Notes on Electromagnetic Scattering from Rotationally Symmetric Bodies with an Impedance Boundary Condition

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FOREWORD

This report was prepared by the Radiation Laboratory of the Department of Electrical Engineering of The University of Michigan under the direction of Dr. Raymond F. Goodrich, Principal Investigator and Burton A. Harrison, Contract Manager. The work was performed under Contract AF 04(694)-834, "Investigation of Re-entry Vehicle Surface Fields (SURF)". The work was administered under the direction of the Air Force Ballistic Systems Division, Norton Air Force Base, California 92409, by Major A. Aharonian BSYDF and was monitored by Mr. Henry J. Katzman of the Aerospace Corporation.

The publication of this report does not constitute Air Force Approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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ABSTRACT

An analysis is presented of the problem of the numerical computation of electromagnetic scattering from rotationally symmetric boundaries satisfying an impedance boundary condition. This work extends the work of P. Schweitzer for perfectly conducting boundaries and includes his results as a special case.

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INTRODUCTION

This report investigates the scattering of a monochromatic electromagnetic plane wave from a rotationally symmetric object satisfying an impedance boundary condition. It is actually an extension of the work done by Paul Schweitzer at Lincoln Laboratories* and which includes his analysis as a particular case.

Many sections of this report have been taken mutatis mutandis from P. Schweitzer's report. It seemed unnecessary to alter his presentation since one could hardly hope to surpass him in clarity.

I am indebted to my colleagues Dr. Olov Einarsson, Dr. Vaughan Weston, and Dr. Raymond Goodrich of the Radiation Laboratory for their assistance in formulating the boundary value problem and for many helpful discussions on the physics of the impedance boundary condition.

*P. Schweitzer, "Electromagnetic Scattering from Rotationally Symmetric Perfect Conductors," Lincoln Laboratory, Project Report PA-88 (BMRS), February 1965.

Section I: The Geometry

We consider a body whose axis of symmetry coincides with the z-axis. The body extends from $z = 0$ to $z = L$. The surface is given by $\sqrt{x^2 + y^2} = f(z)$. More generally, one could parameterize by arc length instead of z if the surface were re-entry, i.e., if $f(z)$ was a multiple-valued function of z .

A point on the surface is given by two coordinates z and ϕ, where $0 \leqslant z \leqslant L$, and $0 \leqslant \phi \leqslant 2\pi$. In the usual cylindrical coordinates, a point $\bar{r}$ on the surface is given by

$$\bar{r} = z\,\hat{z} + \rho\,\hat{\rho} = f(z)\cos\phi\,\hat{x} + f(z)\sin\phi\,\hat{y} + z\,\hat{z} \tag{1.1}$$

where $\hat{z} = (0,\ 0,\ 1)$ = unit vector in z direction

$$\rho = \sqrt{x^2 + y^2} = f(z) \tag{1.2}$$

$$\hat{\rho} = (\cos\phi,\ \sin\phi,\ 0) = \text{unit vector in x, y plane} \tag{1.3}$$

we can construct, at each point on the surface, the triad $\hat{n}$, $\hat{a}$, $\hat{t}$ of orthogonal unit vectors

$$\hat{n} = (\cos\phi\,\alpha(z),\ \sin\phi\,\alpha(z),\ -f'(z)\,\alpha(z))$$

$$= \text{unit outward normal vector to surface} \tag{1.4}$$

$$\hat{a} = \hat{z} \times \hat{\rho} = (-\sin\phi,\ \cos\phi,\ 0) = \text{unit transverse (azimuthal) vector in direction of increasing } \phi \tag{1.5}$$

$$\hat{t} = \hat{n} \times \hat{a} = (f'(z)\,\alpha(z)\cos\phi,\ f'(z)\,\alpha(z)\sin\phi,\ \alpha(z))$$

$$= \text{unit longitudinal vector, in direction of increasing arc length.} \tag{1.6}$$

where

$$\alpha(z) = \frac{1}{\sqrt{1 + \left[f'(z)\right]^2}} = \text{cosine of angle between t and z} \tag{1.7}$$

For these to exist, f(z) must be a piecewise - differentiable function of z .

The parameterization by z can be replaced by parameterization by the normalized arc length

$$t = t(z) = \frac{\int_0^z dw \sqrt{1 + \left[f'(w)\right]^2}}{C} \tag{1.8}$$

where

$$C = \int_0^L dw \sqrt{1 + \left[f'(w)\right]^2}$$

$$= \text{half of transverse circumference.} \tag{1.9}$$

As z runs from 0 to L, t runs from 0 to 1 . The surface element is

$$dz = d z\, d\phi\; f(z) \sqrt{1 + \left[f'(z)\right]^2} = C\; dt\; d\phi\; f(z) \tag{1.10}$$

The $\hat{n}$, $\hat{a}$, $\hat{t}$ triad is related to $\hat{x}$, $\hat{y}$, $\hat{z}$ by

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\phi\, \alpha(z) & -\sin\phi & f'(z)\, \alpha(z) \cos\phi \\ \sin\phi\, \alpha(z) & \cos\phi & f'(z)\, \alpha(z) \sin\phi \\ -f'(z)\, \alpha(z) & 0 & \alpha(z) \end{bmatrix} \begin{bmatrix} \hat{n} \\ \hat{a} \\ \hat{t} \end{bmatrix} \tag{1.11}$$

$$\begin{bmatrix} \hat{n} \\ \hat{a} \\ \hat{t} \end{bmatrix} = \begin{bmatrix} \cos\phi\, \alpha(z) & \sin\phi\, \alpha(z) & -f'(z)\, \alpha(z) \\ -\sin\phi & \cos\phi & 0 \\ f'(z)\, \alpha(z) \cos\phi & f'(z)\, \alpha(z) \sin\phi & \alpha(z) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{1.12}$$

These two matrices are transposes of each other because the linear transformation is orthogonal.

Section II: The Incident Electric Field and Polarization Vectors

We use the convention real $\left[\ldots e^{-i\omega t}\right]$.

The incident radiation will be a plane wave given by

$$\bar{E}^{(0)}(r) = \bar{\epsilon}\, e^{i\bar{k}_0\cdot\bar{r}} \tag{2.1}$$

$\bar{\epsilon}$ points in the direction of polarization, and $\bar{k}_0$ is the propagation vector.

$$|\bar{k}_0| = k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda} \tag{2.2}$$

$$\bar{k}_0\cdot\bar{\epsilon} = 0 \tag{2.3}$$

With no loss of generality, $\bar{k}_0$ lies in the x, z plane and makes an angle θ_0 with the z axis

$$\bar{k}_0 = (k_0 \sin\theta_0,\ 0,\ k_0\cos\theta_0)\ . \tag{2.4}$$

For objects of rotational symmetry, there are two very special directions of incident polarization, ϵ_I and ϵ_{II}, such that the directly-backscattered electric field has the same polarization as the incident electric field (i.e., no depolarization).

These are given by

$$\hat{\epsilon}_I \sim \bar{k}_0 \text{ x unit vector in direction of axis of symmetry} \tag{2.5}$$

$$\hat{\epsilon}_{II} \sim \bar{k}_0 \times \hat{\epsilon}_I \tag{2.6}$$

$\hat{\epsilon}_{II}$ lies in the plane of incidence, while $\hat{\epsilon}_I$ is perpendicular to the plane of incidence. More concretely

$$\hat{\epsilon}_I = \hat{y} = (0, 1, 0) \tag{2.7}$$

$$\hat{\epsilon}_{II} = (-\cos\theta_0, 0, \sin\theta_0) \tag{2.8}$$

The general case can therefore be written

$$\bar{\epsilon} = a_I \hat{\epsilon}_I + a_{II} \hat{\epsilon}_{II} \tag{2.9}$$

Finally, the incident polarization can be decomposed along triad $\hat{n}$, $\hat{a}$, $\hat{t}$ at any point (z, ϕ) on the surface of the object

$$\begin{aligned}
\bar{\epsilon} &= a_I \hat{\epsilon}_I + a_{II} \hat{\epsilon}_{II} \\
&= a_I \Big\{ (\hat{\epsilon}_I \cdot \hat{x}) \left[(\hat{x}\cdot\hat{n})\, \hat{n} + (\hat{x}\cdot\hat{a})\, \hat{a} + (\hat{x}\cdot\hat{t})\, \hat{t} \right] \\
&\quad + (\hat{\epsilon}_I \cdot \hat{y}) \left[(\hat{y}\cdot\hat{n})\, \hat{n} + (\hat{y}\cdot\hat{a})\, \hat{a} + (\hat{y}\cdot\hat{t})\, \hat{t} \right] \\
&\quad + (\hat{\epsilon}_I \cdot \hat{z}) \left[(\hat{z}\cdot\hat{n})\, \hat{n} + (\hat{z}\cdot\hat{a})\, \hat{a} + (\hat{z}\cdot\hat{t})\, \hat{t} \right] \\
&\quad + \text{similar terms involving } a_{II} \text{ and } \epsilon_{II} .
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\bar{\epsilon} &= a_I \begin{bmatrix} \hat{n} & \hat{a} & \hat{t} \end{bmatrix} \begin{bmatrix} \sin\phi\, \alpha(z) \\ \cos\phi \\ \sin\phi\, \alpha(z)\, f'(z) \end{bmatrix} \\
&\quad + a_{II} \begin{bmatrix} \hat{n} & \hat{a} & \hat{t} \end{bmatrix} \begin{bmatrix} -\cos\theta_0 \cos\phi\, \alpha(z) - \sin\theta_0\, \alpha(z)\, f'(z) \\ \cos\theta_0 \sin\phi \\ -\cos\theta_0 \cos\phi\, \alpha(z)\, f'(z) + \sin\theta_0\, \alpha(z) \end{bmatrix}
\end{aligned} \tag{2.11}$$

Section III: Decomposition of the Incident Electric Field into Cylindrical Modes

In this section, we give expressions for the decomposition of the incident electric field, evaluated on the surface of the conductor, into cylindrical modes.

The incident electric field is

$$\bar{E}^{(0)}(\bar{r}) = \bar{\epsilon}\, e^{i\bar{k}_0 \cdot \bar{r}}$$

On the surface,

$$\bar{r} = (f(z)\cos\phi\ ,\ f(z)\sin\phi,\ z) \tag{3.1}$$

$$\bar{k}_0 = (k_0 \sin\theta_0,\ 0,\ k_0\cos\theta_0)\ . \tag{3.2}$$

Note propagation vector is, with no loss of generality, in the plane $y = 0$, i.e. , $\phi_0 = 0$.

Therefore

$$\bar{k}_0 \cdot \bar{r} = k_0 f(z)\sin\theta_0 \cos\phi + k_0 z\cos\theta_0 \tag{3.3}$$

Using the expansion

$$e^{ix\cos\phi} = \sum_{m=0}^{\infty} \epsilon_m\, i^m J_m(x)\cos m\phi \tag{3.4}$$

where

$$\epsilon_m = 2 - \delta_{m0}\ ,$$

We can write

$$\overline{E}^{(0)}(\overline{r}) = \overline{\epsilon}\, e^{i k_0 z \cos\theta_0} \cdot$$

$$\sum_{m=0}^{\infty} \epsilon_m \, i^m \, J_m (k_0 \, f(z) \sin\theta_0) \cos m\phi \tag{3.5}$$

$\overline{r}$ on surface.

Inserting (2.11) into (3.5) and using the trigonometric identities

$$\cos m\phi \cos\phi = \frac{\cos(m+1)\phi + \cos(m-1)\phi}{2} \tag{3.6}$$

$$\cos m\phi \sin\phi = \frac{\sin(m+1)\phi - \sin(m-1)\phi}{2} \tag{3.7}$$

we obtain

$$\overline{E}^{(0)}(\overline{r}) = a_I \sum_{m=0}^{\infty} \left\{ E^{mIt}(z) \sin m\phi \, \hat{t} + E^{mI\phi}(z) \cos m\phi \, \hat{a} \right\}$$

$$+ a_{II} \sum_{m=0}^{\infty} \left\{ E^{mIIt}(z) \cos m\phi \, \hat{t} + E^{mII\phi}(z) \sin m\phi \, \hat{a} \right\}$$

$$+ (\overline{E}^{(0)}(\overline{r}) \cdot \hat{n}) \, \hat{n} \quad , \quad \overline{r} \text{ on surface} \tag{3.8}$$

$E^{m,\, I\, \text{or}\, II,\, t\, \text{or}\, \phi}(z)$ are given by somewhat complicated expressions involving exponentials and Bessel functions. They are given in page 18 of P. Schweitzer's report and it seems pointless to duplicate them here.

Equation (3.8) contains the desired decomposition of the incident electric field into different polarizations (a_I and a_{II}) and different modes (different m). Note how, by our special choice of $\hat{\epsilon}_I$ and $\hat{\epsilon}_{II}$, the various polarizations decouple so nicely. The I mode has an angular dependence ($\sin m\phi\, \hat{t}$, $\cos m\phi\, \hat{a}$) while the II mode has an angular dependence ($\cos m\phi\, \hat{t}$, $\sin m\phi\, \hat{a}$) .

Section IV: The Boundary Value Problem

We consider the following physical situation: We are given an electromagnetic field $\overline{E}^{(0)}$, $\overline{H}^{(0)}$ incident on a body of revolution. The total field satisfies a Leontovich boundary condition

$$(\hat{n} \times \overline{E}) \times \hat{n} = \eta Z \hat{n} \times \overline{H} \tag{4.1}$$

on the surface of the body. η and Z are constants characterizing the body and the medium surrounding the body, respectively.

$$\eta = \frac{1}{\sqrt{\frac{\mu_0}{\mu}\left(\frac{\epsilon}{\epsilon_0} - i\frac{\delta}{\omega\epsilon_0}\right)}} \tag{4.2}$$

$$Z = \sqrt{\frac{\mu_0}{\epsilon_0}} \tag{4.3}$$

The scattered electric field is given in terms of the surface fields on the body by the integral (see J.A. Stratton, Electromagnetic Theory, page 466).

$$\overline{E}^S = -\int_S \left[i\omega\mu\, (\hat{n} \times \overline{H})\, G + (\hat{n} \times \overline{E}) \times \nabla G + (\hat{n}\cdot E)\, \nabla G\right] dS . \tag{4.4}$$

We have assumed that the field vectors contain the time only as a factor $\exp(-i\omega t)$.

$$G = G(\overline{r}, \overline{r}') = \frac{\exp(i k_0 |\overline{r} - \overline{r}'|)}{4\pi|\overline{r} - \overline{r}'|} \tag{4.5}$$

The function $\overline{E}^s$ defined above is discontinuous across S. The nature of the discontinuity can be investigated by the methods used in potential theory, since G becomes equal to $1/4\pi|\overline{r}-\overline{r}'|$ for small values of $|\overline{r}-\overline{r}'|$. Some of the most important results are

1) If $\overline{A}$ is a vector field tangent to S, and $\overline{r}_0$ is a point on S, then the integral

$$\overline{I}(\overline{r}) = \int_S \overline{A}(\overline{r}')\, G(\overline{r}, \overline{r}')\, dS'$$

is continuous for all $\overline{r}$.

2) $$\hat{n}(\overline{r}_0) \times \lim_{\overline{r}\to\overline{r}_0} \int_S \overline{A}(\overline{r}') \times \nabla' G(\overline{r}, \overline{r}')\, dS'$$

$$= \pm \frac{1}{2} A(\overline{r}_0) + \int_S \hat{n}(\overline{r}_0) \times \left[\overline{A}(\overline{r}') \times \nabla' G(\overline{r}_0, \overline{r}')\right] dS' .$$

In the left hand member, the approach $\overline{r} \longrightarrow \overline{r}_0$ is along the normal to S. In the right-hand member, the plus and minus signs correspond respectively, to an approach from the outside and from the inside of S. The integral in the right-hand member can be shown to be convergent.

3) The normal component of $\int_S \overline{A}(\overline{r}') \times \nabla' G(\overline{r}, \overline{r}')\, dS'$ is continuous on S.

The term

$$\overline{E}_3 = \int_S (\hat{n} \cdot \overline{E})\, \nabla G\, d\,S$$

suffers a discontinuity on transition through S equal to $\hat{n} \cdot \Delta\overline{E}_3$, where $\Delta\overline{E}_3$ is the difference of the values outside and inside. The third term in $\overline{E}^s$, therefore, does not affect the transition of the tangential component, but reduces the normal component of $\overline{E}$ to zero.

Since the total electric field $\overline{E}^{(0)} + \overline{E}^{s}$ is zero inside the scatterer, we obtain from (4.4)

$$0 = \hat{n} \times \overline{E}^{(0)} - \lim_{\overline{r} \to \overline{r}_0} \hat{n}(\overline{r}_0) \times \int_S \Big[i\omega\mu\, (\hat{n} \times \overline{H})\, G +$$

$$+ (\hat{n} \times \overline{E}) \times \nabla G + (\hat{n} \cdot \overline{E})\, \nabla G \Big] dS \, ,$$

where the approach is from the inside.

Application of 2) gives

$$0 = \hat{n} \times \overline{E}^{(0)} + \frac{1}{2}\, \hat{n} \times \overline{E} - \hat{n} \times \int_S \Big[i\omega\mu\, (\hat{n} \times \overline{H})\, G +$$

$$+ (\hat{n} \times \overline{E}) \times \nabla G + (\hat{n} \cdot \overline{E})\, \nabla G \Big] dS \qquad (4.6)$$

For a point just outside the surface of the scatterer

$\hat{n} \times \overline{E} = \hat{n} \times (\overline{E}^{(0)} + \overline{E}^{s})$. Hence

$$\hat{n} \times \overline{E} = \hat{n} \times \overline{E}^{(0)} - \lim_{\overline{r} \to \overline{r}_0} \hat{n}(\overline{r}_0) \times \int_S \Big[i\omega\mu\, (\hat{n} \times \overline{H})\, G +$$

$$+ (\hat{n} \times \overline{E}) \times \nabla G + (\hat{n} \cdot \overline{E})\, \nabla G \Big] dS \, ,$$

where the approach is from the outside.

Application of 2) gives

$$\frac{3}{2}\, \hat{n} \times \overline{E} = \hat{n} \times \overline{E}^{(0)} - \hat{n} \times \int_S \Big[i\omega\mu\, (\hat{n} \times \overline{H})\, G +$$

$$+ (\hat{n} \times \overline{E}) \times \nabla G + (\hat{n} \cdot \overline{E})\, \nabla G \Big] dS \qquad (4.7)$$

Adding Eqs. (4.6) and (4.7) gives

$$\hat{n} \times \overline{E} + 2\,\hat{n} \times \int_S \left[i\omega\mu\, (\hat{n} \times \overline{H})\, G + (\hat{n} \times \overline{E}) \times \nabla G + (\hat{n}\cdot\overline{E})\, \nabla G \right] dS = 2\,\hat{n} \times \overline{E}^{(0)} \tag{4.8}$$

Use of the definitions

$$\hat{n} \times \overline{E} = \overline{K}^{*}\,, \quad \hat{n} \times \overline{H} = -\overline{K}\,, \quad \sigma = -\epsilon\, \hat{n}\cdot\overline{E}$$

and the relations

$$\overline{K}^{*} = -\eta Z\, \hat{n} \times \overline{K}\,, \quad \sigma = \frac{1}{i\omega}\, \nabla\cdot\overline{K}$$

in Eq. (4.8) gives

$$-\eta Z\, \hat{n} \times \overline{K} + 2\,\hat{n} \times \int_S \Big[-i\omega\mu\, \overline{K}\, G - \eta Z\, (\hat{n} \times \overline{K}) \times \nabla G$$

$$- \frac{1}{i\omega\epsilon}\, (\nabla\cdot\overline{K})\, \nabla G \Big] dS = 2\,\hat{n} \times \overline{E}^{(0)}.$$

This can be written alternatively as

$$\Big[\frac{1}{2}\, \eta Z\, \overline{K} + \int_S \Big[+i\omega\mu\, \overline{K}\, G + \eta Z\, (\hat{n} \times \overline{K}) \times \nabla G +$$

$$+ \frac{1}{i\omega\epsilon}\, (\nabla\cdot\overline{K})\, \nabla G \Big] dS + \overline{E}^{(0)} \Big]_{tan} = 0 \qquad \overline{r} \text{ on } S\,. \tag{4.9}$$

This system of integral equations constitutes the mathematical formulation of our boundary value problem.

Section V: Reduction of the integral equations for numerical solution

Following Schweitzer we will use the method of Galerkin to solve the integral equations (4.9) .

