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Modeling Sequences of Long Memory Positive Weakly Stationary Random Variables

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Abstract

In this paper we introduce a new class of covariance stationary long-memory models on the positive half-line. The overall structure of the models is related to that of GARCH processes of Engle (1982) and Bollerslev (1986), whereby sequence of random variables of interest have multiplicative shocks structure. Unlike FIGARCH model of Baillie, Bollerslev and Mikkelsen (1996), our models are weakly stationary with non-summable autocovariances and hence belong to the class of long-memory models according to the criteria of McLeod and Hipel (1978). In addition, we are able to ensure positivity of all underlying components of the model, thereby improving on the results of Giraitis, Robinson and Surgailis (2000). Apart from volatility modeling, the class of models introduced in this paper will find applications in high-frequency financial data econometrics.

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Introduction

In this paper we introduce and study a class of covariance stationary long-memory time-series models on the positive half-line. Similarly to the popular GARCH models of Engle (1982) and Bollerslev (1986), our model is defined as the product of a sequence of positive i.i.d. innovations $\{\epsilon_t\}_{t\in\mathbf{Z}}$ and a latent observation driven covariance stationary positive long-memory component $\{\psi_t\}_{t\in\mathbf{Z}}$:

$$X_t = \psi_t \cdot \epsilon_t \,. \tag{1}$$

The model is potentially useful in several applications in econometrics and statistics. In particular, it provides an alternative to FIGARCH models of Baillie, Bollerslev and Mikkelsen (1996) and can also be used to model long-range dependence in high-frequency financial durations data as in Jasiak (1999). In the paper we give conditions for stationarity and existence of first two moments of our model, discuss its extension to the short-memory case and outline estimation and inference procedures.

Since the introduction of GARCH models into econometric literature by Engle (1982) and Bollerslev (1986) there has been a substantial interest in time-series models with multiplicative shocks structure. This approach to the design of time-series models is especially appealing for modeling conditional heteroscedasticity, for it allows researchers to capture variations in the second moment of data in the separate process. Recently, techniques and ideas employed in GARCH models have been utilized for statistical modeling of other positive time-series processes, most notably high-frequency financial durations data in Engle and Russel (1998) and Engle (2000).

The evidence of long-range dependence in the volatility of many financial assets has led to the extension of GARCH models to account for this empirical regularity. Baillie, Bollerslev and Mikkelsen (1996) and Ding and Granger (1996) introduce a class of FIGARCH models. In contrast to GARCH processes, FIGARCH models assign hyperbolic weights on the effects of past shocks in the conditional volatility part of the model using the $(1-L)^d$ polynomial. Analogously, Jasiak (1999) documents substantial persistence in the financial durations data and introduces FIACD model able to pick up this regularity.

However, as shown in Giraitis, Kokoszka and Leipus (2000), weakly stationary version of FIGARCH model in fact implies absolute summability of autocovariances of squared returns or, in the case of FIACD model, financial durations. Therefore, according to McLeod and Hipel (1978), FIGARCH and FIACD models belong to the class of short-memory models. Empirical success of these models in picking up observed long-memory dynamics of volatility and financial durations can be attributed to the non-stationarity of $\{X_t\}_{t\in\mathbf{Z}}$ implied by their standard form.

In addition, when used for modeling sequences of positive random variables such as financial duration, standard form of FIGARCH model implies infinite unconditional first moment of such processes. In practice this is rarely the case. Moreover, it is often desirable to have a model for weakly stationary sequences of positive random variables incorporating long-range dependence.

In this paper we introduce a new approach to modeling sequences of positive random variables $\{X_t\}_{t\in\mathbf{Z}}$ with multiplicative long-memory component as in (1). In contrast to FIGARCH and FIACD models, we are able to ensure weak stationarity of $\{X_t\}_{t\in\mathbf{Z}}$ and have non-summable autocovariances. These characteristics of our model parallel ones of ARFIMA processes of Granger (1980) and Hosking (1981), which are defined on the entire real line.

Attempts to define $\{X_t\}_{t\in\mathbf{Z}}$ that is both weakly stationary and exhibits long-range dependence are presented in the recent paper by Giraitis, Robinson and Surgailis (2000). Authors formulate and study the following model:

$$\begin{aligned} Y_t &= \psi_t^* \cdot z_t \\ X_t &= (Y_t)^2 \\ \psi_t^* &= a + (1-L)^{-d} Y_{t-1}\,, \end{aligned} \tag{2}$$

where $\{z_t\}_{t\in\mathbf{Z}}$ is a sequence of i.i.d. random variables with mean zero and unit variance. Giraitis, Robinson and Surgailis (2000) show that for $0 < d < \frac{1}{2}$ $\{X_t\}_{t\in\mathbf{Z}}$ is weakly stationary with non-summable autocovariances.

In this paper we introduce and study a class of models for covariance stationary long-memory sequences $\{X_t\}_{t\in\mathbf{Z}}$ that are related to Giraitis, Robinson and Surgailis (2000). However, in contrast to the latter, where $\{\psi_t^*\}_{t\in\mathbf{Z}}$ is defined on the entire real line and therefore does not have usual volatility interpretation, our models restrict $\{\psi_t\}_{t\in\mathbf{Z}}$ to lie on the positive half-line as in the class of GARCH models. By formulating ψ_t in terms of the martingale difference sequence $\{X_\tau - \psi_\tau\}_{\tau<t}$ weighted by the coefficients of the $(1-L)^{-d}$ polynomial, we are able to reproduce results of Giraitis, Robinson and Surgailis (2000) pertaining to the weak stationarity and long memory in $\{X_t\}_{t\in\mathbf{Z}}$ without sacrificing volatility interpretation of ψ_t. Our model is also applicable to the financial durations data, where parameter ψ_t is proportional to the conditional intensity of the associated point process and therefore also needs to be positive.

The paper is organized as follows. In section 1 we introduce the long-memory version of our model, derive conditions for stationarity and existence of moments and study the implied autocorrelation structure of $\{X_t\}_{t\in\mathbf{Z}}$. In section 2 we discuss short-memory extension of the model from section 1 and present relevant results. Section 3 describes maximum-likelihood inference in the models. Conclusion summarizes the findings.

1 Sequences of stationary long-memory positive random variables

In this section we introduce the model for stationary[1] long-memory sequences of positive random variables with multiplicative shocks as in (1). We derive conditions for the existence of first two moments of $\{X_t\}_{t\in\mathbf{Z}}$ and study its autocovariance structure.

[1] Unless mentioned otherwise, in the rest of the paper "stationarity" refers to the concept of "weak stationarity".

1.1 The basic model

We define the following model for positive random variables with multiplicative shocks:

$$\begin{aligned} X_t &= \psi_t \cdot \epsilon_t \\ \psi_t &= a + \gamma(1-L)^{-d}(X_{t-1} - \psi_{t-1}) \end{aligned} \tag{3}$$

where $\{\epsilon_t\}_{t\in\mathbf{Z}}$ is a sequence of i.i.d. non-negative random variables with $\mathbb{E}[\epsilon_1] = 1$, and $\gamma, a > 0$ are constants. In this subsection we also assume that $0 < d < 1$. By construction, if $\{\psi_t\}_{t\in\mathbf{Z}}$ is stationary and $\mathbb{E}|\psi_t| < \infty$, sequence of random variables $\{X_t - \psi_t\}_{t\in\mathbf{Z}}$ is well defined and constitutes a sequence of martingale differences, i.e. $\mathbb{E}[X_t - \psi_t|\mathfrak{F}_{t-1}] = 0$, where $\mathfrak{F}_t$ denotes information generated by the process up to time t.

