Plasma Oscillations[1]

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Abstract

The equivalence of the Landau and Van Kampen treatments of the initial value problem for plasma oscillations is demonstrated. Using completeness and orthogonality theorems for the normal modes, an integral representation for the solution of the initial value problem is obtained which is shown to be identical with that obtained by modifying the integration contour in Landau's Laplace Transform solution.

I. INTRODUCTION

The initial value problem for an electronic plasma has been solved by two strikingly different methods. Landau (1) has given a solution using a Laplace Transform technique. Van Kampen (2) has solved the problem by means of a normal mode expansion. Since both approaches have some puzzling features, it is interesting to see the complete identity of the solutions. This is shown below utilizing an orthogonality property which is proved. The results obtained in order to prove the identity are rather interesting mathematically, as they indicate that many of the classical completeness and orthogonality theorems hold for quite pathological operators.

The plan of this paper is as follows. In Section II the derivation of the fundamental equation is sketched, primarily to introduce notation. The equations considered are identical with those treated by Landau and Van Kampen except that the unperturbed distribution is left arbitrary. The normal modes are described in Section III. Next, the adjoint equation and its normal modes are discussed. In Section V, which contains the crux of the proof of identity, an orthogonality theorem is proved for solutions of the original and adjoint equations. The appropriate normalization constants for the orthogonality integrals are also obtained. A completeness theorem is then proved by an almost trivial modification of Van Kampen's method. In Section VII, Van Kampen's normal mode solution of the initial value problem is given. Landau's Laplace Transform approach is slightly changed in Section VIII by the introduction of two-sided transforms. By deforming the integration contours for the solution, we obtain Van Kampen's results. It is then apparent that the Laplace Transform method gives a generating function for the "improper" eigenfunctions of Van Kampen.

II. FUNDAMENTAL EQUATIONS

In our model of the plasma the positive ions will be imagined to be smeared out to form a uniform background. Collisions are omitted and the electron distribution is considered to be but slightly perturbed from a spatially uniform but otherwise arbitrary distribution. Finally, we consider only longitudinal waves in an infinite medium.

Since only the electrons are treated dynamically, we have but a single distribution function $F(\vec{r},\vec{v},t)$ describing the system. Neglecting collisions, this satisfies the Liouville equation

$$\frac{\partial F}{\partial t} + \vec{v}\cdot\nabla F + \vec{a}\cdot\nabla_v F = 0 \quad . \tag{1}$$

Here the local acceleration $\vec{a}$ is given in terms of the electron charge (e), mass (m) and the local electric field $\vec{E}$ by

$$\vec{a} = \frac{e\vec{E}}{m}(\vec{r},t) \quad .$$

The electric field is then determined in terms of the electric charge density $\rho(\vec{r},t)$ by

$$\nabla\cdot\vec{E} = 4\pi\rho \quad , \tag{2}$$

where

$$\rho(\vec{r},t) = e\int F d^3\vec{v} - \rho_o \quad , \tag{3}$$

and ρ_o is the uniform charge density due to the positive ions.

We now linearize the equations, writing

$$F = f_o + f \quad , \tag{4}$$

where f_o is a given time and space independent function of velocity (for example, f_o might be a Maxwell distribution) and f is assumed small. Inserting the "Ansatz" for F in the previous equations and neglecting terms of higher order in the perturbed quantities gives

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \frac{e}{m}\vec{E}\cdot\nabla_v f_o = 0 \quad . \tag{5}$$

Introducing a scalar potential by $\vec{E} = -\nabla\phi$, we have

$$\nabla^2\phi = -4\pi e\int f d^3\vec{v} \quad . \tag{6}$$

Since the coefficients in (5) and (6) are independent of position we can treat the spatial dependence by Fourier decomposition. Restricting our attention to one Fourier component, i.e., writing ϕ, $\vec{E}$, $f \sim e^{i\vec{k}\cdot\vec{r}}$ and omitting an implied index $\vec{k}$, reduces the above to

$$\frac{\partial f}{\partial t} + i\vec{k}\cdot\vec{v}f = i\vec{k}\cdot\nabla_v f_o \frac{4\pi e^2}{mk^2}\int f(\vec{v})d^3\vec{v} \quad , \tag{7}$$

and

$$\phi = \frac{4\pi e}{k^2}\int f d^3\vec{v} \quad . \tag{8}$$

In velocity space we introduce axes along and perpendicular to the vector $\vec{k}$. The components of $\vec{v}$ with respect to these axes are then written as $v_{\|}$ and $\vec{v}_{\perp}$. With this notation we have, for example,

$$\int f d^3\vec{v} = \int f(v_{\|}, \vec{v}_{\perp})\, dv_{\|}\, d\vec{v}_{\perp} \quad . \tag{9}$$