We assume the (unknown) induced current is expanded in terms of the complete set $\{K_n\}$:

$$\overline{K}(\overline{r}) = \sum_{n=1}^{\infty} C_n\, \overline{K}_n(\overline{r})\,, \quad \overline{r} \text{ on } S\,. \tag{5.1}$$

If this expression is inserted in (4.9), we have

$$\left\{ \frac{1}{2} \eta Z \sum_{n=1}^{\infty} C_n \overline{K}_n (\overline{r}) \right.$$

$$+ \sum_{n=1}^{\infty} C_n \int_S \left[i\omega\mu \overline{K}_n (\overline{r}') G + \eta Z (\hat{n} \times K_n (\overline{r}')) \times \nabla G + \right.$$

$$\left. \left. + \frac{1}{i\omega\epsilon} (\nabla \cdot \overline{K}_n (\overline{r}')) \nabla G \right] dS' + \overline{E}^{(0)} \right\}_{tan} = 0$$

If both sides are dotted with $K_m (\overline{r})$ and integrated over the surface of the conductor, we obtain

$$\left\{ \frac{1}{2} \eta Z \sum_{n=1}^{\infty} C_n \int_S dS \, \overline{K}_n (\overline{r}) \cdot \overline{K}_m (\overline{r}) \right.$$

$$+ \sum_{n=1}^{\infty} C_n \int_S dS \, \overline{K}_m (\overline{r}) \cdot \int_S \left[i\omega\mu \overline{K}_n (\overline{r}') G + \eta Z (\hat{n} \times \overline{K}_n (\overline{r}')) \times \nabla G \right.$$

$$\left. \left. + \frac{1}{i\omega\epsilon} (\nabla \cdot \overline{K}_n (\overline{r}')) \nabla G \right] dS' + \int_S \overline{E}^{(0)} \cdot \overline{K}_m (\overline{r}) \, dS \right\} = 0 \, .$$

That is,

$$\sum_{n=1}^{\infty} C_n \left\{ \frac{1}{2} \eta Z \int_S dS \, \overline{K}_n (\overline{r}) \cdot \overline{K}_m (\overline{r}) \right.$$

$$+ \int_S dS \, \overline{K}_m (\overline{r}) \cdot \int_S dS' \left[i\omega\mu \overline{K}_n (\overline{r}') G + \eta Z (\hat{n} \times \overline{K}_n (\overline{r}')) \times \nabla G + \right.$$

$$\left. \left. + \frac{1}{i\omega\epsilon} (\nabla \cdot \overline{K}_n (\overline{r}')) \nabla G \right] \right\} = - \int_S \overline{E}^{(0)} \cdot \overline{K}_m (\overline{r}) \, dS \, . \qquad (5.2)$$

This is of the form

$$\sum_{n=1}^{\infty} C_n T_{mn} = - b_m \tag{5.3}$$

with

$$T_{mn} = \frac{1}{2} \eta Z \int_S dS\, \overline{K}_n(\overline{r}) \cdot \overline{K}_m(\overline{r})$$

$$+ \int_S dS\, \overline{K}_m(\overline{r}) \cdot \int_S dS' \Big[i\omega\mu\, \overline{K}_n(\overline{r}')\, G + \eta Z\, (\hat{n} \times \overline{K}_n(\overline{r})\,) \times \nabla G +$$

$$+ \frac{1}{i\omega\epsilon} (\nabla \cdot \overline{K}_n(\overline{r}')\,) \nabla G \Big] \tag{5.4}$$

$$b_m = \int_S \overline{E}^{(0)} \cdot \overline{K}_m(\overline{r})\, dS \tag{5.5}$$

We wish to transform the term involving the divergence of $\overline{K}_n$. We write

$$I = \int_S dS\, \overline{K}_m(\overline{r}) \cdot \int_S dS' (\nabla \cdot \overline{K}_n(\overline{r}')\,) \nabla G$$

$$= - \int_S dS\, \overline{K}_m(\overline{r}) \cdot \nabla \int_S dS' (\nabla \cdot \overline{K}_n(\overline{r}')\,) G$$

$$= - \int_S dS\, \overline{K}_m(\overline{r}) \cdot \nabla h(\overline{r}) \tag{5.6}$$

where the gradient outside the integral sign operates with respect to the unprimed coordinates. By use of a vector identity

$$I = -\int_S dS \left\{ \nabla \cdot \left[\overline{K}_m(\overline{r})\, h(\overline{r}) \right] - \left[\nabla \cdot K_m(\overline{r}) \right] h(\overline{r}) \right\} .$$

Invoking the result

$$\nabla \cdot \overline{V} = \hat{n} \cdot \nabla \times (\hat{n} \times \overline{V})$$

if

$$\hat{n} \cdot \overline{V} = 0 \quad ,$$

the first integral becomes

$$\int dS\, \nabla \cdot \left[\overline{K}_m(\overline{r})\, h(\overline{r}) \right] = \int dS\, \hat{n} \cdot \nabla \times (\hat{n} \times \overline{K}_m\, h) = \int dv\, \nabla \cdot (\nabla \times (\hat{n} \times \overline{K}_m\, h)) = 0 \quad .$$

Since the first contribution to I vanishes, we have left

$$I = \int_S dS \left[\nabla \cdot \overline{K}_m(\overline{r}) \right] h(\overline{r}) = \int_S dS \left[\nabla \cdot \overline{K}_m(\overline{r}) \right] \int_S dS' \left[\nabla \cdot \overline{K}_n(r') \right] G \quad .$$

Use of this result in Eq. (5.4) gives

$$T_{mn} = \frac{1}{2} \eta Z \int_S dS\, \overline{K}_n(\overline{r}) \cdot \overline{K}_m(\overline{r}) + i\omega\mu \int_S dS\, \overline{K}_m(\overline{r}) \cdot \int_S dS'\, \overline{K}_n(\overline{r}')\, G$$

$$+ \eta Z \int_S dS\, \overline{K}_m(\overline{r}) \cdot \int_S dS'\, (\hat{n} \times \overline{K}_n(\overline{r}')) \times \nabla G$$

$$+ \frac{1}{i\omega\epsilon} \int_S dS\, (\nabla \cdot K_m(\overline{r})) \int_S dS'\, (\nabla \cdot K_n(\overline{r}'))\, G \qquad (5.7)$$

Once T_{mn} and b_m are available, Eq. (5.3) constitutes an infinite system of simultaneous linear equations for C_1, C_2, ...

In practice one approximates $\overline{K}$ by a finite sum

$$\overline{K}(\overline{r}) \approx \sum_{n=1}^{N} C_n \overline{K}_n(\overline{r}) \tag{5.8}$$

and solves the N x N set of truncated equations

$$\sum_{n=1}^{N} T_{mn} C_n = - b_m \qquad m = 1, 2, \ldots \quad N . \tag{5.9}$$

for approximate values of the C's .

To show the reasoning underlying the Galerkin procedure, we note that the error $\overline{e}(\overline{r})$ made in the right hand side by replacing the exact current in (4.9) , rewritten as

$$\Gamma K(\overline{r}) = - \overline{E}^{(0)}(\overline{r}) \quad \text{tangential components} \tag{5.10}$$

by the approximation

$$\overline{K}(\overline{r}) \approx \sum_{i=1}^{N} C_i K_i(\overline{r}) \tag{5.11}$$

is

$$\overline{e}(\overline{r}) = \sum_{i=1}^{N} C_i \Gamma \overline{K}_i(\overline{r}) + \overline{E}^{(0)}(\overline{r}) \tag{5.12}$$

We now require that the N C_i's be chosen in such a way that the error, when averaged with $K_i(r)$ be zero:

$$\int_S dS\, \overline{e}(\overline{r}) \cdot \overline{K}_i(\overline{r}) = 0 \qquad i = 1, 2, \ldots , N . \tag{5.13}$$

These N simultaneous equations are precisely Eqs. (5.9) .

Section VI: Fourier Decomposition of the Kernel

We proceed next to expand the Green's function

$$G = G(\bar{r}, \bar{r}') = \frac{e^{ik_0|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \tag{6.1}$$

in a Fourier series.

$$R = |\bar{r}-\bar{r}'| = \sqrt{(z-z')^2 + [f(z)]^2 + [f(z')]^2 - 2 f(z) f(z') \cos(\phi - \phi')} \tag{6.2}$$

We write

$$k_0^{-1} G(\bar{r}, \bar{r}') = \frac{1}{4\pi^2} \sum_{m=0}^{\infty} G_m(z, z') \cos m(\phi - \phi') \tag{6.3}$$

so that

$$G_m(z, z') = \int_0^{\pi} d\theta \, \frac{e^{ik_0 R}}{k_0 R} \cos m\theta \tag{6.4}$$

with

$$R = \sqrt{(z-z')^2 + [f(z)]^2 + [f(z')]^2 - 2 f(z) f(z') \cos\theta} \; . \tag{6.5}$$

G_m has the symmetry properties

$$G_m(z, z') = G_{-m}(z, z') \tag{6.6}$$

$$G_m(z, z') = G_m(z', z) \tag{6.7}$$

Section VII: Fourier Decomposition of the Current

It will be convenient to consider the currents as functions of the normalized arc length t instead of the distance z along the axis of symmetry. The transformation from z to t is one-one.

We write the fourier decompositions of the current as

$$\bar{K}^{I}(\bar{r}) = \sum_{m=0}^{\infty} \left\{ K^{2m-1, I, t}(t) \sin m\phi + K^{2m, I, t}(t) \cos m\phi \right\} \hat{t}$$

$$+ \sum_{m=0}^{\infty} \left\{ K^{2m-1, I, \phi}(t) \sin m\phi + K^{2m, I, \phi}(t) \cos m\phi \right\} \hat{a} \qquad (7.1)$$

and

$$\bar{K}^{II}(\bar{r}) = \sum_{m=0}^{\infty} \left\{ K^{2m-1, II, t}(t) \sin m\phi + K^{2m, II, t}(t) \cos m\phi \right\} \hat{t}$$

$$+ \sum_{m=0}^{\infty} \left\{ K^{2m-1, II, \phi}(t) \sin m\phi + K^{2m, II, \phi}(t) \cos m\phi \right\} \hat{a} \qquad (7.2)$$

The superscripts I and II refer to the two principal directions of polarization discussed in Sections II and III .

The trial shapes introduced in Eq. (5.1) we will assume to be of the form

$$\bar{K}^{III}(\bar{r}) = \sum_{m=0}^{\infty} \sum_{n=1}^{N1} \left\{ C_n^{2m-1, III, t} K_n^{2m-1, III, t}(t) \sin m\phi + \right.$$

$$\left. + C_n^{2m, III, t} K_n^{2m, III, t}(t) \cos m\phi \right\} \hat{t}$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{N2} \left\{ C_n^{2m-1, III, \phi} K_n^{2m-1, III, \phi}(t) \sin m\phi + \right.$$

$$\left. + C_n^{2m, III, \phi} K_n^{2m, III, \phi}(t) \cos m\phi \right\} \hat{a} \qquad (7.3)$$

where III is a generic variable denoting I or II.

The $K_n^{m,\,I\,or\,II,\,t\,or\,\phi}$ are known functions, the shapes of which we will discuss in more detail below. The C's are unknown coefficients, to be obtained by solving the simultaneous equations.

For convenience in setting up the equations on the computer, we make the following simplifications:

1) We set N1 = N2 = N. That is, we use the same number N of trial functions for both the longitudinal and transverse current. This seems reasonable because in the sampling approach we would normally sample both K_t and K_ϕ at the same points.

2) We shall insist that the trial functions $K_n^{m,\,I\,or\,II,\,t\,or\,\phi}$ be independent of polarization (I or II), mode m, and the direction (t or ϕ). This is reasonable since they are usually taken to be say polynomials in t, or exponentials, $e^{j2\pi nt}$, which do not depend on polarization, mode, or direction.

3) We shall insist that the $K_i(t)$'s be dimensionless. The C's will have the dimension of current per unit length, per unit electric field.

4) We shall insist that the K_i's be real. This simplifies the splitting of the quadratures for the b's and T's into real and imaginary parts. The C's will carry the complex behavior of the current.

5) We shall renumber the ordering of the trial shapes so that the t and ϕ directions are adjacent.

Putting this together, we are going to assume for each mode m exactly N trial shapes

$$K_i(t) \qquad i = 1, 2, \ldots N$$

and construct a 4N-parameter trial function

$$\overline{K}^{mI}(\overline{r}) \approx \sum_{i=1}^{N} \left\{ C_{2i-1}^{mI} \sin m\phi \, \hat{t} + C_{2i}^{mI} \cos m\phi \, \hat{a} \right\} K_i(t)$$

$$+ \sum_{i=1}^{N} \left\{ C_{2i+2N-1}^{mI} \cos m\phi \, \hat{t} + C_{2i+2N}^{mI} \sin m\phi \, \hat{a} \right\} K_i(t) \qquad (7.4)$$

and

$$\overline{K}^{mII}(\overline{r}) \approx \sum_{i=1}^{N} \left\{ C_{2i-1}^{mII} \cos m\phi \, \hat{t} + C_{2i}^{mII} \sin m\phi \, \hat{a} \right\} K_i(t)$$

$$+ \sum_{i=1}^{N} \left\{ C_{2i+2N-1}^{mII} \sin m\phi \, \hat{t} + C_{2i+2N}^{mII} \cos m\phi \, \hat{a} \right\} K_i(t) \qquad (7.5)$$

The exact shapes for the $K_i(t)$ will be given later. For the present Eqs. (7.4) and (7.5) completely describe the trial current. Note that C_1, C_3, . . . , C_{4N-1} describe the longitudinal current while C_2, . . . , C_{4N} describe the transverse current.

We will also assume that

6) The $K_i(t)$ are independent of the angle of incidence θ_0 : the C's will carry the dependence on θ_0 . The reason for this is that the T matrix will then be independent of the angle of incidence, and need be calculated only once while investigating several directions of incidence.

From the symmetry of the problem and the linearity of Maxwell's equations, we know that if the incident field has azimuthal dependence sin $m\phi$ or cos $m\phi$, then the components of the induced current will have dependence sin $m\phi$ and cos $m\phi$.

For a given mode then, the 4N-parameter trial function describing the induced current is given by Eqs. (7.4) and (7.5) .

Section VIII: The Variational Equations

Let $\Gamma(\bar{r}, \bar{r}')$ be the dyadic Green's function which expresses the scattered electric field in terms of the surface electric current

$$\bar{E}^{(s)}(\bar{r}) = \int_S \Big[i\omega\mu\, \bar{K}(\bar{r}')\, G(\bar{r}, \bar{r}') + \eta Z\, (\hat{n} \times \bar{K}(r'))\times \nabla G(\bar{r}, \bar{r}') +$$

$$+ \frac{1}{i\omega\epsilon} (\nabla \cdot \bar{K}(\bar{r}'))\, \nabla G(\bar{r}, \bar{r}') \Big] dS' = \int_S \Gamma(\bar{r}, \bar{r}') \cdot \bar{K}(\bar{r}')\, dS' . \qquad (8.1)$$

We assert that the quantity

$$[A] = \Big\{ \frac{1}{2} \eta Z \int dS\, K^2 + 2 \int dS\, \bar{K}(\bar{r}) \cdot \bar{E}^{(0)}(\bar{r}) +$$

$$+ \int dS\, \bar{K}(\bar{r}) \cdot \int dS' \Gamma(\bar{r}, \bar{r}') \cdot \bar{K}(\bar{r}') \Big\} \qquad (8.2)$$

is stationary when the correct K is used.

Proof: Using the symmetry of Γ, we have

$$\delta[A] = \int dS\, \delta \bar{K}(\bar{r}) \cdot \Big[\eta Z\, \bar{K}(\bar{r}) + 2\, E^{(0)}(\bar{r}) +$$

$$+ 2 \int dS' \Gamma(\bar{r}, \bar{r}') \cdot K(\bar{r}') \Big] = 0 .$$

Since the tangential components of $\delta\bar{K}$ may be varied freely $\delta[A] = 0$ if and only if

$$\Big[\frac{1}{2} \eta Z\, \bar{K}(\bar{r}) + \int dS' \Gamma(\bar{r}, \bar{r}') \cdot \bar{K}(\bar{r}') + \bar{E}^{(0)}(\bar{r}) \Big]_{tan} = 0 \qquad (8.3)$$

r on S ,

which is precisely Eq. (4.9)

If the finite approximation (5.8) is inserted into (8.2), we obtain

$$[A] = \left\{ 2 \sum_{n=1}^{N} C_n b_n + \sum_{n=1}^{N} \sum_{m=1}^{N} C_m T_{mn} C_n \right\} \tag{8.4}$$

If we choose C_1 through C_N to make $[A]$ stationary, we get the equations

$$0 = \frac{\partial [A]}{\partial C_m} = 2 b_m + \sum_{n=1}^{N} (T_{mn} + T_{nm}) C_n$$

that is

$$\sum_{n=1}^{N} \left(\frac{T_{mn} + T_{nm}}{2} \right) C_n = - b_m \quad m = 1, 2, \ldots, N. \tag{8.5}$$

If N is sufficiently large, Eqs. (5.9) and (8.5) must lead to the same solution for the C's. In other words the two systems of equations must be identical

$$\frac{T_{mn} + T_{nm}}{2} = T_{mn} \tag{8.6}$$

from which it follows that

$$T_{mn} = T_{nm} \tag{8.7}$$

I have been unable to prove this statement directly for the third term of Eq. (5.7), but hope to return to it later on. Notice that the symmetric character of the fourth term, when written as in Eq. (5.6), is by no means obvious.

If we insert (7.5) into (8.2), but replacing $\overline{E}^{(0)}$ by the incident tangential electric field for this mode-polarization, which is E^{mIII}, we obtain

$$\left[A^{m\,III}\right] = \left\{ \frac{1}{2} \eta Z \int dS\, \overline{K}^{m\,III\,2} + 2 \int dS\, \overline{K}^{m\,III} \cdot \overline{E}^{m\,III} + \right.$$
$$\left. + \int dS\, \overline{K}^{m\,III}(\overline{r}) \cdot \int dS' \Gamma(\overline{r}, \overline{r}') \cdot \overline{K}^{m\,III}(\overline{r}') \right\} \tag{8.8}$$

$$\approx \left\{ 2 \sum_{i=1}^{4N} C_i^{m\,III} \; b_i^{m\,III} + \sum_{i,\,j=1}^{4N} C_i^{m\,III} \; C_j^{m\,III} \; T_{ij}^{m\,III} \right\} \tag{8.9}$$

where

$$b_{2i-1}^{m\,I} = \int dS \; \overline{E}^{mI}(\overline{r}) \cdot \hat{t} \sin m\phi \; K_i(t) \tag{8.10}$$

$$b_{2i+2N-1}^{mI} = \int dS \; \overline{E}^{mI}(\overline{r}) \cdot \hat{t} \cos m\phi \; K_i(t) \tag{8.11}$$

$$b_{2i}^{mI} = \int dS \; \overline{E}^{mI}(\overline{r}) \cdot \hat{a} \cos m\phi \; K_i(t) \tag{8.12}$$

$$b_{2i+2N}^{mI} = \int dS \; \overline{E}^{mI}(\overline{r}) \cdot \hat{a} \sin m\phi \; K_i(t) \tag{8.13}$$

with identical expressions for $b^{m\,II}$ but with all sin $m\phi$ replaced by cos $m\phi$ and viceversa

$$T_{2i-1,\,2j-1}^{mI} = \frac{1}{2} \eta Z \int dS \sin^2 m\phi \; K_i(t) \; K_j(t)$$

$$+ \iint dS \, dS' \left[\sin m\phi \; K_i(t) \; \hat{t} \right] \cdot \Gamma \cdot \left[\sin m\phi' \; K_j(t') \; \hat{t}' \right] \tag{8.14}$$

$$T_{2i-1,\,2j+2N-1}^{mI} = \frac{1}{2} \eta Z \int dS \sin m\phi \cos m\phi \; K_i(t) \; K_j(t)$$

$$+ \iint dS \, dS' \left[\sin m\phi \; K_i(t) \; \hat{t} \right] \cdot \Gamma \cdot \left[\cos m\phi' \; K_j(t') \; \hat{t}' \right] \tag{8.15}$$

$$T_{2i-1,\,2j+2N}^{mI} = \iint dS \, dS' \left[\sin m\phi \; K_i(t) \; \hat{t} \right] \cdot \Gamma \cdot \left[\sin m\phi' \; K_j(t') \; \hat{a}' \right] \tag{8.16}$$

$$T_{2i-1,\,2j}^{mI} = \iint dS \, dS' \left[\sin m\phi \; K_i(t) \; \hat{t} \right] \cdot \Gamma \cdot \left[\cos m\phi' \; K_j(t') \; \hat{a}' \right] \tag{8.17}$$

$$T^{mI}_{2i+2N-1,\, 2j+2N-1} = \frac{1}{2}\eta Z \int dS \cos^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ \iint dS\, dS' \left[\cos m\phi\, K_i(t)\, \hat{t}\right] \cdot \Gamma \cdot \left[\cos m\phi'\, K_j(t')\, \hat{a}'\right] \qquad (8.18)$$

$$T^{mI}_{2i+2N-1,\, 2j+2N} = \iint dS\, dS' \left[\cos m\phi\, K_i(t)\, \hat{t}\right] \cdot \Gamma \cdot \left[\sin m\phi'\, K_j(t')\, \hat{a}'\right] \qquad (8.19)$$

$$T^{mI}_{2i+2N-1,\, 2j} = \iint dS\, dS' \left[\cos m\phi\, K_i(t)\, \hat{t}\right] \cdot \Gamma \cdot \left[\cos m\phi'\, K_j(t')\, \hat{a}'\right] \qquad (8.20)$$

$$T^{mI}_{2i+2N,\, 2j} = \frac{1}{2}\eta Z \int dS \sin m\phi \cos m\phi\, K_i(t)\, K_j(t)$$

$$+ \iint dS\, dS' \left[\sin m\phi\, K_i(t)\, \hat{a}\right] \cdot \Gamma \cdot \left[\cos m\phi'\, K_j(t')\, \hat{a}'\right] \qquad (8.21)$$

$$T^{mI}_{2i+2N,\, 2j+2N} = \frac{1}{2}\eta Z \int dS \sin^2 m\phi\, K_j(t)\, K_i(t)$$

$$+ \iint dS\, dS' \left[\sin m\phi\, K_i(t)\, \hat{a}\right] \cdot \Gamma \cdot \left[\sin m\phi'\, K_j(t')\, \hat{a}'\right] \qquad (8.22)$$

$$T^{mI}_{2i,\, 2j} = \frac{1}{2}\eta Z \int dS \cos^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ \iint dS\, dS' \left[\cos m\phi\, K_i(t)\, \hat{a}\right] \cdot \Gamma \cdot \left[\cos m\phi'\, K_j(t')\, \hat{a}'\right] \qquad (8.23)$$

with similar formulas for $T^{m\,II}_{ij}$ but with all sin $m\phi$ replaced by cos $m\phi$ and vice-versa.

