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THE MECHANICS OF SUSPENSIONS

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THE MECHANICS OF SUSPENSIONS

1. AIM OF THE MECHANICS OF SUSPENSIONS

1.1. A fluid which contains small particles of solids (dust, coal, etc.) or of liquids (droplets of water, oil, paint, or liquid metal) constitutes a suspension. If the fluid is a gas, and if the particles are small, a few microns at most, such a suspension is called an aerosol.

With a suspension having at infinity upstream a velocity U_0, let us place an obstacle, G, in the flow (Fig. 1). The phenomena which we shall

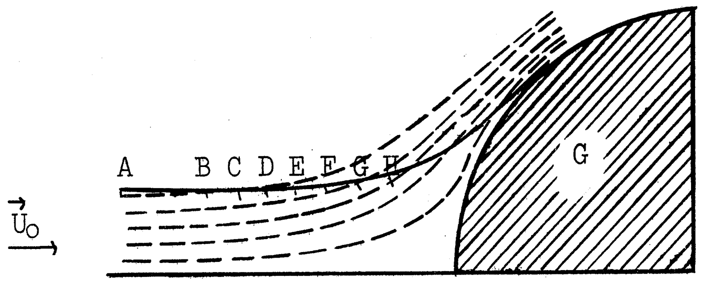

Fig. 1

describe would be the same if the suspension were stationary and the obstacle, G, instead of being immobile, were moving with a velocity of translation U_0.

The streamlines (dotted lines), which are rectilinear at infinity upstream, have some curvature in the neighborhood of the obstacle. The particles follow the streamlines if these are rectilinear but they have trajectories distinct from those of the streamlines in the neighborhood of the obstacle where the dynamic field is no longer uniform.

In what follows we propose to determine analytically and graphically the trajectories of particles in suspension in a nonuniform, but known, dynamic field when the motion of the fluid is two-dimensional or axi-symmetric, in other words, when the streamlines are plane.

1.2. We shall deal later with the applications of this study. It seems worth noting from the outset the most important of these.

1.2.1. If the particles have a specific mass larger than that of the fluid, the trajectories are less curved* than the streamlines; some of them hit the obstacle, G, and if the particles have adhesive properties, they will be caught by it. Such is the case for droplets of tar hitting the walls of a shock purifier; such is also the case for supercooled droplets of water which freeze upon impact (freezing on airplanes).

We shall learn to determine those regions of the surface of the body G which are hit by the particles (surface of catch) and the intensity of the number of particles caught around a given point on the impact surface (coefficient of catch).

1.2.2. Downstream from the surface of catch and in the potential flow of the fluid, there are some regions which are deprived of particles as a result of the deflection of the trajectories (zones of clear air). The knowledge of such zones permits the protection of bodies from the impact of particles while exposing them to the potential flow.

1.2.3. One can also inject small particles into a flow in order to visualize it. Yet, according to what we have just said, solid particles do not follow the streamlines. The Mechanics of Suspensions enables one to find under what conditions the deviation of the trajectories of particles from the streamlines is small enough for this method of visualization to be acceptable.

2. EQUATIONS OF THE MECHANICS OF SUSPENSIONS

2.1. General Equation

2.1.1. We shall make the hypothesis that the particles are too few to disturb the fluid flow, so that everywhere the streamlines are identical to what they would be if the fluid did not have particles in suspension.

* Original article reads "plus tendues" and is presumed to be in error.

We shall also assume that all the particles are identical. In reality such is never the case, but the over-all result will be obtained by superposing the results obtained for particles of various dimensions.

2.1.2. The flow of the fluid is defined by:

1) The characteristics of the fluid, that is, its specific mass, ρ': its kinematic viscosity, ν; and its velocity at infinity upstream, $\vec{U}_0$.

2) The characteristics of the obstacle, that is, its shape and one of its dimensions, D.

Starting from the above information, fluid mechanics allows us to determine theoretically the dynamic field around the obstacle. We suppose that the dynamic field is effectively known.

The Mechanics of Suspensions will now permit us to determine the trajectories of the particles if we define:

3) The characteristics of the particles, that is, their shape, their specific mass, ρ; their volume, V; or their total area, S; or one of their dimensions, d; in the case of a spherical particle, d is the diameter.

2.1.3. Let M be a point in a dynamic field, $\vec{U}'$ the velocity of the fluid at that point, and $\vec{U}$ that of the particle at M (Fig. 2). The relative velocity, $\vec{u}$, of the particle with respect to the fluid is defined by the relation

$$\vec{u} = \vec{U} - \vec{U'} \quad . \tag{1}$$

Let α be the angle between the velocities $\vec{U}$ and $\vec{U'}$; the absolute value of $\vec{u}$ is given by the relation

$$u = \sqrt{U^2 + U'^2 - 2UU' \cos \alpha} \quad . \tag{2}$$

2.1.4. The forces applied to the particle passing the point M are

1) The inertia force, $-\rho V \frac{d\vec{U}}{dt}$;

2) The pressure forces, $-\iint_S \vec{n} p \, dS$, where $\vec{n}$ designates the unit vector normal to the surface and directed from the inside toward the outside of the particle.

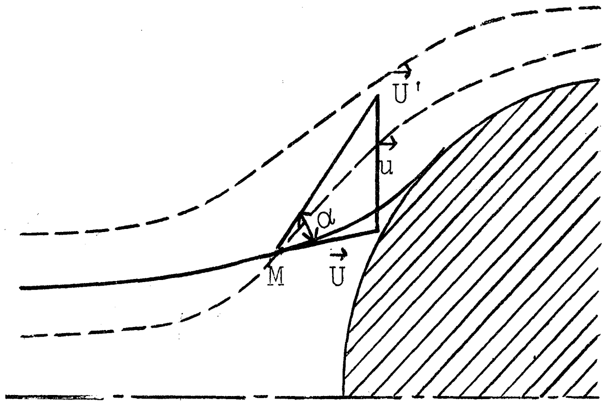

Fig. 2

One can transform the surface integral into a volume integral and write:

$$-\iint_S \vec{n} p \, dS = -\iiint_V \overrightarrow{\text{grad}}\, p \, dV.$$

Because of the small dimensions of the particle, one can replace the volume integral by V $\overrightarrow{\text{grad}}$ p and write:

$$-\iint_S \vec{n} p \, dS = -V \overrightarrow{\text{grad}}\, p.$$

3) The drag forces with which the fluid opposes the notion of the particle moving with the relative velocity $\vec{u}$. Because of the small dimensions of the particle these forces reduce to a resultant $\vec{R}$, which we shall write, by introducing the projected area S', in its classical form

$$\vec{R} = -\frac{C}{2} \rho' S' u \vec{u} . \quad (3)$$

The drag coefficient is a function, $\phi(ud/\nu)$, of the Reynolds number with relative speed u. The shape of the particle being defined, one can also write $\vec{R}$ as

$$\vec{R} = -\frac{\rho' V u}{d} \vec{u} f \left(\frac{ud}{\nu}\right) . \quad (4)$$

The comparison of Equations 3 and 4 shows that

$$f\left(\frac{ud}{\nu}\right) = \frac{S'd}{2V} C . \quad (5)$$

4) The forces of gravity, equal to $\vec{g}$ per unit mass. The gravitational force which is exerted on a particle is then $\rho V \vec{g}$.

To sum it up, the equation of motion of a particle going through a point M can be written

$$-\rho V \frac{d\vec{U}}{dt} - V\,\overrightarrow{\text{grad}}\ p - \frac{\rho' V u}{d}\vec{u}\, f\left(\frac{ud}{\nu}\right) + \rho V \vec{g} = 0$$

or, dividing by V,

$$\rho \frac{d\vec{U}}{dt} + \overrightarrow{\text{grad}}\ p + \frac{\rho' u}{d}\vec{u}\, f\left(\frac{ud}{\nu}\right) - \rho\vec{g} = 0 . \tag{6}$$

The Euler equation for the motion of a fluid is

$$\rho' \frac{d\vec{U}'}{dt} + \overrightarrow{\text{grad}}\ p - \rho'\vec{g} = 0 . \tag{7}$$

Subtracting term by term Equation 6 from 7, one obtains the general equation which defines the motion of a particle as it goes through M.

$$\boxed{\rho \frac{d\vec{U}}{dt} - \rho' \frac{d\vec{U}'}{dt} = -\frac{\rho' u}{d}\vec{u}\, f\left(\frac{ud}{\nu}\right) + (\rho - \rho')\vec{g}.} \tag{8}$$

This is the general equation of the Mechanics of Suspensions[1].

2.2. Possible Simplification of the General Equation

2.2.1. If the relative velocity, u, is very small, the drag force is proportional to u. Therefore,

$$f\left(\frac{ud}{\nu}\right) = \frac{K\nu}{ud}, \tag{9}$$

where K is a constant which depends on the shape of the particle. The expression for the drag is

[1]One can imagine an equation which would be even more general by assuming that the particle is acted on by a field of uniform forces, $\vec{H}$, which may not necessarily be a gravitational field. One has then

$$\rho \frac{d\vec{U}}{dt} - \rho' \frac{d\vec{U}'}{dt} = -\rho' \frac{u}{d}\vec{u}\, f\left(\frac{ud}{\nu}\right) + \rho\vec{H} - \rho'\vec{g} . \tag{8b}$$

In the particular case where $\rho'\vec{g}$ is negligible compared to $\rho\vec{H}$, Equation 8b is written

$$\rho \frac{d\vec{U}}{dt} - \rho' \frac{d\vec{U}'}{dt} = -\frac{\rho' u}{d}\, f\left(\frac{ud}{\nu}\right) + \rho\vec{H} . \tag{8c}$$

$$\vec{R} = -K\rho' \frac{V\nu}{d^2} \vec{u}.$$

2.2.2. When the particle is spherical (fog drop), Equation 5 shows that

$$f\left(\frac{ud}{\nu}\right) = \frac{3}{4} C. \qquad (10)$$

The coefficient of drag, C, varies with the Reynolds Number $R = ud/\nu$ according to the curve of Fig. 3 (logarithmic coordinates).

