which may be of interest for further work deals with the fact that the method derived in this thesis began from the Hilbert Transforms (2.9) and (2.10). However, as pointed out by Bode[5], additional relationships between the real and imaginary parts of a positive real function may be obtained by performing a contour integration (including the whole right-half plane) on different functions of the impedance $Z(p)$ . A number of these relations have been obtained and are given by Bode.

## ACKNOWLEDGMENTS

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## Chapter Five

## APPENDIX

### A. Main Computer Program.

This program gives the desired coefficients of the Fourier series for $R^*(\phi)$, from the knowledge of the real and imaginary parts for $0 \leq \omega \leq 1$. The data is given as the Fourier coefficients ($B_n$ and $C_n$) for the real and imaginary parts in the range $0 \leq \omega \leq 1$; or $-\pi \leq \phi \leq 0$. The result obtained is the first $N + 1$ Fourier coefficients for $R^*(\phi)$, $0 \leq \phi \leq \pi$ (corresponding to $1 \leq \omega \leq \infty$).

```
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 K(25),A(25),B(45),C(45),R(90),X(90)
      REAL*8 BB(45,25),Y(25),Z(45),BC1(25)
      DATA PI/3.14159265358979/
      CALL FCVTHB(1)
10    PRINT 999
999   FORMAT ('1')
C *** GET R,X AND NCAP
      CALL RXFUN(R,X,NCAP,NBC,&500)
      NB1 = NBC+1
      NB2 = NB1+1
      NB3 = NBC+NB1
      ASEP = PI/NB3
      NNC = NB3 + 2
      NP1 = NCAP+1
      NC2 = NCAP/2
      WRITE (9,1003) (I,R(I),X(I),I=1,NB1)
1003  FORMAT ('-    I'6X'R(I)'11X'X(I)'/1X/(I5,2G15.7))
      DO 100 I=NB2,NB3
      R(I) = R(NNC-I)
      X(I) = X(NNC-I) * (-DCOS(ASEP*(I-1)))
100   X(NNC-I) = X(I)
C *** GET B AND C
      CALL FORIT(R,NBC,NBC,B,Z,IER)
      IF (IER.EQ.0) GO TO 110
105   PRINT 1001, IER
1001  FORMAT ('-  ***  IER = 'I5)
      GO TO 10
110   PRINT 1002, (I,B(I),Z(I),I=1,NB1)
1002  FORMAT ('-    I'6X'B(I)'11X'Z(I)'/1X/(I5,2G15.7))
      CALL FORIT(X,NBC,NBC,C,Z,IER)
      IF (IER.NE.0) GO TO 105
      PRINT 1004, (I,C(I),Z(I),I=1,NB1)
1004  FORMAT ('-    I'6X'C(I)'11X'Z(I)'/1X/(I5,2G15.7))
C *** CALCULATE BB AND Y IF NEEDED
      TFAC = .125D0
      DO 150 N=1,NP1
      TFAC = 4.DD*TFAC
      BB(N,N) = TFAC
      IF (N.EQ.1) BB(1,1) = 1.D0
      IS = N+1
      FI1 = -TFAC*(N+N)
      FN = N-1
      DO 150 I=IS,NB1
      BB(I,N) = FI1
      FI = I-1
150   FI1 = -FI1*(FI+FN)*(FI+1.D0)/((FI-FN+1.D0)*FI)
      Y(1) = 1.DC
      BC1(1) = 1.D0
      DO 160 I=1,NP1
      FI1 = I
      FI = FI1+FI1
      BC1(I+1) = BC1(I)*(FI+FI-2.D0)/FI1
160   Y(I+1) = Y(I)*(FI-1.D0)/FI
      WRITE (9,1009)
```

```
 1009  FORMAT ('-  BB')
       DO 170 N=1,NP1
 170   WRITE (9,1006) N,(BB(I,N),I=N,NB1)
 1006  FORMAT ('0'I5,4G15.7/(6X,4G15.7))
       WRITE (9,1020) (I,Y(I),BC1(I),I=1,NP1)
 1020  FORMAT ('-   I'6X'Y(I)'10X'BC1(I)'/1X/(I5,2G15.7))
 C *** CALCULATE R(N)
 200   DO 250 N=1,NP1
       R(N) = 0.DO
       DO 250 I=1,NP1
 250   R(N) = R(N) + BB(I+N-1,N)*B(I+N-1)
       PRINT 1005, (I,R(I),I=1,NP1)
 1005  FORMAT ('-   I'6X'R(I)'/1X/(I5,G15.7))
 C *** CALCULATE K(N)
       TFAC = .5DO
       DO 330 N=1,NO2
       TFAC = TFAC*4.DO
       K(N) = 0.DO
       FN = N-1
       FTN = FN+FN
       DO 310 I=1,N
 310   K(N) = K(N) + R(N-I+1)*Y(I)
       S1 = 0.DO
       TIF = 2.DO
       TIJ = 1.DO
       DO 325 J=1,NP1
       S2 = 0.DO
       JM1 = J-1
       FJ = JM1
       IF (J.EQ.1) GO TO 320
       TIS = TIF*(FJ+FN+1.DO)/(FJ+1.DO)*TIJ
       FJN = FJ+FTN+2.DO
       DO 315 I=1,JM1
       FI = I-1
       FII = FI+FI+1.DO
       FJI = FJ-FI
       TIS = TIS*(FJI+1.DO)/FJI
       S2 = S2 + TIS*BC1(I)
       TIS = -TIS*((FJN+FI)*(FJI-1.DO)/((FTN+FII+2.DO)
      & *(FTN+FII+3.DO)))
 315   CONTINUE
       TIJ = TIJ*FJN/FJ
 320   S1 = S1 + C(J+N)*(BC1(J)+S2)
       TIF = -TIF
 325   CONTINUE
       K(N) = K(N) + TFAC*S1
 330   CONTINUE
       PRINT 1007, (I,K(I),I=1,NO2)
 1007  FORMAT ('-   I'6X'K(I)'/1X/(I5,G15.7))
 C *** CALCULATE A(N)
       TFAC = 1.DO
       DO 450 N=1,NO2
       SUM = 0.
       IF (N.EQ.1) GO TO 415
       JS = N-1
```

```
         FN = JS+JS
         FI1 = FN/TFAC
         DO 410 J=1,JS
         SUM = SUM + FI1*A(N-J)
         FJ = J
 410     FI1 = FI1*(FN-FJ)/(FJ+1.D0)
 415     A(N) = TFAC*(K(N)-SUM)
 450     TFAC = TFAC*4.D0
         PRINT 1008, (I,A(I),I=1,NO2)
 1008    FORMAT ('-    I'6X'A(I)'/1X/(I5,G15.7))
 C *** DO PART 4 IF DESIRED
 C       AN END OF FILE WILL SKIP IT
         PRINT 1010
 1010    FORMAT ('-  PART 4 ?')
         READ (1,1011,END=10) Z(1)
 1011    FORMAT (A8)
         PRINT 1012
 1012    FORMAT ('-'5X'PHI'10X'R*(PHI)'/1X)
         DO 600 I=1,20
         T = 1.D0/I
         PHI = 2.D0*DARCOS(T)
         SUM = 0.D0
         DO 610 N=1,NO2
 610     SUM = SUM + A(N)*DCOS((N-1)*PHI)
 600     PRINT 1013, PHI,SUM
 1013    FORMAT (2G15.7)
         GO TO 10
 500     CALL SYSTEM
         END
```