Section IX: Evaluation of the b's

The b's are given by the integrals (8.10) - (8.13). In order to evaluate the angular integrals involved in calculating the b's, we will need the results

$$\int_0^{2\pi} d\phi \sin m\phi \, e^{-ix\cos\phi} = 0 \qquad (9.1)$$

$$\int_0^{2\pi} d\phi \cos m\phi \, e^{-ix\cos\phi} = 2\pi i^m J_{m(-x)}$$

$$= 2\pi (-i)^m J_{m(x)} \qquad (9.2)$$

m integral.

In order to calculate the integrals over t involved in evaluation of the b's it is convenient to define three sets of d-functions by

$$\begin{bmatrix} d_i^{m,1}(\theta_0) \\ d_i^{m,2}(\theta_0) \\ d_i^{m,3}(\theta_0) \end{bmatrix} = \frac{i\omega\mu C}{2} i^m \int_0^1 dt' \, f(z') \, e^{ik_0 z' \cos\theta_0} K_i(t) \; . \; J_m(k_0 f(z') \sin\theta_0) \; . \begin{bmatrix} 1/2 \\ \alpha(z') f'(z')/2 \\ \alpha(z') \end{bmatrix}$$

$$i = 1, 2, \ldots, N . \qquad (9.3)$$

These are all functions of the single angle variable θ_0 . Using the relations

$$\sin(\pi - \theta_0) = \sin\theta_0$$

$$\cos(\pi - \theta_0) = -\cos\theta_0$$

it is easy to verify that

$$d_i^{m,p}(\pi-\theta_0) = (-)^{m+1}\left[d_i^{m,p}(\theta_0)\right]^* \qquad i = 1, 2, \ldots, N \quad p = 1, 2, 3 \tag{9.4}$$

so that, when evaluating the d's for a range of θ_0, it is sufficient to consider only the set $\theta_0 \in \left[0, \pi/2\right]$.

We now possess the tools for evaluating both the angular integrals and t integrals involved in the calculation of the b's .

The b's depend on the angle of incidence θ_0 of the incident electric field, and will be denoted by $b(\theta_0)$. We now proceed to evaluate the b's .

From equation (8.10) we have

$$b_{2i-1}^{mI}(\theta_0) = \int dS\, \bar{E}^{mI}(\bar{r})\cdot\hat{t}\sin m\phi\, K_i(t) \tag{9.5}$$

Invoking mode orthogonality,

$$= \int dS \sum_{m'=0}^{\infty} \bar{E}^{m'I}(\bar{r})\cdot\hat{t}\sin m\phi\, K_i(t)$$

$$= \int dS\, \hat{\epsilon}_I\, e^{i\bar{k}_0\cdot\bar{r}}\cdot\hat{t}\sin m\phi\, K_i(t) \tag{9.6}$$

From (2.7) and (1.12) we have

$$\hat{\epsilon}_I\cdot\hat{t} = f'(z)\,\alpha(z)\sin\phi \tag{9.7}$$

so that (9.6) becomes, upon use of (3.3)

$$b_{2i-1}^{mI}(\theta_0) = C\int_0^1 dt \int_0^{2\pi} d\phi\, f(z)\, e^{ik_0 z\cos\theta_0}\, f'(z)\,\alpha(z)\ .$$

$$e^{ik_0 f(z)\sin\theta_0\cos\phi}\,\sin\phi\,\sin m\phi\;\; K_i(t)$$

Inserting

$$\sin \phi \sin m\phi = \frac{\cos (m-1)\phi - \cos (m+1)\phi}{2}$$

and performing the angle integral via (9.2), this becomes

$$b^{mI}_{2i-1}(\theta_0) = 2\pi C \int_0^1 dt\, f(z)\, e^{ik_0 z \cos\theta_0} \frac{f'(z)\,\alpha(z)}{2}$$

$$\left[i^{m-1} J_{m-1}(k_0 f(z) \sin\theta_0) - i^{m+1} J_{m+1}(k_0 f(z) \sin\theta_0)\right] K_i(t)$$

Evaluating the t integral by (9.3), we obtain the result

$$b^{mI}_{2i-1}(\theta_0) = \frac{4\pi}{i\omega\mu} \left[d_i^{m-1,2}(\theta_0) - d_i^{m+1,2}(\theta_0)\right]. \tag{9.8}$$

Similarly,

$$b^{mI}_{2i+2N-1}(\theta_0) = \int dS\, \bar{E}^{mI}(\bar{r})\cdot\hat{t} \cos m\phi\, K_i(t) \tag{9.9}$$

$$= \int dS\, e^{i\bar{k}_0\cdot\bar{r}}\, \hat{\epsilon}_I\cdot\hat{t} \cos m\phi\, K_i(t)$$

$$= C \int_0^1 dt \int_0^{2\pi} d\phi\, f(z)\, e^{ik_0 z \cos\theta_0} f'(z)\, \alpha(z)\, K_i(t)$$

$$e^{ik_0 f(z) \sin\theta_0 \cos\phi} (\sin (m+1)\phi - \sin (m-1)\phi) \frac{1}{2} = 0$$

where we used Eq. (9.1). We have then the result

$$b^{mI}_{2i+2N-1}(\theta_0) \equiv 0. \tag{9.10}$$

Similarly,

$$b^{mI}_{2i+2N}(\theta_0) = \int dS\, \overline{E}^{mI} \cdot \hat{a} \sin m\phi\, K_i(t) \tag{9.11}$$

$$= \int dS\, e^{i\overline{k}_0 \cdot \overline{r}}\, \hat{\epsilon}_I \cdot \hat{a} \sin m\phi\, K_i(t)$$

$$\hat{\epsilon}_I \cdot \hat{a} = \cos \phi \tag{9.12}$$

So that

$$b^{mI}_{2i+2N}(\theta_0) = C \int_0^1 dt \int_0^{2\pi} d\phi\, f(z)\, e^{i k_0 z \cos\theta_0} K_i(t)$$

$$e^{i k_0 f(z) \sin\theta_0 \cos\phi} \cos\phi \sin m\phi$$

$$= C \int_0^1 dt \int_0^{2\pi} d\phi\, f(z)\, e^{i k_0 z \cos\theta_0} K_i(t)$$

$$\frac{1}{2}\, e^{ik_0 f(z) \sin\theta_0 \cos\phi} \left[\sin(m+1)\phi + \sin(m-1)\phi\right] = 0$$

where we have again used Eq. (9.1).

We have then the result

$$b^{mI}_{2i+2N}(\theta_0) = 0\ . \tag{9.13}$$

Similarly,

$$b_{2i}^{mI}(\theta_0) = \int dS\, \bar{E}^{mI}(\bar{r}) \cdot \hat{a} \cos m\phi\, K_i(t)$$

$$= \int dS\, e^{i \bar{k}_0 \cdot \bar{r}}\, \hat{\epsilon}_I \cdot \hat{a} \cos m\phi\, K_i(t)$$

$$b_{2i}^{mI}(\theta_0) = C \int_0^1 dt \int_0^{2\pi} d\phi\, f(z)\, e^{i k_0 z \cos\theta_0}\, K_i(t)$$

$$\frac{1}{2}\, e^{i k_0 f(z) \sin\theta_0 \cos\phi} \left[\cos(m-1)\phi + \cos(m+1)\phi\right]$$

$$= 2\pi C \int_0^1 dt\, f(z)\, e^{i k_0 z \cos\theta_0}\, K_i(t)\, \frac{1}{2}\,.$$

$$\left[i^{m-1} J_{m-1}(k_0 f(z) \sin\theta_0) + i^{m+1} J_{m+1}(k_0 f(z) \sin\theta_0)\right] \tag{9.14}$$

or finally,

$$b_{2i}^{mI}(\theta_0) = \frac{4\pi}{i\omega\mu}\left[d_i^{m-1,\,1}(\theta_0) + d_i^{m+1,\,1}(\theta_0)\right]. \tag{9.15}$$

Similarly using

$$\hat{\epsilon}_{II} \cdot \hat{t} = \sin\theta_0\, \alpha(z) - \cos\theta_0\, f'(z)\, \alpha(z) \cos\phi$$

we calculate

$$b^{m\,II}_{2i+2N-1} = \int dS\, \bar{E}^{m\,II} \cdot \hat{t} \sin m\phi\, K_i(t)$$

$$= \int dS\, e^{i\bar{k}_0 \cdot \bar{r}}\, \hat{\epsilon}_{II} \cdot \hat{t} \sin m\phi\, K_i(t)$$

$$= C \int_0^1 dt \int_0^{2\pi} d\phi\, f(z)\, e^{i k_0 z \cos\theta_0}\, e^{i k_0 f(z) \sin\theta_0 \cos\phi}$$

$$(\sin\theta_0\, \alpha(z) - \cos\theta_0\, f'(z)\, \alpha(z) \cos\phi) \sin m\phi\, K_i(t) = 0$$

$$(9.16)$$

we have then

$$b^{m\,II}_{2i+2N-1}(\theta_0) \equiv 0 \; . \tag{9.17}$$

$$b^{m\,II}_{2i-1}(\theta_0) = \int dS\, \bar{E}^{m\,II}(\bar{r}) \cdot \hat{t} \cos m\phi\, K_i(t)$$

$$= \int dS\, e^{i\bar{k}_0 \cdot \bar{r}}\, (\hat{\epsilon}_{II} \cdot \hat{t}) \cos m\phi\, K_i(t)$$

$$= \int dS\, e^{i\bar{k}_0 \cdot \bar{r}}\, K_i(t) \; .$$

$$\left[\sin\theta_0\, \alpha(z) \cos m\phi - \cos\theta_0\, f'(z)\, \alpha(z)\, \frac{\cos(m-1)\phi + \cos(m+1)\phi}{2}\right]$$

$$= 2\pi C \int_0^1 dt\, f(z)\, e^{i k_0 z \cos\theta_0}\, K_i(t) \; .$$

$$\Big\{ \sin\theta_0\, \alpha(z)\, i^m J_m(k_0 f(z) \sin\theta_0)$$

$$- \cos\theta_0 \frac{f'(z)\, \alpha(z)}{2} \Big[i^{m-1} J_{m-1}(k_0 f(z) \sin\theta_0)$$

$$+ i^{m+1} J_{m+1}(k_0 f(z) \sin\theta_0) \Big] \Big\} \tag{9.18}$$

or finally

$$b_{2i-1}^{m\,II}(\theta_0) = \frac{4\pi}{i\,\omega\mu}\left\{\sin\theta_0\, d_i^{m,\,3}(\theta_0)\right.$$

$$\left. - \cos\theta_0\left[d_i^{m-1,\,2}(\theta_0) + d_i^{m+1,\,2}(\theta_0)\right]\right\} \tag{9.19}$$

Similarly, using

$$\hat{\epsilon}_{II}\cdot\hat{a} = \cos\theta_0\,\sin\phi \tag{9.20}$$

we calculate

$$b_{2i}^{m\,II}(\theta_0) = \int dS\,\overline{E}^{m\,II}(\overline{r})\cdot\hat{a}\,\sin m\phi\,K_i(t)$$

$$= \int dS\,e^{i\overline{k}_0\cdot\overline{r}}\,(\hat{\epsilon}_{II}\cdot\hat{a})\,\sin m\phi\,K_i(t)$$

$$= \cos\theta_0\int dS\,e^{i\overline{k}_0\cdot\overline{r}}\,K_i(t)\,\frac{\cos(m-1)\,\phi - \cos(m+1)\,\phi}{\sigma}$$

$$= \cos\theta_0\,2\pi C\int_0^1 dt\,f(z)\,e^{i\,k_0\,z\cos\theta_0}\,K_i(t)\,\frac{1}{2}\ .$$

$$\left[i^{m-1}J_{m-1}(k_0 f(z)\sin\theta_0) - i^{m+1}J_{m+1}(k_0 f(z)\sin\theta_0)\right] \tag{9.21}$$

or finally

$$b_{2i}^{m\,II}(\theta_0) = \cos\theta_0\,\frac{4\pi}{i\,\omega\mu}\left[d_i^{m-1,\,1}(\theta_0) - d_i^{m+1,\,1}(\theta_0)\right]. \tag{9.22}$$

Analogously

$$b^{m\,II}_{2i+2N}(\theta_0) = \int dS\ \bar{E}^{m\,II}\cdot\hat{a}\cos m\phi\ K_i(t)$$

$$= \int dS\ e^{i\bar{k}_0\cdot\bar{r}}\ (\hat{\epsilon}_{II}\cdot\hat{a})\cos m\phi\ K_i(t)$$

$$= \cos\theta_0 \int dS\ e^{i\bar{k}_0\cdot\bar{r}}\ \sin\phi\cos m\phi\ K_i(t) = 0 \qquad (9.23)$$

and we have

$$b^{m\,II}_{2i+2N}(\theta_0) \equiv 0 \qquad (9.24)$$

Our results, equations (9.8, 9.10, 9.13, 9.15, 9.17, 9.19, 9.22, and 9.24) explicitly show how the b's may be evaluated in terms of the d's . The d's are given by the one-dimensional quadratures in (9.3) and are presumed to be known. Hence, the b's can be easily calculated.

Using (9.4), equations (9.8, 9.10, 9.13, 9.15, 9.17, 9.19, 9.22, and 9.24) imply that

$$b^{m\,I}_{i}(\pi-\theta) = (-)^{m+1}\left[b^{m\,I}_{i}(\theta)\right]^{*} \qquad (9.25)$$

$$b^{m\,II}_{i}(\pi-\theta) = (-)^{m}\left[b^{m\,II}_{i}(\theta)\right]^{*} \qquad (9.26)$$

so that the b(θ) 's need be evaluated only for $\theta \in [0,\ \pi/2]$.

Section X: The d-functions

The d-functions have been defined by equation (9.3). The relation

$$d_i^{m,p}(\pi - \theta_0) = (-)^{m+1}\left[d_i^{m,p}(\theta_0)\right]^* \tag{10.1}$$

has been derived, showing the need to evaluate the d's only for the argument lying in $[0, \pi/2]$.

Using the relation

$$J_{-m}(x) = (-)^m J_m(x) \qquad m \text{ integral}, \tag{10.2}$$

we deduce

$$d_i^{-m,p}(\theta_0) = d_i^{m,p}(\theta_0) \tag{10.3}$$

in particular,

$$d_i^{-1,p}(\theta_0) = d_i^{1,p}(\theta_0) \tag{10.4}$$

This relation will be needed when attempting to evaluate $d_i^{m-1,p}$ with $m = 0$.

For $K_i(t)$ of the form to be discussed later - namely piecewise linear in t - the integrals in (9.3) seem to be completely intractable analytically. Even for the simplest geometries of interest, cones and spheres, there has been no success in obtaining closed expressions for these integrals.

Consequently these integrals will have to be done numerically - numerical quadratures, done preferably by a Gaussian quadrature scheme, should cause no difficulty since the integrand is well-behaved.

The only complex quantity within the integrand in (9.3) is the $e^{i k_0 z' \cos\theta_0}$, which can be easily split into its real and imaginary parts. Hence, both the real and imaginary parts of d are calculable by one-dimensional quadratures.

The numerical quadratures should be fast, except the numerical generation and/or table lookup of exponentials, f(z), and Bessel functions, since $K_i(t)$ will be a highly localized function of t.

Section XI: Curvilinear Coordinates on the Scatterer

We construct a set of orthogonal curvilinear coordinates which, on the surface of the body S reduces to the ϕ, t coordinates used so far. The three coordinates would be

$$\xi_1 = \phi \tag{11.1}$$

$$\xi_2 = t \tag{11.2}$$

$$\xi_3 = \xi_3 \text{ (in direction of outward normal to S)} \tag{11.3}$$

and the corresponding unit vectors $(\hat{u}_1, \hat{u}_2, \hat{u}_3) = (\hat{a}, \hat{t}, \hat{n})$ form a right-handed triad. The surface of the conductor is given by $\xi_3 = \xi_{30}$ = constant.

The element of arc length, using the orthogonality of the coordinate system is

$$(dS)^2 = (h_1\, d\,\xi_1)^2 + (h_2\, d\,\xi_2)^2 + (h_3\, d\,\xi_3)^2 \tag{11.4}$$

where, for $\xi_3 = \xi_{30}$,

$$h_1 = \rho = \rho\,(\xi_2) \tag{11.5}$$

$$h_2 = C \tag{11.6}$$

$$h_3 = h_3\,(\xi_3) \tag{11.7}$$

If an arbitrary vector $\overline{v}$ is decomposed along the triad as

$$\overline{v} = v_1 \hat{a} + v_2 \hat{t} + v_3 \hat{n} = v_\phi \hat{a} + v_t \hat{t} + v_n \hat{n} \qquad (11.8)$$

where

$$v_i = v_i (\xi_1, \xi_2, \xi_3) , \qquad (11.9)$$

then we can compute its divergence and curl via

$$\text{div } \overline{v} = \frac{1}{h_1 h_2 h_3} \sum_{n=1}^{3} \frac{\partial}{\partial \xi_n} \left[h_1 h_2 h_3 \frac{v_n}{h_n} \right] . \qquad (11.10)$$

$$\text{curl } \overline{v} = \frac{1}{h_1 h_2 h_3} \sum_{\substack{\ell mn \\ 123 \\ 231 \\ 312}} h_\ell \hat{u}_\ell \left[\frac{\partial}{\partial \xi_m} (h_m v_m) \right] . \qquad (11.11)$$

If $v_n = 0$, (11.12)

then on the surface of the conductor we obtain from (11.10)

$$\text{div } \overline{v} = \frac{1}{\rho} \frac{\partial v_\phi}{\partial \phi} + \frac{1}{\rho c} \frac{\partial}{\partial t} (\rho v_t) \qquad (11.13)$$

Section XII: Explicit formulae for the T matrices

The T matrices have been defined by (8.14 - 8.23, 8.7) . Using Eq. (5.7) we have

$$T^{mI}_{2i-1,\,2j-1} = \frac{1}{2}\,\eta Z \int dS\,\sin^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ \iint dS\, dS'\,(i\omega\mu)\,\sin m\phi\,\sin m\phi'\, K_i(t)\, K_j(t')\,(\hat{t}\cdot\hat{t}')\, G$$

$$+ \eta Z \iint dS\, dS'\,(\sin m\phi)\, K_i(t)\;\hat{t}\cdot\left[(\hat{n}' \times \hat{t}') \times \nabla' G\right]\sin m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS'\left[\nabla\cdot\left[\hat{t}\,\sin m\phi\, K_i(t)\right]\right]\left[\nabla'\cdot\left[\hat{t}'\,\sin m\phi'\, K_j(t')\right]\right] G \qquad (12.1)$$

$$T^{mI}_{2i-1,\,2j+2N-1} = \iint dS\, dS'\,(i\omega\mu)\,\sin m\phi\,\cos m\phi'\, K_i(t)\, K_j(t')\,(\hat{t}\cdot\hat{t}')\, G$$

$$+ \eta Z \iint dS\, dS'\,(\sin m\phi\, K_i(t))\;\hat{t}\cdot\left[(\hat{n}' \times \hat{t}') \times \nabla' G\right]\cos m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS'\left[\nabla\cdot\left[\hat{t}\,\sin m\phi\, K_i(t)\right]\right]\left[\nabla'\cdot\left[\hat{t}'\,\cos m\phi'\, K_j(t')\right]\right] G \qquad (12.2)$$

$$T^{mI}_{2i-1,\,2j+2N} = i\omega\mu \iint dS\, dS'\,\sin m\phi\,\sin m\phi'\, K_i(t)\, K_j(t')\,(\hat{t}\cdot\hat{a}')\, G$$

$$+ \eta Z \iint dS\, dS'\,(\sin m\phi\, K_i(t))\;\hat{t}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right]\sin m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS'\left[\nabla\cdot\left[\hat{t}\,\sin m\phi\, K_i(t)\right]\right]\left[\nabla'\cdot\left[\hat{a}'\,\sin m\phi'\, K_j(t')\right]\right] G \qquad (12.3)$$

$$T^{mI}_{2i-1,\,2j} = i\omega\mu \iint dS\, dS' \left[\sin m\phi\, K_i(t) \cos m\phi'\, K_j(t')\right] (\hat{t}\cdot\hat{a}')\, G$$

$$+ \eta Z \iint dS\, dS' \left(\sin m\phi\, K_i(t)\right) \hat{t}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS' \left[\nabla\cdot\left[\hat{t} \sin m\phi\, K_i(t)\right]\right] \left[\nabla'\cdot\left[\hat{a}' \cos m\phi'\, K_j(t')\right]\right] G \tag{12.4}$$

$$T^{mI}_{2i+2n-1,\,2j+2N-1} = \frac{1}{2}\, \eta Z \int dS \cos^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ i\omega\mu \iint dS\, dS' \cos m\phi \cos m\phi'\, K_i(t)\, K_j(t')\, (\hat{t}\cdot\hat{t}')\, G$$

$$+ \eta Z \iint dS\, dS' \cos m\phi\, K_i(t)\, \hat{t}\cdot\left[(\hat{n}' \times \hat{t}') \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS' \left[\nabla\cdot(\cos m\phi\, K_i(t)\, \hat{t})\right] \left[\nabla'\cdot(\cos m\phi'\, K_j(t')\, \hat{t}')\right] G \tag{12.5}$$

$$T^{mI}_{2i+2N-1,\,2j+2N} = i\omega\mu \iint dS\, dS' \cos m\phi \sin m\phi'\, K_i(t)\, K_j(t')\, (\hat{t}\cdot\hat{a}')\, G$$

$$+ \eta Z \iint dS\, dS' \cos m\phi\, K_i(t)\, \hat{t}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \sin m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS' \left[\nabla\cdot(\cos m\phi\, K_i(t)\, \hat{t})\right] \left[\nabla'\cdot(\sin m\phi'\, K_j(t')\, \hat{a}')\right] G \tag{12.6}$$

$$T^{mI}_{2i+2N-1,\ 2j} = i\omega\mu \iint dS\, dS'\, \cos m\phi \cos m\phi'\, K_i(t)\, K_j(t')\, (\hat{t}\cdot\hat{a}')\, G$$

$$+ \eta Z \iint dS\, dS'\, \cos m\phi\, K_i(t)\, \hat{t}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS' \left[\nabla\cdot\left[\cos m\phi\, K_i(t)\, \hat{t}\right]\right] \left[\nabla'\cdot\left[\cos m\phi'\, K_j(t')\, \hat{a}'\right]\right] G \qquad (12.7)$$

$$T^{mI}_{2i+2N,\ 2j} = i\omega\mu \iint dS\, dS'\, \sin m\phi \cos m\phi'\, K_i(t)\, K_j(t')\, (\hat{a}\cdot\hat{a}')$$

$$+ \eta Z \iint dS\, dS'\, \sin m\phi\, K_i(t)\, \hat{a}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS' \left[\nabla\cdot\left[\sin m\phi\, K_i(t)\, \hat{a}\right]\right] \left[\nabla'\cdot\left[\cos m\phi'\, K_j(t')\, \hat{a}'\right]\right] G \qquad (12.8)$$

$$T^{mI}_{2i+2N,\ 2j+2N} = \frac{1}{2}\, \eta Z \int dS\, \sin^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ i\omega\mu \iint dS\, dS'\, \sin m\phi \sin m\phi'\, K_i(t)\, K_j(t')\, (\hat{a}\cdot\hat{a}')\, G$$

$$+ \eta Z \iint dS\, dS'\, \sin m\phi\, K_i(t)\, \hat{a}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \sin m\phi'\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS' \left[\nabla\cdot\left[\sin m\phi\, K_i(t)\, \hat{a}\right]\right] \left[\nabla'\cdot\left[\sin m\phi'\, K_j(t')\, \hat{a}'\right]\right] G \qquad (12.9)$$

$$T^{mI}_{2i,\,2j} = \frac{1}{2}\,\eta\,Z\int dS\,\cos^2 m\phi\,K_i(t)\,K_j(t)$$

$$+\,i\omega\mu\iint dS\,dS'\,\cos m\phi\,\cos m\phi'\,K_i(t)\,K_j(t')\,(\hat{a}\cdot\hat{a}')$$

$$+\,\eta\,Z\iint dS\,dS'\,\cos m\phi\,K_i(t)\,\hat{a}\cdot\left[(\hat{n}'\times\hat{a}')\times\nabla' G\right]\cos m\phi'\,K_j(t')$$

$$+\,\frac{1}{i\omega\epsilon}\iint dS\,dS'\left[\nabla\cdot(\cos m\phi\,K_i(t)\,\hat{a})\right]\left[\nabla'\cdot(\cos m\phi'\,K_j(t')\,\hat{a}')\right]G\,. \qquad (12.10)$$

Since

$$\int dS = C\int_0^1 dt\,f(z)\int_0^{2\pi} d\phi\,. \qquad (12.11)$$

terms of the form

$$\int dS\,\sin m\phi\,\cos m\phi\,K_i(t)\,K_j(t)$$

integrate to zero, and have not been explictly included above. Equations for T^{mII}_{ij} follow from those for T^{mI}_{ij} by replacing all sin $m\phi$ by cos $m\phi$ and viceversa.