2.2.3. In the case where the relative velocity, u, is small and where the particle is spherical, it is known that the drag is given by Stokes' Law

$$\vec{R} = -3\pi\nu\rho' \vec{u} d .$$

By comparing this relation to the relation of Equation 4, one finds that

$$\boxed{f\left(\frac{ud}{\nu}\right) = \frac{18\nu}{ud}} \qquad (11)$$

and

$$C = \frac{4}{3} f\left(\frac{ud}{\nu}\right) = \frac{24\nu}{ud}. \qquad (12)$$

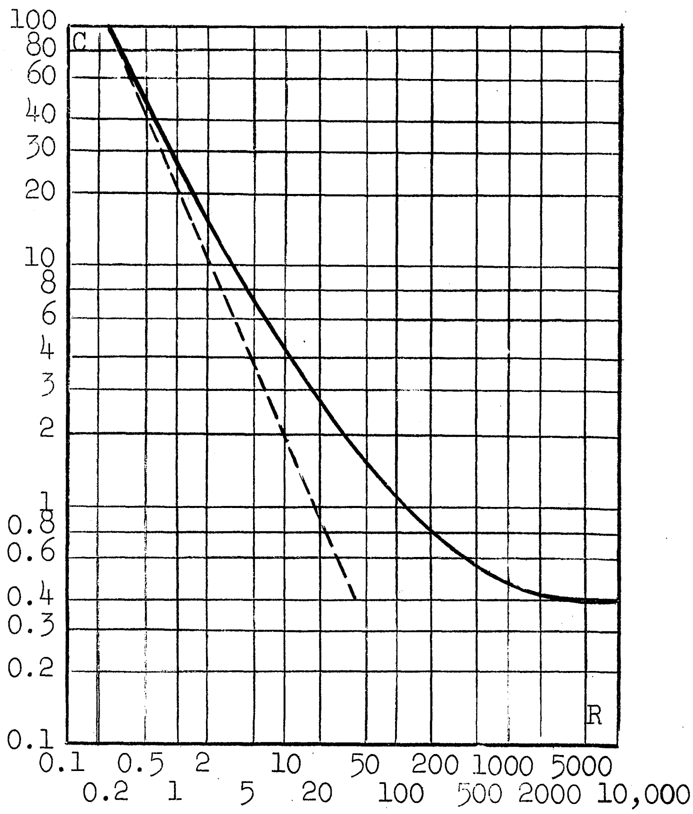

Fig. 3

On Fig. 3 has been traced in dotted lines the straight line expressed by Stokes' Law. One sees that beyond $R = 1$ the drag line deviates notably from Stokes' Law; from $R = 10$, the drag coefficient is already twice that given by Stokes' Law. This law can therefore be applied only to relatively small velocities.

2.2.4. In most applications, and in particular in a case of an aerosol, the specific mass, ρ' of the fluid is small compared to the specific mass of the particles; to the contrary, the absolute values of the velocities $\vec{U}$ and $\vec{U}'$ and the absolute values of their derivatives, $d\vec{U}/dt$ and $d\vec{U}'/dt$, are of the same order of magnitude. Therefore, under Formula 8, the term representing inertia $\rho' dU'/dt$ is negligible compared to the term representing inertia $\rho dU/dt$. Likewise, the term $\rho'\vec{g}$ is negligible compared to $\rho\vec{g}$.

The general equation is then simplified and becomes

$$\boxed{\rho \frac{d\vec{U}}{dt} = -\rho' \frac{u\vec{u}'}{d} f\left(\frac{ud}{\nu}\right) + \rho\vec{g}} \qquad (13)$$

2.2.5. We shall show later that as soon as the velocities become appreciable, the term allowing for gravitation is negligible. The general equation is simplified further and becomes

$$\boxed{\rho \frac{d\vec{U}}{dt} = -\rho' \frac{u\vec{u}'}{d} f\left(\frac{ud}{\nu}\right)} \qquad (14)$$

Of course, in either Formula 13 or 14, one can use the value of the function $f(ud/\nu)$ defined in the particular case 2.2.1, 2.2.2, and 2.2.3.

3. SIMILARITY RELATIONS IN THE MECHANICS OF SUSPENSIONS

The study of similarity in the mechanics of suspensions is of great interest for the interpretation of experiments on models and, as we shall see later, for the construction of trajectories.

3.1. General Conditions

3.3.1. Let us vary the quantities which characterize the particles and the fluid, retaining the shape of the particle and of the obstacle, according to the following conditions:

- the dimensions of the obstacle are multiplied by a factor Λ (similarity relationship between the obstacle and the dynamic field);
- the dimensions of the particle are multiplied by a factor λ;
- the specific mass of the fluid is multiplied by a factor Δ';
- the specific mass of the particles is multiplied by a factor Δ;
- the kinematic viscosity of the fluid is multiplied by a factor γ;
- the velocity of the fluid at infinity upstream is multiplied by Λ/Θ.

In general, such variations result in important modifications in the network of trajectories and streamlines.

3.1.2. Let us imagine that the preceding variations modify neither the relative positions of the trajectories of the particles nor those of the streamlines. In other words, the velocity field is in the second flow similar to what it was in the first one. It follows from Equations 1 and 2 that the magnitudes of all the velocities are multiplied by Λ/Θ and, therefore, the accelerations by Λ/Θ^2. Equation 8, therefore, becomes

$$\Delta\rho \frac{\Lambda}{\Theta^2}\frac{dU}{dt} - \Delta'\rho' \frac{\Lambda}{\Theta^2}\frac{dU'}{dt} = -\frac{\Delta'\rho'}{\lambda d}\frac{\Lambda^2}{\Theta^2} uu\, f\left(\frac{\Delta\lambda}{\Theta\chi}\frac{ud}{\nu}\right) + (\Delta\rho - \Delta'\rho')\vec{g}. \quad (15)$$

If the following conditions are imposed on the multiplier:

$$\Delta = \Delta'; \quad \Lambda = \lambda; \quad \frac{\Lambda^2}{\Theta\chi} = 1; \quad \frac{\Lambda}{\Theta^2} = 1, \quad (16)$$

it is seen that Equation 15 is identical to Equation 8[1].

3.1.3. Let us consider two geometrically similar obstacles of which D and D_1 are two characteristic lengths. Let us place these obstacles in fluids of kinematic viscosities ν and ν_1, of specific masses ρ' and ρ_1' and with velocities at infinity $\vec{U}_0$ and $\vec{U}_{01}$ similarly oriented with respect to the corresponding obstacles. Let us assume that particles of similar shapes are in suspension in each of these fluids; d and d_1 are two characteristic lengths of these particles, and ρ and ρ_1 are their specific masses respectively.

If the equations

$$\boxed{\frac{\rho}{\rho'} = \frac{\rho_1}{\rho_1'}; \quad \frac{D}{d} = \frac{D_1}{d_1}; \quad \frac{U_0 D}{\nu} = \frac{U_{01}D_1}{\nu}; \quad \frac{U_0^2}{D} = \frac{U_{01}^2}{D_1},} \quad (17a)$$

which follows from the conditions in (16), are satisfied, the networks formed by the streamlines and the trajectories of the particles around each of the obstacles are geometrically similar.

[1]In Equation 8c, given in Footnote 1 of Paragraph 2.1.4, let us imagine that the field H is in addition multiplied by the factor K. This equation becomes

$$\Delta\rho \frac{\Lambda}{\Theta^2}\frac{d\vec{U}}{dt} - \Delta'\rho' \frac{\Lambda}{\Theta^2}\frac{d\vec{U'}}{dt} = -\frac{\Delta'\rho'}{\lambda d}\frac{\Lambda^2}{\Theta^2} u\vec{u}\, f\left(\frac{\Delta\lambda}{\Theta\chi}\frac{ud}{\nu}\right) + \Delta K\rho\vec{H} \quad (15c)$$

This equation is identical to Equation 8c if

$$\Delta = \Delta'; \quad \Lambda = \lambda; \quad \frac{\Lambda^2}{\Theta\chi} = 1; \quad K = \frac{\Lambda}{\Theta^2}. \quad (16c)$$

It is immediately seen that the system in (17a) is equivalent to the system

$$\frac{U_0 D}{\nu} = \frac{U_{01} D_1}{\nu_1}; \quad \frac{U_0 d}{\nu} = \frac{U_{01} d_1}{\nu_1}; \quad \frac{\rho}{\rho'} = \frac{\rho_1}{\rho'_1}; \quad \frac{U_0^2}{D} = \frac{U_{01}^2}{D_1} \qquad (17b)$$

or else

$$\frac{U_0 d}{\nu} = \frac{U_{01} d_1}{\nu_1}; \quad \frac{D}{d} = \frac{D_1}{d_1}; \quad \frac{\rho}{\rho'} = \frac{\rho_1}{\rho'_1}; \quad \frac{U_0^2}{d} = \frac{U_{01}^2}{d_1} .$$

These similarity conditions show that the flow remains similar to itself if, having varied arbitrary ρ and D (or d), we choose the four remaining quantities ρ', U_0, d, and ν in such a way that the conditions in (17) are satisfied[1].