Recall that in the similar model of Giraitis, Robinson and Surgailis (2000) $\{\psi_t^*\}_{t\in\mathbf{Z}}$ is allowed to become negative, which invalidates its interpretation as the standard deviation in the framework of GARCH models. In proposition 1 we show that, under suitable restrictions on γ and d in model (3), $\{\psi_t\}_{t\in\mathbf{Z}}$ stays on the positive half-line with probability one. We need the following preliminary results:

Lemma 1 *Define recursively:*

$$\begin{aligned} \eta_1 &:= \theta_2 - \theta_1\theta_1 \\ \eta_2 &:= \theta_3 - \theta_2\theta_1 - \theta_1\eta_1 \\ \eta_3 &:= \theta_4 - \theta_3\theta_1 - \theta_2\eta_1 - \theta_1\eta_2 \\ &\vdots \end{aligned}$$

where $\{\theta_j\}_{j\geq 1}$ *are from the expansion of the polynomial* $(1-L)^{-d} = 1 + \sum_{j=1}^{\infty}\theta_j L^j$ *with* $0 < d < 1$*. Then* $\{\eta_j\}_{j\geq 1} \subseteq \mathbf{R}_+$*.*

PROOF From Hosking (1981) the recursive expression for $\{\theta_j\}_{j\geq 1}$ is given by $\theta_j = \theta_{j-1}\frac{j-1+d}{j}$ and $\theta_1 = d$. Simple calculations show that:

$$\begin{aligned} \eta_1 &= \theta_1\left(\frac{\theta_2}{\theta_1} - d\right) = d\frac{(1-d)}{2} > 0 \\ \eta_2 &= \eta_1\left(\frac{\theta_3 - \theta_2\theta_1}{\eta_1} - d\right) = \ldots = d\frac{(1-d)(2-d)}{6} > 0 \\ \eta_3 &= \eta_2\left(\frac{\theta_4 - \theta_3\theta_1 - \theta_2\eta_1}{\eta_2} - d\right) = \ldots = d\frac{(1-d)(2-d)(3-d)}{24} > 0 \\ &\vdots \end{aligned}$$

By induction $\eta_j = d\prod_{1\leq k\leq j}\frac{(k-d)}{k+1}$ and hence $\{\eta_j\}_{j\geq 1} \subseteq \mathbf{R}_+$. □

Lemma 2 *For given sequences* $\{u_j\}_{j\geq 1} \subseteq \mathbf{R}$*,* $\{\theta_j\}_{j\geq 1} \subseteq \mathbf{R}_+$ *and* $\{\pi_j\}_{j\geq 1} \subseteq \mathbf{R}_+$*, such that* $\left|\sum_{j=1}^{\infty}\theta_j u_j\right| < \infty$ *and* $\pi_j < \theta_j$ *for all* $j \geq 1$*,* $\left|\sum_{j=1}^{\infty}\pi_j u_j\right| \leq \left|\sum_{j=1}^{\infty}\theta_j u_j\right| < \infty$*.*

PROOF From $\left|\sum_{j=1}^{\infty}\theta_j u_j\right| < \infty$ follows that $\sum_{j=1}^{\infty}\theta_j u_j^+ < \infty$ and $\sum_{j=1}^{\infty}\theta_j u_j^- < \infty$, where:

$$u_j^+ = \begin{cases} u_j & \text{if } u_j \geq 0 \\ 0 & \text{if } u_j < 0 \end{cases} \qquad u_j^- = \begin{cases} -u_j & \text{if } u_j \leq 0 \\ 0 & \text{if } u_j > 0 \end{cases}$$

Then $0 \leq \sum_{j=1}^{n} \pi_j u_j^+ \leq \sum_{j=1}^{n} \theta_j u_j^+$ and $0 \leq \sum_{j=1}^{n} \pi_j u_j^- \leq \sum_{j=1}^{n} \theta_j u_j^-$ for all finite n, and by the monotone convergence theorem $\lim_{n\to\infty} \sum_{j=1}^{n} \pi_j u_j^+ \leq \lim_{n\to\infty} \sum_{j=1}^{n} \theta_j u_j^+ < \infty$ and $\lim_{n\to\infty} \sum_{j=1}^{n} \pi_j u_j^- \leq \lim_{n\to\infty} \sum_{j=1}^{n} \theta_j u_j^- < \infty$. Hence, the result follows from:

$$0 \leq \left|\sum_{j=1}^{\infty} \pi_j u_j\right| = \sum_{j=1}^{\infty} \pi_j u_j^+ + \sum_{j=1}^{\infty} \pi_j u_j^- \leq \sum_{j=1}^{\infty} \theta_j u_j^+ + \sum_{j=1}^{\infty} \theta_j u_j^- = \left|\sum_{j=1}^{\infty} \theta_j u_j\right| < \infty$$

□

Proposition 1 *If $\gamma \leq d$, then for any sequence $\{\epsilon_t\}_{t\in\mathbf{Z}} \subseteq \mathbf{R}_+ \cup \{0\}$ in model (3) $\{\psi_t\}_{t\in\mathbf{Z}} \subseteq \mathbf{R}_+$.*

PROOF It is enough to show that for $\psi_t > 0$ and any sequence $\{\epsilon_\tau\}_{\tau\geq t}$ the resulting sequence $\{\psi_\tau\}_{\tau\geq t}$, recursively defined from (3), is positive.

Define $m_k := a + \gamma \sum_{j=0}^{\infty} \theta_{j+k} \psi_{t-1-j}(\epsilon_{t-1-j} - 1)$. Then model (3) induces the following recursion:

$$\begin{aligned}
\psi_t &= m_0 \\
\psi_{t+1} &= m_1 + \gamma\theta_0\psi_t(\epsilon_t - 1) \\
\psi_{t+2} &= m_2 + \gamma \textstyle\sum_{j=0}^{1} \theta_j \psi_{t+1-j}(\epsilon_{t+1-j} - 1) \\
&= (m_2 - \theta_1 m_1) + (\theta_1 - \gamma)\psi_{t+1} + \gamma\psi_{t+1}\epsilon_{t+1} \\
\psi_{t+3} &= m_3 + \gamma \textstyle\sum_{j=0}^{2} \theta_j \psi_{t+2-j}(\epsilon_{t+2-j} - 1) \\
&= \ldots = (m_3 - \theta_1 m_2 - \eta_1 m_1) + (\theta_1 - \gamma)\psi_{t+2} + \eta_1\psi_{t+1} + \gamma\psi_{t+2}\epsilon_{t+2} \\
\psi_{t+4} &= m_4 + \gamma \textstyle\sum_{j=0}^{3} \theta_j \psi_{t+3-j}(\epsilon_{t+3-j} - 1) \\
&= \ldots = (m_4 - \theta_1 m_3 - \eta_1 m_2 - \eta_2 m_1) + (\theta_1 - \gamma)\psi_{t+3} + \eta_1\psi_{t+2} + \eta_2\psi_{t+1} + \gamma\psi_{t+3}\epsilon_{t+3} \\
&\;\vdots
\end{aligned}$$

where coefficients $\{\theta_j\}_{j\geq 0}$ are from the expansion of the polynomial in model (3) $(1-L)^{-d} = \sum_{j=0}^{\infty} \theta_j L^j$ and $\{\eta_j\}_{j\geq 1} \subseteq \mathbf{R}_+$ is defined as in Lemma 1. By induction, for $j \geq 2$:

$$\begin{aligned}
\psi_{t+j} &= (m_j - \theta_1 m_{j-1} - \textstyle\sum_{k=1}^{j-2} \eta_k m_{j-1-k}) \\
&\quad + (\theta_1 - \gamma)\psi_{t+j-1} + \textstyle\sum_{k=1}^{j-2} \eta_k \psi_{t+j-1-k} + \gamma\psi_{t+j-1}\epsilon_{t+j-1}
\end{aligned} \tag{4}$$

It is left to show that $\psi_{t+1} > 0$ when $\psi_t = m_0 > 0$ and that expression in terms of $\{m_k\}_{k=1}^{j}$ in (4) is positive for all $j \geq 2$.