We now proceed to simplify these expressions. Let us first note that when $\eta = 0$, Eq. (4.9) reduces to the well known boundary condition for a perfect conductor.

The T-matrix elements (12.1 - 12.10) are made up of terms independent of η and terms containing η as a factor. It follows then that the part independent of η should correspond to the T-matrix elements for the perfectly conducting case. These T-matrix elements have been calculated explicitly by P. Schweitzer. (See Schweitzer's report, page 68). We shall write here then for future reference:

$$T^{mI,\ p.c.}_{2i-1,\ 2j-1} = i\omega\mu \iint dS\, dS'\, G \Big\{ \sin m\phi \sin m\phi' (\hat{t}\cdot\hat{t}')$$

$$K_i(t)\, K_j(t') \Big\} + \frac{1}{i\omega\epsilon} \iint dS\, dS'\, G\, \mathrm{div}\left[\hat{t}\, K_i(t) \sin m\phi\right]$$

$$\mathrm{div}\left[\hat{t}'\, K_j(t') \sin m\phi'\right] . \qquad (12.12)$$

$$T^{mI,\ p.c.}_{2i-1,\ 2j} = i\omega\mu \iint dS\, dS'\, G \sin m\phi \cos m\phi' (\hat{t}\cdot\hat{a}')\, K_i(t)\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS'\, G\, \mathrm{div}\left[\hat{t}\, K_i(t) \sin m\phi\right] \mathrm{div}\left[\hat{a}'\, K_j(t') \cos m\phi'\right] \qquad (12.13)$$

$$T^{mI,\ p.c.}_{2i,\ 2j-1} = i\omega\mu \iint dS\, dS'\, G \cos m\phi \sin m\phi' (\hat{a}\cdot\hat{t}')\, K_i(t)\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS'\, G\, \mathrm{div}\left[\hat{a}\, K_i(t) \cos m\phi\right] \mathrm{div}\left[\hat{t}'\, K_j(t') \sin m\phi'\right] \qquad (12.14)$$

$$T^{mI,\ p.c.}_{2j,\ 2j} = i\omega\mu \iint dS\, dS'\, G \cos m\phi \cos m\phi' (\hat{a}\cdot\hat{a}')\, K_i(t)\, K_j(t')$$

$$+ \frac{1}{i\omega\epsilon} \iint dS\, dS'\, G\, \mathrm{div}\left[\hat{a}\, K_i(t) \cos m\phi\right] \mathrm{div}\left[\hat{a}'\, K_j(t') \cos m\phi'\right] \qquad (12.15)$$

Similar expressions hold for T^{mII}_{ij} but with all $\sin m\phi$ replaced by $\cos m\phi$, and viceversa. The superscript "p.c." denotes "perfect conductor". Use of these results in Eqs. (12.1 - 12.10) gives

$$T^{mI}_{2i-1,\ 2j-1} = \frac{1}{2}\,\eta\, Z \int dS\ \sin^2 m\phi\ K_i\,(t)\ K_j\,(t)$$

$$+\ \eta\, Z \iint dS\ dS'\ \sin m\phi\ K_i\,(t)\ \hat{t}\cdot\left[(\hat{n}'\ x\ \hat{t}')\ x\ \nabla'\ G\right] \sin m\phi'\ K_j\,(t')$$

$$+\ T^{mI,\ p.c.}_{2i-1,\ 2j-1} \qquad (12.16)$$

$$T^{mI}_{2i-1,\ 2j+2N-1} = \eta\, Z \iint dS\ dS'\ (\sin m\phi\ K_i\,(t)\,)\ \hat{t}\cdot\left[(\hat{n}'\ x\ \hat{t}')\ x\ \nabla'\ G\right]$$

$$\cos m\phi'\ K_j\,(t') \qquad (12.17)$$

The other two terms in (12.17) are identically zero. They are independent of η and they do not occur in the perfectly conducting case, hence they must vanish identically. This can be checked by direct calculation.

$$T^{mI}_{2i-1,\ 2j+2N} = \eta\, Z \iint dS\ dS'\ (\sin m\phi\ K_i\,(t)\,)\ \hat{t}\cdot\left[(\hat{n}'\ x\ \hat{a}')\ x\ \nabla'\ G\right]$$

$$\sin m\phi'\ K_j\,(t') \qquad (12.18)$$

$$T^{mI}_{2i-1,\ 2j} = \eta\, Z \iint dS\ dS'\ (\sin m\phi\ K_i\,(t))\ \hat{t}\cdot\left[(\hat{n}'\ x\ \hat{a}')\ x\ \nabla'\ G\right]$$

$$\cos m\phi'\ K_j\,(t') + T^{mI,\ p.c.}_{2i-1,\ 2j} \qquad (12.19)$$

$$T^{mI}_{2i+2N-1,\ 2j+2N-1} = \frac{1}{2}\,\eta\, Z \int dS\ \cos^2 m\phi\ K_i\,(t)\ K_j\,(t)$$

$$+\ \eta\, Z \iint dS\ dS'\ \cos m\phi\ K_i\,(t)\ \hat{t}\cdot\left[(\hat{n}'\ x\ \hat{t}')\ x\ \nabla'\ G\right] \cos m\phi'\, K_j\,(t') + T^{mII,\ p.c.}_{2i-1,\ 2j-1}$$

$$(12.20)$$

$$T^{mI}_{2i+2N-1,\ 2j+2N} = \eta Z \iint dS\, dS'\, \cos m\phi\, K_i(t)\, \hat{t}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right]$$

$$\sin m\phi'\, K_j(t') + T^{m\,II,\ p.c.}_{2i-1,\ 2j} \qquad (12.21)$$

$$T^{mI}_{2i+2N-1,\ 2j} = \eta Z \iint dS\, dS'\, \cos m\phi\, K_i(t)\, \hat{t}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

(12.22)

$$T^{mI}_{2i+2N,\ 2j} = \eta Z \iint dS\, dS'\, \sin m\phi\, K_i(t)\, \hat{a}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

(12.23)

$$T^{mI}_{2i+2N,\ 2j+2N} = \frac{1}{2}\, \eta\, z \int dS\, \sin^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ \eta Z \iint dS\, dS'\, \sin m\phi\, K_i(t)\, \hat{a}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \sin m\phi'\, K_j(t') + T^{m\,II,\ p.c.}_{2i,\ 2j}$$

(12.24)

$$T^{mI}_{2i,\ 2j} = \frac{1}{2}\, \eta\, z \int dS\, \cos^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ \eta Z \iint dS\, dS'\, \cos m\phi\, K_i(t)\, \hat{a}\cdot\left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

$$+ T^{mI,\ p.c.}_{2i,\ 2j} \qquad (12.25)$$

Notice that the T matrix is of the form

$$T^{I} = \begin{bmatrix} T^{I,\ p.c.} & 0 \\ 0 & T^{II,\ p.c.} \end{bmatrix} + \eta\, T^{I'} \qquad (12.26)$$

where $T^{I'}$ arises from the impedance boundary condition.

Similarly

$$T^{II} = \begin{bmatrix} T^{II,\,p.c.} & 0 \\ 0 & T^{I,\,p.c.} \end{bmatrix} + \eta\, T^{II'} \qquad (12.27)$$

As pointed out by Schweitzer

$$T^{mI,\,p.c.}_{ij} = (-)^{i+j}\, T^{m\,II,\,p.c.}_{ij} \quad , \quad m \geqslant 1 \qquad (12.28)$$

$$\left[T^{m\,I,\,p.c.}\right]^{-1}_{ij} = (-)^{i+j} \left[T^{m\,II,\,p.c.}\right]^{-1}_{ij} \quad , \quad m \geqslant 1 \qquad (12.29)$$

so that only $T^{m\,I,\,p.c.}$ has to be calculated.

For $m = 0$

$T^{0I,\,p.c.}_{ij} = 0$ unless i and j are both even,

and

$T^{0\,II,\,p.c.}_{ij} = 0$ unless i and j are both odd.

Note also that for either polarization the b-column matrix is of the form

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2N} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (12.30)$$

We now proceed to obtain explicit expressions for the elements of T' .

Section XIII: The T' Matrix

We will write down here the expressions for the elements of the $T^{I'}$ matrix.

$$T^{mI'}_{2i-1,\, 2j-1} = \frac{1}{2} Z \int dS \sin^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ Z \iint dS\, dS' \sin m\phi\, K_i(t)\, \hat{t} \cdot \left[(\hat{n}' \times \hat{t}') \times \nabla' G\right] \sin m\phi'\, K_j(t') \tag{13.1}$$

$$T^{mI'}_{2i-1,\, 2j} = Z \iint dS\, dS' \sin m\phi\, K_i(t)\, \hat{t} \cdot \left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t') \tag{13.2}$$

$$T^{mI'}_{2i,\, 2j} = \frac{1}{2} Z \int dS \cos^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ Z \iint dS\, dS' \cos m\phi\, K_i(t)\, \hat{a} \cdot \left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t') \tag{13.3}$$

$$T^{mI'}_{2i-1,\, 2j+2N-1} = Z \iint dS\, dS' (\sin m\phi\, K_i(t))\, \hat{t} \cdot \left[(\hat{n}' \times \hat{t}') \times \nabla' G\right]$$

$$\cos m\phi'\, K_j(t') \tag{13.4}$$

$$T^{mI'}_{2i-1,\, 2j+2N} = Z \iint dS\, dS' (\sin m\phi\, K_i(t))\, \hat{t} \cdot \left[(\hat{n}' \times \hat{a}') \times \nabla' G\right]$$

$$\sin m\phi'\, K_j(t') \tag{13.5}$$

$$T^{mI'}_{2i+2N,\, 2j} = Z \iint dS\, dS' \sin m\phi\, K_i(t)\, \hat{a} \cdot \left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t') \tag{13.6}$$

$$T^{mI'}_{2i+2N-1,\ 2j+2N-1} = \frac{1}{2} Z \int dS \cos^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ Z \iint dS\, dS' \cos m\phi\, K_i(t)\, \hat{t} \cdot \left[(\hat{n}' \times \hat{t}') \times \nabla' G\right] \cos m\phi'\, K_j(t') \tag{13.7}$$

$$T^{mI'}_{2i+2N-1,\ 2j+2N} = Z \iint dS\, dS' \cos m\phi\, K_i(t)\, \hat{t} \cdot \left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \sin m\phi'\, K_j(t') \tag{13.8}$$

$$T^{mI'}_{2i+2N-1,\ 2j} = Z \iint dS\, dS' \cos m\phi\, K_i(t)\, \hat{t} \cdot \left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \cos m\phi'\, K_j(t') \tag{13.9}$$

$$T^{mI'}_{2i+2N,\ 2j+2N} = \frac{1}{2} Z \int dS \sin^2 m\phi\, K_i(t)\, K_j(t)$$

$$+ Z \iint dS\, dS' \sin m\phi\, K_i(t)\, \hat{a} \cdot \left[(\hat{n}' \times \hat{a}') \times \nabla' G\right] \sin m\phi'\, K_j(t') \tag{13.10}$$

The expressions for $T^{mII'}_{ij}$ are given by Eqs. (13.1) - (13.10) but replacing all sin $m\phi$ by cos $m\phi$ and viceversa.

We have

$$\int dS = C \int_0^1 dt\, f(z) \int_0^{2\pi} d\phi$$

so that

$$\int dS \sin^2 m\phi\, K_i(t)\, K_j(t) = C \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t) \cdot$$

$$\int_0^{2\pi} d\phi \sin^2 m\phi = C\pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t) \tag{13.11}$$

and

$$\int dS \cos^2 m\phi \, K_i(t) \, K_j(t) = C\pi \int_0^1 dt \, f(z) \, K_i(t) \, K_j(t) \tag{13.12}$$

Also since

$$\hat{t} = \hat{n} \times \hat{a}$$

and $\hat{n} \times \hat{t} = - \hat{a}$,

we have

$$T^{mI'}_{2i-1,\, 2j-1} = \frac{1}{2} C Z \pi \int_0^1 dt \, f(z) \, K_i(t) \, K_j(t)$$

$$- Z \iint dS \, dS' \sin m\phi \, K_i(t) \, \hat{t} \cdot \left[\hat{a}' \times \nabla' G \right] \sin m\phi' \, K_j(t') \tag{13.13}$$

$$T^{mI'}_{2i-1,\, 2j} = Z \iint dS \, dS' \sin m\phi \, K_i(t) \, \hat{t} \cdot \left[\hat{t}' \times \nabla' G \right] \cos m\phi' \, K_j(t') \tag{13.14}$$

$$T^{mI'}_{2i,\, 2j} = \frac{1}{2} C Z \pi \int_0^1 dt \, f(z) \, K_i(t) \, K_j(t)$$

$$+ Z \iint dS \, dS' \cos m\phi \, K_i(t) \, \hat{a} \cdot \left[\hat{t}' \times \nabla G \right] \cos m\phi' \, K_j(t') \tag{13.15}$$

$$T^{mI'}_{2i-1,\, 2j+2N-1} = - Z \iint dS \, dS' \left(\sin m\phi \, K_i(t) \right) \hat{t} \cdot \left[\hat{a}' \times \nabla' G \right] \cos m\phi' \, K_j(t') \tag{13.16}$$

$$T^{mI'}_{2i-1,\, 2j+2N} = Z \iint dS \, dS' \left(\sin m\phi \, K_i(t) \right) \hat{t} \cdot \left[\hat{t}' \times \nabla' G \right] \sin m\phi' \, K_j(t') \tag{13.17}$$

$$T^{mI'}_{2i+2N,\,2j} = Z \iint dS\, dS'\, \sin m\phi\, K_i(t)\, \hat{a}\cdot\left[\hat{t}' \times \nabla' G\right] \cos m\phi'\, K_j(t') \tag{13.18}$$

$$T^{mI'}_{2i+2N-1,\,2j+2N-1} = \frac{1}{2} Z C \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t)$$

$$- Z \iint dS\, dS'\, \cos m\phi\, K_i(t)\, \hat{t}\cdot\left[\hat{a}' \times \nabla' G\right] \cos m\phi'\, K_j(t') \tag{13.19}$$

$$T^{mI'}_{2i+2N-1,\,2j+2N} = Z \iint dS\, dS'\, \cos m\phi\, K_i(t)\, \hat{t}\cdot\left[\hat{t}' \times \nabla' G\right] \sin m\phi'\, K_j(t') \tag{13.20}$$

$$T^{mI'}_{2i+2N-1,\,2j} = Z \iint dS\, dS'\, \cos m\phi\, K_i(t)\, \hat{t}\cdot\left[\hat{t}' \times \nabla' G\right] \cos m\phi'\, K_j(t') \tag{13.21}$$

$$T^{mI'}_{2i+2N,\,2j+2N} = \frac{1}{2} C Z \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t)$$

$$+ Z \iint dS\, dS'\, \sin m\phi\, K_i(t)\, \hat{a}\cdot\left[\hat{t}' \times \nabla' G\right] \sin m\phi'\, K_j(t') \tag{13.22}$$

The gradient of a scalar function in curvilinear coordinates is

$$\nabla f = \frac{\hat{u}_1}{h_1} \frac{\partial f}{\partial \xi_1} + \frac{\hat{u}_2}{h_2} \frac{\partial f}{\partial \xi_2} + \frac{\hat{u}_3}{h_3} \frac{\partial f}{\partial \xi_3} \; . \tag{13.23}$$

In our case we have

$(\hat{u}_1, \hat{u}_2, \hat{u}_3) = (\hat{a}, \hat{t}, \hat{n})$ and

$\xi_1 = \phi$

$\xi_2 = t$

$\xi_3 = \xi_3$ (in direction of outward normal)

Hence

$$\nabla G = \frac{\hat{a}}{\rho}\frac{\partial G}{\partial \phi} + \frac{\hat{t}}{C}\frac{\partial G}{\partial t} + \frac{\hat{n}}{h_3}\frac{\partial G}{\partial \xi_3}$$

$$\hat{t} \times \nabla G = -\frac{\hat{n}}{\rho}\frac{\partial G}{\partial \phi} + \frac{\hat{a}}{h_3}\frac{\partial G}{\partial \xi_3}\,, \tag{13.24}$$

$$\hat{a} \times \nabla G = \frac{\hat{n}}{C}\frac{\partial G}{\partial t} - \frac{\hat{t}}{h_3}\frac{\partial G}{\partial \xi_3}\,. \tag{13.25}$$

We also have

$$\hat{t}\cdot\hat{t}' = \alpha(z)\,\alpha(z')\left[1 + f'(z)\,f'(z')\cos(\phi - \phi')\right] \tag{13.26}$$

$$\hat{t}\cdot\hat{a}' = \alpha(z)\,f'(z)\sin(\phi - \phi') \tag{13.27}$$

$$\hat{a}\cdot\hat{a}' = \cos(\phi - \phi') \tag{13.28}$$

$$\hat{t}\cdot\hat{n}' = \alpha(z)\,\alpha(z')\left[f'(z)\cos(\phi - \phi') - f'(z')\right] \tag{13.29}$$

$$\hat{a}\cdot\hat{n}' = -\alpha(z')\sin(\phi - \phi') \tag{13.30}$$

These results follow easily from Eq. (1.12). Hence

$$\hat{t}\cdot\left[\hat{a}' \times \nabla' G\right] = \frac{\hat{t}\cdot\hat{n}'}{C}\frac{\partial G}{\partial t'} - \frac{\hat{t}\cdot\hat{t}'}{h_3'}\frac{\partial G}{\partial \xi_3'}$$

$$= \frac{1}{C}\,\alpha(z)\,\alpha(z')\left[f'(z)\cos(\phi - \phi') - f'(z')\right]\frac{\partial G}{\partial t'}$$

$$- \frac{1}{h_3'}\,\alpha(z)\,\alpha(z')\left[1 + f'(z)\,f'(z')\cos(\phi - \phi')\right]\frac{\partial G}{\partial \xi_3'}\,. \tag{13.31}$$

$$\hat{t}\cdot\left[\hat{t}' \times \nabla' G\right] = -\frac{\hat{t}\cdot\hat{n}'}{\rho'}\frac{\partial G}{\partial \phi'} + \frac{\hat{t}\cdot\hat{a}'}{h_3'}\frac{\partial G}{\partial \xi_3'}$$

$$= -\frac{1}{\rho'}\alpha(z)\,\alpha(z')\left[f'(z)\cos(\phi-\phi') - f'(z')\right]\frac{\partial G}{\partial \phi'}$$

$$+\frac{1}{h_3'}\alpha(z)\,f'(z)\sin(\phi-\phi')\frac{\partial G}{\partial \xi_3'} \quad . \tag{13.32}$$

$$\hat{a}\cdot\left[\hat{t}' \times \nabla' G\right] = -\frac{\hat{a}\cdot\hat{n}'}{\rho'}\frac{\partial G}{\partial \phi'} + \frac{\hat{a}\cdot\hat{a}'}{h_3'}\frac{\partial G}{\partial \xi_3'}$$

$$= \frac{1}{\rho'}\alpha(z')\sin(\phi-\phi')\frac{\partial G}{\partial \phi'} + \frac{1}{h_3'}\cos(\phi-\phi')\frac{\partial G}{\partial \xi_3'} \quad . \tag{13.33}$$

It is convenient at this point to introduce the scalar function $G_m(z, z')$ defined by

$$G_m(z, z') = \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \, \frac{e^{ik_0 R}}{4\pi k_0 R}\, e^{im(\phi-\phi')} \tag{13.34}$$

where

$$R = \left|\bar{r} - \bar{r}'\right| = \sqrt{(z-z')^2 + \left[f(z)\right]^2 + \left[f(z')\right]^2 - 2f(z)\,f(z')\cos(\phi-\phi')} \quad . \tag{13.35}$$

More simply,

$$G_m(z, z') = \int_0^{\pi} d\theta \, \frac{e^{ik_0 R}}{k_0 R} \cos m\theta \tag{13.36}$$

where

$$R = \sqrt{(z-z')^2 + \left[f(z)\right]^2 + \left[f(z')\right]^2 - 2\,f(z)\,f(z')\cos\theta} \quad . \tag{13.37}$$

Equation (13.36) is seen to be identical with the Fourier coefficient (6.4) defined in Section VI .