B. Subroutine Forit.

The following subroutine obtains the Fourier coefficients $B_n$ and $C_n$ of Eqs. (2.52) and (2.53). The data is given as the value of the real $(R(\phi))$ and imaginary $(X(\phi))$ parts, for $0 \leq \omega \leq 1$, $(-\pi \leq \phi \leq 0)$ at $4N + 5$ equally spaced frequencies in the $\phi$ range. As a result one obtains the first $2N + 3$ Fourier coefficients for $R(\phi)$ and $\cos\frac{\phi}{2} X(\phi)$ .

```
      SUBROUTINE FCRIT(FNT,N,M,A,B,IER)
      IMPLICIT REAL*8 (A-H,O-Z)
      CIPENSICN A(1),B(1),FNT(1)

      FORIT COMPUTES FCURIER COEFICIENTS FROM A TABULATED SET OF DATA POINTS
      THESE FUNCTICN VALUES ARE IN 'FNT' AND FNT(I) MUST HAVE THE VALUE AT
        2*PI*(I-1)/(2*N+1), I=1,2,...,2*N+1
      'M' IS THE CESIRED ORDER OF THE COEFICIENTS, SO M+1 ARE RETURNED IN
        'A' ANC 'B'.  (B(1) = 0., ALWAYS)
      'FNT', 'A' AND 'B' ARE ALL REAL*8
      'IER' HAS RETURN CODE:
        C - ACK
        1 - 'N' LESS THEN 'M'
        2 - M LESS THAN ZERO

      IER = 2
      IF (M.LT.C) RETURN
      IER=1
      IF (N.LT.M) RETURN
      IER=0
      NC = N+N+1
      CCEF = 2.CC/NC
      CCNST = 3.14159265358979*COEF
      S1 = DSIN(CCNST)
      C1 = DCOS(CONST)
      C = 1.CO
      S = 0.CO
      J = 1
      FNTZ = FNT(1)
      B(1) = C.CC
   70 U2 = C.CO
      U1 = C.CO
      I = NC
   75 UZ = FNT(I)+2.DO*C*U1-U2
      U2 = U1
      U1 = UZ
      I = I - 1
      IF (I.GT.1) GC TC 75
      A(J) = CCEF*(FNTZ+C*U1-U2)
      B(J) = CCEF*S*U1
      IF (J.GT.M) GC TC 100
      J = J + 1
      Q = C1*C - S1*S
      S = C1*S + S1*C
      C = Q
      GC TC 70
  100 A(1) = A(1)*.5D0
      RETURN
      END
```