1. Write $\psi_{t+1} = m_1 - \gamma\psi_t + \gamma\psi_t\epsilon_t$. It is sufficient to show that $\psi_{t+1} \geq m_1 - \gamma\psi_t = m_1 - \gamma m_0 > 0$ when $0 < m_0 < \infty$. Then:

$$m_1 - \gamma m_0 = a(1-\gamma) + \gamma \sum_{j=0}^{\infty} (\theta_{j+1} - \gamma\theta_j) u_j \,,$$

where $u_j := \psi_{t-1-j}(\epsilon_{t-1-j} - 1)$ and $\{u_j\}_{j\geq 0} \subseteq \mathbf{R}$ is a given sequence. Since, by Hosking (1981), $0 \leq \theta_{j+1} - \gamma\theta_j = \theta_j \left(\frac{j+d}{j+1} - \gamma\right) \leq \theta_j(1-\gamma)$ for all $j \geq 0$ if $\gamma \leq d$, by Lemma 2 we get:

$$0 \leq \left|\sum_{j=0}^{\infty} (\theta_{j+1} - \gamma\theta_j) u_j\right| \leq (1-\gamma) \left|\sum_{j=0}^{\infty} \theta_j u_j\right| < \infty \,.$$

Since we are only interested in the case when $\sum_{j=0}^{\infty}(\theta_{j+1}-\gamma\theta_j)u_j < 0$, using the inequality above we conclude that:

$$\psi_{t+1} \geq a(1-\gamma) + \gamma(1-\gamma)\sum_{j=0}^{\infty}\theta_j u_j = (1-\gamma)\psi_t > 0\,.$$

2. We demonstrate positivity of $m_j - \theta_1 m_{j-1} - \sum_{k=1}^{j-2}\eta_k m_{j-1-k}$ for the case $j = 3$. Other cases are handled analogously. Note that by Lemma 2 sequence $\{|m_k|\}_{k\geq 0}$ is finite and non-increasing. Write:

$$m_3 - \theta_1 m_2 - \eta_1 m_1 = (1-\theta_1-\eta_1)a + \gamma\sum_{j=0}^{\infty}(\theta_{j+3} - \theta_1\theta_{j+2} - \eta_1\theta_{j+1})u_j\,,$$

where $\{u_j\}_{j\geq 0} \subseteq \mathbf{R}$ is defined as previously. Then, by Hosking (1981), we have the following inequality:

$$\begin{aligned}&\theta_{j+3} - \theta_1\theta_{j+2} - \eta_1\theta_{j+1} = \\ &\quad\theta_{j+1}\left(\tfrac{(j+1+d)(j+2+d)}{(j+3)(j+2)} - \theta_1\tfrac{(j+1+d)}{(j+2)} - \eta_1\right) < \theta_{j+1}(1-\theta_1-\eta_1)\,,\end{aligned}$$

and hence by Lemma 2 the inequality:

$$0 \leq \left|\sum_{j=0}^{\infty}(\theta_{j+3} - \theta_1\theta_{j+2} - \eta_1\theta_{j+1})u_j\right| \leq (1-\theta_1-\eta_1)\left|\sum_{j=0}^{\infty}\theta_{j+1}u_j\right| < \infty$$

From here, positivity of $m_3 - \theta_1 m_2 - \eta_1 m_1$ follows from the argument in 1 and positivity of m_1.

Therefore, from previous, condition $\theta_1 - \gamma = d - \gamma > 0$ is sufficient for the sequence $\{\psi_\tau\}_{\tau\geq t}$ from the recursion (4) to be positive for any $\{\epsilon_\tau\}_{\tau\geq t} \subseteq \mathbf{R}_+ \cup \{0\}$. □

Note that Proposition 1 shows positivity of the sequence $\{\psi_t\}_{t\in\mathbf{Z}}$ from model (3) for any sequence $\{\epsilon_t\}_{t\in\mathbf{Z}} \subseteq \mathbf{R}_+ \cup \{0\}$, not necessarily random. For random sequences $\{\epsilon_t\}_{t\in\mathbf{Z}}$, almost sure positivity of $\{\psi_t\}_{t\in\mathbf{Z}}$ is a trivial consequence of Proposition 1.

1.2 Stationarity and moments of the model

In order to study stationarity and moments of $\{X_t\}_{t\in\mathbf{Z}}$ from model (3) we now develop an alternative representation of $\{\psi_t\}_{t\in\mathbf{Z}}$ in terms of the Volterra series; see Priestley (1988) and Giraitis, Kokoszka and Leipus (2000). Recursive substitution into expression for ψ_t yields:

$$\begin{aligned}\psi_t &= a + \gamma(1-L)^{-d}(\psi_{t-1}\epsilon_{t-1} - \psi_{t-1}) = a + \gamma\textstyle\sum_{j_1=1}^{\infty}\theta_{j_1-1}\psi_{t-j_1}(\epsilon_{t-j_1}-1) \\ &= a + a\gamma\textstyle\sum_{j_1=1}^{\infty}\theta_{j_1-1}(\epsilon_{t-j_1}-1) \\ &\quad +\gamma^2\textstyle\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\theta_{j_1-1}\theta_{j_2-1}\psi_{t-j_1-j_2}(\epsilon_{t-j_1}-1)(\epsilon_{t-j_1-j_2}-1) \\ &= a + a\gamma\textstyle\sum_{j_1=1}^{\infty}\theta_{j_1-1}(\epsilon_{t-j_1}-1) \\ &\quad +a\gamma^2\textstyle\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\theta_{j_1-1}\theta_{j_2-1}(\epsilon_{t-j_1}-1)(\epsilon_{t-j_1-j_2}-1) \\ &\quad +\gamma^3\textstyle\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\sum_{j_3=1}^{\infty}\theta_{j_1-1}\theta_{j_2-1}\theta_{j_3-1}\psi_{t-j_1-j_2-j_3}(\epsilon_{t-j_1}-1)(\epsilon_{t-j_1-j_2}-1)(\epsilon_{t-j_1-j_2-j_3}-1) \\ &= \ldots = a + a\textstyle\sum_{l=1}^{\infty}\gamma^l\sum_{j_1=1\ldots j_l=1}^{\infty}\theta_{j_1-1}\cdots\theta_{j_l-1}(\epsilon_{t-j_1}-1)\cdots(\epsilon_{t-j_1-\ldots-j_l}-1)\,.\end{aligned}$$