3.2. Simplification of the Similarity Conditions

3.2.1. When gravitation is negligible, the last term of Equation 15 disappears and, as a result, the last condition in (16) is superfluous. The first three are enough for Equation 15 to be identical to Equation 8. Similarity conditions are therefore reduced to

$$\boxed{\frac{\rho}{\rho'} = \frac{\rho_1}{\rho'_1}; \quad \frac{D}{d} = \frac{D_1}{d_1}; \quad \frac{U_0 D}{\nu} = \frac{U_{01} D_1}{\nu_1}} \qquad (18)$$

3.2.2. For an aerosol, assuming gravity negligible as well as the term $\rho' \, dU'/dt$, the last terms of each member of the relation in Equation 15 disappear. To ensure similarity, the only conditions required are

$$\Delta = \frac{\Delta' \Delta}{\lambda}; \quad \frac{\Delta \lambda}{\Theta \chi} = 1$$

which can be written

$$\boxed{\frac{D}{d}\,\frac{\rho'}{\rho} = \frac{D_1}{d_1}\,\frac{\rho'_1}{\rho_1}; \quad \frac{U_0 d}{\nu} = \frac{U_{01} d_1}{\nu_1}} \qquad (19)$$

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[1]In the case of a uniform field $\vec{H}$, the size of which varies (footnote 1 of paragraph 2.1.4), the conditions of similarity are retained according to Equation 16c (footnote 1 of paragraph 3.1.2),

$$\frac{\rho}{\rho'} = \frac{\rho'}{\rho'_1}; \quad \frac{D}{d} = \frac{D_1}{d_1}; \quad \frac{U_0 D}{\nu} = \frac{U_{01} D_1}{\nu_1}; \quad \frac{dH}{U_0^2} = \frac{d_1 H_1}{U_0^2} \qquad (17c)$$

One derives easily from the above relations

$$\boxed{\frac{H}{H_1} = \left(\frac{D_1}{D}\right)^3 \left(\frac{\nu}{\nu_1}\right)^2 \left(\frac{d_1}{d}\right)^3 \left(\frac{\nu}{\nu_1}\right)^2} \qquad (17d)$$

3.3. A Few Conclusions

From the conditions of (17), (18), and (19) one can make certain conclusions concerning the possibility of experiments with models.

3.3.1. Assuming gravitation is not negligible, let us define the value of D. The second equation in (17) gives the dimension, d, of the particle; the fourth gives the velocity, U_0; and, finally, the third gives the kinematic viscosity, ν . This last factor is hard to modify, so that experiments with scaled-down models are practically impossible if gravitation cannot be neglected.

3.3.2. Let us assume that gravitation is negligible and define the value of D. The second equation in (18) gives the value of d; the third equation gives the value of U_0/ν . It is then possible to define the value of ν and to modify U_0 in order to realize similarity.

The case of an aerosol is the same as the latter, since we have assumed above that the specific masses ρ and ρ' remain constant, which makes Equation 19 coincide with Equation 18.

3.3.3. Therefore, let us consider the fluid of defined characteristics (ρ' and ν constants) and a solid of constant specific mass, ρ. If the dimensions of the obstacle are divided by Λ , the flow will be similar to itself if the dimensions of the particles are divided by Λ and the velocity at infinity upstream is multiplied by Λ [1].

4. GRAPHICAL DETERMINATION OF TRAJECTORIES

4.1. Conditions of Application

We shall develop a graphical method which permits one to trace, from point to point, the trajectory of particles in suspension in a fluid

[1]Consider the case where the factors D and ρ' vary together and where we are dealing with an aerosol with negligible gravitation (the case of air models flying at variable altitudes). It is easy to see that in order to have similarity of flow, when D is divided by Λ and ρ' by Δ', one must, according to relation 19, divide d by the product $\Lambda\,\Delta'$ and multiply U_0 by the product $\Lambda\,\Delta'$.

in motion. This method as presented assumes that the following conditions hold.

4.1.1. First Condition. The fluid has a negligible specific mass compared to that of the particles. We are referring to an aerosol such as a fog, a cloud, a sand storm, etc. In addition, the velocities are large enough for gravitation to be negligible.

We have seen that under these conditions the motion of a particle is defined everywhere by the vectorial equation

$$\rho \frac{d\vec{U}}{dt} = - \frac{\rho' u \vec{u}}{d} f\left(\frac{ud}{\nu}\right) . \qquad (14)$$

In addition, the similarity conditions are defined by the two relations

$$\frac{U_0 d}{\nu} = \frac{U_{01} d_1}{\nu_1} ; \qquad \frac{\rho' D}{\rho d} = \frac{\rho_1' D_1}{\rho_1 d_1} . \qquad (19)$$

Thus, all the characteristics of the flow are defined by the two dimensionless numbers $U_0 d/\nu$ and $\rho' D/\rho d$.

4.1.2. Second Condition. The trajectories of the particles are contained in planes. This second condition holds if the obstacle is a cylinder of infinite span of which the generating lines are perpendicular to the velocity at infinity, $\vec{U}_0$. All that is required then is to trace the trajectories in a plane perpendicular to the generating lines.

This condition is also satisfied if the obstacle is a solid of revolution, the velocity at infinity being parallel to the axis of rotation. It is only necessary then to trace the trajectories in a meridian plane.

4.2. Principle of the Method

4.2.1. Let us consider the trajectory which goes through point M (Fig. 4). Let us project the vectorial equation (14) onto the tangent MU to the trajectory, oriented in the direction of the vector $\vec{U}$ and onto the normal MN to the trajectory obtained from MT by a rotation of + $\pi/2$ around M.

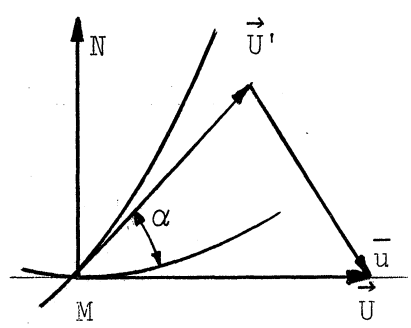


Fig. 4

We thus obtain the intrinsic equation of motion of a particle going through M.

Let us call $d\ell$ the element of arc of the trajectory, α the algebraic angle $\widehat{\vec{U}, \vec{U}'}$ which forms the two velocity vectors at M, and R the radius of curvature of the trajectory at the point M (R positive when the center of curvature goes through the half-normal MN, negative in other cases).

The intrinsic equations are written

$$U \frac{dU}{d\ell} = \frac{\rho'}{\rho d} (U' \cos \alpha - U) u \, f(\frac{ud}{\nu}) \qquad (20)$$

and

$$\frac{U^2}{R} = \frac{\rho'}{\rho d} U' \sin \alpha \; u \; f(\frac{ud}{\nu}) , \qquad (21)$$

where, as above,

$$\vec{u} = \vec{U} - \vec{U}'. \qquad (1)$$

4.2.2. Let us assume that at point A, we know velocity $\vec{U}'$ and $\vec{U}$ of the fluid and of the particle. Equation 1 then gives us the relative velocity, $\vec{u}$; its absolute value is

$$u = \sqrt{U^2 + U'^2 - 2UU' \cos \alpha} \qquad (2)$$

Equation 21 gives the algebraic value R of the radius of curvature of the trajectory at A and, therefore, the position of the corresponding center of curvature. We can trace the element AB of the trajectory by equating it at point A to the arc of a circle of the same curvature (Fig. 1).

To know the absolute value $U_B = U_A + \Delta U$ of the velocity of the particle going through point B, one need only substitute in Equation 20 (considered as a difference equation) the value of the arc AB for the differential element $d\ell$.

The velocity vector of the particle at point B is as determined and since we know from the aerodynamic field the velocity vector $\vec{U}'_B$ of the fluid at point B, we can construct, starting from B, a new arc BC, and so on from point to point.

4.2.3. The preceding reasoning assumes that at point A, where we start the construction of the trajectory vector, $\vec{U}$ is known. This is never the case. To begin with, it is assumed that at point A (far enough from the obstacle for the velocity $\vec{U}'_A$ of the fluid to be very little different from its velocity at infinity) the particle has kept the velocity $\vec{U}_0$ which it had far away.

By starting from different points A at different distances away from the obstacle, we have been able to confirm that the error made at the start is readily noticed. The method has a kind of autocorrection of local errors which is worth stressing. Let us assume that at point C of the trajectory (Fig. 1) we have used too high a radius of curvature. As a result the trajectory is too straight and the angle which it makes with the stream-lines going through D is too large; Formula 21 shows that the radius of curvature computed at point D is too low. This deviation by error which one gets at point D compensates in part for the deviation to excess which we had at point C.

4.3. Practical Computation of the Radius of Curvature at a Point

4.3.1. If the aerosol is fog, the particles are spherical droplets of diameter d, and to every value of the velocity u corresponds a value of the function $f(ud/\nu)$ given by the curve of Fig. 3. Thus, knowledge at one point of vectors $\vec{U}$ and $\vec{U'}$ determines the numerical values of U, U', u, α, and $f(ud/\nu)$. Equation 21, therefore, permits us to calculate readily the radius of curvature, R, at this point. The following remarks show that a computation of this radius is often simplified.

4.3.2. If the absolute values of $\vec{U}$ and $\vec{U'}$ are close to each other, a condition which holds if the point in question is not too close to the obstacle, the absolute value of the relative velocity $\vec{u}$ is, according to Equation 2, equal to

$$u = 2U' \left| \sin \frac{\alpha}{2} \right| .$$

Relation 21 then leads to the following expression of the radius of curvature, which involves only the values of U' and of α:

$$R = \frac{\rho d}{2\rho' \sin \alpha \sin \frac{\alpha}{2}} \cdot \frac{1}{f\left(\frac{2U'd \left| \sin \frac{\alpha}{2} \right|}{\nu}\right)} \tag{22}$$

4.3.3. If one is still far from the obstacle and if the angle α is small enough for its sine to be equated to its value in radians, Equation 22 simplifies further to give

$$R = \frac{\rho d}{\rho' \alpha^2} \cdot \frac{1}{f\left(\frac{U'd \left| \alpha \right|}{\nu}\right)} . \tag{23}$$

4.3.4. Finally, if the velocity u is itself small, Stokes' Law is applicable and one can write, according to (9), (10), and (12),

$$R = \frac{\rho d^2 U'}{18\rho' \nu |\alpha|} . \qquad (24)$$

This last formula can be used as long as the Reynolds Number ud/ν is smaller than unity.

4.3.5. If, in the course of the construction of a trajectory, Formula 24 is proven to be continually applicable, α remaining small, the trajectory follows the streamlines close enough for U and U' to be used constantly one for the other; only Formula 24 is involved in the construction while Formula 20 is not.

Let us imagine, in that case, one obstacle around which we have two flows with the same streamlines and differing only by velocities at infinity U_0 and U_{01}. Consider two spherical particles of the same density, and of diameters d and d_1, one of each placed in two flows and in the same position. If the relation is satisfied, Formula 24 shows that the two trajectories coincide.

As a result, if one has constructed, making use only of Formula 24, a field of trajectories corresponding to a velocity U_0 and of particles of diameter d, the same network of trajectories will be obtained for a flow velocity $U_{01} < U_0$ and particles of diameter d_1 so that the relation $U_0 d^2 = U_{01} d_1^2$ is satisfied[1].

