THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE
ANN ARBOR

CORRELATION BETWEEN NOISE VOLTAGES IN ANTENNA SYSTEMS

Technical Report No. 88

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Project 2262

TASK ORDER NO. EDG-10
CONTRACT NO. DA-36-039 sc-63203
SIGNAL CORPS, DEPARTMENT OF THE ARMY
DEPARTMENT OF ARMY PROJECT NO. 3-99-04-042
SIGNAL CORPS PROJECT 194B

October 1958

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ABSTRACT

To calculate the signal-to-noise ratio of the signal resulting when a number of antenna voltages are combined, it is necessary to know the degree of correlation between the individual noise components. In this report mathematical expressions are derived for the correlation between two noise voltages, induced in two antennas placed a known distance apart, assuming that the intensity of the noise arriving from each direction is known.

The correlation coefficient is also derived for the noise voltages generated in two channels of a direction finder.

The results are applied, as an example, to the case of a two-element rotatable Adcock antenna.

CORRELATION BETWEEN NOISE VOLTAGES IN ANTENNA SYSTEMS

1. INTRODUCTION

In the course of a study of the effect of random noise on direction finding systems, some mathematical expressions were developed for the correlation coefficient of two noise voltages in certain situations. It is the purpose of this memorandum to summarize these results. A possible application of these expressions is the calculation of noise in direction finding systems, when antenna noise is dominating. An example of such a calculation is given in Section 4.

The fact that correlation exists between two random noise voltages indicates that these two voltages are not independent of each other. The correlation coefficient is a measure of the degree of dependence of the two noise voltages. To be able to calculate for instance the rms value of the noise voltage that results when the two noise voltages in question are combined, one must know the value of the correlation coefficient. To clarify this statement a few definitions and examples will be given here. These fundamental relations can also be found in Reference 1.

When e_1 and e_2 denote the instantaneous values of two random noise voltages, their correlation coefficient (C) is defined as:

$$C = \frac{\overline{e_1 \cdot e_2}}{\sqrt{\overline{e_1^2} \cdot \overline{e_2^2}}} \qquad (1)$$

A bar placed over a quantity denotes the time average of that quantity.

$\overline{e_1^2}$ and $\overline{e_2^2}$ are proportional to the average noise powers of the two signals, and the rms values of the two noise voltages can be written as:

$$\sqrt{\overline{e_1^2}} \quad \text{and} \quad \sqrt{\overline{e_2^2}}\,.$$

When the two noise voltages are added, a new random noise voltage is formed whose instantaneous value (e) is given by: $e = e_1 + e_2$. For the average noise power of this voltage (e) we find:

$$\overline{e^2} = \overline{(e_1 + e_2)^2} = \overline{e_1^2} + \overline{e_2^2} + 2\,\overline{e_1 \cdot e_2} =$$

$$\overline{e_1^2} + \overline{e_2^2} + 2\,C\sqrt{\overline{e_1^2} \cdot \overline{e_2^2}}$$

This shows that when two independent (uncorrelated) noise voltages ($C = 0$) are added, the average power of the resulting noise voltage is equal to the sum of the average powers of its components. This is a well known result.

When the two voltages e_1 and e_2 have the same rms value $(\overline{e_1^2} = \overline{e_2^2})$, their sum-voltage ($e = e_1 + e_2$) will have an average noise power proportional to:

$$\overline{e^2} = \overline{(e_1 + e_2)^2} = 2\,\overline{e_1^2}\,(1 + C) \qquad (2)$$

and their difference-voltage ($e' = e_1 - e_2$) will have an average noise power proportional to:

$$\overline{e'^2} = \overline{(e_1 - e_2)^2} = 2\,\overline{e_1^2}\,(1 - C) \qquad (3)$$

These examples show the importance of the correlation coefficient when noise voltages are combined.

In this memorandum expressions for the correlation between two noise voltages are derived for the following two cases:

(1) The noise voltages generated in two identical antennas that are placed a known distance apart. The noise spectrum is assumed to be a narrow rectangular band (see Section 2).

(2) The noise voltages generated in two crossed antenna systems; such a configuration is common in radio direction finders. A case of special interest for a two-channel direction finder occurs, when the orientation between the antenna systems is exactly 90° (see Section 3).

In Section 4 the results of Section 2 (see 1 above) are applied to a two-element rotatable Adcock direction finder to show the result of antenna spacing on the signal-to-noise ratio at the receiver input.

2. TWO ANTENNAS IN A NOISE FIELD

2.1 The General Case

In this section a general expression is derived for the correlation coefficient of two noise voltages generated in two identical antennas which are placed a distance d apart.

For the derivation of this expression the following assumptions and definitions will be made:

(1) Usually a radio receiver having a narrow bandwidth, will be used with the antenna system. Therefore we shall only consider noise having a narrow bandwidth.

(1a) It is assumed that the noise has a narrow "rectangular" frequency spectrum, extending from $\omega_1 = 2\pi f_1$ to $\omega_2 = 2\pi f_2$ and having a center frequency $\omega_0 = (\omega_1 + \omega_2)/2$.

(1b) It is further assumed that the bandwidth is small with respect to the center frequency: $(\omega_2 - \omega_1) < 0.01\ \omega_0$.

(2) Random noise signals are arriving from different directions.

(2a) The noise arriving from a small angular segment $d\alpha$ in the direction α, contributes an amount $f(\alpha)\cdot d\alpha$ to the average noise power at the output of each antenna, where $f(\alpha)$ is a known function.

(2b) It is assumed that there is no correlation between noise signals arriving from different directions.

(3) The two antennas, whose noise outputs are compared, are assumed to be identical, and to have no mutual coupling whatever.

(3a) If the two antennas do not have an omnidirectional pattern, then it is assumed that the directional patterns of the two antennas are oriented in the same direction. This will make $f(\alpha)$ as defined in (2a) equal for the two antennas.

(3b) The two noise voltages generated in the antennas will have the same average power, in other words, they have the same rms value. This will be shown in Section 2.1.3.