C. Subroutine to go From The $\omega$ Domain to the $\phi$ Domain.

The following subroutine transforms any function from the $\omega$ - domain ($0 \leq \omega \leq 1$) to the $\phi$ - domain ($-\pi \leq \phi \leq 0$) by the transformation $\omega = \cos \frac{\phi}{2}$ . It also obtains $4N + 5$ points of the function equally spaced in the $\phi$ - domain.

```
      SUBROUTINE RXFUN(R,X,NCAP,NBC)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION R(90),X(90)
      DATA PI/3.14159265358979/
 10   PRINT 1000
 1000 FORMAT ('  NCAP = ?')
      READ (1,1001,END=500) NCAP
 1001 FORMAT (I5)
      IF (NCAP.EQ.0) GO TO 10
      PRINT 1002,NCAP
 1002 FORMAT ('-NCAP = 'I5)
      NCAP = NCAP+NCAP+2
      NBC = NCAP+NCAP
      NP2 = NBC + 1
      ASEP = PI/(NP2+NBC)
      DO 100 I=1,NP2
      CP = DCOS(ASEP*(I-1))
      CP2 = CP*CP
      R(I) = 1.D0 - CP/2.D0
      X(I) = (DLOG((2.D0-CP)/(2.D0+CP)) +
     & CP*DLOG((4.D0-CP2)/CP2)/2.D0)/PI
 100  CONTINUE
      RETURN
 500  RETURN 1
      END
```

D. Subroutine Used to Read Points From Subroutine C Into the Subroutine Forit.

```
      SUBROUTINE RXFUN(R,X,NCAP,NBC)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION R(90),X(90)
      DATA PI/3.14159265358979/
10    PRINT 1000
1000  FORMAT ('  NCAP = ?')
      READ (1,1001,END=500) NCAP
1001  FORMAT (I5)
      IF (NCAP.EQ.0) GO TO 10
      PRINT 1002,NCAP
1002  FORMAT ('-NCAP = 'I5)
      NCAP = NCAP+NCAP+2
      NBC = NCAP+NCAP
      NP2 = NBC + 1
      ASEP = PI/(NP2+NBC)
      AGD = ASEP*360./PI
      PRINT 1003, NP2,AGD
1003  FORMAT ('-ENTER 'I2' R AND X POINTS SEPARATED '
     &/' BY 'G15.7' DEGREES')
      READ (1,1004) (R(I),X(I),I=1,NP2)
1004  FORMAT (2G10.10)
      RETURN
500   RETURN 1
      END
```

E.

To show that:

$$\int_0^1 \frac{d\lambda}{(\lambda^2 - \omega^2)\sqrt{1 - \lambda^2}} \equiv 0 \quad \text{for } 0 \leq \omega \leq 1 \tag{E.1}$$

make the following change of variable:

$$\lambda^2 = \frac{x^2}{1 + x^2} \tag{E.2}$$

$$\frac{d\lambda}{\sqrt{1 - \lambda^2}} = \frac{dx}{(1 + x^2)} \tag{E.3}$$

So that (E.1) becomes:

$$\int_0^\infty \frac{dx}{(1 - \omega^2)\left[x^2 - \frac{\omega^2}{1 - \omega^2}\right]} \equiv 0 \tag{E.4}$$

Note that (E.1) must be evaluated in the Cauchy principal value sense; i.e.

$$\int_0^1 dx = \lim_{\epsilon \to 0} \left[ \int_0^{\omega - \epsilon} dx + \int_{\omega + \epsilon}^1 dx \right] \tag{E.5}$$