4.3.6. In order to know which one of the preceding formulas should be used, it is important to determine the approximation that can be tolerated in the computation of the radius of curvature. This approximation causes not only an error in construction of the corresponding small arc of a circle but also an error in the angle of the tangent to the end of the arc and the tangent to the streamline which goes through that end; this last error has an influence of a computation of a radius of curvature of the following arc.

Let MM' be the arc of a circle with which we replace an element of trajectory (Fig. 5). Let us call the angle at the corresponding center ω, which is also the acute angle between the tangents of both ends of the arc. Let us use two coordinate axes, Mx' and My', the first tangent at M to the arc of circle and oriented in the direction of motion and the second perpendicular and oriented toward the center. If we define $MM' = \ell = \omega R$, the coordinates of M' are

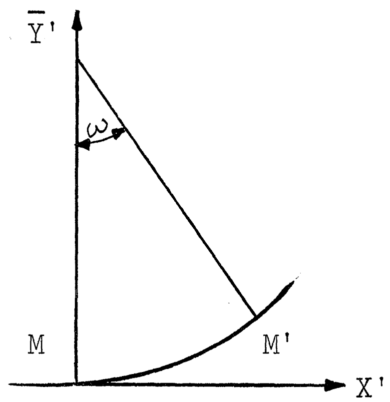


Fig. 5

[1]This last remark has been made by Mr. Le Gallo, of ONERA, while using this method.

$$x' = R \sin \omega; \quad y' = R(1 - \cos \omega).$$

The length ℓ of the arc being satisfied, an error ΔR made in the computation of the radius of curvature causes an error:

$$\Delta \omega = -\frac{1}{R^2} \Delta R$$

on the angle, ω , and, as a consequence the following errors on the coordinates of M':

$$\Delta x' = (\sin \omega - \omega \cos \omega)\Delta R \approx \frac{\omega^3}{2} \Delta R = \frac{\ell^3}{2R^3} \Delta R$$

$$\Delta y' = (1 - \cos \omega - \omega \sin \omega)\Delta R \approx -\frac{\omega^2}{2} \Delta R = -\frac{\ell^2}{2R^2} \Delta R.$$

Let us determine the importance of these errors by two numerical applications. Let us take, for instance, $R = 56$ cm, and $\ell = 1$ cm. If the radius R is calculated to approximately 1 cm, the errors $\Delta x'$, $\Delta y'$, and $\Delta \omega$ are negligible, since

$$|\Delta x'| < \frac{1}{3 \cdot 10^5} \text{ cm}; \quad |\Delta y'| < \frac{1}{6000} \text{ cm}; \quad |\Delta \omega| < \frac{1}{3000} \text{ radian.}$$

Let us consider now the case where $R = 10$ cm, while $\ell = 2$ cm. If the approximation in the radius R is 1 mm, the errors in $\Delta x'$, $\Delta y'$, and $\Delta \omega$ remain acceptable, since

$$|\Delta x'| < \frac{2}{5000} \text{ cm}; \quad |\Delta y'| < \frac{2}{1000} \text{ cm}; \quad |\Delta \omega| < \frac{2}{1000} \text{ radian.}$$

Therefore, as long as the radius of curvature is large ($R > 25$ cm), it can be computed with an approximation of 1 cm; when it is around 20 cm, it is necessary to compute it with an approximation of only a few mm.

4.4. Case When Gravitational Forces Are No Longer Negligible

4.4.1. We shall determine now the trajectories of particles in a different case from that studied in paragraph 4.1.

1) We shall assume that the fluid has a negligible specific mass compared to that of the particles, but we shall keep the gravitational term of the particles. The vectorial equation of motion is then everywhere

$$\rho \frac{d\vec{U}}{dt} = -\rho' \frac{u\vec{u}}{d} f\left(\frac{ud}{\nu}\right) + \rho\vec{g}. \tag{13}$$

2) The streamlines are planes and, furthermore they are in vertical planes.

This last condition holds if the obstacle is a cylinder, of which the generating lines perpendicular to the velocity at infinity upstream are horizontal, and equally, if the obstacle is a body of revolution of which the axis parallel to the velocity at infinity is vertical. The trajectories are obviously in the same planes as the streamlines.

4.4.2. Let us consider a vertical plane containing a group of streamlines (Fig. 6); let us state that in that plane the trigonometric sense has the positive sense of rotation. Let us define

$$(\vec{U}, \vec{U'}) = \alpha, \qquad (\vec{U}, \vec{g}) = \beta.$$

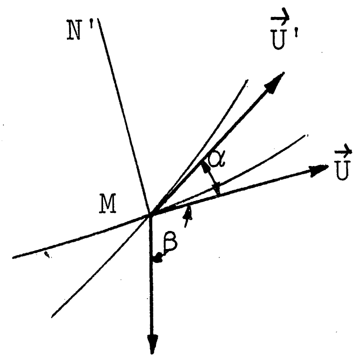


Fig. 6

Let us project the two sides of Equation 13 on the tangent MT to the velocity of the particles oriented positively in the direction of that velocity and on the normal MN deduced from MT by rotation of the angle, $\pi/2$, around point M. We have

$$U \frac{dU}{d\ell} = \frac{\rho'}{\rho d} (U'\cos \alpha - U)\, u\, f\left(\frac{ud}{\nu}\right) + g \cos \beta \tag{25}$$

and

$$\frac{U^2}{R} = \frac{\rho'}{\rho d} U' \sin \alpha\, u\, f\left(\frac{ud}{\nu}\right) + g \sin \beta. \tag{26}$$

R is the radius of curvature of the trajectory (positive if the center of curvature is on the positive semi-straight line MN; negative, in the opposite case).

We define

$$\frac{1}{R_1} = \frac{\rho'}{\rho d} \frac{U'u}{U^2} \sin \alpha\, f\left(\frac{ud}{\nu}\right); \quad \frac{1}{R_2} = \frac{g}{U^2} \sin \beta. \tag{27}$$

Relation 26 can be written

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad . \tag{28}$$

Hence the theorem:

The curvature $1/R$ of the trajectory at a point M is equivalent to the curvature $1/R_1$ which it would have if gravity were neglected, plus the curvature $1/R_2$ which it would have if the velocity $\vec{U}$ and $\vec{U'}$ of the particle and of the fluid had the same direction[1].

Gravitation thus introduces in the computation of R a correction term which depends only on the velocity U and on the angle β.

4.4.3. The construction of a trajectory is accomplished as in the case where gravitation is negligible (4.2.2).

Knowing the velocities $\vec{U}$ and $\vec{U'}$ at a given point A, one measures on the drawing the angles α and β. Then the quantities appearing in the second terms of Formulas 25 and 26 are known and one can, in particular compute 1/R. An element AB of the trajectory can be drawn by comparing it to an arc of a circle with radius R and then continuing step by step.

4.4.4. We are now able to evaluate the error made when gravitation is neglected. Formula 28 can be written

$$\frac{R_1 - R}{R} = \frac{R_1}{R_2} = \frac{R_1 g}{U^2} \sin \beta. \qquad (29)$$

The relative error $(R_1 - R)/R$ which is made on the radius of curvature R when it is replaced by R_1 repeatedly diminishes when the velocity U increases.

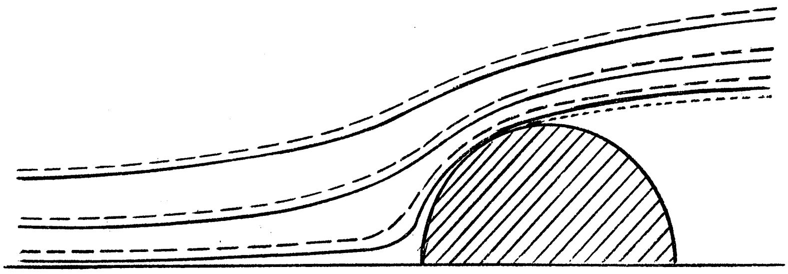
Let us illustrate this point by a few numerical examples, assuming $g = 10 \text{ m/s}^2$, $\beta = \pi/2$, and R_1 and R_2 to have the same sign (the most unfavorable case). For $U = 1$ m/s and $R_1 = 8$ cm, $(R_1 - R)/R = 0.8$. It would be absurd, in this case, to neglect gravity.

$$\text{For } U = 10 \text{ m/s} \quad \begin{cases} R_1 = 80 \text{ cm}, \dfrac{R_1 - R}{R} = 0.08 \\ R_1 = 8 \text{ cm}, \dfrac{R_1 - R}{R} = 0.008. \end{cases}$$

The error of 6.5 cm on a radius of the order of 85 cm is not negligible. However, one can tolerate an error of 0.65 mm on a radius of 8 cm.

$$\text{For } U = 30 \text{ m/s} \quad \begin{cases} R_1 = 80 \text{ cm}, \dfrac{R_1 - R}{R} = 0.009 \\ R_1 = 8 \text{ cm}, \dfrac{R_1 - R}{R} = 0.0009. \end{cases}$$

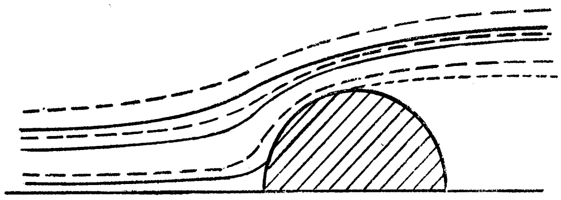
[1]The curves are algebraic numbers, the symbols for which have been defined above.



