THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE
ANN ARBOR

THE SYNTHESIS OF NONUNIFORM TRANSMISSION LINES

Technical Report No. 116
2899-41-T

Cooley Electronics Laboratory
Department of Electrical Engineering

By Pieter G. Cath

Approved by: Charles B. Sharpe

Project 2899

TASK ORDER NO. EDG-4
CONTRACT NO. DA-36-039 sc-78283
SIGNAL CORPS, DEPARTMENT OF THE ARMY
DEPARTMENT OF ARMY PROJECT NO. 3A99-06-001-01

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The University of Michigan

January 1961

ACKNOWLEDGMENTS

The author is deeply indebted to the Chairman of his Doctoral Committee, Professor C. B. Sharpe, who not only aroused his interest in the problem of nonuniform lines, but who also was a source of constant encouragement and of valuable suggestions during the course of the research.

The author also wishes to thank the other members of his Doctoral Committee for their helpful comments concerning the material.

Finally, the author would like to thank Miss A. M. Rentschler, Mrs. A. M. Brockus, Mr. R. E. Graham, and Mr. E. J. DeRienzo who prepared the manuscript for publication.

A part of the research reported in this dissertation was supported by the U. S. Army Signal Corps.

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (Continued)

LIST OF SYMBOLS

Symbol	Description	Defined or first used in equation
$C_n(s)$	Function generated by the Fourier transform of cos $n\pi y$	(4.17)
$C_{ne}(s)$	$C_{ne}(s) = C_n(s)$ for even n	(4.17)
$C_{no}(s)$	$C_{no}(s) = C_n(s)$ for odd n	(4.18)
E_i	Equation replacing one equation (E_r) of the old reference	(5.33)
E_k	Equation denoting the kth plane in n-dimensional space ($1 \leq k \leq m$)	(5.7)
E_0	Denotes constraint equation	(5.21)
E_r	Plane of the old reference to be replaced by E_i	(5.35)
$F(s)$	Function to be approximated	(5.2)
$F[\rho]$	Complex Fourier transform of ρ	(4.1)
$G(s)$	Fourier transform of $\rho(y)$	(4.13)
$I(x)$	Current in transmission line at the point x	(3.1)
R_1	Real part of Z_1	(4.10)
$S_n(s)$	Function generated by the Fourier transform of sin $n\pi y$	(4.28)
$S_{ne}(s)$	$S_{ne}(s) = S_n(s)$ for even n	(4.28)
$S_{no}(s)$	$S_{no}(s) = S_n(s)$ for odd n	(4.29)
$U(x)$	Voltage in transmission line at the point x	(3.2)
$U_n(s)$	Transformation of $C_n(s)$	(A.2)
$V_n(s)$	Transformation of $S_n(s)$	(A.12)
$Z_0(y)$	Characteristic-impedance function	(3.5)
Z_{01}	$Z_{01} = Z_0(0)$; characteristic impedance at input of the nonuniform line	(4.5)
Z_{02}	$Z_{02} = Z_0(y=1)$; characteristic impedance at receiving end of the nonuniform line	(3.23)

Symbol	Description	Defined or first used in equation
Z_1	$Z_1 = Z_1(s)$; internal impedance of generator	(4.5)
Z_2	$Z_2 = Z_2(s)$; load impedance	(4.6)
a_{kj}	Direction numbers of the plane E_k $(1 \leq j \leq n)$	(5.5)
a_n	Coefficients of the cosine terms in expansion (4.12)	(3.31)
a_{ne}	$a_{ne} = a_n$ for even n	(4.21)
a_{no}	$a_{no} = a_n$ for odd n	(4.21)
b_n	Coefficients of the sine terms in expansion (4.12)	(3.27)
b_{ne}	$b_{ne} = b_n$ for even n	(4.32)
b_{no}	$b_{no} = b_n$ for odd n	(4.32)
c_k	Constant term in equation E_k	(5.6)
d_i	Diameter of center conductor	(4.50)
d_o	Inside diameter of outer conductor	(4.50)
$f(s)$	Function approximating $F(s)$	(5.1)
$f_j(s)$	Approximating functions $(1 \leq j \leq n)$	(5.1)
$h(s)$	Approximation error	(5.2)
h	Reference error	(5.18)
h'	Reference error of the new reference	(5.30)
h_k	Error, or residue, of equation E_k	(5.8)
h'_k	Residues belonging to the new reference	(5.37)
ℓ	Length of nonuniform transmission line	(3.20)
ne	Stands for "only even values of n"	(4.17)
no	Stands for "only odd values of n"	(4.18)
$\bar{n}_k$	Vector normal to the plane E_k	(5.9)