For many purposes, for instance, to calculate ϕ, we are only interested in the function $g(v_{\|})$ defined by

$$\int f(v_{\|}, \vec{v}_{\perp})\, d\vec{v}_{\perp} = g(v_{\|}) \tag{10}$$

Integrating both sides of (7) over all $\vec{v}_{\perp}$ then yields

$$\frac{\partial g}{\partial t} + ikvg(v,t) = -ik\eta(v) \int_{-\infty}^{\infty} g(v')dv' \tag{11}$$

with

$$\eta(v_{\|}) = -\frac{4\pi e^2}{mk^2} \frac{\partial}{\partial v_{\|}} \int f_o(v_{\|}, \vec{v}_{\perp})d\vec{v}_{\perp} \quad . \tag{12}$$

Many authors (1,2) have noted that, for an isotropic distribution $f_o(v)$, (12) simplifies to

$$\eta(v) = \frac{8\pi^2 e^2}{mk^2} v f_o(v) \quad .$$

However, we will here make no use of this simplification.

Finally we note for future reference that ϕ is given in terms of $g(v)$ by

$$\phi = \frac{4\pi e}{k^2} \int_{-\infty}^{\infty} g(v')dv' \quad . \tag{13}$$

Our basic problem will be to solve (11) given $g(v,o)$. The electric field at any time t is then to be obtained using (13).

III. NORMAL MODES

Following Van Kampen (2) closely, we first consider the normal modes corresponding to (11). The only difference from his work is that we will (a) not specify $\eta(v)$ and (b) prove some orthogonality relations which make the normal mode expansions trivial.

Let us look for solutions of (11) which vary exponentially with time, i.e., $g \sim e^{-i\omega t}$. Label solutions by

$$\nu = \omega/k \quad .$$

Then our "eigenfunctions" will have to satisfy the equation

$$(v-\nu)g_\nu(v) = -\eta(v) \int_{-\infty}^{\infty} g_\nu(v')dv' \quad . \tag{14}$$

It is convenient to classify the eigenvalues and eigenfunctions in four groups.

CLASS 1a

For these solutions ν is real and $\eta(\nu) \neq 0$. In later work we will say that such points ν are members of the set Σ. It may be noted that these are the only solutions for the Maxwell distribution and hence were the ones considered by Van Kampen (2).

Normalizing the solutions so that

$$\int_{-\infty}^{\infty} g_\nu(v')dv' = 1 \tag{15}$$

shows that

$$g_\nu(v) = -P\frac{\eta(v)}{v-\nu} + \lambda(\nu)\,\delta(v-\nu) \quad . \tag{16}$$

Here δ denotes the Dirac delta function and the P signifies that the principal value integral is to be taken when integrating the expression for $g_\nu(v)$ with respect to velocity. The normalization condition (15) gives

$$\lambda(\nu) = 1 + P\int_{-\infty}^{\infty}\frac{\eta(v)dv}{v-\nu} \quad . \tag{17}$$

CLASS 1b

Here we have ν real and $\eta(\nu) = 0$. We can proceed exactly as above, obtaining

$$g_\nu(v) = -P\frac{\eta(v)}{v-\nu} + \lambda(\nu)\,\delta(v-\nu) \quad , \tag{18}$$

where again

$$\lambda(\nu) = 1 + \int_{-\infty}^{\infty}\frac{\eta(v)dv}{v-\nu} \quad . \tag{19}$$

In (18) the principal value sign is of course now not necessary. However, it is convenient to carry it along since it serves to remind us that in integrating <u>with respect to ν</u> we are to omit the points of Class 1c.