(4) The distance between the two antennas "d" is assumed to be smaller than 5λ, where λ is the wavelength corresponding to the center frequency ω_0 of the noise band.

The expression for the correlation coefficient in the general case will be derived in Section 2.1.3. Sections 2.1.1 and 2.1.2 are devoted to deriving expressions that form the basis for the derivation in Section 2.1.3.

2.1.1 Single Noise Source. Consider two antennas, whose output voltages are V_1 and V_2 respectively. They are placed a distance d apart and receive a signal radiated by a single noise source. The noise signal is arriving from a direction α which is the angle between the wavefront and the line connecting the two antennas (Fig. 1).

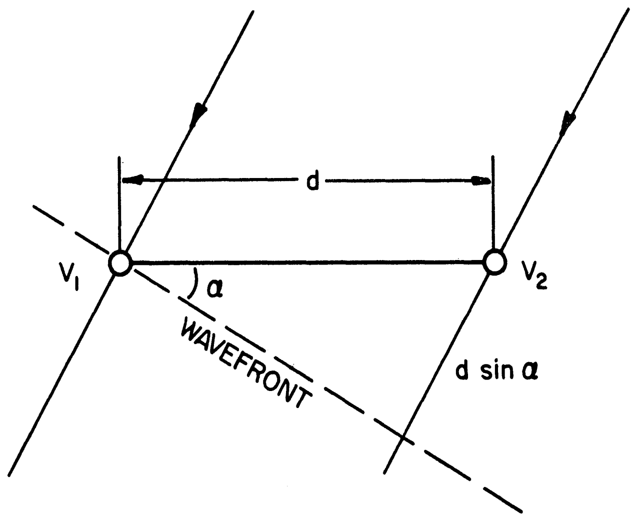

Figure 1 Two Antennas and Wavefront

Both antennas now receive the same noise voltage except for a time delay τ, which is a function of the angle of arrival α. From Fig. 1 it can be seen that the signal is received later by antenna 1, and the time delay τ is given by

$$\tau = \frac{d \sin \alpha}{c} = \frac{2\pi d \sin \alpha}{\omega_o \lambda}$$

The noise is confined to a narrow frequency band according to assumption (1). Let the bandwidth be $w_\omega = (\omega_2 - \omega_1)$. Then the correlation coefficient $C(\tau)$ of the two noise voltages V_1 and V_2, which are equal except for the time delay τ, is found by using the autocorrelation function for noise:

$$C(\tau) = \cos \omega_o \tau \frac{\sin(w_\omega \tau/2)}{w_\omega \tau/2}$$

(see Ref. 1). It follows from assumptions (1b) and (4),

$$w_\omega \tau/2 < 5\,\pi/100$$

and consequently the following approximation is accurate to within 1/2%:

$$\frac{\sin(w_\omega \tau/2)}{w_\omega \tau/2} \cong 1$$

With this approximation we find the following expression for the correlation coefficient $C(\tau)$:

$$C(\tau) = \cos \omega_o \tau \ .$$

This can be written as a function of the angle α:

$$C(\alpha) = \cos\left[\frac{2\pi d}{\lambda}\sin\alpha\right] \qquad (4)$$

This expression gives the magnitude of the correlation coefficient of the two antenna voltages V_1 and V_2 when a single noise signal is received from a direction α.

2.1.2 n Noise Sources. Let us now consider the two antennas of the previous section placed in a noise field generated by n different noise sources. These n noise sources are assumed to be uncorrelated and to contribute the same amount of average noise power to the output of the antennas. This last restriction will be removed later in this section.

Let the j^{th} noise source $(1 \le j \le n)$ contribute an instantaneous voltage e_{1j} to the output of antenna no. 1. Then the total instantaneous voltage E_1 of antenna 1 will be equal to:

$$E_1 = \sum_{j=1}^{n} e_{1j} \; .$$

Similarly we define the voltage e_{2j} and E_2 for antenna 2, so that:

$$E_2 = \sum_{j=1}^{n} e_{2j}$$

In general there will be some correlation between e_{1j} and e_{2j} as we have already seen in the previous section (2.1.1). If this correlation coefficient is C_j, then by definition:

$$C_j = \frac{\overline{e_{1j} \cdot e_{2j}}}{\sqrt{\overline{e_{1j}^2} \cdot \overline{e_{2j}^2}}} \qquad (1)$$

It will now be shown in this section that the correlation between E_1 and E_2 can be found by taking the arithmetrical average of all the correlation coefficients C_j. However to show this we must first establish the following relations.

Because we assumed that each of the n noise sources contributes the same amount of average noise power to the antenna outputs, and because of assumption (3a), the following quantities are equal

$$\overline{e_{1j}^2} = \overline{e_{2j}^2} = \overline{e_{1k}^2} = \overline{e_{2k}^2} = \overline{e^2}$$

which defines the new quantity $\overline{e^2}$. Having established this, we can now write:

$$C_j = \frac{\overline{e_{1j} \cdot e_{2j}}}{\overline{e^2}}$$

Since the n noise sources were assumed to be uncorrelated we know that

$$\overline{e_{1j} \cdot e_{2k}} = 0 \quad \text{and}$$

$$\overline{e_{1j} \cdot e_{1k}} = 0 \quad \text{when } j \neq k.$$

These last expressions allow us to calculate the average noise powers of the antenna voltages E_1 and E_2:

$$\overline{E_1^2} = \overline{\left[\sum_{j=1}^{n} e_{1j}\right]^2} = \sum_{j=1}^{n} \overline{e_{1j}^2} = n.\overline{e^2}$$

In the same fashion we find that $\overline{E_2^2}$ is equal to

$$\overline{E_2^2} = \sum_{j=1}^{n} \overline{e_{2j}^2} = n.\overline{e^2}$$

and this shows that also:

$$\overline{E_1^2} = \overline{E_2^2}$$

With these results the correlation between E_1 and E_2 can be calculated immediately.

$$C = \frac{\overline{E_1 \cdot E_2}}{\overline{E_1^2}} = \frac{\overline{(\Sigma\, e_{1j}) \cdot (\Sigma\, e_{2j})}}{(\Sigma\, \overline{e^2})}$$

$$= \frac{\Sigma\, \overline{e_{1j} \cdot e_{2j}}}{\Sigma\, \overline{e^2}} = \frac{\Sigma\, C_j \cdot \overline{e^2}}{n.\overline{e^2}} = \frac{\sum_{j=1}^{n} C_j}{n} \qquad (5)$$

and this shows that the correlation coefficient C can be found by taking the arithmetical average of the individual correlation coefficients C_j, provided that all noise sources contribute the same amount of average noise power to the antenna signals.