Symbol	Description	Defined or first used in equation
s	Frequency variable; $s = \frac{\ell}{\lambda}$	(3.21)
s_k	kth sampling point of the variable s	(5.3)
u_n	Transformation of a_n	(A.2)
v_n	Transformation of b_n	(A.12)
x	Independent variable of length along the transmission line	(3.1)
x_j	Coefficients in the expansion (5.1) corresponding to the jth coordinate in n-dimensional space	(5.1)
y	Normalized variable of length; $y = \frac{x}{\ell}$	(3.20)
$y(x)$	Shunt admittance per unit length at the point x	(3.1)
$z(x)$	Series impedance per unit length at the point x	(3.2)
$\Gamma(x)$	Reflection-coefficient function	(3.7)
Γ	$\Gamma = \Gamma(s)$; reflection coefficient at the input of the nonuniform transmission line	(3.17)
Γ_o	Load reflection referred back to input of line	(3.24)
$\Gamma_1(s)$	Function of $\Gamma(s)$ and $\Gamma_o(s)$	(4.4)
Γ_2	$\Gamma_2 = \Gamma_2(s)$; reflection coefficient at the load	(3.23)
$\beta(x)$	Imaginary part of $\gamma(x)$	(3.18)
$\gamma(x)$	Propagation-constant function	(3.6)
λ	Wavelength of electromagnetic wave	(3.21)
λ_k	Non-zero constants $(1 \leq k \leq n+1)$	(5.10)
λ_k'	Constants belonging to the new reference	(5.35)
μ_k	Constant coefficients	(5.33)
$\rho(y)$	Reflection-distribution function	(3.13)

ABSTRACT

A general synthesis procedure has been developed by which matching sections can be synthesized. A matching section is a section of lossless nonuniform transmission line of finite length that provides a match between a generator with complex internal impedance and a complex load impedance, such that maximum power transfer is achieved over a specified range of frequencies. The values of the internal impedance of the generator and the load impedance can be given either in equation form or in the form of measurements. Special cases that can be treated with the general method include impedance transformers, for which generator and load impedance are both real, and driving point impedances that must exhibit a certain behavior over a specified range of frequencies. Existing methods to synthesize impedance transformers have been extended.

The synthesis procedure is based upon approximate solutions to the equations describing nonuniform transmission lines that have appeared in the literature. Using these approximate solutions, the matching problem is reduced to the problem of finding a real function, the reflection-distribution function, whose complex Fourier transform approximates a complex function, determined by the generator and load impedances. The reflection-distribution function must be identically zero outside a specified interval.

The reflection-distribution function is found by expansion into a trigonometric series and subsequently determining the coefficients in this expansion. A method is developed by which the complex function to be approximated is first separated in real and imaginary parts. The coefficients in the trigonometric expansion are then determined such that these real and imaginary parts are approximated separately in a discrete Chebyshev sense. By discrete Chebyshev sense is meant that the maximum magnitude of the approximation error is minimum at a discrete number of sampling points.

The approximation process makes use of the theory of discrete Chebyshev approximation. The theory of discrete Chebyshev approximation, subject to constraints, has been treated. The constraints arise from the necessity to control the characteristic-impedance level at the terminals of the uniform transmission line.