CLASS 1c

These are exceptional cases of Class 1b which occur when ν is real, $\eta(\nu) = 0$ and $\lambda(\nu)$ also vanishes. These can only occur for a finite number of points ν_i where $i = 1,2,\cdots m$. These solutions are

$$g_{\nu_i}(v) \equiv g_i(v) = \frac{-\eta(v)}{v-\nu_i} \quad , \tag{20}$$

where again

$$\int_{-\infty}^{\infty} g_i(v)dv = 1 \quad . \tag{21}$$

CLASS 2

These are solutions for complex ν. Again taking

$$\int_{-\infty}^{\infty} g_\nu(v) = 1 \quad ,$$

we have

$$g_\nu(v) = \frac{-\eta(v)}{v-\nu} \quad . \tag{22}$$

The normalization condition then again determines a finite discrete set of points ν_j where $j = m+1, m+2, \cdots n$.

The varieties of solutions can then be summarized as follows: We have a continuum of solutions for all real ν such that not simultaneously

$$\eta(\nu_i) = 0 = \lambda(\nu_i) \equiv 1 + \int_{-\infty}^{\infty} \frac{\eta(v)dv}{v-\nu_i} \quad . \tag{23}$$

We have a discrete set of solutions for ν_i such that either ν_i is complex and

$$\int_{-\infty}^{\infty} \frac{\eta(v)dv}{v-\nu_i} = 1$$

or ν_i is real with $\eta(\nu_i) = \lambda(\nu_i) = 0$.

There are two important points to note.

(a) All solutions have been normalized so that

$$\int_{-\infty}^{\infty} g_\nu(v)dv = 1 \quad . \tag{24}$$

(b) We assume, for simplicity only, that all roots ν_i are simple.

IV. THE ADJOINT EQUATION

Along with our fundamental (11), it is useful to consider an adjoint equation which we take to be

$$\frac{\partial \tilde{g}}{\partial t} + ikv\ \tilde{g}(v) = -ik \int_{-\infty}^{\infty} \eta(v')\ \tilde{g}(v')dv' \quad . \tag{25}$$

The utility of studying this equation is shown in the next section. Here we show that to each eigenvalue ν of the fundamental Eq. (11) there is an eigenfunction of the adjoint Eq. (25). Thus for each ν of the previous section we consider the equation.

$$(v-\nu)\ \tilde{g}_\nu(v) = -\int_{-\infty}^{\infty} \eta(v')\ \tilde{g}_\nu(v')dv' \quad . \tag{26}$$

To exhibit the solutions we parallel the decomposition of ν into classes.

CLASS 1a

ν real, $\eta(\nu) \neq 0$. We normalize $\tilde{g}_\nu$ so that

$$\int_{-\infty}^{\infty} \eta(v')\ \tilde{g}_\nu(v')dv' = 1 \quad . \tag{27}$$

Then (26) becomes

$$(v-\nu)\ \tilde{g}_\nu(v) = -1 \quad , \tag{28}$$

with the solution

$$\tilde{g}_\nu(v) = -P\frac{1}{v-\nu} + \tilde{\lambda}(\nu)\ \delta(v-\nu) \quad , \tag{29}$$

from which we obtain, using the normalization condition (27),

$$\tilde{\lambda}(\nu)\ \eta(\nu) = 1 + P\int_{-\infty}^{\infty} \frac{\eta(v)dv}{v-\nu} = \lambda(\nu) \quad .$$

$$\therefore \quad \tilde{\lambda}(\nu) = \frac{\lambda(\nu)}{\eta(\nu)} \quad . \tag{30}$$

CLASS 1b

This class is characterized by: ν real, $\eta(\nu) = 0$, $\lambda(\nu) \neq 0$. For solutions of this class we can take

$$\tilde{g}_{\nu}(v) = \delta(v-\nu) \quad , \tag{31}$$

since inserting in the right-hand side of (26) gives

$$\int_{-\infty}^{\infty} \tilde{g}_{\nu}(v)\ \eta(v)dv = \eta(\nu) = 0 \quad .$$

[Hence, $\tilde{g}_{\nu}(v)$ given by (31) does indeed satisfy (26).]