As mentioned earlier, this restriction will now be removed. Let the j^{th} noise source contribute an average power equal to P_j, to the antenna signal. This noise signal can then be thought of as the sum of the unit contributions of a number of P_j uncorrelated noise sources of equal power. As was shown in the introduction, noise powers can be added if the noise signals involved are uncorrelated. All of these P_j noise sources are situated at the location of the j^{th} noise source and each contributes unity average power to the antenna outputs. Also the correlation coefficient C_j as defined above, is the same for each of these P_j noise sources. By this reasoning the problem has now been reduced to one in which all noise sources contribute an equal amount of power to the antenna outputs. Equation 5 is applicable to this situation. Using Eq 5 we find the following value for the correlation coefficient of the two antenna voltages:

$$C = \frac{\sum_{j=1}^{n} P_j \cdot C_j}{\sum_{j=1}^{n} P_j} \qquad (6)$$

This then gives the correlation coefficient for the more general case that the n noise sources do not have equal strength.

2.1.3 Arbitrary Noise Distribution. With Eq 6, developed in the previous paragraph, it will only be a small step to calculate the correlation coefficient for the case that the noise distribution is arbitrary. Let the noise distribution be given by the function $f(\alpha)$ as defined in Section 2 (assumption (2a)). Then it will be clear that $f(\alpha) \cdot d\alpha$ corresponds to P_j in the previous section (2.1.2). The assumption that noise signals arriving from different directions are uncorrelated (assumption (2b)), corresponds to the statement that the j^{th} and k^{th} noise sources ($j \neq k$) are uncorrelated.

Since noise signals arriving from different directions are uncorrelated the average noise power output of antenna 1 will be:

$$\overline{E_1^2} = \int_{-\pi}^{\pi} f(\alpha)\cdot d\alpha$$

The average noise power at the output of antenna 2 will be given by the same expression so that $\overline{E_1^2} = \overline{E_2^2}$ (see assumptions (3) and (3a)).

The correlation between the two antenna voltages E_1 and E_2 can now be found by substituting $f(\alpha)\cdot d\alpha$ for P_j in Eq 6:

$$C = \frac{\int_{-\pi}^{\pi} f(\alpha)\cdot C(\alpha)\cdot d\alpha}{\int_{-\pi}^{\pi} f(\alpha)\cdot d\alpha}$$

In Section 2.1.1 an expression was derived for $C(\alpha)$. Using this expression we find:

$$C = \frac{\int_{-\pi}^{\pi} f(\alpha)\ \cos\ (x\sin\alpha)\ d\alpha}{\int_{-\pi}^{\pi} f(\alpha)\ d\alpha} \qquad (7)$$

where $x = \frac{2\pi d}{\lambda}$

This expression will now be written in a different form, by using the Fourier expansion of the function $f(\alpha)$:

$$f(\alpha) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos n\alpha + b_n \sin n\alpha)$$

where:
$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha)\cdot d\alpha$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha)\ \cos(n\alpha)\ d\alpha$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha)\ \sin(n\alpha)\ d\alpha$$

$(n=1, 2, 3, \ldots, \infty)$

Substituting this expression for $f(\alpha)$ in Eq 7 and at the same time reversing the order of integration and summation, yields:

$$C = \frac{\frac{1}{2}\pi a_o \int_{-\pi}^{\pi} \cos(x\sin\alpha)\, d\alpha}{\pi a_o} +$$

$$\frac{1}{\pi a_o} \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(n\alpha) \cos(x\sin\alpha)\, d\alpha +$$

$$\frac{1}{\pi a_o} \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(n\alpha) \cos(x\sin\alpha)\, d\alpha$$

This expression can be simplified by using the following equalities:

(1) $$\int_{-\pi}^{\pi} \sin(n\alpha) \cos(x\sin\alpha)\, d\alpha = 0,$$

because the integrand is an odd function of the variable α.

(2) $$\int_{-\pi}^{\pi} \cos(n\alpha) \cos(x\sin\alpha)\, d\alpha = 2 \int_{0}^{\pi} \cos(n\alpha) \cos(x\sin\alpha)\, d\alpha,$$

because the integrand is an even function of α.

(3) $$\int_{0}^{\pi} \cos(n\alpha) \cos(x\sin\alpha)\, d\alpha = \pi J_n(x)$$ for even values of n

($n = 0, 2, 4, \ldots$). J_n is the Bessel function of order n.

When n is odd ($n = 1, 3, 5, \ldots$):

$$\int_{0}^{\pi} \cos(n\alpha) \cos(x\sin\alpha)\, d\alpha = 0$$

For the proof of these relations see Reference 2.

Using these equalities, we find the following expression for C:

$$C = J_o(x) + \frac{2}{a_o} \sum_{n=1}^{\infty} a_{2n} J_{2n}(x) \qquad \left(x = \frac{2\pi d}{\lambda}\right) \qquad (7a)$$

This then is the desired expression for the correlation coefficient between the noise voltages at the output of two antennas placed a distance d apart.

2.2 Special Cases

In the next two paragraphs Eq 7a will be applied to two special cases.

In Section 2.2.1 two omnidirectional antennas, which are placed in an isotropic noise field, will be considered.