The angular integrals occurring in the T' matrix elements can now be performed

$$\iint d\phi\, d\phi'\, G \cos m\phi \cos m\phi' = \left[1 + \delta_{m,0}\right] \frac{k_0 G_m}{2} \tag{13.38}$$

$$\iint d\phi\, d\phi'\, G \cos m\phi \cos m\phi' \cos(\phi - \phi') =$$

$$= \left[1 + \delta_{m0}\right] \frac{k_0}{4} \left[G_{m-1} + G_{m+1}\right] \tag{13.39}$$

$$\iint d\phi\, d\phi'\, G \cos m\phi \sin m\phi' \sin(\phi - \phi') =$$

$$= \left[1 + \delta_{m0}\right] \frac{k_0}{4} \left[G_{m+1} - G_{m-1}\right] \tag{13.40}$$

$$\iint d\phi\, d\phi'\, G \sin m\phi \sin m\phi' = \left[1 - \delta_{m0}\right] \frac{k_0 G_m}{2} \tag{13.41}$$

$$\iint d\phi\, d\phi'\, G \sin m\phi \sin m\phi' \cos(\phi - \phi') = \left[1 - \delta_{m0}\right] \frac{k_0}{4} \left[G_{m-1} + G_{m+1}\right] \tag{13.42}$$

$$\iint d\phi\, d\phi'\, G \sin m\phi \cos m\phi' \sin(\phi - \phi') = \left[1 - \delta_{m0}\right] \frac{k_0}{4} \left[G_{m-1} + G_{m+1}\right] \tag{13.43}$$

Eqs. (13.38) - (13.43) have been quoted from Schweitzer.

Let us consider the expression

$$I = \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \left[\frac{\partial}{\partial \phi'} \frac{e^{i k_0 R}}{4\pi k_0 R}\right] e^{im(\phi - \phi')} \tag{13.44}$$

Integrating by parts we have

$$I = \int_0^{2\pi} d\phi \left[\frac{e^{ik_0R}}{4\pi k_0 R} e^{im(\phi - \phi')} \right]_0^{2\pi}$$

$$- \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \frac{e^{ik_0R}}{4\pi k_0 R} \frac{\partial}{\partial \phi'} e^{im(\phi - \phi')}$$

$$= i m \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \frac{e^{ik_0R}}{4\pi k_0 R} e^{im(\phi - \phi')}$$

$$= i m \, G_m(z, z') \ ,$$

that is

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \left[\frac{\partial}{\partial \phi'} \frac{e^{ik_0R}}{4\pi k_0 R} \right] e^{im(\phi - \phi')} = i m \, G_m(z, z') \ . \tag{13.45}$$

Also

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \left[\frac{\partial}{\partial \xi_3'} \frac{e^{ik_0R}}{4\pi k_0 R} \right] e^{im(\phi - \phi')} = \frac{\partial}{\partial \xi_3'} G_m(z, z') \tag{13.46}$$

and

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \left[\frac{\partial}{\partial t'} \frac{e^{ik_0R}}{4\pi k_0 R} \right] e^{im(\phi - \phi')} = \frac{\partial}{\partial t'} G_m(z, z') \tag{13.47}$$

we have then from Eqs. (13.13) - (13.22)

$$T^{m I'}_{2i-1,\,2j-1} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t)$$

$$- \frac{1}{4}\, C\, Z\, K_0 \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, f(z')\, \alpha(z)\, \alpha(z')\, f'(z)$$

$$K_i(t)\, K_j(t')\, \frac{\partial}{\partial t'} \left[G_{m-1} + G_{m+1}\right]\Big]$$

$$+ \frac{1}{2}\, C\, Z\, k_0 \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, f'(z')\, f(z')\, \alpha(z)\, \alpha(z')$$

$$K_i(t)\, K_j(t')\, \frac{\partial}{\partial t'}\, G_m \Big]$$

$$+ \frac{1}{2}\, C^2\, Z\, k_0 \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, f(z')\, \alpha(z)\, \alpha(z')\, \frac{1}{h_3'}$$

$$K_i(t)\, K_j(t')\, \frac{\partial}{\partial \xi_3'}\, G_m \Big]$$

$$+ \frac{1}{4}\, C^2\, Z\, k_0 \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, f(z')\, f'(z)\, f'(z')\, \alpha(z)\, \alpha(z')$$

$$\frac{1}{h_3'}\, K_i(t)\, K_j(t')\, \frac{\partial}{\partial \xi_3'} \left[G_{m-1} + G_{m+1}\right]\Big] \qquad (13.48)$$

$$T^{mI'}_{2i-1, 2j} = - Z C^2 \frac{k_0}{4} \left[1 - \delta_{m0}\right] \cdot$$

$$\int_0^1 dt \int_0^1 dt' \Big[f(z) f(z') \alpha(z) \alpha(z') r'(z) \frac{1}{\rho'} K_i(t) K_j(t')$$

$$\left[(m+1) G_{m+1} + (m-1) G_{m-1}\right]\Big] + Z C^2 m \frac{k_0}{2} \left[1 - \delta_{m0}\right] \cdot$$

$$\int_0^1 dt \int_0^1 dt' \left[f(z) f(z') \alpha(z) \alpha(z') f'(z') \frac{1}{\rho'} K_i(t) K_j(t') G_m \right]$$

$$+ Z C^2 \frac{k_0}{4} \left[1 - \delta_{m0}\right] \cdot$$

$$\int_0^1 dt \int_0^1 dt' \Big[f(z) f(z') \alpha(z) f'(z) K_i(t) K_j(t') \frac{1}{h_3'} \cdot$$

$$\frac{\partial}{\partial \xi_3'} \left[G_{m-1} + G_{m+1}\right]\Big] \tag{13.49}$$

$$T^{mI'}_{2i, 2j} = \frac{1}{2} C Z \pi \int_0^1 dt\, f(z) K_i(t) K_j(t)$$

$$+ Z C^2 \frac{k_0}{4} \int_0^1 dt \int_0^1 dt' \Big[f(z) f(z') \frac{1}{\rho'} \alpha(z') K_i(t) K_j(t') \cdot$$

$$\left[1 + \delta_{m0}\right] \left[(m+1) G_{m+1} - (m-1) G_{m-1}\right]\Big]$$

$$+ Z C^2 \frac{k_0}{4} \left[1 + \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z) f(z') \frac{1}{h_3'} K_i(t) K_j(t')$$

$$\frac{\partial}{\partial \xi_3'} \left[G_{m-1} + G_{m+1}\right]\Big] \tag{13.50}$$

$$T^{mI'}_{2i-1,\,2j+2N-1} = -Z\iint dS\, dS' \left(\sin m\phi\, K_i(t)\right) \hat{t}\cdot\left[\hat{a}' \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

$$\hat{t}\cdot\left[\hat{a}' \times \nabla' G\right] = \frac{1}{C}\,\alpha(z)\,\alpha(z')\left[f'(z)\cos(\phi-\phi') - f'(z')\right]\frac{\partial G}{\partial t'}$$

$$-\frac{1}{h_3'}\,\alpha(z)\,\alpha(z')\left[1 + f'(z)\, f'(z')\cos(\phi-\phi')\right]\frac{\partial G}{\partial \xi_3'}$$

$$\therefore$$

$$T^{mI'}_{2i-1,\,2j+2N-1} \equiv 0 \tag{13.51}$$

$$T^{mI'}_{2i-1,\,2j+2N} = Z\iint dS\, dS' \left(\sin m\phi\, K_i(t)\right) \hat{t}\cdot\left[\hat{t}' \times \nabla' G\right] \sin m\phi'\, K_j(t')$$

$$\hat{t}\cdot\left[t' \times \nabla' G\right] = -\frac{1}{\rho'}\,\alpha(z)\,\alpha(z')\left[f'(z)\cos(\phi-\phi') - f'(z')\right]\frac{\partial G}{\partial \phi'} +$$

$$+\frac{1}{h_3'}\,\alpha(z)\, f'(z)\sin(\phi-\phi')\,\frac{\partial G}{\partial \xi_3'}$$

$$\therefore\quad T^{mI'}_{2i-1,\,2j+2N} \equiv 0 \tag{13.52}$$

$$T^{mI'}_{2i\underline{+}2N,\,2j} = Z\iint dS\, dS' \sin m\phi\, K_i(t)\, \hat{a}\cdot\left[\hat{t}' \times \nabla' G\right] \cos m\phi'\, K_j(t')$$

$$\hat{a}\cdot\left[\hat{t}' \times \nabla' G\right] = \frac{1}{\rho'}\,\alpha(z')\sin(\phi-\phi')\,\frac{\partial G}{\partial \phi'} + \frac{1}{h_3'}\cos(\phi-\phi')\,\frac{\partial G}{\partial \xi_3'}$$

$$\therefore\quad T^{mI'}_{2i+2N,\,2j} \equiv 0 \tag{13.53}$$

$$
\begin{aligned}
T^{m\,I'}_{2i+2N-1,\ 2j+2N-1} &= \frac{1}{2}\ Z\,C\,\pi \int_0^1 dt\ f(z)\ K_i(t)\ K_j(t) \\
&- Z\,C\,\frac{k_0}{4}\left[1+\delta_{m0}\right]\int_0^1 dt \int_0^1 dt' \Big[K_i(t)\ K_j(t')\ \alpha(z)\ \alpha(z')\ f'(z) \\
&\quad f(z)\ f(z')\ \frac{\partial}{\partial t'}\left[G_{m-1}+G_{m+1}\right]\Big] \\
&+ Z\,C\,\frac{k_0}{2}\left[1+\delta_{m0}\right]\int_0^1 dt \int_0^1 dt' \Big[f(z)\ f(z')\ K_i(t)\ K_j(t') \\
&\quad \alpha(z)\ \alpha(z')\ f'(z')\ \frac{\partial}{\partial t'}\ G_m\Big] \\
&+ Z\,C^2\,\frac{k_0}{2}\left[1+\delta_{m0}\right]\int_0^1 dt \int_0^1 dt'\ f(z)\ f(z')\ K_i(t)\ K_j(t') \\
&\quad \frac{1}{h_3'}\ \alpha(z)\ \alpha(z')\ \frac{\partial}{\partial \xi_3'}\ G_m \\
&+ Z\,C^2\left[1+\delta_{m0}\right]\frac{k_0}{4}\int_0^1 dt \int_0^1 dt'\ f(z)\ f(z')\ K_i(t)\ K_j(t') \\
&\quad \frac{1}{h_3'}\ \alpha(z)\ \alpha(z')\ f'(z)\ f'(z')\ \frac{\partial}{\partial \xi_3'}\left[G_{m-1}+G_{m+1}\right]. \qquad (13.54)
\end{aligned}
$$

$$T^{mI'}_{2i+2N-1,\ 2j+2N} = Z\ C^2 \frac{1}{4} k_0 \left[1 + \delta_{m0}\right] \int_0^1 dt \int_0^1 dt'\ f(z)\ f(z')$$

$$\frac{1}{\rho'} \alpha(z)\ \alpha(z')\ f'(z)\ K_i(t)\ K_j(t') \left[(m+1)\ G_{m+1} + (m-1)\ G_{m-1}\right]$$

$$- Z\ C^2\ m \left[1 + \delta_{m0}\right] \frac{k_0}{2} \int_0^1 dt \int_0^1 dt'\ f(z)\ f(z') \frac{1}{\rho'}\ \alpha(z)\ \alpha(z')\ f'(z')$$

$$K_i(t)\ K_j(t')\ G_m$$

$$+ Z\ C^2 \left[1 + \delta_{m0}\right] \frac{k_0}{4} \int_0^1 dt \int_0^1 dt'\ f(z)\ f(z') \frac{1}{h_3'}\ \alpha(z)\ f'(z)$$

$$K_i(t)\ K_j(t') \frac{\partial}{\partial \xi_3'} \left[G_{m+1} - G_{m-1}\right] \qquad (13.55)$$

$$T^{mI'}_{2i+2N-1,\ 2j} = Z \iint dS\ dS'\ \cos m\phi\ K_i(t)\ \hat{t} \cdot \left[\hat{t}' \times \nabla' G\right] \cos m\phi'\ K_j(t')$$

$$\hat{t} \cdot \left[\hat{t}' \times \nabla' G\right] = -\frac{1}{\rho'} \alpha(z)\ \alpha(z') \left[f'(z) \cos(\phi - \phi') - f'(z') \frac{\partial G}{\partial \phi'} + \right.$$

$$+ \frac{1}{h_3'} \alpha(z)\ f'(z) \sin(\phi - \phi') \frac{\partial G}{\partial \xi_3'}$$

$\therefore$

$$T^{mI'}_{2i+2N-1,\ 2j} \equiv 0 \qquad (13.56)$$

$$T^{mI'}_{2i+2N,\ 2j+2N} = \frac{1}{2} C Z \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t)$$

$$+ Z C^2 k_0 \frac{1}{4} \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt'\, f(z)\, f(z')\, K_i(t)\, K_j(t') \frac{1}{\rho'} \alpha(z')$$

$$\left[(m+1)\, G_{m+1} - (m-1)\, G_{m-1}\right]$$

$$+ Z C^2 k_0 \frac{1}{4} \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt'\, f(z)\, f(z')\, K_i(t)\, K_j(t') \frac{1}{h_3'}$$

$$\frac{\partial}{\partial \xi_3'} \left[G_{m-1} + G_{m+1}\right] \tag{13.57}$$

In view of equations (13.51, 13.52, 13.53, 13.56) and Eq. (8.7), we see that the T' matrix is of the form

$$T^{I'} = \begin{bmatrix} T_A^{I'} & 0 \\ 0 & T_B^{I'} \end{bmatrix} \tag{13.58}$$

and

$$T^{II'} = \begin{bmatrix} T_A^{II'} & 0 \\ 0 & T_B^{II'} \end{bmatrix} = \begin{bmatrix} T_B^{I'} & 0 \\ 0 & T_A^{I'} \end{bmatrix} \tag{13.59}$$

What we have in fact shown is that the problem admits of a lower dimensional representation than that assumed by Eqs. (7.4) and (7.5) . In other words we have

$$\bar{K}^{mI}(\bar{r}) \approx \sum_{i=1}^{N} \left\{ C^{mI}_{2i-1} \sin m\phi\, \hat{t} + C^{mI}_{2i} \cos m\phi\, \hat{a} \right\} K_i(t) \tag{13.60}$$

and

$$\overline{K}^{mII}(\overline{r}) \approx \sum_{i=1}^{N} \left\{ C_{2i-1}^{mII} \cos m\phi\, \hat{t} + C_{2i}^{mII} \sin m\phi\, \hat{a} \right\} K_i(t) . \tag{13.61}$$

A direct proof of this would be desirable.

For the T matrix we have

$$T^{I} = T^{I,\, p.c.} + \eta\, T^{I'} \tag{13.62}$$

$$T^{II} = T^{II,\, p.c.} + \eta\, T^{II'} \tag{13.63}$$

and

$$T^{mI'}_{2i-1,\, 2j-1} = \frac{1}{2} C Z \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t)$$

$$- C^2 Z \frac{k_0}{4} \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \left[f(z)\, f(z')\, f'(z)\, \alpha(z) \right.$$

$$\left. K_i(t)\, K_j(t') \frac{\partial}{\partial z'} (G_{m-1} + G_{m+1}) \right] \tag{13.64}$$

$$T^{mI'}_{2i-1,\, 2j} = - C^2 Z \frac{k_0}{4} \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \left\{ f(z)\, \alpha(z)\, \alpha(z') \right.$$

$$\left[f'(z) \left[(m+1)\, G_{m+1} + (m-1)\, G_{m-1} \right] - 2 f'(z')\, G_m\, m \right.$$

$$\left. \left. + f(z')\, f'(z)\, f'(z') \frac{\partial}{\partial z'} (G_{m-1} - G_{m+1}) \right] K_i(t)\, K_j(t') \right\} \tag{13.65}$$

$$T^{mI'}_{2i,\,2j} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\; f(z)\, K_i(t)\, K_j(t)$$

$$+ C^2 Z \frac{k_0}{4}\left[1 + \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, K_i(t)\, K_j(t')$$

$$\Big\{ \alpha(z') \left[(m+1)\, G_{m+1} - (m-1)\, G_{m-1}\right] - f(z')\, f'(z')\, \alpha(z')\, .$$

$$\frac{\partial}{\partial z'} (G_{m-1} + G_{m+1}) \Big\} \Big] \qquad (13.66)$$

$$T^{mII'}_{2i-1,\,2j-1} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\; f(z)\, K_i(t)\, K_j(t)$$

$$- C^2 Z \frac{k_0}{4}\left[1 + \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, f(z')\, f'(z)\, \alpha(z)$$

$$K_i(t)\, K_j(t') \frac{\partial}{\partial z'} (G_{m-1} + G_{m+1}) \Big] \qquad (13.67)$$

$$T^{m\,II'}_{2i-1,\,2j} = C^2 Z \frac{k_0}{4}\left[1 + \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big\{ f(z)\, \alpha(z)\, \alpha(z')$$

$$\Big[f'(z) \left[(m+1)\, G_{m+1} + (m-1)\, G_{m-1}\right] - 2\, f'(z')\, G_m \quad m$$

$$+ f(z')\, f'(z)\, f'(z') \frac{\partial}{\partial z'} (G_{m-1} - G_{m+1}) \Big] K_i(t)\, K_j(t') \Big\} \qquad (13.68)$$

$$T^{m\,II'}_{2i,\,2j} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t)$$

$$+ C^2\, Z\, \frac{k_0}{4} \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, \alpha(z')\, K_i(t)\, K_j(t')$$

$$\Big\{ (m+1)\, G_{m+1} - (m-1)\, G_{m-1} - f(z')\, f'(z')\, \frac{\partial}{\partial z'} (G_{m-1} + G_{m+1}) \Big\} \Big] \qquad (13.69)$$

Going from Eqs. (13.48) - (13.57) to Eqs. (13.64) - (13.69), we made use of the results

$$\rho' = f(z') \qquad (13.70)$$

$$1 + \left[f'(z')\right]^2 = \frac{1}{\alpha(z')^2}$$

and

$$\frac{\partial}{\partial t'} = C\, \hat{t}' \cdot \nabla' \qquad (13.71)$$

$$\frac{\partial}{\partial \xi_3'} = h_3'\, \hat{n} \cdot \nabla' \qquad (13.72)$$

From Eqs. (1.12) we have

$$\hat{t}' = f'(z')\, \alpha(z')\, \cos \phi'\, \hat{x} + f'(z')\, \alpha(z')\, \sin \phi'\, \hat{y} + \alpha(z')\, \hat{z}$$

and

$$\hat{n}' = \cos \phi'\, \alpha(z')\, \hat{x} + \sin \phi'\, \alpha(z')\, \hat{y} + (-f'(z')\, \alpha(z'))\, \hat{z}$$

this together with

$$\nabla' = \hat{x}\, \frac{\partial}{\partial x'} + \hat{y}\, \frac{\partial}{\partial y'} + \hat{z}\, \frac{\partial}{\partial z'}$$

and the fact that $G_m = G_m(z, z')$, gives the results

$$\frac{\partial G_m}{\partial t'} = C\,\alpha(z')\,\frac{\partial G_m}{\partial z'}\;, \qquad (13.73)$$

$$\frac{\partial G_m}{\partial \xi_3'} = -\,h_3'\,f'(z')\,\alpha(z')\,\frac{\partial G_m}{\partial z'} \qquad (13.74)$$

the b-column matrix is now, of course, made up of the elements b_1^{III} through b_{2N}^{III} only.

Section XIV: Explicit Choice of the Trial Functions

The usual choice of the trial functions $K_i(t)$ in a variational principle is of shapes closely related to the expected shape of the current; in this way only a few shapes are needed and N will be small.

The unfortunate part is that each of the matrix elements involves a laborious quadrature over the entire unit square

$$\int_0^1 dt \int_0^1 dt'$$

These quadratures require an unacceptable amount of time. If $N \sim 20$, the T's will be 40 x 40 symmetric matrices. $\frac{40 \times 41}{2} = 820$ of the matrix elements would have to be evaluated for each mode m .

We conclude that global trial functions are unacceptable, since the T quadratures are too time-consuming.

For this reason we shall pick localized trial functions so that the integrals go, not over the entire unit square, but over a small subportion.