The result of this investigation is a general synthesis procedure which extends the methods presently available. Not only matching between real impedances, but also matching between complex impedances can be achieved.

Several examples have been given of the synthesis of impedance transformers and matching sections.

CHAPTER I

INTRODUCTION

The subject of nonuniform transmission lines has attracted considerable interest with the development of microwave techniques during recent years. Synthesis of conventional networks becomes increasingly difficult at high frequencies because of the complications caused by parasitic elements. Nonuniform lines offer a very attractive solution to this problem, since the upper frequency is limited only by the occurrence of higher modes when the transverse dimensions of the line are comparable with the wavelength.

An exact solution of the differential equations describing the nonuniform line is possible in only a few special cases. This has led to the development of a number of approximate solutions. The most widely used of these is the method developed by Bolinder (Refs. 2, 3), because it provides the best approximation with regard to accuracy and simplicity. The requirement, however, that the nonuniform line be matched at the receiving end, has restricted the use of Bolinder's method to the synthesis of impedance transformers. An impedance transformer is a nonuniform line that provides a match between two real impedances of different value.

Recently, Orlov (Ref. 9) and Sharpe (Ref. 11) have developed an approximate solution which is valid when an arbitrary mismatch exists at the receiving end of the line. Their solution makes it possible to consider the synthesis of matching sections for which both the load impedance and the internal impedance of the generator are complex quantities.

The subject of this dissertation is the development of a synthesis procedure based upon the approximate solution by Orlov and Sharpe. The procedure involves the construction of a bounded real function, which is identically zero outside a specified interval, whose complex Fourier transform approximates a given complex function. The theory of discrete Chebyshev approximation will prove to be a very powerful tool in this approximation process.

The present method is particularly suited for the synthesis of nonuniform lines whose behavior must be controlled over a given range of frequencies. The method is completely general and can therefore also be applied to the synthesis of impedance transformers and driving point impedances.

The treatment in this dissertation is subdivided into several chapters. The next chapter, Chapter II, contains a brief statement of the problem. In Chapter III a review is given of the pertinent literature. The literature is subdivided into two principal categories. In the first the approximate solutions to the differential equations of nonuniform lines are developed, in the second, some of these approximate solutions are applied to the synthesis of impedance transformers. In Chapter IV, the general matching problem is considered and reduced to an approximation problem, which can be solved using the theory of discrete Chebyshev approximation, as developed in Chapter V. Chapter VI is devoted to examples of the synthesis of impedance transformers and matching sections. Conclusions are given in Chapter VII.

CHAPTER II

STATEMENT OF THE PROBLEM

The principal problem considered in this study is the synthesis of matching sections. By a matching section is meant a nonuniform line, of finite length, which can be inserted between a generator and a load to obtain maximum energy transfer over a given range of frequencies. Both the internal impedance of the generator, Z_1, and the load impedance, Z_2, are complex quantities, which are functions of frequency. The circuit configuration is shown in Fig. 2.1.