Define $M(l,t) := \sum_{j_1=1 \ldots j_l=1}^{\infty} \theta_{j_1-1} \cdots \theta_{j_l-1} (\epsilon_{t-j_1} - 1) \cdots (\epsilon_{t-j_1-\ldots-j_l} - 1)$. Then we can write:

$$\psi_t = a + a \sum_{l=1}^{\infty} \gamma^l M(l,t)\,. \tag{5}$$

We use this equivalent representation of $\{\psi_t\}_{t \in \mathbf{Z}}$ to prove the following proposition:

Proposition 2 *If* $\left(\gamma^2 \mathbb{E}\left[(\epsilon_1 - 1)^2\right] \sum_{j=1}^{\infty} \theta_{j-1}^2\right) < 1$ *and conditions of the Proposition 1 are satisfied then the sequence* $\{\psi_t\}_{t \in \mathbf{Z}}$ *defined by (3) is weakly stationary with the moments:*

$$\begin{aligned} \mathbb{E}[\psi_t] &= a \\ \mathbb{E}\left[(\psi_{t+k} - a)(\psi_t - a)\right] &= \frac{a^2 \gamma^2 \mathbb{E}\left[(\epsilon_1 - 1)^2\right]}{1 - \gamma^2 \mathbb{E}[(\epsilon_1 - 1)^2] \sum_{j=1}^{\infty} \theta_{j-1}^2} \sum_{j=1}^{\infty} \theta_{j-1} \theta_{j+k-1}\,. \end{aligned}$$

PROOF In order to prove that $\{\psi_t\}_{t \in \mathbf{Z}} \subseteq \mathrm{L}^2$ we use its representation given in (5).

We first prove that $\{M(l,t)\}_{l \geq 1} \subseteq \mathrm{L}^2$. Observe the following recursive equality for the sequence $\{M(l,t)\}_{l \geq 1}$, analogous to the one in Kokoszka and Leipus (2000):

$$M(l,t) = \sum_{j=1}^{\infty} \theta_{j-1} (\epsilon_{t-j} - 1) M(l-1, t-j)\,, \tag{6}$$

with $M(0,t) := 1$ for all $t \in \mathbf{Z}$.

Consider the case $l = 1$. Then, since the proposition implies that $\mathbb{E}\left[(\epsilon_1 - 1)^2\right] < \infty$ and $0 < d < \frac{1}{2}$, and $\{\epsilon_t\}_{t \in \mathbf{Z}}$ is i.i.d. , $M(1,t)$ is stationary and square integrable by the well-known results from the literature on long-memory processes; see e.g. Brockwell and Davis (1991) Theorem 13.2.1.

Assume $M(l-1,t)$ is stationary and square integrable. Define:

$$M_n(l,t) := \sum_{j=1}^{n} \theta_{j-1} (\epsilon_{t-j} - 1) M(l-1, t-j)\,.$$

We need to show that $\{M_n(l,t)\}_{n \geq 1} \subseteq \mathrm{L}^2$ and it converges in L^2 as $n \to \infty$. Using linearity of the integral:

$$\begin{aligned} \mathbb{E}\left[M_n(l,t)^2\right] &= \sum_{j=1}^{n} \sum_{k=1}^{n} \theta_{j-1} \theta_{k-1} \mathbb{E}\left[(\epsilon_{t-j} - 1)(\epsilon_{t-k} - 1) M(l-1, t-j) M(l-1, t-k)\right] \\ &= \sum_{j=1}^{n} \theta_{j-1}^2 \mathbb{E}\left[(\epsilon_{t-j} - 1)^2\right] \mathbb{E}\left[M(l-1, t-j)^2\right] \\ &= \mathbb{E}\left[(\epsilon_1 - 1)^2\right] \mathbb{E}\left[M(l-1, 1)^2\right] \sum_{j=1}^{n} \theta_{j-1}^2 < \infty \end{aligned}$$

since, by construction, for $m = j \wedge k$, $M = j \vee k$ for $j,k \in \{1 \ldots n\}$, $(\epsilon_{t-m} - 1)$ is independent of $(\epsilon_{t-M} - 1) M(l-1, t-m) M(l-1, t-M)$ and $M(l-1,t)$ is assumed to be stationary.

Now, we show that $\lim_{n,m \to \infty} \|M_n(l,t) - M_m(l,t)\|_2 = 0$. For $n < m$ it follows from:

$$\begin{aligned} \|M_n(l,t) - M_m(l,t)\|_2 &= \left(\mathbb{E}\left[\left(\sum_{j=n+1}^{m} \theta_{j-1} (\epsilon_{t-j} - 1) M(l-1, t-j)\right)^2\right]\right)^{\frac{1}{2}} \\ &= \left(\sum_{j=n+1}^{m} \theta_{j-1}^2 \mathbb{E}\left[(\epsilon_{t-j} - 1)^2\right] \mathbb{E}\left[M(l-1, t-j)^2\right]\right)^{\frac{1}{2}} \\ &= \left(\mathbb{E}\left[(\epsilon_1 - 1)^2\right] \mathbb{E}\left[M(l-1, 1)^2\right]\right)^{\frac{1}{2}} \left(\sum_{j=n+1}^{m} \theta_{j-1}^2\right)^{\frac{1}{2}} \to 0 \text{ as } n,m \to \infty \end{aligned}$$

using the same independence and stationarity argument as above, and that for $0 < d < \frac{1}{2}$ $\sum_{j=1}^{\infty} \theta_{j-1}^2 < \infty$. Hence, by completeness of L^2, $M_n(l,t) \to M(l,t)$ in L^2 as $n \to \infty$ for all $t \in \mathbf{Z}$.

Finally, stationarity of $M(l,t)$ follows from (6) by the i.i.d. assumption on $\{\epsilon_t\}_{t\in\mathbf{Z}}$ and stationarity of $M(l-1,t)$.

Hence, $\{M(l,t)\}_{l\geq 1}$ is the sequence of square integrable random variables. They have means independent of l and given by:

$$\begin{aligned}\mathbb{E}\left[M(l,t)\right] &= \lim_{n\to\infty} \mathbb{E}\left[M_n(l,t)\right] \\ &= \sum_{j=1}^{\infty} \theta_{j-1}\mathbb{E}\left[\epsilon_{t-j}-1\right]\mathbb{E}\left[M(l-1,t-j)\right] = 0\,.\end{aligned}$$

Their variances are defined by the recursion:

$$\begin{aligned}\mathbb{E}\left[M(l,t)^2\right] &= \lim_{n\to\infty} \mathbb{E}\left[M_n(l,t)^2\right] \\ &= \mathbb{E}\left[(\epsilon_1-1)^2\right]\mathbb{E}\left[M(l-1,1)^2\right]\sum_{j=1}^{\infty} \theta_{j-1}^2\,,\end{aligned}$$

from where $\mathbb{E}\left[M(l,t)^2\right] = \left(\mathbb{E}\left[(\epsilon_1-1)^2\right]\sum_{j=1}^{\infty}\theta_{j-1}^2\right)^l$. Moreover, elements of the sequence $\{M(l,t)\}_{l\geq 1}$ are mutually orthogonal in L^2. Consider covariance of $M(1,t)$ and $M(l,t)$ for $l \geq 1$:

$$\begin{aligned}\mathbb{E}\left[M(1,t)M(l,t)\right] &= \lim_{n\to\infty} \mathbb{E}\left[M_n(1,t)M_n(l,t)\right] \\ &= \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} \theta_{j_1-1}\theta_{j_2-1} \\ &\quad \mathbb{E}\left[(\epsilon_{t-j_1}-1)(\epsilon_{t-j_2}-1)M(l-1,t-j_2)\right]\,.\end{aligned}$$

For $j_1 \neq j_2$ and $m = j_1 \wedge j_2$, $M = j_1 \vee j_2$ for $j_1, j_2 \in \{1\ldots n\}$, $(\epsilon_{t-m}-1)$ is independent of the rest of the terms under the expectation operator, giving zero expectation. For $j_1 = j_2$ and using stationarity assumption the expectation is given by $\mathbb{E}\left[(\epsilon_1-1)^2\right]\mathbb{E}\left[M(l-1,1)\right]$, which is zero for $l > 1$.