A convenient choice of trial function is obtained by approximating the $K^{m\,III\,t}(t)$ and $K^{m\,III\,\phi}(t)$ by piecewise-linear functions of **t**. We break the unit interval into N points, $0 = t_1 < t_2 < \ldots < t_N = 1$, and sample the current at these N points. We define

$$C^{m\,III}_{2i-1} = K^{m\,III\,t}(t_i) \qquad i = 1, 2, \ldots, N \tag{14.1}$$

$$C^{m\,III}_{2i} = K^{m\,III\,\phi}(t_i) \qquad i = 1, 2, \ldots, N \tag{14.2}$$

= sampled values of the current.

and draw straight lines connecting these sampled values.

The choice of piecewise-linear trial current is equivalent to the statement

$$K^{m\,III\,t}(t) \simeq \sum_{i=1}^{N} C^{m\,III}_{2i-1} K_i(t) \tag{14.3}$$

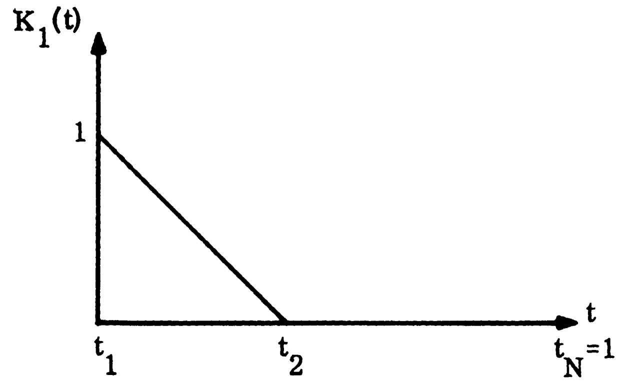
$$K^{m\,III\,\phi}(t) \simeq \sum_{i=1}^{N} C^{m\,III}_{2i} K_i(t) \tag{14.4}$$

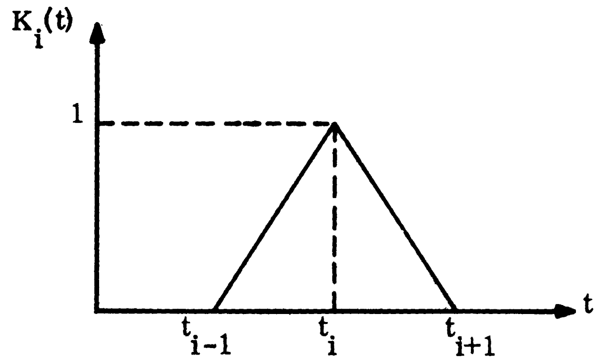
where

$$K_1(t) = \begin{cases} \dfrac{t_2 - t}{t_2 - t_1} & 0 = t_1 \leqslant t \leqslant t_2 \\ 0 & \text{elsewhere} \end{cases} \tag{14.5}$$

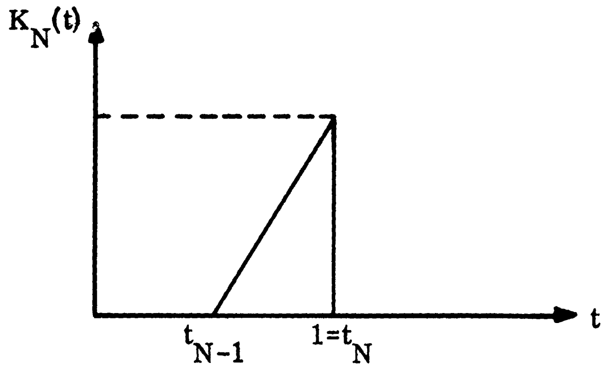
$$K_i(t) = \begin{cases} \dfrac{t - t_{i-1}}{t_i - t_{i-1}} & t_{i-1} \leqslant t \leqslant t_i \\ \dfrac{t_{i+1} - t}{t_{i+1} - t_i} & t_i \leqslant t \leqslant t_{i+1} \\ 0 \quad \text{elsewhere} & 2 \leqslant i \leqslant N - 1 \end{cases} \tag{14.6}$$

$$K_N(t) = \begin{cases} \dfrac{t - t_{N-i}}{t_N - t_{N-i}} & t_{N-1} \leqslant t \leqslant t_N = 1 \\ 0 & \text{elsewhere} \end{cases} \tag{14.7}$$





$2 \leqslant i \leqslant N-1$



This choice of trial function has the following convenient properties:

1. Most importantly, the K's are localized; the integrals for the T's are only over a small portion of the unit square.

2. The K_i (t) are simple to generate and to differentiate.

3. The piecewise linear approximation is a good one: If one wished to approximate a function with 10 maxima a 30th degree polynomial and a 30th degree linear fit are probably equally good - - while the T's are much easier to generate in the latter case.

4. If the piecewise - linear approximation is used the C's have a very simple interpretation via (14.1) and (14.2) - - they are the sampled values of the current (and not merely the coefficient of K_i (t) in the sum (14.3) and (14.4)) . Indeed a print-out of the C's, once the variational equations are solved, is precisely a printout of the sampled values of the current, these samplings being carried out at $t_1, t_2, \dots, t_N$.

Section XV: Classification of the Different Types of Matrix Elements

We define the cell C (i, j) as the rectangular region (t, t') with

$$t_i \leqslant t \leqslant t_{i+1} \tag{15.1}$$

$$t_j \leqslant t' \leqslant t_{j+1} \tag{15.2}$$

Most of the T integrals go over 4 cells. For example T_{59} goes over the 4 cells C(3,5) , C(3,6) , C(4,5) and C(4,6) . However since K_1(t) and K_N (t) , are special, some of the T's involve integrals over only 2 cells. For example T_{13} involves integrals over C(1,1) and C(1,2) . Finally T_{11}, T_{12}, T_{21} and T_{22} involve integrals over the single cell C(1,1) whereas $T_{2N-1,\,2N-1}$, $T_{2N-1,\,2N}$, $T_{2N,\,2N-1}$ and $T_{2N,\,2N}$ involve integrals over the single cell C(N-1, N-1) .

Thus the T_{ij} can be classified by whether they involve integrals over 1, 2, or 4 cells.

We next distinguish between integrals which go over non-diagonal cells $C(i, j)$ $i \neq j$ and integrals which go over diagonal cells $C(i, i)$

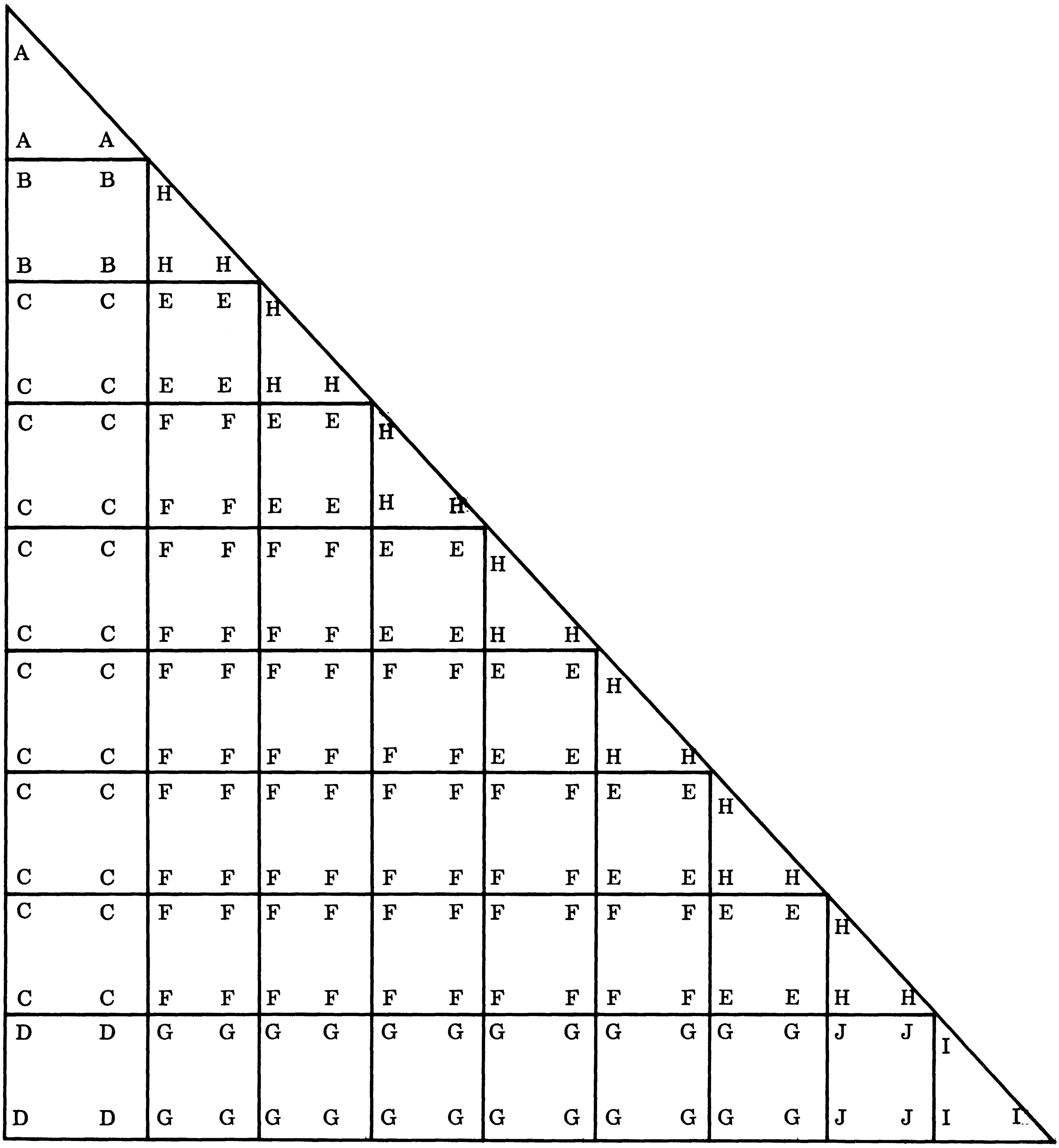
The 3 T's listed above which go solely over the cell $C(1, 1)$ involve purely diagonal cells. In addition, diagonal and almost diagonal terms like T_{33}, T_{34}, T_{43}, and T_{44} involve integrals over 4 cells, two of which are diagonal (here $C(1, 1)$ and $C(2, 2)$) and two of which one non-diagonal (here $C(1, 2)$ and $C(2, 1)$). Some integrals, say for T_{13}, go over two cells, one of which is diagonal and the other of which is non-diagonal. Some integrals, say for T_{53}, go over 4 cells (here $C(1, 2)$, $C(1, 3)$, $C(2, 2)$, $C(2, 3)$), one of which is diagonal and 3 of which are non-diagonal. And finally most of the T's involve integrals over 4 non-diagonal cells.

Thus the T_{ij} can be classified by whether the cells involved are diagonal or non-diagonal.

Putting these classifications together, we find that for $N \geqslant 4$ there are exactly 10 types of T's, which we shall denote as types A, B, C, D, E, F, G, H, I, J.

These 10 types are shown in the next figure for the case N = 9, where T is an 18 x 18 matrix. Since T is symmetric, only the elements below or on the main diagonal need be shown.

The classification is given in Table I. We assume that $N \geqslant 4$.



The Ten Types of Matrix Elements if T is 18 x 18

TABLE I

The Ten Types of T Matrix Elements when $N > 4$.

Type	Description of Matrix Elements	Number of Cells Integrated Over	Cell Numbering	Cell Diagonal?
A	$\left\{ T_{11},\ T_{21},\ T_{22} \right\}$	1	C(1, 1)	yes
B	$\left\{ T_{31},\ T_{32},\ T_{41},\ T_{42} \right\}$	2	C(1, 1) C(2, 1)	yes no
C	$\left\{ T_{2i-1,\,1},\ T_{2i-1,\,2},\ T_{2i,\,1},\ T_{2i,\,2} \right\}$ $3 \leqslant i \leqslant N-1$	2	C(i-1, 1) C(i, 1)	no no
D	$\left\{ T_{2N-1,\,1},\ T_{2N-1,\,2},\ T_{2N,\,1},\ T_{2N,\,2} \right\}$	1	C(N-1, 1)	no
E	$\left\{ T_{2i-1,\,2i-3},\ T_{2i-1,\,2i-2},\ T_{2i,\,2i-3},\ T_{2i,\,2i-2} \right\}$ $3 \leqslant i \leqslant N-1$	4	C(i-1, i-2) C(i-1, i-1) C(i, i-2) C(i, i-1)	no yes no no

Type	Description of Matrix Elements	Number of Cells Integrated Over	Cell Numbering	Cell Diagonal?
F	$\left\{ T_{2i-1,\,2j-1},\ T_{2i-1,\,2j},\ T_{2i,\,2j-1},\ T_{2i,\,2j} \right\}$ $2 \leqslant j \leqslant i-2$ $4 \leqslant i \leqslant N-1$	4	C(i-1, j-1) C(i-1, j) C(i, j-1) C(i, j)	no no no no
G	$\left\{ T_{2N-1,\,2j-1},\ T_{2N-1,\,2j},\ T_{2N,\,2j-1},\ T_{2N,\,2j} \right\}$ $2 \leqslant j \leqslant N-2$	2	C(N-1, j-1) C(N-1, j)	no no
H	$\left\{ T_{2i-1,\,2i-1},\ T_{2i,\,2i-1},\ T_{2i,\,2i} \right\}$ $2 \leqslant i \leqslant N-1$	4	C(i-1, i-1) C(i-1, i) C(i, i-1) C(i, i)	yes no no yes
I	$\left\{ T_{2N-1,\,2N-1},\ T_{2N,\,2N-1},\ T_{2N,\,2N} \right\}$	1	C(N-1, N-1)	yes
J	$\left\{ T_{2N-1,\,2N-3},\ T_{2N-1,\,2N-2},\ T_{2N,\,2N-3},\ T_{2N,\,2N-2} \right\}$	2	C(N-1, N-2) C(N-1, N-1)	no yes

Type Description of Matrix Elements Number of Cells Integrated Over Cell Numbering Cell Diagonal?

Table I must be kept in mind while programming the computer to calculate the matrix elements of T .

Diagonal Cells and Cells Bordering a Diagonal Cell

The T-matrix elements of types A, B, E, H, I and J involve one or more integrals over diagonal cells. These integrals must be done analytically, due to the integrable singularity in the G_m .

Also T-matrix elements involving integrals over cells bordering a diagonal cell require special treatment since the integrand becomes singular on one of the borders of the cell.

This analysis has been carried through in detail by the author for the elements of $T^{m\,p.c.}$. (See Appendix "Evaluation of T-matrix elements involving integration over a diagonal cell and/or integration over a cell bordering a diagonal cell".)

The $T^{m'}$-matrix elements involve integrals of the form

$$\int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t_{i+1}} dt' \, F_1(t') \, F_2(t) \, \frac{\partial}{\partial z'} \, G_m(z, z') \qquad (15.3)$$

and

$$\int_{t_i}^{t_{i+1}} dt \int_{t_{i+1}}^{t_{i+2}} dt' \, F_1(t') \, F_2(t) \, \frac{\partial}{\partial z'} \, G_m(z, z') \qquad (15.4)$$

Remembering the equation

$$t = t(z) = \frac{1}{C} \int_0^z dw \sqrt{1 + \left[f'(w)\right]^2}$$

and

$$\frac{\partial}{\partial z} = \frac{1}{C} \sqrt{1 + \left[f'(z)\right]^2} \; \frac{\partial}{\partial t}$$

we see that (15. 3) and (15. 4) can be put in the form

$$\int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t_{i+1}} dt' \; F_1(t') \, F_2(t) \frac{\partial}{\partial t'} \, G_m(t, t') \tag{15.5}$$

and

$$\int_{t_i}^{t_{i+1}} dt \int_{t_{i+1}}^{t_{i+2}} dt' \; F_1(t') \, F_2(t) \frac{\partial}{\partial t'} \, G_m(t, t') \tag{15.6}$$

where $F_1(t)$ and $F_2(t)$ are different from those of (15. 3) and (15. 4) .

Eqs. (15. 5) and (15. 6) can be reduced to the form considered in the Appendix by an integration by parts.

We have

$$\int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t_{i+1}} dt' \; F_1(t') \, F_2(t) \frac{\partial}{\partial t'} \, G_m(t, t') =$$

$$= \int_{t_i}^{t_{i+1}} dt \; F_2(t) \, F_1(t_{i+1}) \, G_m(t, t_{i+1}) - \int_{t_i}^{t_{i+1}} dt \; F_2(t) \, F_1(t_i) \, G_m(t, t_i)$$

$$- \int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t_{i+1}} dt' \; F_2(t) \, F_1'(t') \, G_m(t, t') \tag{15.7}$$

The first two integrals in the right hand side of (15. 7) can be handled as Eqs. (53) - (55) of the Appendix. The third integral is of the form of Eq. (1) of the Appendix.

An identical analysis is applicable to Eq. (15. 4).

Section XVI: Some Comments on the T'-matrix

For $m \geqslant 1$, we compare (13.64) with (13.67), (13.65) with (13.68), and (13.66) with (13.69) to conclude

$$T_{ij}^{mI'} = (-)^{i+j}\, T_{ij}^{mII'} \qquad m \geqslant 1 \tag{16.1}$$

This result also holds for $T^{m\,p.c.}$ as noted in Eq. (12.28). Hence for the elements of the T-matrix (13.62) or (13.63) one has

$$T_{ij}^{mI} = (-)^{i+j}\, T_{ij}^{mII} \qquad m \geqslant 1 \tag{16.2}$$

As indicated by Schweitzer, this result could have been anticipated by noting that (8.14) and (8.18), (8.17) and (8.19), (8.23) and (8.22) differ only by the replacement of sin $m\phi$ by cos $m\phi$ and viceversa. This replacement could be achieved by the substitution, for $m \geqslant 1$, $\phi = \frac{\pi}{2m} - \phi''$ which sends sin $m\phi$ into cos $m\phi''$, cos $m\phi$ into sin $m\phi''$. The same transformation (which is a shift in the angle origin plus a reflection through the plane of symmetry) sends (neglecting the shift in origin) $\hat{a}$ into $-\hat{a}$ and $\hat{t}$ into $\hat{t}$. This explains why the $T_{2i,\,2j}$ and $T_{2i-1,\,2j-1}$, which involve $\hat{a}\cdot\Gamma\cdot\hat{a}$ and $\hat{t}\cdot\Gamma\cdot\hat{t}$, maintain their sign, whereas $T_{2i-1,\,2j}$ and $T_{2i,\,2j-1}$ which involve $\hat{t}\cdot\Gamma\cdot\hat{a}$ and $\hat{a}\cdot\Gamma\cdot\hat{t}$, undergo a change in sign.

From Eqs. (13.64) - (13.69) we have when m = 0

$$T_{2i-1,\,2j-1}^{0I'} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\; f(z)\, K_i(t)\, K_j(t) \tag{16.3}$$

$$T_{2i-1,\,2j}^{0I'} = 0 \tag{16.4}$$

$$T_{2i,\,2j}^{0I'} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t)$$

$$+ C^2\, Z\, k_0 \int_0^1 dt \int_0^1 dt' \left\{ f(z)\, K_i(t)\, K_j(t')\, \alpha(z') \left[G_1 - f(z')\, f'(z')\, \frac{\partial G_1}{\partial z'} \right] \right\} \tag{16.5}$$

$$T^{0\,II'}_{2i-1,\,2j-1} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t)$$

$$- C^2\, Z\, k_0 \int_0^1 dt \int_0^1 dt' \left[f(z)\, f(z')\, f'(z)\, \alpha(z)\, K_i(t)\, K_j(t') \frac{\partial}{\partial z'}\, G_1 \right]$$

(16.6)

$$T^{0\,II'}_{2i-1,\,2j} = 0 \qquad (16.7)$$

$$T^{0\,II'}_{2i,\,2j} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, K_i(t)\, K_j(t) \qquad (16.8)$$

We notice that the only elements different from zero are

$$T^{0I'}_{2i-1,\,2j-1} = T^{0\,II'}_{2i,\,2j}\,,\quad T^{0\,I'}_{2i,\,2j}\,,\ \text{and}\ T^{0\,II'}_{2i-1,\,2j-1} \qquad (16.9)$$

Hence for m = 0 we have to store only one matrix $T^{0'}$ whose entries are

$$\tilde{T}^{0'}_{2i-1,\,2j-1} = T^{0\,I'}_{2i-1,\,2j-1} \qquad (16.10)$$

$$\tilde{T}^{0'}_{2i,\,2j} = T^{0I'}_{2i,\,2j} \qquad (16.11)$$

$$\tilde{T}^{0'}_{2i-1,\,2j} = \tilde{T}^{0'}_{2j,\,2i-1} = T^{0\,II'}_{2i-1,\,2j-1} \qquad (16.12)$$

from which both $T^{0I'}$ and $T^{0\,II'}$ may be recovered

$$T^{0\,I'}_{ij} = \begin{cases} \tilde{T}^{0'}_{ij} & \text{if both } i \text{ and } j \text{ are odd} \\ \tilde{T}^{0'}_{ij} & \text{if both } i \text{ and } j \text{ are even} \\ 0 & \text{if } i \text{ odd and } j \text{ even or viceversa} \end{cases} \tag{16.13}$$

$$T^{0\,II'}_{ij} = \begin{cases} \tilde{T}^{0'}_{i,\, j+1} & \text{if both } i \text{ and } j \text{ are odd} \\ \tilde{T}^{0'}_{i-1,\, j-1} & \text{if both } i \text{ and } j \text{ are even} \\ 0 & \text{if } i \text{ odd and } j \text{ even or viceversa .} \end{cases} \tag{16.14}$$

Similarly, the result $T^{mI'}_{ij} = (-)^{i+j}\ T^{m\,II'}_{ij}$ for $m \geqslant 1$ allows a great simplification in storage, since only the T's for the II polarization need be put on tape - - to be later withdrawn and inverted.