So that (E.4) becomes :

$$0 \equiv \lim_{\epsilon \to 0} \left[ \int_0^{\frac{\omega - \epsilon}{\sqrt{1 - (\omega - \epsilon)^2}}} dx + \int_{\frac{\omega + \epsilon}{\sqrt{1 - (\omega + \epsilon)^2}}}^{\infty} dx \right] \tag{E.6}$$

Integrating (E.4) one obtains:

$$0 \equiv \frac{1}{1-\omega^2}\left\{ \frac{1}{\dfrac{2\omega}{\sqrt{1-\omega^2}}} \; \mathrm{Ln} \; \frac{\omega - \dfrac{\omega}{\sqrt{1-\omega^2}}}{x + \dfrac{\omega}{\sqrt{1-\omega^2}}} \right\}_0^{\infty} \tag{E.7}$$

Making use of (E.6) one obtains for (E.7):

$$0 = \frac{1}{2\omega\sqrt{1-\omega^2}} \lim_{\epsilon \to 0} \left\{ \mathrm{Ln} \left[ (1) \cdot \frac{\dfrac{\omega-\epsilon}{\sqrt{1-(\omega-\epsilon)^2}} - \dfrac{\omega}{\sqrt{1-\omega^2}}}{\dfrac{\omega-\epsilon}{\sqrt{1-(\omega-\epsilon)^2}} + \dfrac{\omega}{\sqrt{1-\omega^2}}} \cdot \right.\right.$$

$$\left.\left. \cdot \frac{\dfrac{\omega+\epsilon}{\sqrt{1-(\omega+\epsilon)^2}} + \dfrac{\omega}{\sqrt{1-\omega^2}}}{\dfrac{\omega+\epsilon}{\sqrt{1-(\omega+\epsilon)^2}} - \dfrac{\omega}{\sqrt{1-\omega^2}}} \cdot (-1) \right] \right\} \tag{E.8}$$

when the limit is taken one obtains:

$$0 \equiv 0 \;. \quad \text{Q. E. D.}$$

F.

To show that equation (2.64) has no singularities we make use of the fact that $X(\lambda)$ is the imaginary part of a function analytic in the right-half plane, and hence has a power series about the origin. Thus:

$$\lambda X(\lambda) = \sum_{n=1}^{\infty} X_n \lambda^{2n} \tag{F.1}$$

$$\omega X(\omega) = \sum_{n=1}^{\infty} X_n \omega^{2n} \tag{F.2}$$

So that:

$$\lambda X(\lambda) - \omega X(\omega) = \sum_{n=1}^{\infty} X_n (\lambda^{2n} - \omega^{2n}) \tag{F.3}$$

Equation (2.64) then becomes:

$$I = \int_0^1 \frac{\sum_{n=1}^{\infty} X_n (\lambda^{2n} - \omega^{2n}) \, d\lambda}{(\lambda^2 - \omega^2) \sqrt{1 - \lambda^2}} \tag{F.4}$$

But:

$$\frac{\lambda^{2n} - \omega^{2n}}{\lambda^2 - \omega^2} = \sum_{i=1}^{n} \lambda^{2(i-1)} \omega^{2(m-i)} \tag{F.5}$$

then:

$$I = \int_0^1 \sum_{n=1}^{\infty} \frac{X_n}{\sqrt{1 - \lambda^2}} \sum_{i=1}^{n} \lambda^{2(i-1)} \omega^{2(m-i)} d\lambda \tag{F.6}$$

which has no singularities Q. E. D.

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13. ABSTRACT

In certain problems of Network Theory the real and imaginary parts of a driving-point impedance function may be independently given in a band of interest $(\omega_1 \leq \omega \leq \omega_2)$, leaving its continuations onto $(0 \leq \omega \leq \omega_1)$ and $(\omega_2 \leq \omega \leq \infty)$ completely unspecified.

By manipulating the Hilbert Transforms, relating the real and imaginary parts of a function of a complex variable $(p = \sigma + j\omega)$ having no poles on the $j\omega$ axis or in the right-half plane, this report shows that if the continuations exist they are unique and readily obtained.

Three necessary conditions for the existence of the continuations to the given parts (of the driving-point impedance function) are obtained. Further, if these three conditions are satisfied the continuations may be obtained as a Fourier series.

Four known impedances were used as examples and their continuations determined. The results obtained were excellent. The agreement between the Fourier series solution (using the first six terms at most) and the exact expression was good up to the second significant figure.

In the appendix a computer program which obtains the Fourier coefficients of the unknown continuations, from the given real and imaginary parts, is furnished.

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