CLASS 1c

Here $\eta(\nu_i) = 0 = \lambda(\nu_i)$ with ν_i real. Normalizing according to (27) gives

$$\tilde{g}_{\nu_i}(v) \equiv \tilde{g}_i(v) = -\frac{1}{v-\nu_i} \quad . \tag{32}$$

Inserting this expression in the normalization condition yields an identity in virtue of (23) which defines ν_i.

CLASS 2

Here ν is complex. Again we normalize according to (27). Then

$$\tilde{g}_i(v) = -\frac{1}{v-\nu_i} \quad . \tag{33}$$

Again the normalization condition becomes an identity.

V. ORTHOGONALITY PROPERTIES

The eigenfunctions of the adjoint equation which have just been constructed become useful because of the following theorem.

Theorem:

The functions $\tilde{g}_{\nu'}(v)$, $g_\nu(v)$ are orthogonal for $\nu \neq \nu'$.

Proof:

We rewrite (14) and (26) in the forms

$$\nu\, g_\nu(v) \;=\; v\, g_\nu(v) + \eta(v) \int_{-\infty}^{\infty} g_\nu(v')dv' \quad , \tag{14'}$$

and

$$\nu'\, \tilde{g}_{\nu'}(v) \;=\; v\, \tilde{g}_{\nu'}(v) + \int_{-\infty}^{\infty} \eta(v')\, \tilde{g}_{\nu'}(v')dv' \quad . \tag{26'}$$

Multiplying (14') by $\tilde{g}_{\nu'}(v)$, (26') by $g_\nu(v)$, subtracting one from the other, and integrating over all v gives

$$(\nu-\nu') \int_{-\infty}^{\infty} \tilde{g}_{\nu'}(v)\, g_\nu(v)dv \;=\; 0 \quad , \tag{34}$$

which proves the theorem.

To apply this orthogonality theorem to an eigenfunction expansion, it is necessary to know the value of the orthogonality integral for $\nu = \nu'$. Since we have the explicit form of the functions, these may be calculated directly.

There are three cases of interest.

A. DISCRETE EIGENVALUES

We have

$$\int_{-\infty}^{\infty} \tilde{g}_i(v)\ g_j(v) \ = \ \delta_{ij}\, C_i \quad , \tag{35}$$

with

$$C_i \ = \ \int_{-\infty}^{\infty} \frac{\eta(v)dv}{(v-\nu_i)^2} \quad .$$

It may be readily verified that the assumption that ν_i is a simple zero of the characteristic equation

$$1 + \int_{-\infty}^{\infty} \frac{\eta(v)dv}{v-\nu_i} \ = \ 0$$

implies that $C_i \neq 0$.

B. ν IN CLASS 1b

$\eta(\nu) = 0$, $\lambda(\nu) \neq 0$, ν real. We find on substituting from (18) and (31) that

$$\int_{-\infty}^{\infty} \tilde{g}_{\nu'}(v)\ g_\nu(v)dv \ = \ C_\nu\ \delta(\nu-\nu') \tag{36}$$

with

$$C_\nu \ = \ \lambda(\nu) \ \neq \ 0 \quad . \tag{37}$$

C. ν IN CLASS 1a

A little care is needed in performing the integrals since we are integrating the product of two quite singular "functions" - namely, the two principal value functions. However, an application of the Poincaré-Bertrand transformation formula (3) gives the result of (36) where now

$$C_\nu = \frac{\lambda^2(\nu) + \pi^2\eta^2(\nu)}{\eta(\nu)} \neq 0 \quad . \tag{38}$$

VI. COMPLETENESS THEOREM

To use the normal modes of Section III for expansion purposes, it is, of course, important to know that we have found "enough" functions. That this is so is shown by the completeness theorem which follows. The proof is but a slight modification of one given in the special case of a Maxwell distribution by Van Kampen (2).

Theorem:

The functions $g_\nu(v)$, $g_i(v)$ constructed in Section III are complete for functions[2] defined on $-\infty < v < \infty$.