In Section 2.2.2 Eq 7a will be applied to the case of a single noise source. Because this case was already treated in Section 2.1.1, resulting in Eq 1, the application below should be regarded as a verification of our calculations.

2.2.1 Isotropic Noise Field. Consider two omnidirectional antennas placed in an isotropic noise field. In such a field an equal amount of average noise power is received from all directions. Together with the statement that the antennas are omnidirectional, this implies that $f(\alpha)$ is a constant, independent of α.

With $f(\alpha) = K$ = constant, the Fourier coefficients of $f(\alpha)$ can be calculated and we find:

$$a_o = K$$

$$a_n = 0 \quad (n = 1, 2, 3, \ldots, \infty)$$

Substituting these values in Eq 7a, we find that the correlation coefficient of the two antenna voltages is equal to

$$C = J_o(x) \tag{8}$$

$$\text{where } x = \frac{2\pi d}{\lambda}$$

This function is shown in Fig. 2, where d/λ has been chosen as the variable.

If two antennas that do not have an omnidirectional pattern, are placed in an isotropic noise field, Eq 8 does not apply. In this case noise coming from different directions will not contribute the same amount of average noise power to the antenna output, and consequently $f(\alpha)$ is not a constant. $f(\alpha)$ in this case is determined by the antenna pattern as will be apparent from the definition of $f(\alpha)$ (assumption (2a), together with assumption (3a)).

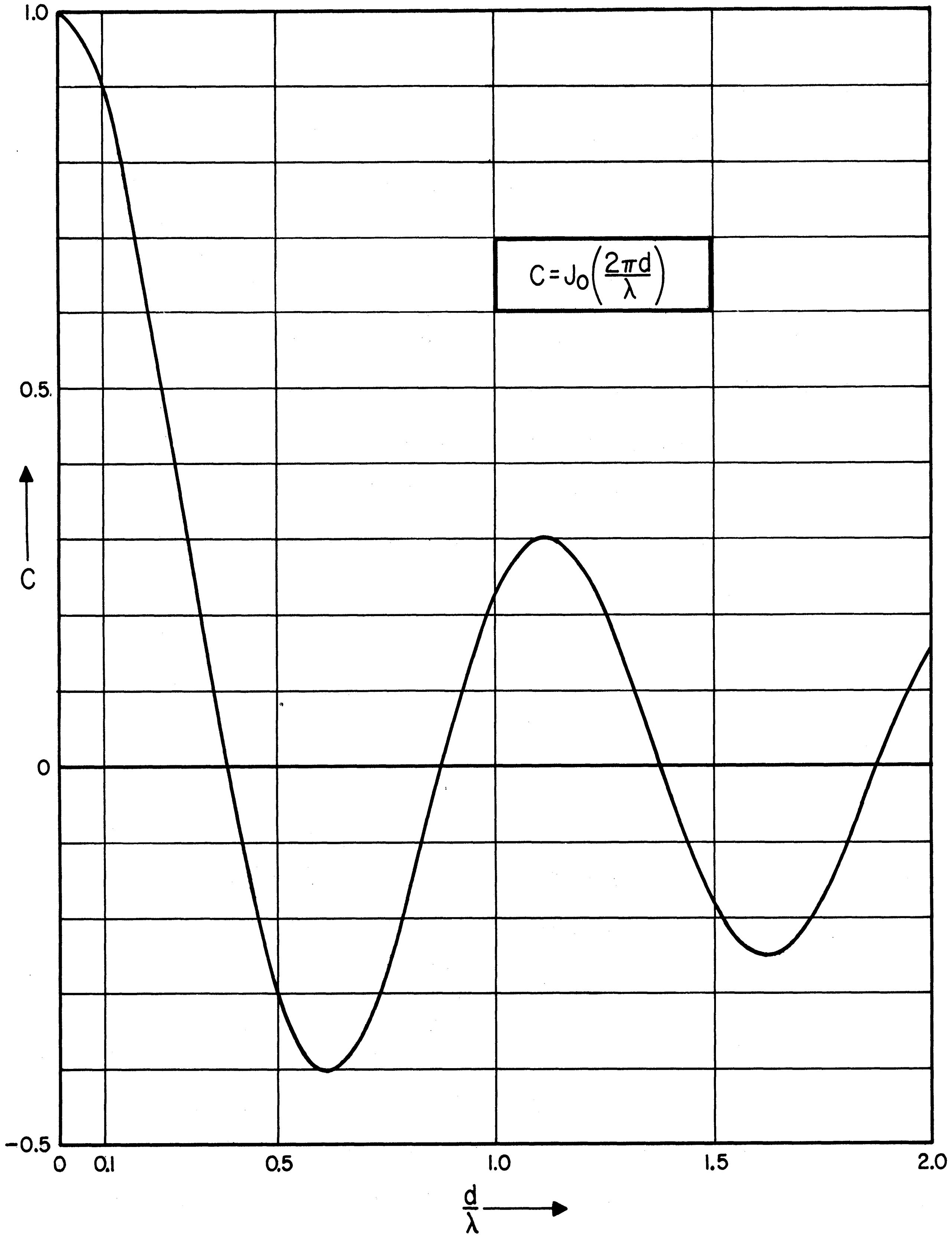

FIG. 2 CORRELATION COEFFICIENT IN ISOTROPIC NOISE FIELD

2.2.2 Single Noise Source. It has already been shown (Section 2.1.1) that the correlation coefficient of the two antenna voltages is equal to

$$C(\alpha) = \cos(x\sin\alpha)$$

when only one noise signal is arriving from a direction α. In this paragraph it will be demonstrated that the application of Eq 7a to this situation leads to the same result.