For the perfect conductor we have the results

$$T^{0I,\ p.c.}_{ij} = 0 \text{ unless } i \text{ and } j \text{ are both even}$$

and $$T^{0II,\ p.c.}_{ij} = 0 \text{ unless } i \text{ and } j \text{ are both odd.}$$

Hence for $m = 0$ we store on tape the matrix elements $T^{0,\ I,\ p.c.}_{2i,\ 2j}$ and $T^{0\,II,\ p.c.}_{2i-1,\ 2j-1}$ (which are the only matrix elements of $T^{0\,I\ p.c.}$ and $T^{0\,II\ p.c.}$ which are non-zero). That is, we construct the matrix

$$\tilde{T}^{0,\ p.c.}_{ij} = \begin{cases} T^{0I,\ p.c.}_{ij} & \text{if } i \text{ and } j \text{ are both even} \\ T^{0\,II,\ p.c.}_{ij} & \text{if } i \text{ and } j \text{ are both odd} \end{cases} \tag{16.15}$$

from which $T^{0I,\ p.c.}$ and $T^{0\,II,\ p.c.}$ may be obtained

$$T_{ij}^{0I,\ p.c.} = \begin{cases} \tilde{T}_{ij}^{0,\ p.c.} & \text{if } i \text{ and } j \text{ are both even} \\ 0 & \text{otherwise} \end{cases} \tag{16.16}$$

$$T_{ij}^{0\,II,\ p.c.} = \begin{cases} \tilde{T}_{ij}^{0,\ p.c.} & \text{if } i \text{ and } j \text{ are both odd} \\ 0 & \text{otherwise} \end{cases} \tag{16.17}$$

In this fashion we can store on tape, for <u>each</u> m , a <u>single</u> matrix T^m from which both T^{mI} and T^{mII} may be recovered.

More formally, for each m we shall compute a symmetric matrix T_{ij}^m

$$T_{ij}^m = T_{ji}^m \tag{16.18}$$

$$T^m = \tilde{T}^{m,\ p.c.} + \eta\, \tilde{T}^{m'} \tag{16.19}$$

$$T_{ij}^m = T_{ij}^{m\,II,\ p.c.} + \eta\, T_{ij}^{m\,II'} \qquad m \geqslant 1 \tag{16.20}$$

from which both T^{mI} and $T^{m\,II}$ may be recovered by

$$T_{ij}^{m\,II} = \begin{cases} \tilde{T}_{i\,j}^{0,\ p.c.} + \eta\, \tilde{T}_{i,\ j+1}^{0'} & \text{if both } i \text{ and } j \text{ are odd} \\ \eta\, \tilde{T}_{i-1,\ j-1}^{0'} & \text{if both } i \text{ and } j \text{ are even} \\ 0 & \text{if } i \text{ odd and } j \text{ even or viceversa} \\ T_{ij}^m & m \geqslant 1 \end{cases}$$

and

$$T^{mI}_{ij} = \begin{cases} \tilde{T}^{0,\,p.c.}_{ij} + \eta\, \tilde{T}^{0'}_{ij} & \text{if both i and j are even} \\ \eta\, \tilde{T}^{0'}_{ij} & \text{if both i and j are odd} \\ 0 & \text{if i odd and j even or viceversa} \\ (-)^{i+j}\, T^{m}_{ij} & m \geqslant 1 \;. \end{cases}$$

The result (16.2) is extremely convenient for two reasons. We have already discussed how it halves, for $m \geqslant 1$, the generation and storage of the T matrices for the two polarizations.

Second, (16.2) also implies that a single inversion for each $m \geqslant 1$, gives both $\left[T^{m\,I}\right]^{-1}$ and $\left[T^{m\,II}\right]^{-1}$. This cuts the labor in inversion in half.

More concretely, it is easy to show that

$$\left[T^{m\,I}\right]^{-1}_{ij} = (-)^{i+j} \left[T^{m\,II}\right]^{-1}_{ij} \qquad m \geqslant 1 \;. \tag{16.21}$$

The proof, due to Schweitzer, is:

$$\sum_k \left[T^{m\,I}\right]^{-1}_{ik} T^{m\,I}_{kj} = \sum_k (-)^{i+k} \left[T^{m\,II}\right]^{-1}_{ik} (-)^{k+j}\, T^{m\,II}_{kj}$$

$$= (-)^{i+j} \left\{ \left[T^{m\,II}\right]^{-1} T^{m\,II} \right\}_{ij} = (-)^{i+j}\, \delta_{ij} = \delta_{ij} \;. \tag{16.22}$$

Consequently the inversion of $T^{m\,II}$ for $m \geqslant 1$ automatically yields by (16.21) the inverse of $T^{m\,I}$.

At this point the reader can turn to Section XXXIII of P. Schweitzer's report, for the consideration of the sequence used in generating the T matrix elements, and the desired storage scheme.

Section XVII: The case when the surface impedance η is a function of the arc length: $\eta = \eta(t)$.

It is easy to see, by merely following the steps of the derivation, that Eq. (4.9) is valid even when η is a function of the arc length t .

Similarly all the statements regarding the variational principle are also valid.

The only change brought about is the fact that η has to be kept under the sign of integration.

The T matrix is now

$$T^{m\,III} = T^{m\,III,\,p.c.} + T^{m\,III'} \tag{17.1}$$

The elements of $T^{m\,III'}$ are

$$T^{m\,I'}_{2i-1,\,2j-1} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, \eta(t)\, K_i(t)\, K_j(t)$$

$$- C^2 Z \frac{k_0}{4} \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, f(z')\, f'(z)\, \alpha(z)\, \eta(t')$$

$$K_i(t)\, K_j(t') \frac{\partial}{\partial z'} (G_{m-1} + G_{m+1}) \Big] . \tag{17.2}$$

$$T^{m\,I'}_{2i-1,\,2j} = - C^2 Z \frac{k_0}{4} \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big\{ f(z)\, \alpha(z)\, \alpha(z')\, \eta(t')$$

$$\Big[f'(z) \Big[(m+1)\, G_{m+1} + (m-1)\, G_{m-1}\Big] - 2\, f'(z')\, m\, G_m$$

$$+ f(z')\, f'(z)\, f'(z') \frac{\partial}{\partial z'} (G_{m-1} - G_{m+1}) \Big]\, K_i(t)\, K_j(t') \Big\} \tag{17.3}$$

$$T^{m\,I'}_{2i,\,2j} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, \eta(t)\, K_i(t)\, K_j(t)$$

$$+ C^2 Z \frac{k_0}{4} \left[1 + \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, K_i(t)\, K_j(t')\, \eta(t')$$

$$\Big\{ \alpha(z') \left[(m+1)\, G_{m+1} - (m-1)\, G_{m-1}\right] - f(z')\, f'(z')\, \alpha(z')$$

$$\frac{\partial}{\partial z'} (G_{m-1} + G_{m+1}) \Big\} \Big] \tag{17.4}$$

$$T^{m\,II'}_{2i-1,\,2j-1} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, \eta(t)\, K_i(t)\, K_j(t)$$

$$- C^2 Z \frac{k_0}{4} \left[1 + \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, f(z')\, f'(z)\, \alpha(z)\, \eta(t')$$

$$K_i(t)\, K_j(t') \frac{\partial}{\partial z'} (G_{m-1} + G_{m+1}) \Big] \tag{17.5}$$

$$T^{m\,II'}_{2i-1,\,2j} = C^2 Z \frac{k_0}{4} \left[1 + \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big\{ f(z)\, \alpha(z)\, \alpha(z')$$

$$\eta(t') \Big[f'(z) \left[(m+1)\, G_{m+1} + (m-1)\, G_{m-1}\right] - 2\, f'(z')\, m\, G_m$$

$$+ f(z')\, f'(z)\, f'(z') \frac{\partial}{\partial z'} (G_{m-1} - G_{m+1}) \Big] K_i(t)\, K_j(t') \Big\} \tag{17.6}$$

$$T^{mII'}_{2i,\,2j} = \frac{1}{2}\, C\, Z\, \pi \int_0^1 dt\, f(z)\, \eta(t)\, K_i(t)\, K_j(t)$$

$$+ C^2 Z \frac{k_0}{4} \left[1 - \delta_{m0}\right] \int_0^1 dt \int_0^1 dt' \Big[f(z)\, \alpha(z')\, K_i(t)\, K_j(t')\, \eta(t')$$

$$\Big\{ (m+1)\, G_{m+1} - (m-1)\, G_{m-1} - f(z')\, f'(z') \frac{\partial}{\partial z'} (G_{m-1} + G_{m+1}) \Big\} \Big]. \tag{17.7}$$

APPENDIX

EVALUATION OF T-MATRIX ELEMENTS INVOLVING INTEGRATION OVER A DIAGONAL CELL AND/OR INTEGRATION OVER A CELL BORDERING A DIAGONAL CELL

Introduction.

In P. Schweitzer's analysis of electromagnetic scattering from rotationally symmetric bodies the T-matrix elements of types A, B, E, H, I and J (see Schweitzer's report, pages 90-92) involve one or more integrals over diagonal cells, and/or over cells bordering a diagonal cell. These integrals must be treated analytically due to the integrable singularity in G_m.

A detailed treatment of this analysis is presented in the following pages. The method used essentially imitates that used by TRG with only minor modifications dictated by the difference in approach.*

The values for the components of the surface field are very sensitive to the values of their diagonal matrix elements, which involve precisely the integral over the singularity. For this reason it is necessary that the analysis here presented be dealt with extreme care, since errors in this part will be strongly reflected in the answers.

* M.G. Andreasen "Scattering from Rotationally Symmetric Metallic Bodies," TRG Final Report.

Evaluation of T-matrix elements involving integration over a diagonal cell and/or integration over a cell bordering a diagonal cell

A typical integral over a diagonal cell is of the form

$$T^m = \int_{t_i}^{t_{i+1}} dt' \; F_1(t') \int_{t_i}^{t_{i+1}} dt \; F_2(t) \; G_m(t, t') \quad , \tag{1}$$

where

$$G_m(z, z') = \int_0^{\pi} d\theta \; \frac{e^{jk_oR}}{k_oR} \cos m\,\theta \tag{2}$$

and

$$R = \sqrt{(z - z')^2 + \left[f(z)\right]^2 + \left[f(z')\right]^2 - 2\, f(z)\, f(z') \cos\theta} \; . \tag{3}$$

In order to do integral (1) we expand $F_2(t)$ in a Taylor series about $t = t'$ and keep only the linear part

$$F_2(t) \approx F_2(t') + F_2'(t')\,(t - t') \tag{4}$$

and (1) becomes

$$T^m = \int_{t_i}^{t_{i+1}} dt' \; F_1(t') \, F_2(t') \int_{t_i}^{t_{i+1}} dt \; G_m(t, t') \; +$$

$$\int_{t_i}^{t_{i+1}} dt' \; F_1(t') \, F_2'(t') \int_{t_i}^{t_{i+1}} dt \; (t - t') \, G_m(t, t') \quad . \tag{5}$$

We will treat the integrals

$$I_n^m = \int_{t_i}^{t_{i+1}} dt\,(t - t')^n\, G_m(t, t') \quad , \quad n = 0 \text{ or } 1 \tag{6}$$

analytically and then do

$$T^m = \int_{t_i}^{t_{i+1}} dt'\, F_1(t')\, F_2(t')\, I_0^m + \int_{t_i}^{t_{i+1}} dt'\, F_1(t')\, F_2'(t')\, I_1^m \tag{7}$$

by numerical quadratures.

From the geometry of the body it is seen that k_oR can be approximated by

$$k_oR = \sqrt{\left[f_2(t) - f_2(t')\right]^2 + 4\, f_1(t')\, f_3(t')\, \sin^2\left(\frac{\theta}{2}\right) \left[f_2(t) - f_2(t')\right] + 4\, f_1^2(t')\, \sin^2\left(\frac{\theta}{2}\right)}$$

(8)

The functions $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$ are defined in P. Schweitzer's report, page 180.

We expand $f_2(t)$ in a power series about $t = t_i$ and neglect all terms of second or higher order

$$f_2(t) = f_2(t_i) + f_2'(t_i)\,(t - t_i) \tag{9}$$

so that

$$k_oR = \sqrt{\left[f_2'(t_i)\right]^2 (t - t')^2 + 4\, f_1(t')\, f_3(t')\, \sin^2\left(\frac{\theta}{2}\right) f_2'(t_i)\,(t - t') + 4 f_1^2(t')\, \sin^2\left(\frac{\theta}{2}\right)}$$

(10)

From (6) we then have

$$I_n^m = \int_{t_i}^{t_{i+1}} dt\,(t - t')^n\, G_m(t,t')$$

$$= \int_{t_i}^{t_{i+1}} dt\,(t - t')^n \int_0^{\pi} d\theta\, \frac{e^{jk_oR}}{k_oR} \cos m\theta$$

$$= \int_0^{\pi} d\theta \cos m\,\theta \int_{t_i}^{t_{i+1}} dt\,(t - t')^n\, \frac{e^{jk_oR}}{k_oR}$$

$$= \int_0^{\pi} d\theta \cos m\theta \int_{t_i - t'}^{t_{i+1} - t'} ds\; s^n\, \frac{e^{jk_oR}}{k_oR} \tag{11}$$

where

$$k_oR = \sqrt{\left[f_2'(t_i)\right]^2 s^2 + 4 f_1(t')\, f_3(t')\, f_2'(t_i) \sin^2\left(\frac{\theta}{2}\right) s + 4 f_1^2(t') \sin^2\left(\frac{\theta}{2}\right)} \tag{12}$$

We want to introduce this expression in the amplitude factor of the wave function $\exp(+jk_oR)/k_oR$. In the phase factor, we shall introduce the following expression which is obtained by expanding k_oR in a power series of s and retaining only the first two terms of the series

$$k_oR = 2\, f_1(t') \sin\left(\frac{\theta}{2}\right) + f_3(t')\, f_2'(t_i) \sin\left(\frac{\theta}{2}\right) s\ . \tag{13}$$

We can write

$$e^{jk_o R} \approx e^{j 2 f_1(t') \sin(\frac{\theta}{2})} \left(1 + j\, f_3(t')\, f_2'(t_i) \sin(\frac{\theta}{2})\, s\right) \tag{14}$$

Let

$$\Gamma = \left| f_2'(t_i) \right| \tag{15}$$

$$\Lambda = 4\, f_1(t')\, f_3(t')\, f_2'(t_i) \sin^2(\frac{\theta}{2}) \tag{16}$$

$$\Omega = 2\, f_1(t') \sin(\frac{\theta}{2}) \tag{17}$$

so that

$$k_o R = \sqrt{\Gamma^2 s^2 + \Lambda s + \Omega^2} \tag{18}$$

and

$$I_n^m = \int_0^\pi d\theta\, e^{j\Omega} \cos m\theta \int_{t_i - t'}^{t_{i+1} - t'} ds\, s^n \frac{1}{\sqrt{\Gamma^2 s^2 + \Lambda s + \Omega^2}}$$

$$+ j f_3(t')\, f_2'(t_i) \int_0^\pi d\theta\, e^{j\Omega} \cos m\theta \sin(\frac{\theta}{2}) \int_{t_i - t'}^{t_{i+1} - t'} ds \frac{s^{n+1}}{\sqrt{\Gamma^2 s^2 + \Lambda s + \Omega^2}} \tag{19}$$

Let

$$\psi_n = \int_{t_i - t'}^{t_{i+1} - t'} ds \frac{s^n}{\sqrt{\Gamma^2 s^2 + \Lambda s + \Omega^2}} \tag{20}$$

$$I_n^m = \int_0^{\pi} d\theta \; e^{j\Omega} \cos m\theta \, \psi_n + j f_3(t') f_2'(t_i) \int_0^{\pi} d\theta \; e^{j\Omega} \cos m\theta \sin(\frac{\theta}{2}) \, \psi_{n+1} \quad . \tag{21}$$

When I_1^m is evaluated ψ_2 can be neglected

$$\psi_o = \int_{t_i - t'}^{t_{i+1} - t'} \frac{ds}{\sqrt{\Gamma^2 s^2 + \Lambda s + \Omega^2}} =$$

$$\frac{1}{\Gamma} \log \left| 2\Gamma \sqrt{\Gamma^2 s^2 + \Lambda s + \Omega^2} + 2\Gamma^2 s + \Lambda \right| \Bigg|_{t_i - t'}^{t_{i+1} - t'}$$

$$= \frac{1}{\Gamma} \log \left| \frac{2\Gamma\sqrt{\Gamma^2 (t_{i+1} - t')^2 + \Lambda (t_{i+1} - t') + \Omega^2} + 2\Gamma^2 (t_{i+1} - t') + \Lambda}{2\Gamma\sqrt{\Gamma^2 (t_i - t')^2 + \Lambda (t_i - t') + \Omega^2} + 2\Gamma^2 (t_i - t') + \Lambda} \right| \tag{22}$$

$$\psi_1 = \int_{t_i - t'}^{t_{i+1} - t'} ds \frac{s}{\sqrt{\Gamma^2 s^2 + \Lambda s + \Omega^2}} = \frac{1}{\Gamma^2} \Bigg[\sqrt{\Gamma^2 (t_{i+1} - t')^2 + \Lambda (t_{i+1} - t') + \Omega^2}$$

$$- \sqrt{\Gamma^2 (t_i - t')^2 + \Lambda (t_i - t') + \Omega^2} - \frac{\Lambda}{2} \psi_0 \Bigg] . \tag{23}$$

Let us first consider I_0^m

$$I_o^m = \int_0^{\eta_m} d\theta \, e^{j\Omega} \cos m\theta \, \psi_o + \int_{\eta_m}^{\pi} d\theta \, e^{j\Omega} \cos m\theta \, \psi_o$$

$$+ j \, f_3(t') \, f_2'(t_i) \int_0^{\pi} d\theta \, e^{j\Omega} \cos m\theta \sin\left(\frac{\theta}{2}\right) \psi_1 \tag{24}$$

where

$$\eta_o = \frac{\pi}{20} \, , \quad \eta_m = \frac{\pi}{20m} \quad m = 1, 2, \ldots \tag{25}$$

For $\theta \approx 0$ $\sin \theta \approx \theta$ and we have

$$\Lambda = f_1(t') \, f_3(t') \, f_2'(t_i) \, \theta^2 \tag{26}$$

$$\Omega = f_1(t') \, \theta \, . \tag{27}$$

Let

$$t_i - t' = \beta_i \quad , \quad f_1(t') \, f_3(t') \, f_2'(t_i) = \xi \tag{28}$$

$$f_1(t') \, f_3(t') \, f_2'(t_i) \, \beta_i + f_1^2(t') = \gamma_i \tag{29}$$

$$f_1(t') \, f_3(t') \, f_2'(t_i) \, \beta_{i+1} + f_1^2(t') = \gamma_{i+1} \tag{30}$$

Then

$$\psi_o = \frac{1}{\Gamma} \log \left| \frac{2\Gamma\sqrt{\Gamma^2 \beta_{i+1}^2 + \gamma_{i+1} \, \theta^2} + 2\Gamma^2 \beta_{i+1} + \xi \theta^2}{2\Gamma\sqrt{\Gamma^2 \beta_i^2 + \gamma_i \, \theta^2} + 2\Gamma^2 \beta_i + \xi \theta^2} \right| \tag{31}$$

Introduce the approximations

$$e^{j\Omega} = 1 + j f_1(t') \theta \tag{32}$$

$$\cos m\theta = 1 - \frac{1}{2} m^2 \theta^2 \tag{33}$$

Define

$$\Delta I_0^m = \int_0^{\eta_m} d\theta\, e^{j\Omega} \cos m\theta\, \psi_0 = \int_0^{\eta_m} d\theta\, g(\theta)\, \psi_0 \tag{34}$$

where

$$g(\theta) = 1 + j f_1(t') \theta - \frac{1}{2} m^2 \theta^2 - j \frac{1}{2} m^2 f_1(t') \theta^3 \tag{35}$$

Then

$$I_0^m = \Delta I_0^m + \int_{\eta_m}^{\pi} d\theta\, e^{j\Omega} \cos m\theta\, \psi_0 + j f_3(t') f_2'(t_i) \int_0^{\pi} d\theta\, e^{j\Omega} \cos m\theta \sin(\frac{\theta}{2}) \psi_1 \tag{36}$$

To do integral (34) we will consider several cases depending on whether β_i and/or β_{i+1} are positive, negative or zero.

$$\beta_i = t_i - t' = t_{i+1} - t' - (t_{i+1} - t_i) = \beta_{i+1} - (t_{i+1} - t_i)$$

<u>**Case I**</u> $\beta_i = 0$ which implies $\beta_{i+1} > 0$.