Proof:

We want to show that an "arbitrary"[2] function $g(v)$ can be written in the form

$$g(v) = \sum_i a_i g_i(v) + \int A(\nu)\, g_\nu(v) d\nu \quad . \tag{39}$$

The orthogonality results of the previous section show that if it is possible to write $g(v)$ in the form of (39), then

$$a_i = \frac{1}{C_i} \int \tilde{g}_i(v)\, g(v) dv \quad . \tag{40}$$

Then to prove (39), it is sufficient to show that it is always possible to solve the equation

$$g'(v) = \int A(\nu)\, g_\nu(v) d\nu \tag{41}$$

for $A(\nu)$ when

$$g'(v) = g(v) - \sum_i a_i g_i(v) \tag{42}$$

with a_i given by (40).

Inserting the expressions for $g_\nu(v)$ from (16) and (18), we see that (41), which we are to solve, is

$$g'(v) = \lambda(v)\, A(v) + \eta(v)\, P\int \frac{A(\nu)d\nu}{\nu - v} \tag{43}$$

where

$$\lambda(v) = 1 + P\int_{-\infty}^{\infty} \frac{\eta(v')dv'}{v' - v} \quad . \tag{44}$$

Let us now define three functions of a complex variable z by the equations

$$N(z) = \frac{1}{2\pi i}\int \frac{A(\nu)d\nu}{\nu - z} \quad , \tag{45}$$

$$Q(z) = \frac{1}{2\pi i}\int \frac{\eta(v')dv'}{v' - z} \quad , \tag{46}$$

$$M(z) = \frac{1}{2\pi i}\int \frac{g'(v')dv'}{v' - z} \quad . \tag{47}$$

Here we have assumed that the unknown function $A(\nu)$ exists and is well-behaved. Then it is readily seen that these three functions are analytic in the complex plane with branch cuts along the real axis. Moreover, they vanish at infinity at least as fast as $1/z$. (This, of course, assumes that $\int A(\nu)d\nu$, $\int \eta(v')dv'$, $\int g'(v')dv'$ all exist.)

Let $N^{\pm}(v)$, $Q^{\pm}(v)$, $M^{\pm}(v)$ denote the limits of these functions as z approaches the real axis from above and below, respectively. Then we have

$$N^+(v) - N^-(v) = A(v) \quad , \tag{48}$$

$$\pi i \, [N^+(v) + N^-(v)] = P \int \frac{A(\nu) d\nu}{\nu - v} \quad , \tag{49}$$

$$Q^+(v) - Q^-(v) = \eta(v) \quad , \tag{50}$$

$$\pi i \, [Q^+(v) + Q^-(v)] = P \int \frac{\eta(\nu) d\nu}{\nu - v} \quad , \tag{51}$$

and hence

$$\lambda(v) = 1 + \pi i \, (Q^+ + Q^-) \quad . \tag{52}$$

Finally we have

$$M^+(v) - M^-(v) = g'(v) \quad . \tag{53}$$

Inserting these expressions into the integral equation (43) gives

$$\{N^+(v) \, [1 + 2\pi i \, Q^+(v)] - M^+(v)\} - \{N^-(v) \, [1 + 2\pi i \, Q^-(v)] - M^-(v)\} = 0 \; . \tag{54}$$

Hence, to summarize, if we can find a function $N(z)$ which is analytic in the complex plane with cuts along the real axis, which vanishes at infinity and whose jump along the real axis satisfies (54), we can find $A(v)$ using (48).

Let us suppose such an $N(z)$ exists. Then the function

$$J(z) = N(z) \, [1 + 2\pi i \, Q(z)] - M(z) \tag{55}$$

has the properties:

(a) It is analytic in the cut plane.

(b) It has zero discontinuities along the cuts.

(c) It vanishes at infinity.

From properties (a) and (b), we see $J(z)$ is analytic everywhere.

Since it is zero at infinity, it is then zero everywhere (Liouville's Theorem).