When a single noise signal is arriving from a direction α_o, $f(\alpha) = 0$ whenever $\alpha \neq \alpha_o$. However $f(\alpha_o)$ is infinitely large in such a way that $\int_{-\pi}^{\pi} f(\alpha)\cdot d\alpha = P$, where P is the average noise power at the outputs of the antennas. In other words $f(\alpha)$ assumes the character of the Dirac delta-function. For the Fourier coefficients* of this function we find:

$$a_o = P/\pi \quad \text{and}$$

$$a_n = \frac{P}{\pi}\cos(n\alpha_o)$$

These values are substituted in Eq 7a, giving

$$C(\alpha) = J_o(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos(2n\alpha_o),$$

and it can easily be shown that this is equal to:

$$C(\alpha) = \cos(x\sin\alpha)$$

To show this we use the generating function for Bessel functions:

$$G(t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)\cdot t^n$$

(See Reference 2)

For t we now substitute $t = \exp(j\alpha_o) = \cos\alpha_o + j\sin\alpha_o$.

Then:

$$G\left[\exp(j\alpha_o)\right] = \exp(jx\sin\alpha_o) = \sum_{n=-\infty}^{+\infty} J_n(x)\exp(j\,n\alpha_o)$$

When the real part of both sides of this equation is taken, it is found that:

* The Fourier series representation of the delta function does not converge. The representation of $C(\alpha)$ by a series of Bessel functions using these Fourier coefficients, does converge.

$$\mathrm{Re}\left\{ G\left[\exp(j\alpha_o)\right]\right\} = \cos(x\sin\alpha_o) = \sum_{n=-\infty}^{+\infty} J_n(x)\cos(n\alpha_o)$$

Because $\cos(n\alpha_o) = \cos(-n\alpha_o)$ and because $J_{-n}(x) = (-1)^n J_{+n}(x)$, the equation above reduces to:

$$\cos(x\sin\alpha_o) = J_o(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cdot\cos(2n\alpha_o),$$

which completes our proof.

3. TWO CHANNELS OF A DIRECTION FINDER

In this section the correlation between noise voltages generated in two channels of a direction finder will be studied. Consider 4 antennas (11, 12, 21, 22) as indicated in Figure 3. The lines 11-12 and 21-22 bisect each other at the point O and we shall call the angle between these lines φ. These antennas may be the four masts of a 4-element Adcock in which case the angle $\varphi = 90^o$. But when the four antennas are part of a different antenna system, the angle φ may have other values.

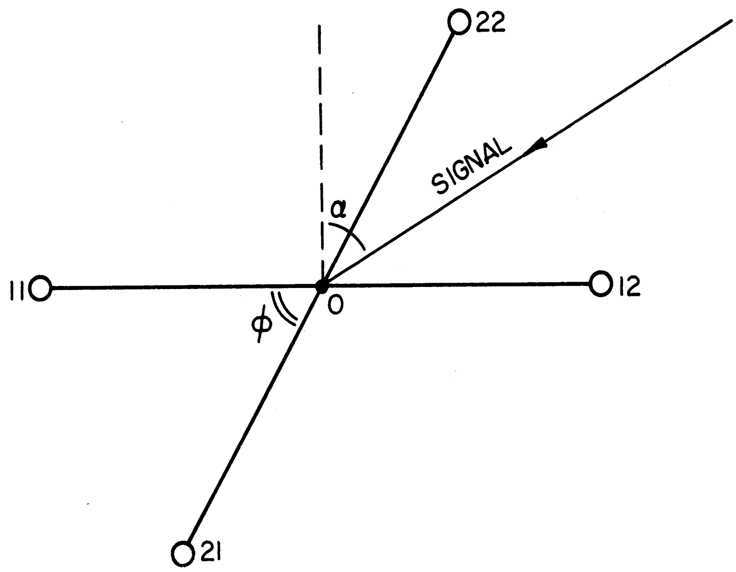

Figure 3 Geometry of Two Channel Antenna System

In direction finders the voltage of channel 1 (E_1) is obtained by taking the difference of the voltages of antennas 11 and 12, so that:

$$E_1 = E_{11} - E_{12}$$

Similarly: $$E_2 = E_{21} - E_{22}$$

As is well known, the antenna pattern resulting from doing this is approximately a cosine pattern. This approximation is valid provided the distance between the antennas is reasonably small, e.g. $< \lambda/8$. We shall assume in the following derivations that both channels under consideration have a cosine pattern.

The noise power received from a small segment $d\alpha$ in the direction α*, will be given by the function $g(\alpha) \cdot d\alpha$. Note that this function $g(\alpha)$ is in general not the same as the function $f(\alpha)$, as defined in Section 2, because $g(\alpha)$ does not take into account the directivity pattern of the antenna system (compare assumptions (2a) and (3a)).

As before it is assumed that noise signals arriving from different directions are uncorrelated (compare with assumption (2b)).

3.1 Arbitrary Noise Distribution

3.1.1 General Case. In this paragraph a general expression will be derived for the correlation coefficient of the noise voltages generated in two antenna systems. As described in Section 3, both antenna systems will have a cosine pattern, and also the noise power per radian, arriving from the direction α, is given by the function $g(\alpha)$.

To derive the general expression, it will first be necessary to consider the correlation coefficient in the case of n discrete uncorrelated noise sources.

Let the j^{th} noise source be located in the direction α_j. It will generate the voltage e_{1j} in channel 1 and the voltage e_{2j} in channel 2. These

* The angle α is defined as the angle between the direction of arrival of the signal and the perpendicular to the line 11-12, connecting the two antennas of the first channel.

voltages will be equal to (See Fig. 3):

$$e_{1j} = e_{\alpha_j} \sin \alpha_j$$

$$e_{2j} = e_{\alpha_j} \sin(\alpha_j + \varphi)$$

The quantity $\overline{e^2_{\alpha_j}}$ is proportional to the average power of the j^{th} noise source, and it corresponds to $g(\alpha_j)\cdot d\alpha$. The proportionality constant will be neglected in these derivations because it is the same for all noise sources and will therefore cancel out in the final equation.