$U_o = 125$ m/s $\quad d = 2\mu \quad \frac{U_oD}{\nu} = 336000 \quad \frac{d}{D} = 0.5\cdot10^{-4}$

- - - - - - Streamlines
——— Particle trajectories
- - - - - - - - - Real boundaries of the wake

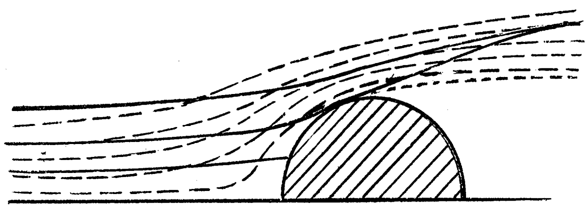
Fig. 7



$U_o = 125$ m/s $\quad d = 3\mu \quad \frac{d}{D} = 0.75\cdot10^{-4}$

$\frac{U_oD}{\nu} = 336000$

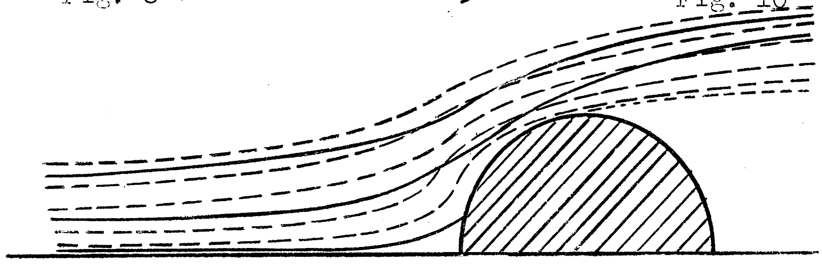
Fig. 8



$U_o = 125$ m/s $\quad d = 12\mu \quad \frac{d}{D} = 3\cdot10^{-4}$

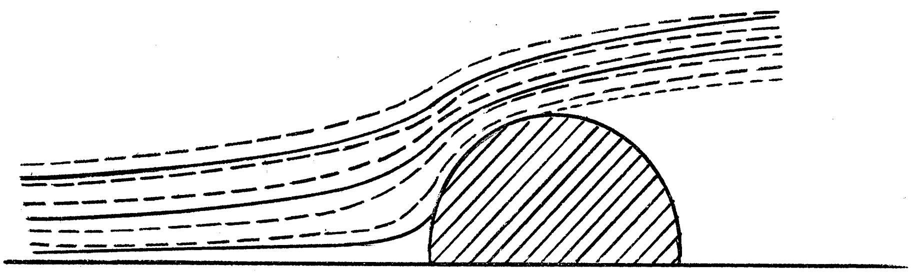
$\frac{U_oD}{\nu} = 336000$

Fig. 10



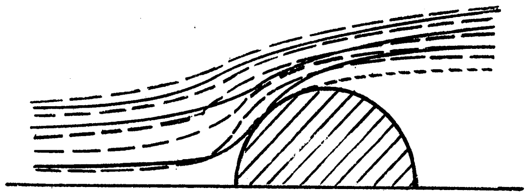
$U_o = 125$ m/s $\quad d = 5\mu \quad \frac{U_oD}{\nu} = 336000 \quad \frac{d}{D} = 1.25\cdot10^{-4}$

Fig. 9



$U_o = 5$ m/s $\quad d = 12\mu \quad \frac{U_oD}{\nu} = 13400 \quad \frac{d}{D} = 3\cdot10^{-4}$

Fig. 11



$U_0 = 10$ m/s $\quad d = 12\mu \quad \frac{U_0 D}{\nu} = 26800 \quad \frac{d}{D} = 3 \cdot 10^{-4}$

Fig. 12

It is possible to tolerate an error of 0.72 cm on a radius of 80 cm and, consequently, an error of 0.07 mm on a radius of 8 cm.

To sum up, to neglect gravity for velocities inferior to 1 m/s would lead to absurd results. On the contrary, it is permissible to neglect gravity when velocities are greater than 20 m/s, especially when the radii of curvature of trajectories are smaller.

5. APPLICATION OF MECHANICS OF SUSPENSIONS TO VISUALIZATION

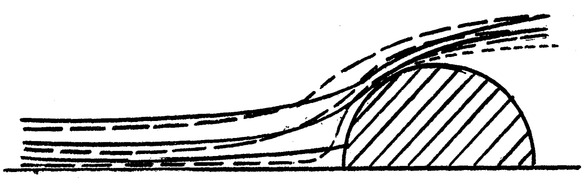
5.1. The method of visualization of streamlines of suspensions of particles cannot rigorously give a picture in the neighborhood of an obstacle since, as we have seen, the trajectories of particles are distinct from streamlines.

We want to learn in which cases the deviation of the trajectories of particles from streamlines is small enough for this method of visualization to be of interest. With this goal in mind, we have drawn (solid lines) the trajectories of aluminum particles, assumed spherical and put on suspension in a flow, with a wake, around a cylinder of revolution of infinite span, the velocity of the fluid at infinity being perpendicular to the generating lines of the cylinder.

The various cases which were studied are summed up in the table on the following page.

5.2. Let us examine Figs. 7, 8, 9, and 10 corresponding to a velocity of $U_0 = 125$ m/s and to a cylinder of diameter $D = 4$ cm. These figures show us the influence of particle size.

Figure	U_0, m/s	d, Microns	$R = \frac{U_0 D}{\nu}$	$\frac{d}{D}$
7	125	2	336,000	0.5×10^{-4}
8	125	3	336,000	0.75×10^{-4}
9	125	5	336,000	1.25×10^{-4}
10	125	12	336,000	3×10^{-4}
11	5	12	13,400	3×10^{-4}
12	10	12	26,800	3×10^{-4}
13	25	12	67,000	3×10^{-4}



$U_0 = 25$ m/s $\quad d = 12\mu \quad \frac{U_0 D}{\nu} = 67000 \quad \frac{d}{D} = 3 \cdot 10^{-4}$

Fig. 13

In the case of Fig. 7, where the particles have a diameter of 2 microns, the deviation of the trajectories from the streamlines is insignificant. It can be said that with such fine particles, visualization is good, even for high Reynolds Numbers.

As Fig. 8 shows, when the diameter of the particle increases from 2 to 3 microns, the catch is already appreciable, but on the whole, visualization remains good. For a diameter of 5 microns the network of trajectories is quite distinct from that of the streamlines. We find downstream an apparent wake of greater width than the real wake. In other words, there exists a region of clear air belonging to the potential flow.

With particles of 12 microns, the discrepancy between the two networks is considerable and visualization is very poor.

5.3. Figs. 10, 11, 12, and 13 refer to particles of diameter d = 12 microns and to a cylinder of diameter D = 4 cm. They show the influence of the velocity U_0. If U_0 = 5 m/s, visualization is good. Streamlines and trajectories differ but little.

If U_0 = 10 m/s, the deficiency between the network of streamlines and that of trajectories is already considerable. One finds again an apparent wake of greater width than the actual wake. If U_0 = 25 m/s visualization is really poor.

5.4. The particles in suspension in fluids are not homogeneous; both their sizes and their shapes necessarily differ. As a result, if two particles are placed in identical conditions the drag force will not be the same. In order to elevate this influence, when one considers first one particle and then another in the same stream, we have superposed in the same drawing the trajectories of particles of 2 microns and 5 microns in diameter for a velocity U_0 = 125 m/s (Fig. 14); then, the trajectories of two other particles of 2 microns and 12 microns, respectively, in diameter at the same flow velocity (Fig. 15).

Fig. 14 shows that in the neighborhood of the obstacle the trajectories mix or cross; in particular, two trajectories are seen intersecting each other twice. From this must result a certain lack of definition in the visualization, which might lead one to believe in the presence of a turbulence which in reality does not exist. Let us observe, on the other hand, that a slight turbulence cannot alter trajectories because of the inertia of the particles. We conclude that visualization by the suspension of particles may lead to erroneous interpretations in the matter of turbulence[1].

Fig. 15, where the trajectories of particles of very different dimensions (2 microns and 12 microns) have been superposed, shows that with a very large heterogeneity of particles the aspect of a suspension must be very ill defined and visualization makes no sinse.

6. APPLICATION OF THE MECHANICS OF SUSPENSIONS TO CHRONOPHOTOGRAPHIC MEASUREMENTS

6.1. The measurement of the velocity $\vec{U'}$ at a given point of a rapid flow can be made by successive photographs of particles in suspension in a stream. This process assumes first of all that the trajectories are

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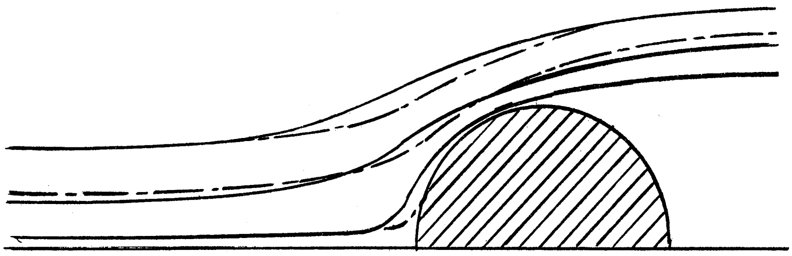
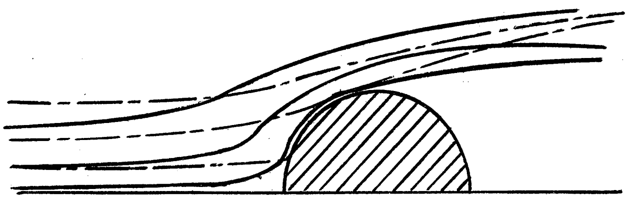
[1] One might also wonder what happens to the particles which strike the obstacle and which are more numerous the greater the dimension, d, is. Carried away by the air stream, they must increase the density of the trajectories in the neighborhood of the tangent trajectory. To this fact may be due the precision with which one may see the apparent wake in certain photographs of aerosols in motion.



U_o = 125 m/s ——— d = 2μ —·—·— d = 5μ

Fig. 14



U_o = 125 m/s ——— d = 2μ —·—·— d = 12μ

Fig. 15

considered with the streamlines and we know that such a result requires, rigorously, a uniform dynamic field.

Let us then imagine a cylindrical fluid stream where the velocity is equal to $\vec{U_0}$ at every point. Let us release a solid particle without initial velocity at a given point 0 of this fluid flow; its trajectory will be coincident with a streamline, but the absolute value of its velocity will reach the value U_0 only at point A situated downstream of 0. Chronophotography as a means of measuring the velocity of a stream can be used only from point A on. It is therefore useful to know the order of magnitude of the distance OA.