Hence

$$N(z) = \frac{M(z)}{1 + 2\pi i \; Q(z)} \quad . \tag{56}$$

Thus, if a well-behaved solution $A(\nu)$ of our equation exists, the corresponding $N(z)$ is given by (56). Conversely, if $N(z)$ given by (56) is analytic in the cut plane and vanishes at infinity, then the solution $A(\nu)$ is given by (48). Therefore the question of the existence of a solution $A(\nu)$ comes down to whether $N(z)$ of (56) has the requisite properties. Since $M(z)$, $Q(z)$ have the properties of analyticity and vanishing at infinity, the only singularities of $N(z)$ in the cut plane that can occur are poles due to the zeros of the denominator. Where do these occur? From the definition of (46), we see that

$$1 + 2\pi i \; Q(z) = 1 + \int_{-\infty}^{\infty} \frac{\eta(v')dv'}{v'-z} \quad . \tag{57}$$

The zeros of this quantity are just our ν_i. Hence a solution $A(\nu)$ exists provided the numerator in (56) vanishes at these points ν_i. Thus sufficient conditions for a solution are:

$$M(\nu_i) = 0 \quad ,$$

i.e.,

$$\int \frac{g'(v')dv'}{v'-\nu_i} = 0 \qquad i = 1,2,\ldots n \quad . \tag{58}$$

Inserting the expression for $g'(v')$ from (42), we find that the conditions are

$$\int \frac{g(v')dv'}{v' - \nu_i} = \sum_j a_j \int \frac{g_j(v)dv}{v - \nu_i} \quad . \tag{59}$$

But

$$\frac{1}{v - \nu_i} = - \tilde{g}_i \quad ,$$

and thus, using the orthogonality properties of the discrete solutions, we have

$$\int \frac{g(v')dv'}{v'-\nu_i} = \sum_j a_j \int -\tilde{g}_i g_j dv = - a_i C_i \ , \tag{60}$$

or

$$a_i = \frac{1}{C_i} \int \tilde{g}_i(v) \ g(v_i)dv$$

$$= \frac{\int \tilde{g}_i(v) \ g(v)dv}{\int \tilde{g}_i(v) \ g_i(v)dv} \quad . \tag{61}$$

However, from (40) we see that these conditions are indeed satisfied. (It is fortunate that we were also able to show that $C_i \neq 0$.)

There is also a useful corollary of this theorem. It can be stated in the following form:

<u>Corollary</u>:

The functions $g_i(v)$ $(i = 1,2,\cdots n)$ and $g_\nu(v)$ $(\nu \in \Sigma)$ are complete for functions $g(v)$ defined on Σ.

<u>Proof</u>:

A straightforward proof as above is apparent. A slightly simpler proof can, however, be constructed. Let $g(v)$ be equal to the given function in Σ and zero in $\bar{\Sigma}$. (Here $\bar{\Sigma}$ is the set of real points not in

Σ and not a ν_i.) Then the expansion coefficients $A(\nu)$ for $\nu \in \bar{\Sigma}$ are:

$$A(\nu) = \frac{1}{C_\nu} \int \tilde{g}_\nu(v)\; g(v)dv$$

$$= \frac{1}{C_\nu} \int \delta(v-\nu)\; g(v)dv = \frac{g(\nu)}{C_\nu} \equiv 0 \qquad (\nu \in \bar{\Sigma}) \; . \tag{62}$$

VII. SOLUTION OF THE INITIAL VALUE PROBLEM À LA VAN KAMPEN

The problem of interest is to find the solution of (11) with a prescribed initial value $g(v,o)$. Let us use the completeness result and expand in terms of eigenfunctions.

$$g(v,o) = \sum_i a_i g_i(v) + \int A(\nu)\; g_\nu(v)d\nu \; . \tag{63}$$

Using the orthogonality property with respect to the adjoint functions, we have

$$a_i = \frac{1}{C_i} \int \tilde{g}_i(v)\; g(v,o)dv = \frac{-\int \frac{g(v,o)dv}{v-\nu_i}}{\int \frac{\eta(v)dv}{(v-\nu_i)^2}} , \tag{64}$$

and

$$A(\nu) = \frac{1}{C_\nu} \int \tilde{g}_\nu(v)\; g(v,o)dv$$

$$= \left\{ \frac{1}{\lambda^2(\nu) + \pi^2\eta^2(\nu)} \left[\lambda(\nu)g(\nu,o) - \eta(\nu)\; P \int \frac{\eta(v)g(v,o)dv}{v-\nu} \right] \right\} . \tag{65}$$