Next, assume that $M(p-1,t)$ is orthogonal w.r.t. other elements of $\{M(l,t)\}_{l\geq 1}$. Then covariance of $M(p,t)$ and $M(l,t)$ for $l \geq 1$ is given by:

$$\begin{aligned}\mathbb{E}\left[M(p,t)M(l,t)\right] &= \lim_{n\to\infty} \mathbb{E}\left[M_n(p,t)M_n(l,t)\right] \\ &= \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} \theta_{j_1-1}\theta_{j_2-1} \\ &\quad \mathbb{E}\left[(\epsilon_{t-j_1}-1)(\epsilon_{t-j_2}-1)M(p-1,t-j_2)M(l-1,t-j_2)\right]\,.\end{aligned}$$

Similar argument shows that for $p \neq l$ this covariance is zero. Hence, by induction sequence $\{M(l,t)\}_{l\geq 1}$ is orthogonal in L^2.

Now we show that $\{\psi_t\}_{t\in\mathbf{Z}} \subseteq \mathrm{L}^2$. We follow the same line of proof as above. Using (5) define:

$$\psi_{t,n} := a + a\sum_{l=1}^{n} \gamma^l M(l,t)\,.$$

We need to show that $\{\psi_{t,n}\}_{n\geq 1} \subseteq \mathrm{L}^2$ and that it converges in L^2 as $n \to \infty$.

Square integrability of $\{\psi_{t,n}\}_{n\geq 1}$ follows immediately from square integrability of $\{M(l,t)\}_{l\geq 1}$ shown above. By Minkowski's inequality for $n < m$ we can write:

$$\begin{aligned} \|\psi_{t,n} - \psi_{t,m}\|_2 &= \left(\mathbb{E}\left[\left(a\sum_{l=n+1}^{m}\gamma^l M(l,t)\right)^2\right]\right)^{\frac{1}{2}} \\ &\leq a\sum_{l=n+1}^{m}\left(\mathbb{E}\left[\left(\gamma^l M(l,t)\right)^2\right]\right)^{\frac{1}{2}} \\ &= a\sum_{l=n+1}^{m}\left(\gamma^2\mathbb{E}\left[(\epsilon_1-1)^2\right]\sum_{j=1}^{\infty}\theta_{j-1}^2\right)^{\frac{l}{2}} \to 0 \text{ as } n,m\to\infty \end{aligned}$$

since $\sum_{l=1}^{\infty}\left(\gamma^2\mathbb{E}\left[(\epsilon_1-1)^2\right]\sum_{j=1}^{\infty}\theta_{j-1}^2\right)^l < \infty$ by assumption. Hence, by completeness of L^2, $\psi_{t,n}\to\psi_t$ in L^2 as $n\to\infty$ for all $t\in\mathbf{Z}$.

Stationarity of $\{\psi_t\}_{t\in\mathbf{Z}}$ follows from (5) and stationarity of $M(l,t)$ for given $l\geq 1$.

Finally, we consider moments of $\{\psi_t\}_{t\in\mathbf{Z}}$. The first moment is given by:

$$\mathbb{E}[\psi_t] = \lim_{n\to\infty}\psi_{t,n} = a + a\sum_{l=1}^{\infty}\gamma^l\mathbb{E}[M(l,t)] = a\,.$$

The following auxiliary result is used in the derivation of the auto-covariance function of $\{\psi_t\}_{t\in\mathbf{Z}}$:

$$\begin{aligned} \mathbb{E}\left[M(p,t+k)M(l,t)\right] &= \lim_{n\to\infty}\mathbb{E}\left[M_n(p,t+k)M_n(l,t)\right] \\ &= \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\theta_{j_1-1}\theta_{j_2-1} \\ &\quad \mathbb{E}\left[(\epsilon_{t+k-j_1}-1)(\epsilon_{t-j_2}-1)M(p-1,t+k-j_1)M(l-1,t-j_2)\right] \\ &= \mathbb{E}\left[(\epsilon_1-1)^2\right]\sum_{j=1}^{\infty}\theta_{j-1}\theta_{j+k-1}\mathbb{E}\left[M(p-1,t-j)M(l-1,t-j)\right]\,, \end{aligned}$$

where last equality is justified by the fact that for $(j_1-k)\neq j_2$ and $m=(j_1-k)\wedge j_2$, $M=(j_1-k)\vee j_2$ for $j_1,j_2\in\{1\ldots n\}$, $(\epsilon_{t-m}-1)$ is independent of the rest of the terms under the expectation operator, producing zero expectation. By orthogonality of $\{M(l,t)\}_{l\geq 1}$ $\mathbb{E}\left[M(p-1,t-j)M(l-1,t-j)\right]$ will be different from zero only when $p=l$. Therefore we get:

$$\mathbb{E}\left[M(p,t+k)M(l,t)\right] = \begin{cases} \mathbb{E}\left[(\epsilon_1-1)^2\right]\mathbb{E}\left[M(l-1,1)^2\right]\sum_{j=1}^{\infty}\theta_{j-1}\theta_{j+k-1} & \text{when } p=l \\ 0 & \text{when } p\neq l \end{cases}$$

Using this result and expression for the variance of $M(l,1)^2$, the auto-covariance function of $\{\psi_t\}_{t\in\mathbf{Z}}$ is given by:

$$\begin{aligned} \mathbb{E}\left[(\psi_{t+k}-a)(\psi_t-a)\right] &= \lim_{n\to\infty}\mathbb{E}\left[(\psi_{t+k,n}-a)(\psi_{t,n}-a)\right] \\ &= a^2\sum_{p=1}^{\infty}\sum_{l=1}^{\infty}\gamma^{l+p}\mathbb{E}\left[M(p,t+k)M(l,t)\right] \\ &= a^2\gamma^2\mathbb{E}\left[(\epsilon_1-1)^2\right]\left(\sum_{j=1}^{\infty}\theta_{j-1}\theta_{j+k-1}\right)\left(\sum_{l=0}^{\infty}\gamma^{2l}\mathbb{E}\left[M(l,1)^2\right]\right) \\ &= \frac{a^2\gamma^2\mathbb{E}\left[(\epsilon_1-1)^2\right]}{1-\gamma^2\mathbb{E}[(\epsilon_1-1)^2]\sum_{j=1}^{\infty}\theta_{j-1}^2}\sum_{j=1}^{\infty}\theta_{j-1}\theta_{j+k-1}\,. \end{aligned}$$

□

1.3 Discussion

It is a well-established empirical fact that volatility and trading intensity of many financial time series exhibits substantial persistence over time; see Andersen, Bollerslev, Diebold and

Labys (2001) and Jasiak (1999) among many others. Model (3) proposed in this section is designed to account for this empirical regularity and, as such, competes with FIGARCH model of Baillie, Bollerslev and Mikkelsen (1996) and Ding and Granger (1996), and linear ARCH model of Giraitis, Robinson and Surgailis (2000). In this subsection we briefly discuss relative merits of the three models.