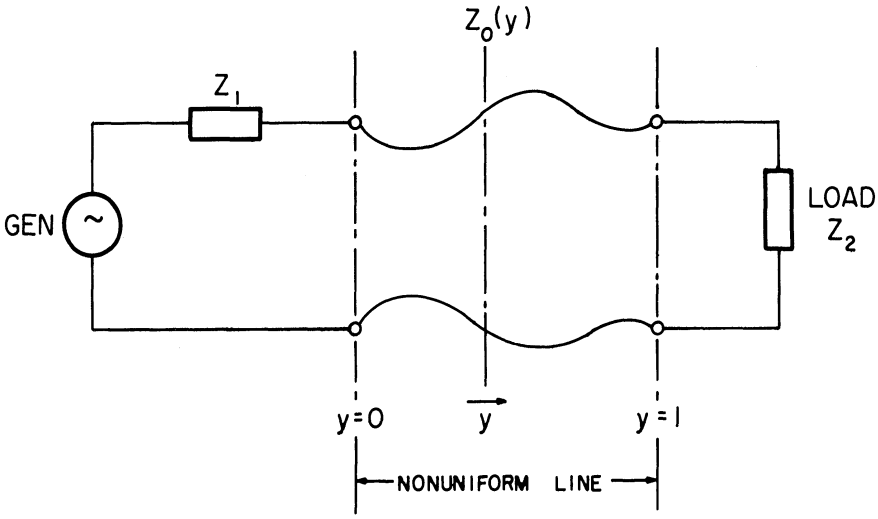

Fig. 2.1 Circuit configuration for the matching problem.

The synthesis problem can now be stated as follows:

> given the internal impedance (Z_1) of the generator, the load impedance (Z_2), and the frequency range over which they are to be matched, synthesize the matching section.

The matching section that will be synthesized is a section of nonuniform transmission line subject to the following restrictions:

a. The length, ℓ, of the nonuniform line is finite.

b. The nonuniform line is lossless.

c. The taper is continuous.

The last restriction is equivalent to the requirement that the characteristic-impedance function, $Z_0(y)$ be continuous. It will be assumed that $Z_0(y)$ does not go to zero or infinity at any place in the line.

The synthesis is completed when the characteristic-impedance function has been determined. The characteristic-impedance function, in turn, is uniquely determined by the reflection-distribution function, $\rho(y)$, which is defined by the relationship

$$\rho(y) \equiv \frac{1}{2} \frac{dZ_0(y)}{Z_0(y)\, dy} = \frac{1}{2} \frac{d \ln Z_0(y)}{dy} \tag{2.1}$$

Because of the requirement (b) that the nonuniform line be lossless, the function $\rho(y)$ is a real function. Because of the requirement (a) that the nonuniform line is of finite length, the function $\rho(y)$, for mathematical convenience, is defined to be identically zero outside the interval (0,1). The requirement that the taper be continuous implies that the function $\rho(y)$ is bounded.

The synthesis problem can be reduced to finding a function $\rho(y)$ such that its complex Fourier transform, $G(s)$, approximates a complex function $\Gamma_1(s)$. The function $\Gamma_1(s)$, as will be shown, can be determined from the impedances $Z_1(s)$ and $Z_2(s)$, which are given. The variable s is defined by $s = \ell/\lambda$, where ℓ is the length of the matching section and λ the wavelength. The variable s, therefore, is dimensionless and proportional to frequency. It will be called the frequency variable.

In general G(s) can only approximate $\Gamma_1(s)$, because not every complex function $\Gamma_1(s)$ is the Fourier transform of a real function which is zero outside the interval (0,1).

The function $\rho(y)$ will be expanded in a trigonometric series:

$$\rho(y) = \sum_{n=0}^{N} [a_n \cos n\pi y + b_n \sin n\pi y] \qquad (0 \leq y \leq 1)$$

$$\rho(y) = 0 \qquad (y < 0;\ y > 1) \tag{2.2}$$

and the coefficients a_n and b_n will be determined such that $\mathrm{Re}\left\{G(s)e^{j2\pi s}\right\}$ approximates $\mathrm{Re}\left\{\Gamma_1(s)e^{j2\pi s}\right\}$, and $\mathrm{Im}\left\{G(s)e^{j2\pi s}\right\}$ approximates $\mathrm{Im}\left\{\Gamma_1(s)e^{j2\pi s}\right\}$ in a discrete Chebyshev sense. A function f(s) is said to approximate a function F(s) in a discrete Chebyshev sense when the maximum of the magnitude of the error

$$\max |h(s)| = \max |F(s) - f(s)| \tag{2.3}$$

is minimum at a discrete set of sampling points.