6.2. Let $U = k U_0$ be the velocity of the particle at a point M defined by $OM = \ell$. Since $\alpha = 0$ and since $u = U_0 - U$, Equation 20 yields

$$U \frac{dU}{d\ell} = \frac{\rho'}{\rho} \frac{(U_0 - U)^2}{d} f \left[\frac{(U_0 - U)d}{\nu}\right]$$

$$k \frac{dk}{d\ell} = \frac{\rho'}{\rho} \frac{(1 - k)^2}{d} f \left[\frac{U_0(1 - k)d}{\nu}\right] .$$

Thus, the distance ℓ traversed as the particle reaches the fraction k of the velocity U_0 is

$$\ell = \frac{\rho d}{\rho'} \int_0^k \frac{k}{1 - k^2} \frac{1}{f\left[(1 - k) \frac{U_0 d}{\nu}\right]} dk. \qquad (30)$$

If we assume that Stokes' Law applies, the integration is made easy and leads to

$$\ell = \frac{\rho d}{18\rho'} \frac{U_0 d}{\nu} \log_e \left[(1 - k) + k\right] .$$

Thus, the length that the particle must traverse in order to reach a given fraction of the stream velocity U_0 is proportional to the velocity U_0, to the square of the diameter d, and to the ratio of the specific masses of the particle and of the fluid.

A numerical application in a case of a spherical aluminum particle 10 microns in diameter placed in an air stream of which the velocity is 30 m/s, under standard conditions, leads to the following table.

k	0.9	0.98	0.99
ℓ	2.8 cm	6 cm	7.2 cm

Stokes' Law is not applicable if, everything else remaining the same, the velocity U_0 is 300 m/s instead of 30 m/s. The computation of the integral appearing in Equation 30 can be made by an approximate method using the experimental curve of Fig. 3. One thus finds that if $k = 0.9$, $\ell = 10$ cm Stokes' Law would have given 25 cm.

6.3. Although it is possible to follow, with the help of Equation 20, the evolution of the velocity of a particle on its trajectory, we have often assumed that in the drawing of trajectories around an obstacle the absolute values of the velocities $\vec{U}$ and $\vec{U'}$ are close together.

We shall show that this assertion is justified as long as the gradient of the velocity $\vec{U'}$ is small and the angle α is relatively small.

1) Let us first assume that in the neighborhood of point A of the field a velocity U'_A of the fluid can be considered constant and that the angle α between the two velocities is zero. Let $\vec{U} = k_A \vec{U'_A}$ be the velocity of the particle at point A. The computation made above (62) immediately shows that the distance ℓ which a particle must traverse for its velocity $\vec{U}$ to the equal to $k\vec{U'_A}$ is

$$\ell = \frac{\rho d}{\rho'} \int_{kA}^{k} \frac{k}{1 - k^2} \frac{1}{f\left[(1 - k) \frac{U'_A d}{\nu}\right]} dk.$$

In the case where Stokes' Law applies, let

$$\ell = \frac{\rho d}{18\rho'} \frac{U'_A d}{\nu} \left[\log_e \frac{1 - k}{1 - k_A} + k - k_A\right].$$

In particular let us consider a droplet of water 10 microns in diameter placed in air under standards conditions. If at point A the fluid has a velocity of 10 m/s and the drop a velocity of 0.75 x 10 m/s, the drop will have, after having gone 1 cm, a velocity of 0.99 x 10 m/s.

One sees from this example that the droplet regains rapidly the velocity of the fluid when the latter does not vary.

2) Let us now seek the influence that the angle α of the two velocities can have on the variations of the absolute value of this velocity U.

Let us therefore assume that at point A the absolute values of the two velocities are practically equal and that the angle α is small. Since $u = 2U \sin \alpha/2$, Equation 20 can be written

$$\frac{dU}{d\ell} = \frac{U\rho'}{\rho d} 2(\cos\alpha - 1) \sin\frac{\alpha}{2} f\left(\frac{2Ud \sin\frac{\alpha}{2}}{\nu}\right)$$

$$\frac{dU}{d\ell} = -4 \frac{U\rho'}{\rho d} \sin^3\frac{\alpha}{2} f\left(\frac{2Ud \sin\frac{\alpha}{2}}{\nu}\right).$$

If the angle α is small enough to be equated to the sine and to the arc

$$\frac{dU}{d\ell} = -\frac{U\rho'}{2\rho d}\alpha^3 f\left(\frac{Ud\alpha}{\nu}\right).$$

If, finally, Stokes' Law is applicable

$$\frac{dU}{d\ell} = \frac{9\rho'\nu}{\rho}\left(\frac{\alpha}{d}\right)$$

In particular, if we consider again the case of the preceding water droplet, $\alpha = 0.4$ radian, dU/d is smaller than 40. After a distance of 1 cm the velocity has changed by less than 40 cm/s, or 4% of its value.

6.4. Finally, let us examine the influence of gravity on the velocity of a particle.

Equation 25 can be written

$$dU = \frac{\rho'}{\rho d}\left(\frac{U'}{U}\cos\alpha - 1\right) u\, f\left(\frac{ud}{\nu}\right) d\ell + \frac{g}{U}\cos\beta\, d\ell,$$

and in this form one sees that dU is the sum of two terms; the first gives the variation of velocity when gravity is neglected; the second measures the correction to be made in order for gravity to be taken into account.

This corrective term, inversely proportional to U, is negligible when speeds are large, especially if the angle β is close to 90°. Thus, with $U = 10$ m/s and $\beta = 96°$, the relative error made in the variation of velocity by neglecting gravity is only 1%.

7. APPLICATION OF THE MECHANICS OF SUSPENSIONS TO CATCH PHENOMENA

7.1. Definition of the Total Coefficients of Catch

7.1.1. At the beginning of our study (1.2.1) we have mentioned the interest created by the study of catch phenomena. Let us make a few definitions.

Let us consider a plane, P, perpendicular to the velocity at infinity upstream and far from the obstacle (Fig. 16). Consider the sections made by the plane, P:

- of the cylinder circumscribing the obstacle parallel to the velocity $\vec{U}_0$; let Σ be the area of this section (projected area).

- of the tube made by the particles which strike the obstacle; let S be the area of that section.

- of the cylinder whose generating lines are parallel to the velocity $\vec{U_0}$ and whose directrix is the curve which bounds, on the obstacle, the region of impact;[1] let σ be the area of that section.

We shall designate the ratio

$$\Gamma_c = \frac{S}{\Sigma} . \tag{31}$$

as the total coefficient of catch relative to the projected area. It is the ratio of the mass of the particles caught by the obstacle in a unit time to the mass of the particles which would flow through the projected area during the same time if the obstacle were removed.

We shall designate the ratio

$$\Gamma_I = \frac{S}{\sigma} . \tag{32}$$

as the total coefficient of catch relative to the region of impact. It is the ratio of the mass of the particles caught during the unit time to that which would hit the region of impact during the same time if the trajectories remained rectilinear up to the obstacle.

7.2. Definition of Local Coefficient of Catch

Let us consider, within the region of impact, an area A simply connected and a point M within this area (Fig. 17).

The trajectories of the particles caught within area A are inside a tube the trace of which on plane P at infinity upstream is a surface area S_1.

The cylinder whose generating lines are parallel to U_0 and whose directrix is the boundary of the area A has for a trace on the plane P a curve which is the boundary of the area σ_1.

We shall designate A the ratio

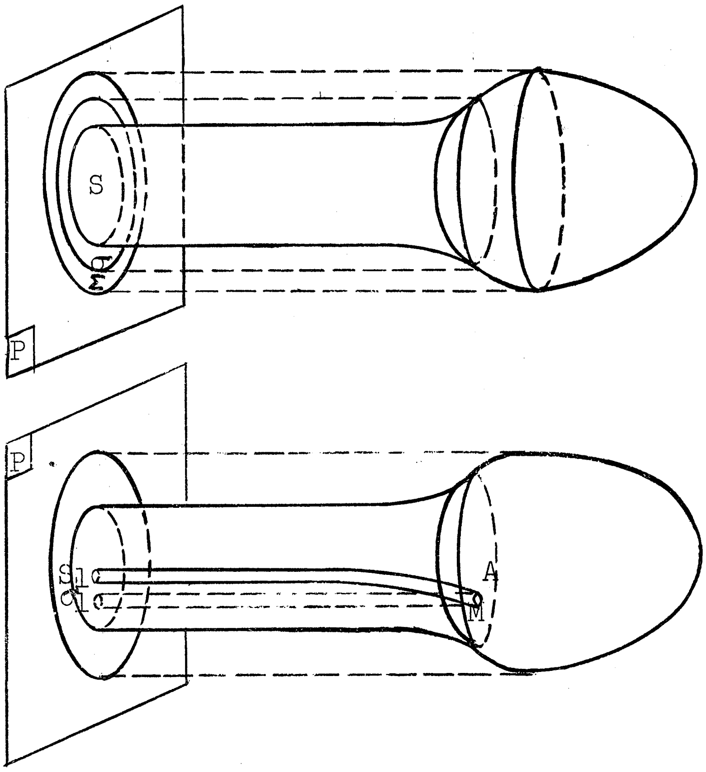
$$\Gamma_A = \frac{S_1}{\sigma_1} \tag{33}$$

[1]The curve is the locus of the points of tangency of the trajectories to the obstacle.

as the mean coefficient of catch relative to area A. The local coefficient of catch, γ, at point M is the limit of this ratio when the area A tends to zero uniformly, while the point M remains inside this area.

7.3. Remarks on Preceding Definitions

7.3.1. The total coefficient of catch relative to the zero of impact is equal to the mean of the local coefficients of catch. The reason we have considered it is that the total coefficient relative to the projected area has more visual meaning.



Figs. 16 and 17

7.3.2. The ratios of the masses of particles which were interjected into the definitions of the coefficients of catch can be replaced by the ratios of the numbers of particles, when these particles are identical.