(It may be noted that (65) is correct for $\nu \in \Sigma$ or $\bar{\Sigma}$.) The solution $g(v,t)$ is then simply obtained from (63) by inserting the appropriate time dependence of the eigenfunctions. Thus

$$g(v,t) = \sum_i a_i \, e^{-ik\nu_i t} g_i(v) + \int A(\nu) \, e^{-ik\nu t} g_\nu(v) d\nu \quad . \quad (66)$$

Furthermore, since

$$\int g_\nu(v)dv = 1 \quad \text{and} \quad \phi(t) = \frac{4\pi e}{k^2} \int g(v,t)dv \ ,$$

we have

$$\phi(t) = \frac{4\pi e}{k^2} \left\{ \sum_i a_i \, e^{-ik\nu_i t} + \int A(\nu) \, e^{-ik\nu t} \, d\nu \right\} \quad . \qquad (67)$$

VIII. THE INITIAL VALUE PROBLEM À LA LANDAU

Let us treat the problem of the previous section by the Laplace Transform method as suggested by Landau (1). It is convenient, however, to use two-sided transforms. Thus we want to solve

$$\frac{\partial g}{\partial t} + ikvg(v,t) = -ik\eta(v) \int_{-\infty}^{\infty} g(v',t)dv' \qquad (11)$$

subject to the condition that $g(v,o)$ is prescribed. Let us decompose $g(v,t)$ into two functions $g_\pm(v,t)$ such that

$$g_+(v,t) = g(v,t) \quad t > 0 \ , \qquad (68)$$

$$\lim_{t \to 0^+} g_+(v,t) = g(v,o) \ ,$$

$$g_+(v,t) = 0 \quad t < 0 \ ,$$

and

$$g_-(v,t) = 0 \quad t > 0 \ , \qquad (69)$$

$$\lim_{t \to 0^-} g_-(v,t) = g(v,o) \ ,$$

$$g_-(v,t) = g(v,t) \quad t < 0 \ .$$

Clearly $g_+(v,t)$ satisfies (11) except in the vicinity of $t = 0$.

Since

$$\int_{0-\epsilon}^{0+\epsilon} \frac{\partial g_+}{\partial t} = g_+(v,+\epsilon) - g(v,-\epsilon) = g(v,o) \quad ,$$

we have the equation

$$\frac{\partial g_+}{\partial t} + ikvg_+(v,t) = g(v,o)\delta(t) - ik\eta(v) \int_{-\infty}^{\infty} g_+(v',t)dv' \quad . \qquad (70)$$

Define the Laplace Transform $h_+(P)$ of $g_+(t)$ by

$$h_+(v,P) = \int_{-\infty}^{\infty} g_+(v,t)\, e^{-Pt}\, dP \quad . \qquad (71)$$

It is assumed, and must later be verified, that Re P can be and is chosen so that the integral exists and that all singularities of the function of a complex variable $h_+(p)$ so defined lie to the left of the value used in (71).

Multiplying (70) by e^{-Pt} and integrating gives

$$(P + ikv)\, h_+(v,P) = g(v,o) - ik\eta(v) \int_{-\infty}^{\infty} h_+(v',P)dv' \quad . \qquad (72)$$

Solving for $h_+(v,P)$ gives

$$h_+(v,P) = \frac{g(v,o)}{P+ikv} - \frac{ik\eta(v)}{P+ikv} \int_{-\infty}^{\infty} h_+(v',P)dv' \quad . \qquad (73)$$

The integral which occurs here may be found by integrating over v and solving the resulting equation. Thus:

$$\int h_+(v',P)dv' = \frac{\displaystyle\int \frac{g(v,o)dv}{P+ikv}}{\left\{1 + ik\displaystyle\int \frac{\eta(v')dv'}{P+ikv'}\right\}} \quad . \qquad (74)$$

Inserting the result into (73) gives $h_+(v,P)$. Finally, for $g_+(v,t)$, we have the inversion formula

$$g_+(v,t) = \frac{1}{2\pi i} \int_{C_+} h_+(v,P)\, e^{Pt}\, dP \quad , \tag{75}$$

where C_+ is a line parallel to the imaginary axis and to the right of all singularities of $h_+(v,P)$.