The total instantaneous voltage generated in channel 1 can be written as:

$$E_1 = \sum_{j=1}^{n} e_{\alpha_j} \sin \alpha_j \tag{9}$$

Similarly we find for channel 2:

$$E_2 = \sum_{j=1}^{n} e_{\alpha_j} \sin(\alpha_j + \varphi) \tag{10}$$

The correlation coefficient of these two noise voltages, by definition, is given by:

$$C = \frac{\overline{E_1 \cdot E_2}}{\sqrt{\overline{E_1^2} \cdot \overline{E_2^2}}} \tag{1}$$

Using Eqs 9 and 10, it is easy to evaluate the quantities $\overline{E_1 \cdot E_2}$, $\overline{E_1^2}$ and $\overline{E_2^2}$. To evaluate these we have to use the assumption that different noise sources are uncorrelated, corresponding to the assumption that noise signals, arriving from different directions are uncorrelated. Because of this assumption all terms of the form $\overline{e_{\alpha_j} \cdot e_{\alpha_k}}$ $(j \neq k)$ are zero.

$$\overline{E_1^2} = \overline{\left[\sum_{j=1}^{n} e_{\alpha_j} \sin \alpha_j\right]^2} = \sum_{j=1}^{n} \overline{e^2_{\alpha_j}} \sin^2 \alpha_j$$

And similarly:

$$\overline{E_2^2} = \sum_{j=1}^{n} \overline{e_{\alpha_j}^2} \sin^2(\alpha_j + \varphi)$$

For the time average of the product of E_1 and E_2 we find:

$$\overline{E_1 \cdot E_2} = \sum_{j=1}^{n} \overline{e_{\alpha_j}^2} \sin \alpha_j \cdot \sin(\alpha_j + \varphi)$$

The transition from the discrete case, with n noise sources, to the continuous case, where the distribution of noise power is given by $g(\alpha)$, can now be made. $g(\alpha) \cdot d\alpha$ is substituted for $\overline{e_{\alpha_j}^2}$ and the summation signs are replaced by integrals. This gives:

$$\begin{aligned} \overline{E_1^2} &= \int_{-\pi}^{\pi} g(\alpha) \sin^2\alpha \, d\alpha \\ &= \frac{1}{2} \int_{-\pi}^{\pi} g(\alpha) \, d\alpha - \frac{1}{2} \int_{-\pi}^{\pi} g(\alpha) \cos 2\alpha \, d\alpha \\ &= \frac{\pi}{2} (a_o - a_2) \end{aligned} \qquad (11)$$

where a_o and a_2 are Fourier cosine coefficients of $g(\alpha)$ defined by

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\alpha) \, d\alpha \text{ and } a_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\alpha) \cos 2\alpha \, d\alpha$$

For $\overline{E_2^2}$ we find:

$$\begin{aligned} \overline{E_2^2} &= \int_{-\pi}^{+\pi} g(\alpha) \sin^2(\alpha + \varphi) \, d\alpha \\ &= \frac{1}{2} \int_{-\pi}^{+\pi} g(\alpha) \, d\alpha - \frac{1}{2} \int_{-\pi}^{+\pi} g(\alpha) \cos 2(\alpha + \varphi) \, d\alpha \\ &= \frac{\pi}{2} a_o - \frac{1}{2} \int_{-\pi}^{+\pi} g(\alpha) \cos 2\alpha \cos 2\varphi \, d\alpha + \frac{1}{2} \int_{-\pi}^{+\pi} g(\alpha) \sin 2\alpha \sin 2\varphi \, d\alpha \\ &= \frac{\pi}{2} (a_o - a_2 \cos 2\varphi + b_2 \sin 2\varphi) \end{aligned} \qquad (12)$$

where b_2 is a Fourier sine coefficient of the function $g(\alpha)$:

$$b_2 = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(\alpha) \sin 2\alpha \, d\alpha$$

And finally the following expression is found for $\overline{E_1 \cdot E_2}$:

$$\overline{E_1 \cdot E_2} = \int_{-\pi}^{+\pi} g(\alpha) \sin\alpha \cdot \sin(\alpha + \varphi) \, d\alpha$$

$$= \frac{1}{2} \int_{-\pi}^{+\pi} g(\alpha) \cos\varphi \, d\alpha - \frac{1}{2} \int_{-\pi}^{+\pi} g(\alpha) \cos (2\alpha + \varphi) \, d\alpha$$

$$= \frac{\pi}{2} a_o \cos\varphi - \frac{1}{2} \int_{-\pi}^{+\pi} g(\alpha) \cos 2\alpha \cos\varphi \, d\alpha$$

$$+ \frac{1}{2} \int_{-\pi}^{+\pi} g(\alpha) \sin 2\alpha \sin\varphi \, d\alpha$$

$$= \frac{\pi}{2} (a_o \cos\varphi - a_2 \cos\varphi + b_2 \sin\varphi) \qquad (13)$$

With Eqs 11, 12, and 13 the correlation coefficient can now be readily found:

$$C = \frac{\overline{E_1 \cdot E_2}}{\sqrt{\overline{E_1^2} \cdot \overline{E_2^2}}} = \frac{(a_o - a_2) \cos\varphi + b_2 \sin\varphi}{\sqrt{(a_o - a_2)(a_o - a_2 \cos 2\varphi + b_2 \sin 2\varphi)}} \qquad (14)$$

Equation 14 then, gives the correlation coefficient of the two noise voltages generated in the two channels of a direction finder when the distribution of noise power is given by $g(\alpha)$.

<u>3.1.2 Perpendicular Antenna Systems.</u> Equation 14 that was derived in Section 3.1.1, gives the correlation coefficient for the more general case that the angle between antenna systems is equal to φ. In two-channel direction finders, using a four element Adcock antenna, this angle φ is usually equal to 90^o. In that case the correlation coefficient can be determined by substituting this value for φ in Eq 14, giving:

$$C_{90^o} = \frac{b_2}{\sqrt{(a_o^2 - a_2^2)}} \tag{15}$$

For the special case that the noise field is isotropic, the Fourier coefficient b_2 becomes zero, and consequently the noise voltages are uncorrelated. This will also be shown in Section 3.2.1.