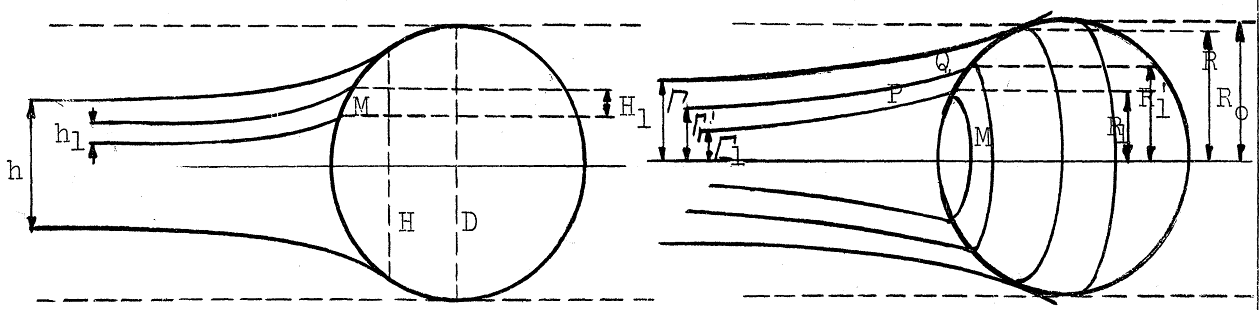
7.3.3. Under the assumption that the particles are few enough to cause no perturbation in the flow of the fluid, the coefficient of catch, everything else being equal, is independent of the number of particles contained in the unit volume of the fluid.

7.4. Application of the Similarity Relations to the Coefficients of Catch

7.4.1. If the two flows are similar, in their streamlines as well as in their particle trajectories, the coefficients of catch are equal for these two flows. They differ, on the other hand, if any one of the similarity conditions fails to be satisfied (17).

Thus, in case gravity forces are negligible (3.2.1), the total coefficient of catch is a function of the three dimensionless numbers, U_0D/ν, D/d, and ρ/ρ'.

$$\Gamma = \psi\left(\frac{U_0D}{\nu}, \frac{D}{d}, \frac{\rho}{\rho'}\right). \qquad (34)$$



Figs. 18 and 19

If, in addition, we are dealing with an aerosol (3.2.2), the expression for the coefficient of catch becomes

$$\Gamma = \psi'\left(\frac{U_0d\rho'}{\mu}, \frac{\rho' D}{\rho d}\right). \qquad (35)$$

7.4.2. It is possible to give some qualitative information on the variations of the functions ψ and ψ'.

1) The relative motion of the particles is hindered by the viscosity of the fluid; the greater this viscosity, the smaller the coefficient of catch which is thus an increasing function of the Reynolds Number $R = U_0D/\nu$ or U_0d/ν . One can thus conclude that the coefficient of catch is an increasing function of the velocity at infinity U_0.

2) When the dimensions of the particles increase, the inertia forces which are proportional to the volume increase faster then the forces which oppose the relative motion of the particles in the fluid. The latter are proportional to a power of the dimension d between 1 and 2. Accordingly the coefficient of catch is an increasing function of the characteristic dimension d of the particles. Hence in Relation 34 it is a decreasing function of the dimensionless number D/d.

3) If, all other things being equal, the specific mass ρ of the particles increases, their inertia forces increase; the coefficient of catch is an increasing function of the specific mass of the particles. Hence in Relation 34 it is an increasing function of the dimensionless number ρ/ρ' or, in Equation 35, of the dimensionless number $\rho'D/\rho d$.

4) The theory yields no direct information on the way in which the coefficient of catch varies when the characteristic dimension D of the obstacle or the specific mass of the fluid ρ' (viscosity being assumed constant) vary. The coefficient Γ is an increasing function of the Reynolds number U_0D/ν (where ρ' and D appear in the numerator) and a decreasing function of the numbers D/d and ρ'/ρ (where ρ' and D also appear in the numerator). The predominant factor cannot be known a priori.

However, Relations 34 and 35 show that the factor D has a smaller influence on the coefficient of catch than the factors U_0 and d. For if we multiply by p the dimension D we modify the coefficient of catch in the same way as if we multiplied simultaneously by p the velocity U_0 and by $1/p$ the dimension d of a particle. Since those two effects cancel each other, the resulting effect is smaller than that of each appearing separately.

Similarly, for the particular case of Equation 25 it is seen that it is equivalent, as far as catch is concerned, to multiply ρ' by p or to multiply simultaneously U_0 and D by p. Since it has been seen that the influence of the velocity U_0 on the coefficient of catch is greater than the influence of the dimension D on that coefficient, when ρ' diminishes the coefficient of catch also diminishes. Thus the intensity of catch has a tendency to decrease slightly as the altitude increases.

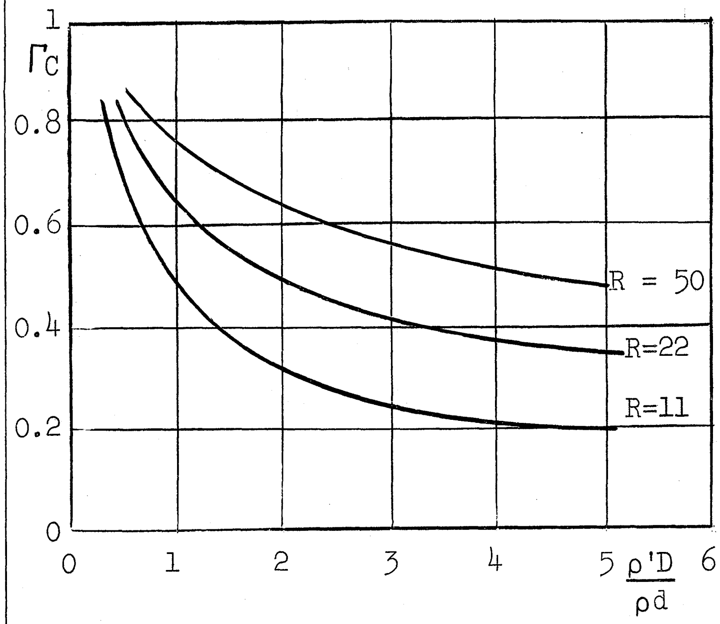
7.5. Graphical Determination of the Catch Coefficients

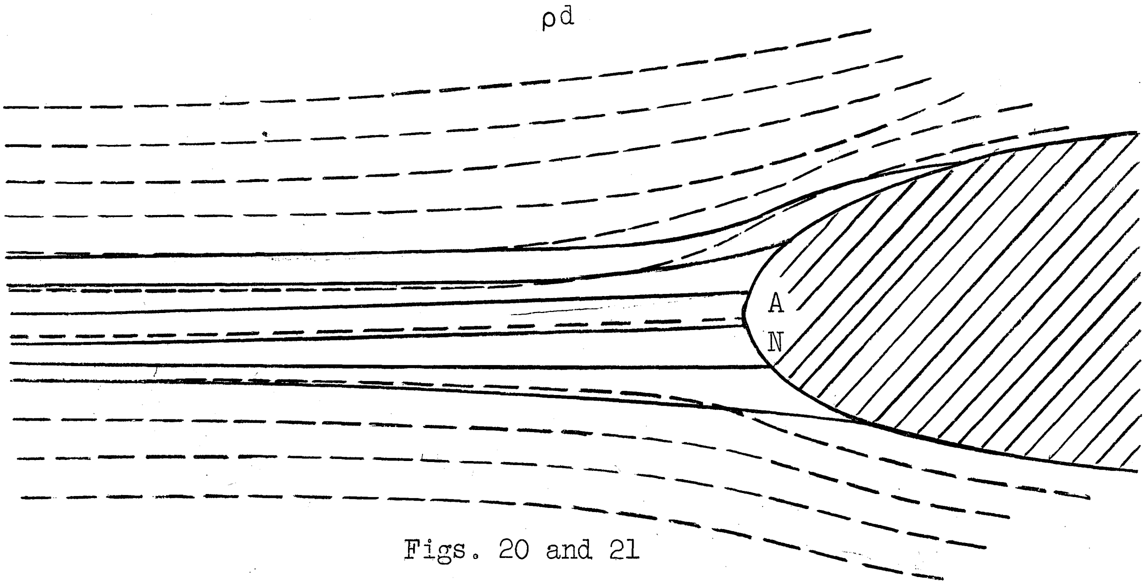
The drawing of the trajectories can be made both for plane flow and for axially symmetric flows (4). Starting from the curves thus obtained it is possible to evaluate the various coefficients of catch.

7.5.1. Let us consider the case of a cylinder of infinite span whose generating lines are normal to the velocity at infinity upstream U_0 (Fig. 18).

The areas S, Σ, σ, S_1, and σ_1 defined above are here rectangles of equal base (the length being arbitrary and chosen on one generating line of the cylinder). The ratio of any two of these areas is thus equal to the ratio of the two heights which can be read on the drawing.

It follows that with the notations shown in the figure, the total coefficient of catch relative to the projected area is h/D; the total coefficient relative to the region of impact is h/H; the mean coefficient around the point M, which can practically be identified with the local coefficient at point M, is h_1/H_1.





Figs. 20 and 21

7.5.2. Let us consider the case of a body of revolution, whose axis is parallel to the velocity at infinity U_0 (Fig. 19).

The streamlines and the trajectories are contained in planes which go through the axis and the network which they form remains identical to itself when one passes from one meridian plane to another.

The areas of impact which we shall consider will therefore be contained in zones generated by meridian arcs. Hence, using the notation shown on Fig. 19, the total coefficient of catch relative to the projected area is $(r/R_0)^2$; the total coefficient relative to the zone of impact is $(r/R)^2$; the local coefficient at point M is the limit of the mean coefficient on the hatched region when r_1' and R_1' respectively tend towards r_1 and R_1; it is represented by $r_1 dr_1/R_1 dR_1 = d(r_1^2)/d(R_1)^2$.[1]

7.5.3. In order to take into account the influence of gravitational forces, the conditions specified in 4.4 must be fulfilled so as to be able to draw trajectories. For instance, we know that if the body of revolution does not have a vertical axis the trajectories are not plane and the method which has been developed does not permit the construction of the trajectories and consequently the graphical determination of the coefficients of catch cannot be made.