Once the values of the coefficients a_n and b_n are known, the reflection-distribution function $\rho(y)$ and, therefore, also the characteristic-impedance function $Z_0(y)$ are completely determined. The determination of the coefficients a_n and b_n, therefore, essentially completes the synthesis of the matching section.

Other problems that will be considered are special cases of the general matching problem that was outlined above. These are the synthesis of impedance transformers (Z_1 and Z_2 both real) and the synthesis of a driving point impedance which approximates a given impedance over a range of frequencies.

CHAPTER III

REVIEW OF THE LITERATURE

3.1 Introduction

The work on continuously tapered lines can be subdivided into two principal categories. To the first belong attempts to find solutions, exact and approximate, to the differential equations describing the behavior of a nonuniform line. The second category consists of synthesis procedures based on these approximate solutions.

The approximate solution that has played a major role in the synthesis of nonuniform lines is the Fourier transform solution. In short, it states that the input reflection coefficient of a nonuniform line equals the Fourier transform of the reflection-distribution function. This solution was proposed by Bolinder (Refs. 2, 3) and was used by many others to synthesize nonuniform lines to act as impedance transformers between real impedances. The reason that only this kind of impedance transformer has been synthesized lies in the fact that the Fourier transform approximation is valid only when the line is properly terminated at the receiving end.

Recently Orlov (Ref. 9) has developed an approximate solution that is valid when an arbitrary mismatch is present at the receiving end of the line. This solution was independently derived by Sharpe (Ref. 11) using a different approach.

In the following paragraphs a review of the pertinent literature will be given. The symbols used by the authors in their original work have been modified, where necessary, to arrive at a uniform notation throughout this review.

3.2 The Differential Equations for Nonuniform Lines

The basic assumption that is always made in the treatment of nonuniform lines is that of the existence of a unique current I(x) and voltage U(x), which at any point in the line satisfy the equations:

$$\frac{dI(x)}{dx} = - y(x)U(x) \tag{3.1}$$

$$\frac{dU(x)}{dx} = - z(x)I(x) \tag{3.2}$$

where y(x) is the shunt admittance and z(x) is the series impedance per unit length.

These equations are valid only when certain conditions are imposed on the electromagnetic field and the line: (a) the mode of wave propagation must be essentially transversal (TEM), which implies that all wavelengths must be large in proportion to the transverse dimensions of the line; (b) there should be no rapid discontinuities in characteristic impedance.

By eliminating U(x) or I(x) from Eqs. 3.1 and 3.2, two second-order differential equations can be obtained.

$$\frac{d^2I(x)}{dx^2} - \frac{d \ln y(x)}{dx} \frac{dI(x)}{dx} - y(x)z(x)I(x) = 0 \tag{3.3}$$

$$\frac{d^2U(x)}{dx^2} - \frac{d \ln z(x)}{dx} \frac{dU(x)}{dx} - y(x)z(x)U(x) = 0 \tag{3.4}$$

When y(x) and z(x) are constant along the line, Eqs. 3.3 and 3.4 reduce to the well-known equations for uniform transmission lines. In general, however, (3.3) and (3.4) cannot be solved except in some special cases such as the exponential line.

It is possible to reduce the order of the differential equations (3.3) and (3.4) by making the proper transformations.

The following quantities are defined:

The characteristic-impedance function:

$$Z_0(x) \equiv \sqrt{\frac{z(x)}{y(x)}} \tag{3.5}$$

The propagation-constant function:

$$\gamma(x) \equiv \sqrt{z(x)\ y(x)} \tag{3.6}$$

The reflection-coefficient function:

$$\Gamma(x) \equiv \frac{\frac{U(x)}{I(x)} - Z_0(x)}{\frac{U(x)}{I(x)} + Z_0(x)} \tag{3.7}$$

Consider the identity:

$$\frac{d\,\frac{U(x)}{I(x)}}{dx} = \frac{1}{I(x)}\ \frac{dU(x)}{dx} - \frac{U(x)}{I(x)^2}\ \frac{dI(x)}{dx} \tag{3.8}$$