3.2 Special Cases

In the following paragraph expressions will be derived for some known noise distributions, using Eq 14. These known distributions will be the isotropicc noise field, and the case of a single noise source.

3.2.1 Isotropic Noise Field. Consider that the antenna system of Fig.3 is placed in an isotropic noise field. In such a field an equal amount of average noise power is received from all directions. This means that $g(\alpha)$ is a constant. The Fourier coefficients a_2 and b_2 will then be equal to zero, so that the following value is found for the correlation coefficient of the noise voltages in the two channels:

$$C_{is} = \cos\varphi \tag{16}$$

It immediately follows that the noise voltages in the two channels of a direction finder, using a four-element Adcock, are uncorrelated, when the antenna system is placed in an isotropic noise field. For in that case $\varphi = 90^o$.

3.2.2 Single Noise Source. In the case that a single noise signal is received, arriving from a direction α_o, the function $g(\alpha)$ assumes the form of the Dirac delta function, as was previously discussed in Section 2.2.2. The Fourier coefficients of this function will be:

$$a_o = a_o$$

$$a_2 = a_o \cos 2\alpha_o$$

$$b_2 = a_o \sin 2\alpha_o$$

Substituting these values in Eq 14, we find for the correlation coefficient:

$$C = \frac{(1 - \cos 2\alpha_0)\cos\varphi + \sin 2\alpha_0 \cdot \sin\varphi}{\sqrt{(1 - \cos 2\alpha_0)(1 - \cos 2\alpha_0 \cdot \cos 2\varphi + \sin 2\alpha_0 \cdot \sin 2\varphi)}}$$

By the use of trigonometric formulae, this expression can be reduced to:

$$C = \pm 1$$

for all values of φ and α_0. That the correlation coefficient is equal to +1 or -1, also follows from the fact that the voltages in the two channels differ only in amplitude, if only one noise signal is received (see Section 3.1.1).

4. SIGNAL TO NOISE RATIO IN A TWO ELEMENT ROTATABLE ADCOCK ANTENNA

In this section the signal to noise ratio at the output of a rotatable Adcock antenna will be considered. This Adcock antenna consists of two identical monopoles, placed a distance d apart. To use this rotatable pair of antennas for direction finding, the difference of the two antenna voltages is taken.

Let a radio signal of frequency ω (corresponding to a wavelength λ) arrive from a direction α (see Fig. 4). Then the two antenna voltages, V_1 and V_2,

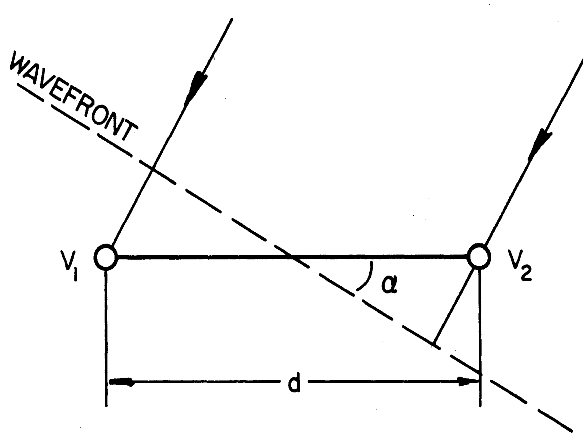

Figure 4 Rotatable Adcock Antenna Configuration

will have the same frequency but different phase angles. Let the phase difference between the two voltages be equal to 2u, which is a function of the direction of arrival α:

$$2u = \frac{2\pi d}{\lambda} \sin \alpha \qquad \text{or}$$

$$u = \frac{\pi d}{\lambda} \sin \alpha.$$

For the difference voltage $V_A = V_1 - V_2$ we find:

$$V_A = A \cos(\omega t - u) - A \cos(\omega t + u)$$

$$V_A = 2A \sin u \cdot \sin \omega t$$

where A is the peak value of the signal amplitude of each of the two monopoles.

The direction of arrival of the signal can be determined by rotating the antenna system until the signal V_A disappears. This occurs when the angle α=0.

If there is noise present, due to an isotropic noise field, the signal-to-noise ratio will go to zero when α does. It will therefore only have meaning to talk about the signal-to-noise ratio when the angle α is different from zero. Indeed, to find the position of the null, the operator will have to rotate the Adcock antenna slightly away from the null position. In the following, the signal-to noise ratio of the Adcock signal V_A will be calculated when the antenna system is moved away from the null position by amounts of 5^o, 10^o and 20^o. This signal-to noise ratio will then be compared to that obtained by a non-directional single monopole, identical to the ones used in the Adcock antenna. Let this antenna be called the monitor antenna.

Assume that the average noise power at the output of this monitor antenna is proportional to $N_M = \overline{E^2}$. The signal output of this antenna will have the same amplitude as each of the monopoles of the Adcock antenna, namely,

$$V_M = A \cos \omega t$$

The signal power of this signal is then equal to:

$$S_M = \frac{A^2}{2}$$

From this it follows that the signal-to-noise ratio of the monitor antenna is equal to:

$$R_M = \frac{S_M}{N_M} = \frac{A^2}{2\overline{E^2}} \tag{17}$$

Let us now consider the signal-to-noise ratio of the output signal of the rotatable Adcock antenna. As was shown above, the output signal will be:

$$V_A = V_1 - V_2 = 2A \sin u \cdot \sin\omega t$$

The signal power is then equal to:

$$S_A = 2\,A^2 \sin^2 u.$$

To calculate the noise power of the difference signal, it is necessary to know the correlation coefficient of the two noise voltages, generated in the two elements of the Adcock antenna. This was pointed out in the introduction. The average noise power at the output of each of the two monopoles is proportional to $\overline{E^2}$, because each of the monopoles is identical to the monitor antenna. The correlation coefficient of the noise signals in the two antennas is equal to

$$C = J_o\left(\frac{2\pi d}{\lambda}\right)$$

because the noise field was assumed isotropic (see Section 2.2.1).