7.6. A Few Results

The graphical method has been used to determine the coefficients of catch of a cylinder of revolution of infinite span placed normal to the velocity at infinity. Fig. 20 gives the total coefficients of catch relative to the projected area as a function of the dimensionless number $\rho' D/\rho d$ for three Reynolds Numbers. The results already specified in 7.4 are found on this graph.

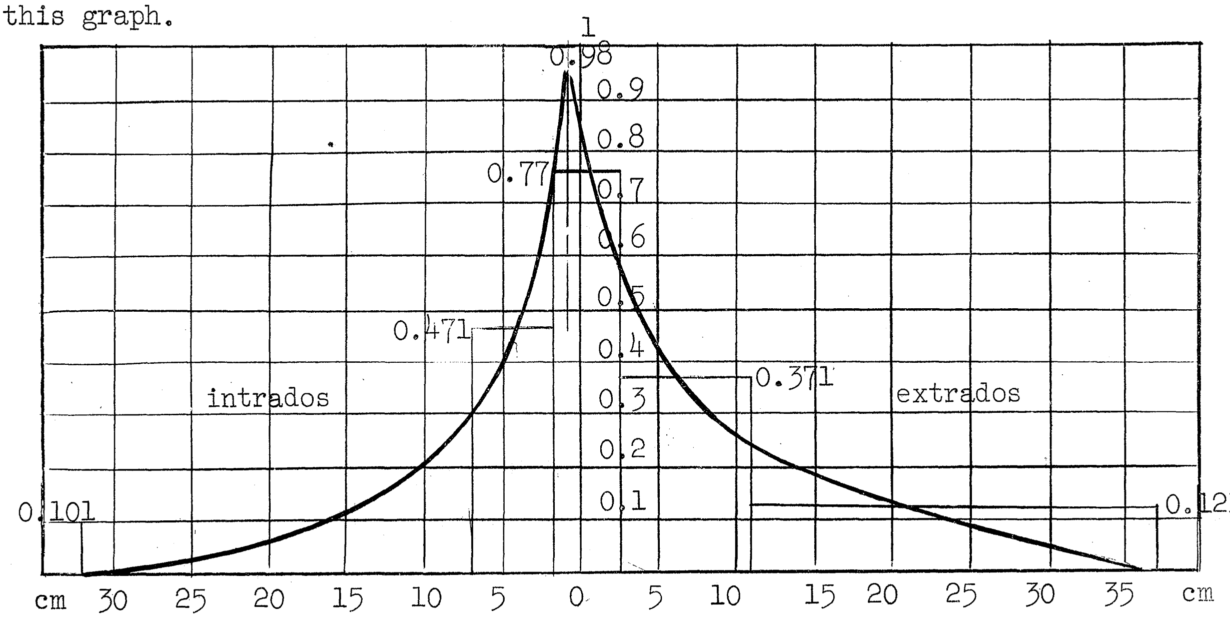


Fig. 22

We have also studied, by the graphical method, catch rates on a Joukowski profile (Göttingen 430). Fig. 21 corresponds to a profile with a 3-meter chord, a 6°33' angle of attack, a 91-m/s velocity, U_0, and a suspension

[1]We have supposed implicitly that the obstacle was limited by a surface simply connected (sphere, ellipsoid, etc.). In the case of the torus (order of connection equal to two), the total coefficients of catch are calculated by formulas anologous to those which give the mean coefficients on a zone of a surface simply connected.

of droplets of water 30 microns in diameter. Curve 22 gives the local coefficient of catch as a function of the curvilinear abscissa with the stagnation point as an origin. The total coefficient of catch Γ_c is 0.32 and the zone of impact extend over about 9% of the chord length.

7.6.2. If the graph of the trajectories are continued beyond their point of contact with the obstacle it is found that some streamlines are between the obstacle and the tangent trajectory. Hence there exists a region of air deprived of particles yet within the potential flow region (Fig. 23). Such a region exists even if the wake is considered: this has been demonstrated both by the graphical method and by experiments.

It is obvious that in the immediate neighborhood of the tangent trajectory, the trajectories crowd each other and mass flow of particles per unit volume becomes very large.

7.6.3. The applications of the above results are varied. Aside from the icing of flying objects, for which our study was undertaken, one could mention the cooling of bodies in motion across clouds, the catching of droplets of fog for study purposes, the operation of shock purifiers, the inhalation of medical aerosols, etc. We shall pass over the details of these applications as well as the numerous experimental works which have been used as a basis for our theory. It should be mentioned that Mr. R. Caron, one of our collaborators, has had the greatest share in this experimental work.

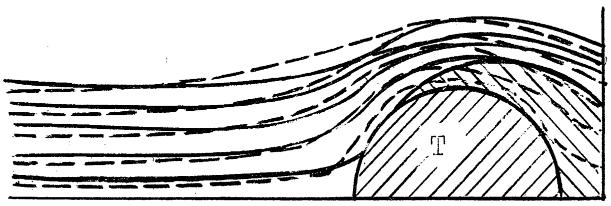


Fig. 23

NOTE BY ICING RESEARCH GROUP, UNIVERSITY OF MICHIGAN

Since there is some controversy concerning the accuracy of the cylinder-catch data in A Mathematical Investigation of Water Droplet Trajectories by Langmuir and Blodgett[1], it is of interest to compare their results with those in the preceding report. Figure 20 of this report is reproduced below with data of Langmuir and Blodgett superimposed.

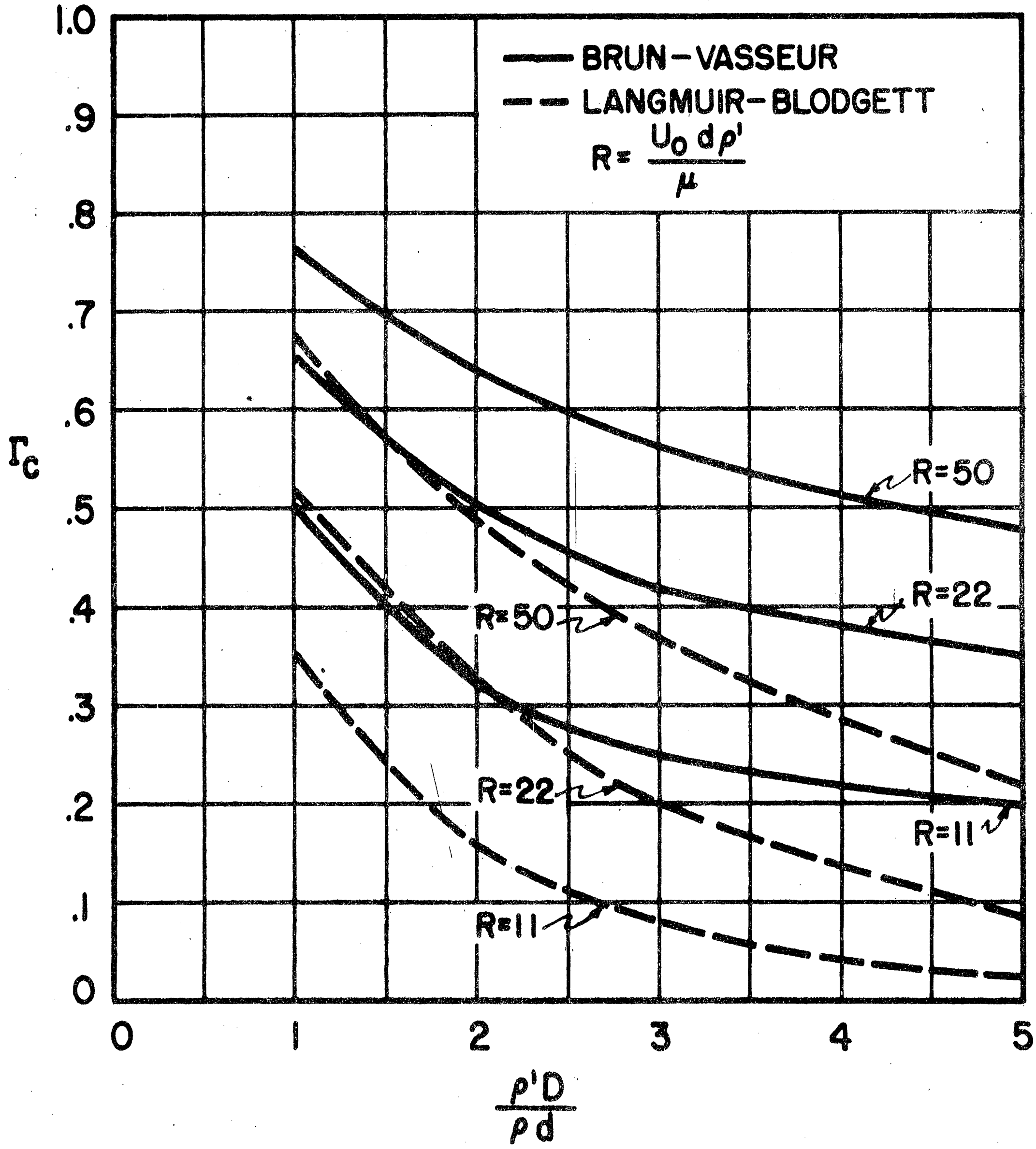


Fig. 20

[1]AAF Technical Report 5418, February, 1946.

The differences are seen to be large and as yet no explanation is available. However, a forthcoming report of the National Advisory Committee for Aeronautics is to contain new cylinder-catch data and will serve as an independent check.

The following table contains a comparison of the sphere-catch data of Langmuir and Blodgett with that of Vasseur[1]. Here Γa denotes local catch at the stagnation point, d/2 is the droplet radius in microns, and the subscripts V and L-B refer to Vasseur and Langmuir-Blodgett, respectively.

d/2	Γc_V	Γc_{L-B}	Γa_V	Γa_{L-B}
12	.06	.09	.12	.27
20	.16	.22	.25	.46
30	.31	.37	.33	.58
40	.46	.48	.50	.69

sphere diameter - 20 cm
air velocity - 90 m/sec
ambient condition - sea level (approx.)

Again, there is as yet no explanation of the differences, but it is hoped that trajectory work now being done at the University of Michigan will clarify the situation in the near future.

[1]"Captation par un Corps de Révolution", La Recherche Aéronautique, Mai-Juin, Numéro 9, 1949.