The simplest form of the FIGARCH model of Baillie, Bollerslev and Mikkelsen (1996) and Ding and Granger (1996) re-expressed in our notation is given by:

$$\begin{aligned} X_t &= \psi_t \cdot \epsilon_t \\ \psi_t &= a + \left(1 - (1-L)^d\right) X_t \,, \end{aligned}$$

where $\{\epsilon_t\}_{t\in\mathbf{Z}}$ is a sequence of i.i.d. non-negative random shocks with $\mathbb{E}[\epsilon_1] = 1$, and $a > 0$ is constant. As shown in Baillie, Bollerslev and Mikkelsen (1996) this form of the FIGARCH model implies infinite first and higher unconditional moments of ψ_t and X_t variables. And even if additional restrictions imposed to ensure weak stationarity of the process, Giraitis, Kokoszka and Leipus (2000) show that the FIGARCH model is essentially short memory according to the criteria of McLeod and Hipel (1978).

It appears that FIGARCH model is not well-suited for modeling long-range dependence in the sequences of positive random variables with multiplicative shocks. In order to ensure summability of $\left(1 - (1-L)^d\right) X_t$ series, where $\{X_t\}_{t\in\mathbf{Z}} \subseteq \mathbf{R}_+$, coefficients of the $1 - (1-L)^d$ polynomial have to be summable. Providing that the second moment exists, this in turn implies absolute summability of the autocovariances of $\{\psi_t\}_{t\in\mathbf{Z}}$ and $\{X_t\}_{t\in\mathbf{Z}}$ sequences and demonstrates short-memory nature of their dynamics.

Giraitis, Robinson and Surgailis (2000) overcome this difficulty by disturbing $\{\psi_t\}_{t\in\mathbf{Z}}$ process using mean zero random variable Y_t as shown in (2). This allows them to introduce much slower decaying structure of the coefficients $(1-L)^{-d}$ in the infinite series representation of their model, while still ensuring L^2 summability. However, as we mentioned previously, variable Y_t in their model lacks the usual volatility parameter interpretation of GARCH models. In addition, variable Y_t is the square root of the variable of interest X_t, which leads to complications in deriving closed-form expressions for the autocovariance function of X_t.

Model (3) in this paper fills the gap between FIGARCH model of Baillie, Bollerslev and Mikkelsen (1996) and the linear ARCH model of Giraitis, Robinson and Surgailis (2000). When applied to volatility modeling, it retains usual GARCH interpretation of the $\{\psi_t\}_{t\in\mathbf{Z}}$ by insuring its positivity, and at the same time allows for weak stationarity of the $\{\psi_t\}_{t\in\mathbf{Z}}$ and $\{X_t\}_{t\in\mathbf{Z}}$ processes. Positivity of the ψ_t parameter also comes as an advantage in the context of modeling trading intensity in high-frequency data.

Parameter d in model (3) has very much the same interpretation as in the class of ARFIMA models of Granger and Joyeux (1980) and Hosking (1981). In particular, using results of Proposition 2, it becomes possible to test for weak stationarity of the volatility or trading intensity process $\{\psi_t\}_{t\in\mathbf{Z}}$ in the real-world datasets.

Another advantage of model (3) lies in the transparency of its dynamic properties. Unlike linear ARCH model of Giraitis, Robinson and Surgailis (2000) and FIEGARCH model of Bollerslev and Mikkelsen (1996), model (3) does not involve non-linear transformations of the underlying processes $\{\psi_t\}_{t\in\mathbf{Z}}$ and $\{X_t\}_{t\in\mathbf{Z}}$ and hence is straightforward to derive autocovariances and other dynamic properties.

2 Combined short- and long-memory model

In this section we extend basic model (3) of section 1 for the sequences of positive long-memory weakly stationary random variables to include simple short-memory component. We again derive conditions for stationarity and existence of first two moments of this extended model.

2.1 Basic model with additional autoregressive component

In this subsection we introduce an extension of the basic long-memory model (3) which includes additional autoregressive component. The models is defined as:

$$\begin{aligned} X_t &= \psi_t \cdot \epsilon_t \\ \psi_t &= a + \gamma(1-\phi L)^{-1}(1-L)^{-d}(X_{t-1} - \psi_{t-1}) \end{aligned} \tag{7}$$

where, in general, restriction $|\phi| < 1$ is assumed and other parameters are as in (3).

Apart from having more flexible structure of autocovariances, one of the reasons for introducing model (7) is to remove the restriction $\gamma \leq d$. As shown in Proposition 1, this restriction is necessary for $\{\psi_t\}_{t\in\mathbf{Z}}$ to stay positive in the basic long-memory specification (3). As we demonstrate later in this subsection, model (7) allows parameter of fractional integration d to become zero such that sequence $\{\psi_t\}_{t\in\mathbf{Z}}$ will retain short-memory dynamics implied by its autoregressive component and satisfy non-negativity condition.

Proposition 3 establishes the set of restrictions on the parameters of (7) sufficient for the sequence $\{\psi_t\}_{t\in\mathbf{Z}}$ implied by the model to stay on the positive half-line with probability one.

Proposition 3 *If $\gamma \leq d + \phi$ and $d(1-d-2\phi) \geq 0$, then for any sequence $\{\epsilon_t\}_{t\in\mathbf{Z}}$ in model (7) $\{\psi_t\}_{t\in\mathbf{Z}} \subseteq \mathbf{R}_+$.*

PROOF Rewrite dynamic equation for ψ_t in model (7) in the following form:

$$\psi_t = a + \gamma \sum_{j=1}^{\infty} \tilde{\theta}_{j-1} \psi_{t-j}(\epsilon_{t-j} - 1)\,. \tag{8}$$

Here, sequence of coefficients $\{\tilde{\theta}_j\}_{j\geq 0}$ comes from the expansion of polynomial $(1-\phi L)^{-1}(1-L)^{-d}$ and is given by:

$$\tilde{\theta}_j = \sum_{k=0}^{j} \phi^k \theta_{j-k}\,,$$

where $\{\theta_j\}_{j\geq 0}$ are as in Proposition 1.

Next, consider sequence $\{\tilde{\eta}_j\}_{j\geq 1}$ defined as:

$$\begin{aligned}\tilde{\eta}_1 &:= \tilde{\theta}_2 - \tilde{\theta}_1\tilde{\theta}_1\\ \tilde{\eta}_2 &:= \tilde{\theta}_3 - \tilde{\theta}_2\tilde{\theta}_1 - \tilde{\theta}_1\tilde{\eta}_1\\ \tilde{\eta}_3 &:= \tilde{\theta}_4 - \tilde{\theta}_3\tilde{\theta}_1 - \tilde{\theta}_2\tilde{\eta}_1 - \tilde{\theta}_1\tilde{\eta}_2\\ &\vdots\end{aligned}$$

Simple calculations show that $\{\tilde{\eta}_j\}_{j\geq 1}$ can be re-expressed in terms of ϕ, θ_1 and the sequence $\{\eta_j\}_{j\geq 1}$ defined in Lemma 1 as follows:

$$\begin{aligned}\tilde{\eta}_1 &:= \eta_1 - \phi\theta_1\\ \tilde{\eta}_2 &:= \eta_2 - \phi\eta_1\\ \tilde{\eta}_3 &:= \eta_3 - \phi\eta_2\\ &\vdots\end{aligned}$$

Using results of Lemma 1 it is easy to see that non-negativity condition for $\tilde{\eta}_1$ is given by $d(1-d-2\phi)\geq 0$, and for $\tilde{\eta}_k$, $k\geq 2$, by $\tilde{\eta}_{k-1}\left(k-d-(k+1)\phi\right)\geq 0$. Hence, it follows that condition $d(1-d-2\phi)\geq 0$ is sufficient for non-negativity of the sequence $\{\tilde{\eta}_j\}_{j\geq 1}$.