We have then

$$\psi_0 = \frac{1}{\Gamma}\log\left|\frac{2\Gamma^2\beta_{i+1}\sqrt{1+\frac{\gamma_{i+1}}{\Gamma^2\beta_{i+1}^2}\theta^2} + 2\Gamma^2\beta_{i+1} + \xi\,\theta^2}{2\Gamma\sqrt{\gamma_i}\,\theta + \xi\,\theta^2}\right|$$

$$\approx \frac{1}{\Gamma}\log\left|\frac{4\Gamma^2\beta_{i+1} + (\gamma_{i+1}/\beta_{i+1})\,\theta^2 + \xi\,\theta^2}{2\Gamma\sqrt{\gamma_i}\,\theta + \xi\,\theta^2}\right|$$

$$\approx \frac{1}{\Gamma}\log\left|\frac{1 + \frac{\gamma_{i+1} + \xi\,\beta_{i+1}}{4\Gamma^2\beta_{i+1}^2}\theta^2}{\frac{\sqrt{\gamma_i}}{2\Gamma\beta_{i+1}}\theta}\right|$$

$$\approx \frac{1}{\Gamma}\,\frac{\gamma_{i+1} + \xi\beta_{i+1}}{4\Gamma^2\beta_{i+1}^2}\,\theta^2 - \frac{1}{\Gamma}\log\left(\frac{\sqrt{\gamma_i}}{2\Gamma\beta_{i+1}}\,\theta\right)$$

and

$$\Delta I_0^m = \frac{\gamma_{i+1} + \xi\beta_{i+1}}{4\Gamma^3\beta_{i+1}^2}\int_0^{\eta_m} d\theta\, g(\theta)\,\theta^2 - \frac{1}{\Gamma}\int_0^{\eta_m} d\theta\, g(\theta)\log\left(\frac{\sqrt{\gamma_i}}{2\Gamma\beta_{i+1}}\,\theta\right).$$

$$\Delta I_0^m = \frac{\gamma_{i+1} + \xi\,\beta_{i+1}}{4\Gamma^3\beta_{i+1}^2}\left[\frac{1}{3}\eta_m^3 + j\,\frac{1}{4}\,f_1(t')\,\eta_m^4\right]$$

$$- \frac{1}{\Gamma}\log\left(\frac{\sqrt{\gamma_i}}{2\Gamma\beta_{i+1}}\,\eta_m\right)\left[\eta_m + j\,\frac{1}{2}\,f_1(t')\,\eta_m^2 - \frac{1}{12}\,m^2\,\eta_m^3\right]$$

$$+ \frac{1}{\Gamma}\left[\eta_m + j\,\frac{1}{4}\,f_1(t')\,\eta_m^2 - \frac{1}{36}\,m^2\,\eta_m^3\right] \qquad (37)$$

<u>Case II</u> $\beta_{i+1} = 0$ which implies $\beta_i < 0$

Then

$$\psi_0 = \frac{1}{\Gamma}\log\left|\frac{2\Gamma\sqrt{\gamma_{i+1}}\,\theta + \xi\,\theta^2}{2\Gamma^2\left|\beta_i\right|\sqrt{1 + \frac{\gamma_i}{\Gamma^2\beta_i^2}\,\theta^2} + 2\Gamma^2\beta_i + \xi\,\theta^2}\right|$$

$$\approx \frac{1}{\Gamma}\log\left|\frac{2\Gamma\sqrt{\gamma_{i+1}} + \xi\,\theta}{\left(\frac{\gamma_i}{\left|\beta_i\right|} + \xi\right)\theta}\right| \approx \frac{\xi}{2\Gamma^2\sqrt{\gamma_{i+1}}}\,\theta - \frac{1}{\Gamma}\log\left(\left|\frac{\gamma_i - \xi\,\beta_i}{2\Gamma\,\beta_i\sqrt{\gamma_{i+1}}}\right|\theta\right)$$

and

$$\Delta I_0^m = \frac{\xi}{2\Gamma^2\sqrt{\gamma_{i+1}}}\int_0^{\eta_m}\theta\,g(\theta)\,d\theta - \frac{1}{\Gamma}\int_0^{\eta_m} g(\theta)\log\left(\left|\frac{\gamma_i + \xi\left|\beta_i\right|}{2\Gamma\,\beta_i\sqrt{\gamma_{i+1}}}\right|\theta\right)d\theta$$

and

$$\Delta I_0^m \approx \frac{\xi}{2\Gamma^2\sqrt{\gamma_{i+1}}}\left[\frac{1}{2}\,\eta_m^2 + j\,f_1(t')\,\frac{1}{3}\,\eta_m^3\right]$$

$$- \frac{1}{\Gamma}\log\left(\left|\frac{\gamma_i + \xi\left|\beta_i\right|}{2\Gamma\beta_i\sqrt{\gamma_{i+1}}}\right|\eta_m\right)\left[\eta_m + j\,f_1(t')\,\frac{\eta_m^2}{2} - \frac{1}{6}\,m^2\,\eta_m^3\right]$$

$$+ \frac{1}{\Gamma}\left[\eta_m + j\,f_1(t')\,\frac{\eta_m^2}{4} - \frac{1}{18}\,m^2\,\eta_m^3\right] \qquad (38)$$

Case III $\beta_i < 0$ which implies $\beta_{i+1} \geqslant 0$. Assume $\beta_{i+1} > 0$ since $\beta_{i+1} = 0$ has already been considered

$$\psi_0 = \frac{1}{\Gamma}\log\left| \frac{2\Gamma^2 \beta_{i+1}\sqrt{1+\frac{\gamma_{i+1}}{\Gamma^2\beta_{i+1}^2}\theta^2} + 2\Gamma^2\beta_{i+1} + \xi\theta^2}{2\Gamma^2|\beta_i|\sqrt{1+\frac{\gamma_i}{\Gamma^2\beta_i^2}\theta^2} + 2\Gamma^2\beta_i + \xi\theta^2} \right|$$

$$\approx \frac{1}{\Gamma}\log\left| \frac{4\Gamma^2\beta_{i+1} + \frac{\gamma_{i+1}}{\beta_{i+1}}\theta^2 + \xi\theta^2}{\frac{\gamma_i}{|\beta_i|}\theta^2 + \xi\theta^2} \right| \approx \frac{1}{\Gamma}\log\left| \frac{1 + \frac{\gamma_{i+1} + \xi\beta_{i+1}}{4\Gamma^2\beta_{i+1}^2}\theta^2}{\frac{\gamma_i + \xi|\beta_i|}{|\beta_i|}\theta^2} \right|$$

$$\approx \frac{\gamma_{i+1} + \xi\beta_{i+1}}{4\Gamma^3\beta_{i+1}^2}\theta^2 - \frac{1}{\Gamma}\log\left(\left|\frac{\gamma_i - \xi\beta_i}{\beta_i}\right|\theta^2\right)$$

and

$$\Delta I_0^m = \int_0^{\eta_m} g(\theta)\,\psi_0\,d\theta = \frac{\gamma_{i+1} + \xi\beta_{i+1}}{4\Gamma^3\beta_{i+1}^2}\int_0^{\eta_m}\theta^2 g(\theta)\,d\theta$$

$$-\frac{2}{\Gamma}\int_0^{\eta_m} g(\theta)\log\left(\sqrt{\left|\frac{\gamma_i - \xi\beta_i}{\beta_i}\right|}\,\theta\right)d\theta$$

$$\Delta I_0^m = \frac{\gamma_{i+1} + \xi\beta_{i+1}}{12\Gamma^3\beta_{i+1}^2}\eta_m^3$$

$$-\frac{1}{\Gamma}\log\left(\left|\frac{\gamma_i + \xi|\beta_i|}{\beta_i}\right|\eta_m^2\right)\left[\eta_m + j\,f_1(t')\frac{\eta_m^2}{2} - \frac{1}{6}m^2\eta_m^3\right]$$

$$+\frac{2}{\Gamma}\left[\eta_m + j\,f_1(t')\frac{\eta_m^2}{4} - \frac{1}{18}m^2\eta_m^3\right] \tag{39}$$

From Eqs. (36) and (23) we have

$$I_0^m = \Delta I_0^m - \frac{1}{2} j \frac{f_3(t')}{f_2'(t_i)} \Delta I_1^m + \int_{\eta_m}^{\pi} d\theta \, e^{j\Omega} \cos m\theta \, \psi_0$$

$$+ j f_3(t') f_2'(t_i) \int_0^{\pi} d\theta \left\{ e^{j\Omega} \cos m\theta \sin\left(\frac{\theta}{2}\right) \cdot \right.$$

$$\left. \frac{1}{\Gamma^2} \left[\sqrt{\Gamma^2 (t_{i+1} - t')^2 + \Lambda (t_{i+1} - t') + \Omega^2} - \sqrt{\Gamma^2 (t_i - t')^2 + \Lambda (t_i - t') + \Omega^2} \right] \right\}$$

$$- \frac{1}{2} j \frac{f_3(t')}{f_2'(t_i)} \int_{\eta_m}^{\pi} d\theta \, e^{j\Omega} \cos m\theta \sin\left(\frac{\theta}{2}\right) \psi_0 \tag{40}$$

where

$$\Delta I_1^m = \int_0^{\eta_m} d\theta \, e^{j\Omega} \cos m\theta \sin\left(\frac{\theta}{2}\right) \psi_0 \approx \int_0^{\eta_m} h(\theta) \, \psi_0 \, d\theta \tag{41}$$

$$h(\theta) = \frac{\theta}{2} + j f_1(t') \frac{\theta^2}{2} - \frac{1}{4} m^2 \theta^3 - j \frac{1}{4} m^2 f_1(t') \theta^4 \; . \tag{42}$$

Equation (41) can be treated in an identical manner to Eq. (34). Cases I, II and III now give

Case I

$$\Delta I_1^m = \frac{\gamma_{i+1} + \xi \beta_{i+1}}{4\Gamma^3 \beta_{i+1}^2} \left[\frac{\eta_m^4}{8} \right]$$

$$- \frac{1}{\Gamma} \log\left(\frac{\sqrt{\gamma_1}}{2\Gamma \beta_{i+1}} \eta_m \right) \left[\frac{1}{4} \eta_m^2 + j \frac{1}{6} f_1(t') \eta_m^3 \right]$$

$$+ \frac{1}{\Gamma} \left[\frac{\eta_m^2}{8} + j f_1(t') \frac{\eta_m^3}{18} \right] \qquad (43)$$

Case II

$$\Delta I_1^m = \frac{\xi \eta_m^3}{12\Gamma^2 \sqrt{\gamma_{i+1}}} - \frac{1}{\Gamma} \log\left(\left| \frac{\gamma_i - \xi \beta_i}{2\Gamma \beta_i \sqrt{\gamma_{i+1}}} \right| \eta_m \right) \left[\frac{\eta_m^2}{4} + j f_1(t') \frac{\eta_m^3}{6} \right]$$

$$+ \frac{1}{\Gamma} \left[\frac{\eta_m^2}{8} + j f_1(t') \frac{\eta_m^3}{18} \right] \qquad (44)$$

Case III

$$\Delta I_1^m = \frac{\gamma_{i+1} + \xi \beta_{i+1}}{32\Gamma^3 \beta_{i+1}^2} \eta_m^4 .$$

$$- \frac{1}{\Gamma} \log\left(\left| \frac{\gamma_i - \xi \beta_i}{\beta_i} \right| \eta_m^2 \right) \left[\frac{1}{4} \eta_m^2 + j f_1(t') \frac{\eta_m^3}{6} \right]$$

$$+ \frac{2}{\Gamma} \left(\frac{\eta_m^2}{8} + j f_1(t') \frac{\eta_m^3}{18} \right) . \qquad (45)$$

All the integrals in Eq. (40) can be done by numerical quadratures. This fact together with the expressions (37) - (39) and (43) - (45) completes the evaluation of $I_0^m(t')$.

Let us now consider $I_1^m(t')$. From Eq. (21) we obtain

$$I_1^m = \frac{1}{\Gamma^2} \int_0^\pi d\theta \, e^{j\Omega} \cos m\theta \left[\sqrt{\Gamma^2 (t_{i+1} - t')^2 + \Lambda(t_{i+1} - t') + \Omega^2} \right.$$

$$\left. - \sqrt{\Gamma^2 (t_i - t')^2 + \Lambda(t_i - t') + \Omega^2} \right] - \frac{1}{2} \frac{f_1(t') \, f_3(t')}{f_2'(t_i)} \Delta I_2^m$$

$$- \frac{2 \, f_1(t') \, f_3(t')}{f_2'(t_i)} \int_{\eta_m}^\pi d\theta \, e^{j\Omega} \cos m\theta \, \sin^2 (\frac{\theta}{2}) \psi_0 \quad , \tag{46}$$

where

$$\Delta I_2^m = 4 \int_0^{\eta_m} d\theta \, e^{j\Omega} \cos m\theta \, \sin^2 (\frac{\theta}{2}) \, \psi_0 \tag{47}$$

$$\approx \int_0^{\eta_m} d\theta \, k(\theta) \, \psi_0$$

where

$$k(\theta) = \frac{1}{4} \theta^2 + j \, f_1(t') \frac{\theta^3}{4} - \frac{1}{8} m^2 \theta^4 \quad . \tag{48}$$

We again have to consider three cases. Proceeding as before we obtain

Case I $\beta_i = 0$

$$\Delta I_2^m = \frac{\gamma_{i+1} + \xi \beta_{i+1}}{4\Gamma^3 \beta_{i+1}^2} \left[\frac{\eta_m^5}{20} \right]$$

$$- \frac{1}{\Gamma} \log\left(\frac{\sqrt{\gamma_i}}{2\Gamma \beta_{i+1}} \eta_m \right) \left[\frac{\eta_m^3}{12} + j f_1(t') \frac{\eta_m^4}{16} - \frac{1}{40} m^2 \eta_m^5 \right]$$

$$+ \frac{1}{\Gamma} \left[\frac{\eta_m^3}{36} + j f_1(t') \frac{\eta_m^4}{64} - \frac{1}{200} m^2 \eta_m^5 \right] \qquad (49)$$

Case II $\beta_{i+1} = 0$

$$\Delta I_2^m = \frac{\xi}{2\Gamma^2 \sqrt{\gamma_{i+1}}} \left[\frac{\eta_m^4}{16} + j f_1(t') \frac{\eta_m^5}{20} \right]$$

$$- \frac{1}{\Gamma} \log\left(\left| \frac{\gamma_i - \xi \beta_i}{2\Gamma \beta_i \sqrt{\gamma_{i+1}}} \right| \eta_m \right) \left[\frac{\eta_m^3}{12} + j f_1(t') \frac{\eta_m^4}{16} - \frac{1}{40} m^2 \eta_m^5 \right]$$

$$+ \frac{1}{\Gamma} \left[\frac{\eta_m^3}{36} + j f_1(t') \frac{\eta_m^4}{64} - \frac{1}{200} m^2 \eta_m^5 \right] . \qquad (50)$$

Case III $\beta_i < 0$

$$\Delta I_2^m = \frac{\gamma_{i+1} + \xi \beta_{i+1}}{4\Gamma^3 \beta_{i+1}^2} \left[\frac{\eta_m^5}{20} \right]$$

$$- \frac{1}{\Gamma} \log\left(\left| \frac{\gamma_i - \xi \beta_i}{\beta_i} \right| \eta_m^2 \right) \left[\frac{\eta_m^2}{12} + j f_1(t') \frac{\eta_m^4}{16} - \frac{m^2}{40} \eta_m^5 \right]$$

$$+ \frac{2}{\Gamma} \left[\frac{\eta_m^3}{36} + j f_1(t') \frac{\eta_m^4}{64} - \frac{m^2}{200} \eta_m^5 \right] \qquad (51)$$

All the integrals in Eq. (46) can be done by numerical quadratures. This fact together with Eqs. (49) - (51) completes the evaluation of $I_1^m(t')$.

Substitution of $I_0^m(t')$ and $I_1^m(t')$ in Eq. (7) and evaluating the integrals by quadratures completes the integration over a diagonal cell.

Integration over a cell bordering a diagonal cell

A typical such integral is of the form

$$T^m = \int_{t_{i+1}}^{t_{i+2}} dt' \; F_1(t') \int_{t_i}^{t_{i+1}} dt \; F_2(t) \; G_m(t, t') \tag{52}$$

Let

$$I^m(t') = \int_{t_i}^{t_{i+1}} dt \; F_2(t) \; G_m(t, t') \tag{53}$$

For $t' = t_{i+1}$ the integrand of $I^m(t')$ becomes infinite and the integral must be treated analytically. For t' other than t_{i+1} the integrand is everywhere well behaved and $I^m(t')$ can be done by ordinary numerical methods.

T^m is given by

$$T^m = \int_{t_{i+1}}^{t_{i+2}} dt' \; F_1(t') \; I^m(t') \; . \tag{54}$$

Proceeding as before we obtain

$$I^m(t_{i+1}) = F_2(t_{i+1}) \; I_0^m(t_{i+1}) + F_2'(t_{i+1}) \; I_1^m(t_{i+1}) \; , \tag{55}$$

where

$$I_0^m(t_{i+1}) = \Delta I_0^m(t_{i+1}) - \frac{1}{2} j \frac{f_3(t_{i+1})}{f_2'(t_{i+1})} \Delta I_1^m(t_{i+1})$$

$$+ \int_{\eta_m}^{\pi} d\theta \, e^{j\Omega} \cos m\theta \, \psi_0$$

$$+ j f_3(t_{i+1}) f_2'(t_{i+1}) \int_0^{\pi} d\theta \left\{ e^{j\Omega} \cos m\theta \sin\left(\frac{\theta}{2}\right) \cdot \right.$$

$$\left. \frac{1}{\Gamma^2} \left[|\Omega| - \sqrt{\Gamma^2 (t_{i+1} - t_i)^2 - \Lambda (t_{i+1} - t_i) + \Omega^2} \right] \right\}$$

$$- \frac{1}{2} j \frac{f_3(t_{i+1})}{f_2'(t_{i+1})} \int_{\eta_m}^{\pi} d\theta \, e^{j\Omega} \cos m\theta \sin\left(\frac{\theta}{2}\right) \psi_0 \tag{56}$$

$$\Omega = 2 f_1(t_{i+1}) \sin\left(\frac{\theta}{2}\right) \tag{57}$$

$$\Gamma = \left| f_2'(t_{i+1}) \right| \tag{58}$$

$$\Lambda = 4 f_1(t_{i+1}) f_3(t_{i+1}) f_2'(t_{i+1}) \sin^2\left(\frac{\theta}{2}\right) \tag{59}$$

$$\psi_0 = -\frac{1}{\Gamma} \log \left| \frac{2\Gamma \sqrt{\Gamma^2 (t_{i+1} - t_i)^2 - \Lambda (t_{i+1} - t_i) + \Omega^2} - 2\Gamma^2 (t_{i+1} - t_i) + \Lambda}{2\Gamma |\Omega| + \Lambda} \right|$$

(60)

and

$$\Delta I_0^m(t_{i+1}) = \frac{\xi}{2\Gamma^2 \sqrt{\gamma_{i+1}}} \left[\frac{1}{2}\,\eta_m^2 + j\,\frac{1}{3}\,f_1(t_{i+1})\,\eta_m^3\right]$$

$$- \frac{1}{\Gamma} \log\left(\left|\frac{\gamma_i + \xi|\beta_i|}{2\Gamma\beta_i\sqrt{\gamma_{i+1}}}\right| \eta_m\right) \left[\eta_m + j\frac{1}{2}\,f_1(t_{i+1})\,\eta_m^2 - \frac{1}{6}\,m^2\,\eta_m^3\right]$$

$$+ \frac{1}{\Gamma}\left[\eta_m + j\,f_1(t_{i+1})\,\frac{\eta_m^2}{4} - \frac{1}{18}\,m^2\,\eta_m^3\right] \tag{61}$$

$$\Delta I_1^m = \frac{\xi\,\eta_m^3}{12\Gamma^2\sqrt{\gamma_{i+1}}}$$

$$- \frac{1}{\Gamma} \log\left(\left|\frac{\gamma_i - \xi\,\beta_i}{2\Gamma\,\beta_i\,\sqrt{\gamma_{i+1}}}\right| \eta_m\right) \left[\frac{\eta_m^2}{4} + j\,f_1(t_{i+1})\,\frac{\eta_m^3}{6}\right]$$

$$+ \frac{1}{\Gamma}\left[\frac{1}{8}\,\eta_m^2 + j\,\frac{1}{18}\,f_1(t_{i+1})\,\eta_m^3\right] \tag{62}$$

and

$$I_1^m(t_{i+1}) = \frac{1}{\Gamma^2}\int_0^{\pi} d\theta\, e^{j\Omega} \cos m\theta \Big[|\Omega|$$

$$- \sqrt{\Gamma^2(t_{i+1} - t_i)^2 - \Lambda(t_{i+1} - t_i) + \Omega^2}\Big] - \frac{1}{2}\,\frac{f_1(t_{i+1})\,f_3(t_{i+1})}{f_2'(t_{i+1})}\,\Delta I_2^m(t_{i+1})$$

$$- \frac{2\,f_1(t_{i+1})\,f_3(t_{i+1})}{f_2'(t_{i+1})}\int_{\eta_m}^{\pi} d\theta\, e^{j\Omega} \cos m\,\theta\,\sin^2\left(\frac{\theta}{2}\right)\psi_0 \tag{63}$$

where

$$\Delta I_2^m(t_{i+1}) = \frac{\xi}{2\Gamma^2\sqrt{\gamma_{i+1}}}\left[\frac{1}{16}\eta_m^4 + j\frac{1}{20}f_1(t_{i+1})\eta_m^5\right]$$

$$-\frac{1}{\Gamma}\log\left(\left|\frac{\gamma_i - \xi\beta_i}{2\Gamma\beta_i\sqrt{\gamma_{i+1}}}\right|\eta_m\right)\left[\frac{\eta_m^3}{12} + j\frac{1}{16}f_1(t_{i+1})\eta_m^4 - \frac{1}{40}m^2\eta_m^5\right]$$

$$+\frac{1}{\Gamma}\left[\frac{1}{36}\eta_m^3 + j\frac{1}{64}f_1(t_{i+1})\eta_m^4 - \frac{1}{200}m^2\eta_m^5\right]. \tag{64}$$

in the above formulas

$$\beta_i = t_i - t_{i+1}$$

$$\xi = f_1(t_{i+1})\, f_3(t_{i+1})\, f_2'(t_{i+1})$$

$$\gamma_i = f_1(t_{i+1})\, f_3(t_{i+1})\, f_2'(t_{i+1})\,\beta_i + f_1^2(t_{i+1})$$

$$\gamma_{i+1} = f_1^2(t_{i+1})$$

Eqs. (56) and (63) when substituted in Eq. (55) complete the evaluation of $I^m(t_{i+1})$. Eq. (54) can now be done by quadratures.

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13. ABSTRACT

An analysis is presented of the problem of the numerical computation of electromagnetic scattering from rotationally symmetric boundaries satisfying an impedance boundary condition. This work extends the work of P. Schweitzer for perfectly conducting boundaries and includes his results as a special case.

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