It follows that the noise power of the Adcock signal N_A is equal to Eq 3:

$$N_A = 2\,\overline{E^2}\,(1 - C)$$

$$= 2\,\overline{E^2}\left[1 - J_o\left(\frac{2\pi d}{\lambda}\right)\right]$$

And consequently the signal-to-noise ratio at the output of the Adcock antenna will be:

$$R_A = \frac{S_A}{N_A} = \frac{A^2 \sin^2 u}{\overline{E^2}\left[1 - J_o\left(\frac{2\pi d}{\lambda}\right)\right]} \tag{18}$$

This signal-to-noise ratio can be expressed in terms of the signal-to-noise ratio of the monitor antenna, giving the following equation:

$$\frac{R_A}{R_M} = \frac{2 \sin^2(\frac{\pi d}{\lambda} \sin\alpha)}{1 - J_o(\frac{2\pi d}{\lambda})} \tag{19}$$

This equation is plotted in Fig. 5 as a function of d/λ for three different values of α: 5°, 10° and 20°.

Figure 5 shows that when $d < 0.2\ \lambda$, as is often the case in a practical direction finder, the signal-to-noise ratio of the antenna signal is almost independent of d. In other words, not much is gained or lost by making d larger or smaller. It must be kept in mind however, that Fig. 5 applies only to the output signal of the antenna system and that noise originating in the receiver is not taken into account. Because of the receiver noise it will be advantageous to choose the distance between the two antennas as large as practically possible.

5. CONCLUSION

Mathematical expressions have been derived for the correlation coefficient of two noise voltages generated in two antennas, which are placed a known distance apart. It is shown that this correlation coefficient depends on the intensity of the incident noise as a function of direction.

An expression for the correlation coefficient of two noise voltages generated in two channels of a direction finder has also been derived.

These expressions are useful to calculate the noise output of different antenna systems, because the noise power resulting when a number of noise voltages are combined, strongly depends on the amount of correlation between these noise voltages. This point is illustrated in Section 4 of the report, where the equations developed in Section 2 were applied to the case of a rotatable 2-element Adcock direction finder.

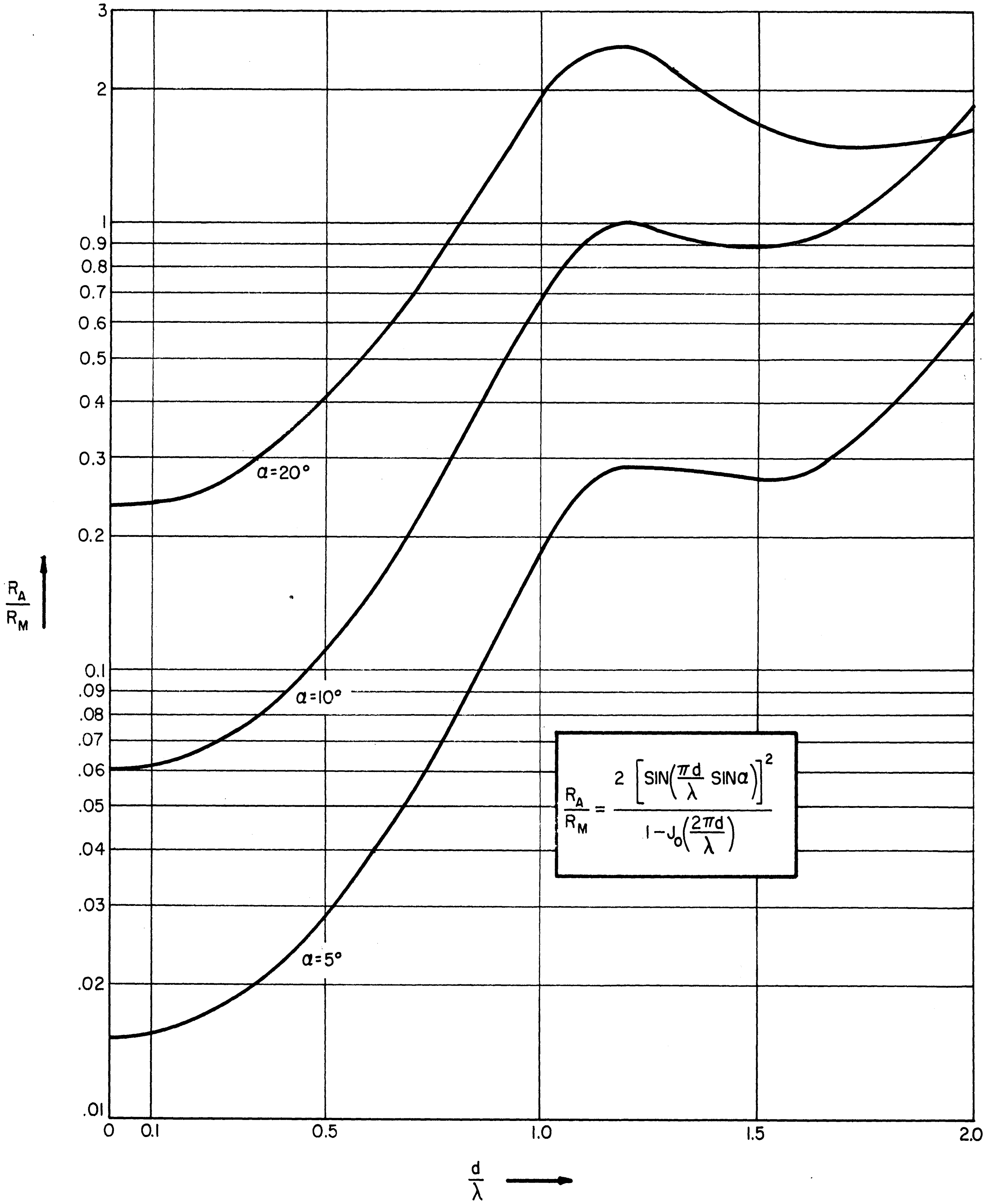

FIG. 5 SIGNAL-TO-NOISE RATIO IN ADCOCK ANTENNA

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