Result $\{\psi_t\}_{t\in\mathbf{Z}}\subseteq\mathbf{R}_+$ is then established using the same line of proof as in Proposition 1 using representation of ψ_t given in (8). □

Note that non-negativity condition $d(1-d-2\phi)\geq 0$ in Proposition 3 is quite restrictive for positive values of ϕ when d is different from zero. For negative values of ϕ, restriction $\gamma\leq d+\phi$ is likely to be binding and implies that parameter d must be different from zero.

Proposition 3 provides only sufficient conditions for positivity of $\{\psi_t\}_{t\in\mathbf{Z}}$. In fact, condition $\{\tilde{\eta}_j\}_{j\geq 1}\subseteq\mathbf{R}_+\cup\{0\}$ in Proposition 3 is not necessary and some lower-order $\tilde{\eta}_j$ can be negative without violating positivity of $\{\psi_t\}_{t\in\mathbf{Z}}$. Analytical derivation of necessary and sufficient conditions for this case is likely to be complicated. In practice, however, non-negativity of $\{\psi_t\}_{t\in\mathbf{Z}}$ can be checked by the impulse response analysis of model (7).

2.2 Conditions for weak stationarity of the model

As in subsection 1.2, additional restrictions on the parameters of model (7) must be placed in order to ensure existence of the weakly stationary solution. They are shown in Proposition 4:

Proposition 4 *If* $\left(\gamma^2\mathbb{E}\left[(\epsilon_1-1)^2\right]\sum_{j=1}^{\infty}\tilde{\theta}_{j-1}^2\right)<1$ *and conditions of the Proposition 3 are satisfied then the sequence* $\{\psi_t\}_{t\in\mathbf{Z}}$ *defined by (7) is weakly stationary with the moments:*

$$\begin{aligned}\mathbb{E}[\psi_t] &= a\\ \mathbb{E}\left[(\psi_{t+k}-a)(\psi_t-a)\right] &= \frac{a^2\gamma^2\mathbb{E}\left[(\epsilon_1-1)^2\right]}{1-\gamma^2\mathbb{E}[(\epsilon_1-1)^2]\sum_{j=1}^{\infty}\tilde{\theta}_{j-1}^2}\sum_{j=1}^{\infty}\tilde{\theta}_{j-1}\tilde{\theta}_{j+k-1}\,.\end{aligned}$$

Proof We show that under assumptions of the proposition the sequence $\{\tilde{\theta}_j\}_{j\geq 0}$ in (8) is square summable. The following representation is useful:

$$\tilde{\theta}_j^2=\left(\sum_{k=0}^{j}\phi^k\theta_{j-k}\right)^2=\sum_{k=0}^{j}\phi^{2k}\theta_{j-k}^2+2\phi\sum_{k=0}^{j-1}\phi^{2k}\theta_{j-k}\tilde{\theta}_{j-k-1}\,,$$

where last sum is zero for $j = 0$. Hence, we need to show that $\sum_{j=0}^{\infty}\sum_{k=0}^{j}\phi^{2k}\theta_{j-k}^2 < \infty$ and $\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\phi^{2k}\theta_{j-k}\tilde{\theta}_{j-k-1} < \infty$. Theorem 3.50 in Rudin (1976) gives sufficient conditions for the convergence of such series. In both cases, $\sum_{j=0}^{\infty}\phi^{2j}$ converges absolutely for $|\phi| < 1$. For $0 \leq d < \frac{1}{2}$, implied by the proposition, $\sum_{j=0}^{\infty}\theta_j^2$ is also absolutely convergent, and hence $\sum_{j=0}^{\infty}\sum_{k=0}^{j}\phi^{2k}\theta_{j-k}^2 < \infty$.

It is left to show that $\sum_{j=0}^{\infty}|\theta_{j+1}\tilde{\theta}_j| < \infty$. Using recursive structure of the sequence $\{\theta_j\}_{j\geq 1}$ given in Lemma 1 and assumption $0 \leq d < \frac{1}{2}$ it is easy to derive the following inequality for all $j \geq 0$:

$$0 \leq |\theta_{j+1}\tilde{\theta}_j| \leq \sum_{k=0}^{j}|\phi^k|\theta_{j-k}^2 \,.$$

By theorem 3.50 in Rudin (1976) the series $\sum_{j=0}^{\infty}\sum_{k=0}^{j}|\phi^k|\theta_{j-k}^2$ converges, thereby establishing $\sum_{j=0}^{\infty}|\theta_{j+1}\tilde{\theta}_j| < \infty$.

Other results of the proposition are established using the same arguments as in Proposition 2 and square summability of $\{\tilde{\theta}_j\}_{j\geq 0}$. □

2.3 Discussion

Compound model (7), which extends basic long-memory specification (3) by including additional autoregressive component, demonstrates that the class of models for stationary sequences of long-memory positive random variables introduced in this paper is a natural generalization of GARCH processes of Engle (1982) and Bollerslev (1986) to the case of non-summable autocovariances. In particular, model (7) nests short-memory GARCH(1,1) model as special case when $d = 0$.

Indeed, model (3) can be seen as an evolution of the short-memory GARCH(1,1) as follows. Write GARCH(1,1) using our notation:

$$\begin{aligned} X_t &= \psi_t \cdot \epsilon_t \\ \psi_t &= a + \gamma(1-\phi L)^{-1}(X_{t-1} - \psi_{t-1})\,, \end{aligned}$$

where the usual assumptions about coefficients a, γ and ϕ, and sequence $\{\epsilon_t\}_{t\in\mathbf{Z}}$ hold. In this form ψ_t is expressed as the infinite sum of exponentially weighted martingale differences $\{X_\tau - \psi_\tau\}_{\tau<t}$, provided that $0 < \phi < 1$, and positivity of $\{\psi_t\}_{t\in\mathbf{Z}}$ is guaranteed when $\gamma \leq \phi$.

Polynomial $(1-\phi L)^{-1}$ in GARCH(1,1) model produces sequence of absolutely summable coefficients, which in turn results in summable autocovariances of the process $\{\psi_t\}_{t\in\mathbf{Z}}$. From this prospective, long-memory model (7) is the evolution of the GARCH(1,1) model above, where polynomial $(1-\phi L)^{-1}$ is replaced by $(1-L)^{-d}$, giving square summable coefficients and non-summable autocovariances of $\{\psi_t\}_{t\in\mathbf{Z}}$. The compound model (7) introduced in this section can be regarded as GARCH(2,1) with one of the autoregressive polynomials replaced by $(1-L)^{-d}$.

3 Maximum likelihood inference

This section provides brief overview of the statistical inference for model (3) in section 1 and model (7) in section 2.