Similarly we can show that

$$\int h_-(v',P)dv' = \frac{-\int \frac{g(v,o)dv}{P+ikv}}{\left\{1 + ik\int \frac{\eta(v')dv'}{P+ikv'}\right\}} \quad , \tag{76}$$

and

$$g_-(v,t) = \frac{1}{2\pi i} \int_{C_-} h_-(v,P)\, e^{Pt}\, dP \quad , \tag{77}$$

where the line C_- is parallel to the imaginary axis and to the left of all singularities of h_-.

For simplicity we restrict the comparison with the calculations of the preceding section to the expressions for the scalar potential. Using (74) through (77), we here have found that

$$\phi(t) = \frac{4\pi e}{k^2}\left[\frac{1}{2\pi i}\int_{C_+} \frac{e^{Pt}\left(\int \frac{g(v,o)dv}{P+ikv}\right) dP}{\left\{1 + ik\int \frac{\eta(v')dv'}{P+ikv'}\right\}} - \frac{1}{2\pi i}\int_{C_-} \frac{e^{Pt}\left(\int \frac{g(v,o)dv}{P+ikv}\right) dP}{\left\{1 + ik\int \frac{\eta(v')dv'}{P+ikv'}\right\}}\right] . \tag{78}$$

Now let us deform the contours $C_\pm$ to contours $C_\pm'$ which run parallel to and just to the right and left of the imaginary axis, respectively. Clearly the only singularities which are encountered arise from zeros of the denominator. These occur where

$$1 + \int \frac{\eta(v')dv'}{v' + P/ik} = 0 \quad . \tag{79}$$

Alternately, we can say that the integrands in (78) have simple poles at points $P = -ik\nu_i$ where

$$1 + \int \frac{\eta(v')dv'}{v' - \nu_i} = 0 \qquad (i = m, m+1, \cdots n) \quad . \tag{80}$$

Applying the Cauchy Residue Theorem, we readily find that the

$$\text{Contribution to } \phi(t) \text{ from Complex Poles} = \frac{4\pi e}{k^2} \sum_{i=m+1}^{n} e^{-ik\nu_i t} a_i \ , \tag{81}$$

with a_i given by (64).

We still must find the result of the integrations over $C_\pm'$. First there are contributions from the poles at ν_i with $i = 1, 2, \cdots m$. Just as above, these give

$$\text{Contribution of Imaginary Poles} = \frac{4\pi e}{k^2} \sum_{i=1}^{m} e^{-ik\nu_i t} a_i \ . \tag{82}$$

Finally we must consider the integrals over $C_\pm'$ where there are no poles. On $C_\pm'$ we have

$$P = -i\nu k \pm \epsilon \quad . \tag{83}$$

Hence on $C_\pm'$ we find

$$1 + ik \int \frac{\eta(v')dv'}{P+ikv'} \rightarrow \lambda(\nu) \pm i\pi\,\eta(\nu) \quad , \tag{84}$$

and

$$\int \frac{g(v,o)dv}{P+ikv} \rightarrow \frac{1}{ik} \int \left[P\,\frac{1}{v-\nu} \pm \pi i\,\delta(v-\nu) \right] g(v,o)dv \quad . \tag{85}$$

Hence we obtain that the

$$\text{Contribution of Imaginary Axis to } \phi(t) = \frac{4\pi e}{k^2} \int A(\nu)\, e^{-ik\nu t}\, d\nu \quad , \tag{86}$$

with $A(\nu)$ as given by (65).

Adding the contributions of (81), (82), and (86), we find that $\phi(t)$ is precisely given by (67) of the preceding section.

FOOTNOTES

[1]Supported in part by the Office of Naval Research, U. S. Navy Department.

[2]For full mathematical rigor we should state conditions on the function to be expanded. A sufficient condition is that it satisfy a Hölder condition. However, on examining the proof given, it seems to hold for considerably weaker conditions on the function.

REFERENCES

[1] L. Landau, Journal of Physics, 10, 25 (1946).

[2] N. G. Van Kampen, Physica, 21, 949 (1955).

[3] Cf., for example, N. I. Muskhelishvili, "Singular Integral Equations," p. 57, Noordhoff, Groningen, Holland (1953).