In sections 1 and 2 a new class of models for sequences of positive stationary long-memory random variables is introduces. One of the fundamental assumptions in this models is the i.i.d. property of the multiplicative shocks $\{\epsilon_t\}_{t\in\mathbf{Z}}$. However, no parametric assumptions on the distribution of the shocks is made. It is clear that the conditional and marginal distributions of the variable of interest X_t is going to depend on the distributional properties of the shocks. At the same time, particular characteristics of the distribution of X_t are likely to be substantially different from application to application.

In the context of short-memory GARCH processes and related ACD and FIACD models QML estimator seems to be the preferred choice. It usually allows to impose only mild moment conditions on the sequence of shocks $\{\epsilon_t\}_{t\in\mathbf{Z}}$ and therefore is attractive in many empirical contexts. However, rigorous discussion of the theoretical properties of the QML estimator has so far been limited only to the cases of ARCH(q) model in Weiss (1986) and GARCH(1,1) model in Lee and Hansen (1994) and Lumsdain (1996).

Consider a sequence of non-negative random variables $\{X_t\}_{t\in\mathbf{Z}}$ with conditional expectations given by the sequence $\{\psi_t\}_{t\in\mathbf{Z}}$, where $\{\psi_t\}_{t\in\mathbf{Z}}$ describes all dynamic properties of $\{X_t\}_{t\in\mathbf{Z}}$, such as in the models (3) or (7). Then the pseudo-likelihood function for the QML estimator based on the likelihood function for the sequence of i.i.d. exponential random variables is given by:

$$\mathcal{L}_T(\boldsymbol{\theta}) = -\sum_{\tau=1}^{T}\left(\log\hat{\psi}_\tau(\boldsymbol{\theta}) + \frac{x_\tau}{\hat{\psi}_\tau(\boldsymbol{\theta})}\right). \tag{9}$$

In this expression parameter vector is given by $\boldsymbol{\theta} = (a, \gamma, d)$ for model (3) and $\boldsymbol{\theta} = (a, \gamma, \phi, d)$ for model (7). Let the vector of true parameters be denoted $\boldsymbol{\theta}_0$, consisting of a_0, γ_0, d_0 and possibly ϕ_0.

One of the difficulties in estimating long memory models is that at every time point $t \in \mathbf{Z}$ the process $\{X_t\}_{t\in\mathbf{Z}}$ depends on its own infinite past history $\{X_\tau\}_{\tau<t}$. In finite-samples time-domain estimation methods this history is not observed, and hence the issue of proper start-up conditions becomes relevant. In particular, $\{\hat{\psi}_\tau\}_{\tau=1}^T$ can only approximate the sequence $\{\psi_\tau\}_{\tau=1}^T$, even when evaluated at $\boldsymbol{\theta}_0$. In the context of linear ARFIMA processes these difficulties are highlighted in Chung and Baillie (1993) among others.

Let $\hat{\psi}_\tau$ be truncated version of ψ_t as follows:

$$\begin{aligned}\hat{\psi}_1(\boldsymbol{\theta}) &= a\\ \hat{\psi}_\tau(\boldsymbol{\theta}) &= a + \gamma\textstyle\sum_{j=1}^{\tau-1}\theta_{j-1}(x_{\tau-j} - \hat{\psi}_{\tau-j})\end{aligned} \tag{10}$$

and coefficients $\{\theta_j\}_{j\geq 0}$ depend on the assumed parametric form of the model. QML estimates $\hat{\boldsymbol{\theta}}_T$ are then obtained by maximizing (9) and (10) subject to the set of constraints derived in sections 1 and 2.

QML estimator based on (9) together with (10) is similar to the one used for modeling financial durations data; see Engle and Russel (1998) and Engle (2000). Consistency and asymptotic normality of the QML estimator for ACD models follows directly from the known result on the estimation of short-memory GARCH processes. However, these results are likely to be of little use in the case of long-memory models (3) and (7) because of their infinite dependence on its own past history and non-summable coefficients. So far, relatively little is known about time-domain estimation of the non-linear processes with long-range dependence similar to those proposed in this paper.

One of the potentially fruitful ideas for the time-domain inference in the class of models put forward in this paper is to initialize *both* the assumed data-generating mechanism and the estimator using $\epsilon_t = 1$ for $t \leq 0$. This will in turn imply that $X_t = \psi_t = a$ for $t \leq 0$. This condition is similar to one used in Tanaka (1999), among others, for the time-domain estimation of possibly non-stationary ARFIMA processes. Main advantage of this approach is seen to be in the possibility of obtaining one estimator for both stationary and non-stationary long-memory processes possessing properties similar to those in the short-memory case, i.e. $\sqrt{T}$-consistency and asymptotic normality.

A limited Monte-Carlo study was carried out by the author in order to understand the properties of the QML estimator (9) and (10) on the synthetic data from the conditional generating mechanism described above. Preliminary results suggest normality of the estimator for parameters γ, ϕ and d and various values of $d_0 \in [0, 1)$. It is also apparent that the distribution of the parameter a is different in the intervals $0 \leq d_0 < \frac{1}{2}$ and $\frac{1}{2} < d_0 < 1$, reflecting the boundary between the stationary and non-stationary cases. Further properties of the estimator are currently under investigation.

Conclusion

In this paper we introduced a class of model for sequences of weakly stationary positive random variables with non-summable autocovariances. Models feature multiplicative structure of innovations and therefore related to the GARCH processes of Engle (1982) and Bollerslev (1986). A class of models with similar structure and dynamic properties, referred to as linear ARCH models, was recently introduced by Giraitis, Robinson and Surgailis (2000). In comparison to linear ARCH processes our models have two important advantages:

- Linear ARCH models involve instantaneous non-linear mapping of the underlying processes which substantially complicates derivation of the autocovariance function for the sequence of random variables of interest. Class of models introduced in this paper does not involve non-linear transforms and therefore easy to handle analytically.

- Linear ARCH models are not nested with short-memory GARCH processes. Moreover, due to the structure of linear ARCH models, interpretation of its core components is very different from those in GARCH models. In this paper we propose a class of models that extends GARCH models to the long-memory case and has similar interpretation of the basic underlying elements.

Another popular in econometric literature model that accounts for long-range dependence commonly found in the volatility of many financial time-series is FIGARCH model of Baillie, Bollerslev and Mikkelsen (1996). However, as shown in Giraitis, Kokoszka and Leipus (2000), FIGARCH model is essentially short-memory when conditions for the existence of the second moment are satisfied. This is due to the fast decay and summability of the coefficients in the ARCH(∞) representation of the model. In contrast to FIGARCH , both linear ARCH process of Giraitis, Robinson and Surgailis (2000) and the class of models proposed in this paper are able to produce weakly stationary sequences of positive random variables with non-summable autocovariances.

Models introduced in this paper have several potential applications in empirical studies in a variety of fields. From the perspective of the dynamic heteroscedasticity, our models offer a framework that naturally extends short-memory GARCH models, including finite and infinite second moment. In the former case, the autocovariances are non-summable, which satisfies the long-memory criterion of McLeod and Hipel (1978). In the paper we also show a method to obtain the compound model which nests both the basic long-memory and GARCH models. This gives researchers a tool to distinguish between the cases of summable and non-summable autocovariances in the volatility process of real-world financial data.

Another potentially attractive area of application for the models is econometrics of high-frequency financial data. Time-series dynamics of durations between adjacent observations in transactions data is complicated and often exhibits long-range dependence. However, direct application of FIGARCH models to this data implies infinite first moment of durations — feature that is rarely true in real-world datasets. Models proposed in this paper do not have this shortcoming and allow researches to test between the cases of short- and long-memory in financial durations data.

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