Substituting (3.1) and (3.2) into (3.8), one obtains

$$\frac{d\,\frac{U(x)}{I(x)}}{dx} = -\,z(x) + y(x)\left[\frac{U(x)}{I(x)}\right]^2 \tag{3.9}$$

From (3.7) it follows that

$$\frac{U(x)}{I(x)} = Z_0(x)\ \frac{1+\Gamma(x)}{1-\Gamma(x)} \tag{3.10}$$

Substitution of (3.10) into (3.9) yields

$$\frac{1+\Gamma(x)}{1-\Gamma(x)}\ \frac{dZ_0}{dx} + \frac{Z_0}{[1-\Gamma(x)]^2}\left\{[1-\Gamma(x)]\ \frac{d\,\Gamma(x)}{dx} + [1+\Gamma(x)]\ \frac{d\Gamma(x)}{dx}\right\} = \tag{3.11}$$

$$- z(x) + y(x)\ Z_0(x)^2 \left[\frac{1+\Gamma(x)}{1-\Gamma(x)}\right]^2$$

Using (3.5) and (3.6) and rearranging terms, (3.11) reduces to

$$\frac{d\,\Gamma(x)}{dx} - 2\ \gamma(x)\Gamma(x) + \frac{1}{2}\,[1-\Gamma(x)^2]\ \frac{d\ \ln Z_0(x)}{dx} = 0 \tag{3.12}$$

The reflection-distribution function is now defined as follows:

$$\rho(x) \equiv \frac{1}{2}\,\frac{d\ \ln Z_0(x)}{dx} \tag{3.13}$$

The differential equation (3.12) can then be written as follows:

$$\frac{d\,\Gamma(x)}{dx} - 2\ \gamma(x)\Gamma(x) + [1-\Gamma(x)^2]\ \rho(x) = 0 \tag{3.14}$$

This is the differential equation in reflection coefficient for a nonuniform line. It is a first-order nonlinear equation known as a Riccati equation. It is exact, and if it could be solved it would give exact solutions for the problem of nonuniform lines. In the following paragraphs the approximate solutions to Eq. 3.14 will be presented.

3.2.1 The Approximate Solution for Terminated Lines. Bolinder (Ref. 3) proposed to study those nonuniform lines for which $|\Gamma(x)|^2 \ll 1$, everywhere on the line. This implies that there be no mismatch at the receiving end of the line. With this assumption, the term $\Gamma(x)^2$ can be neglected in Eq. 3.14. One then obtains an approximate differential equation:

$$\frac{d\,\Gamma(x)}{dx} - 2\,\gamma(x)\,\Gamma(x) + \rho(x) = 0 \qquad (3.15)$$

This is a linear first-order differential equation. It can be solved using an integrating factor, $\exp\,[-2 \int_a^x \gamma(\xi)\,d\xi]$, in which the lower limit of integration is arbitrary. The solution that satisfies the boundary conditions is:

$$\Gamma(x) = \int_x^{\ell} \rho(\eta)\; e^{-2 \int_x^{\eta} \gamma(\xi)\,d\xi}\, d\eta \qquad (3.16)$$

The reflection coefficient at the input of the line equals

$$\Gamma \equiv \Gamma(0) = \int_0^{\ell} \rho(x)\; e^{-2 \int_0^{x} \gamma(\xi)\,d\xi}\, dx \qquad (3.17)$$

For lossless lines $\gamma(x) = j\beta(x)$. When the dielectric in the line is homogeneous, $\gamma(x)$ is constant along the line:

$$\gamma(x) = j\beta = \frac{j2\pi}{\lambda} \qquad (3.18)$$

and (3.17) becomes

$$\Gamma = \int_0^{\ell} \rho(x)\; e^{-j2\beta x}\, dx \qquad (3.19)$$

The restriction that the dielectric be homogeneous can easily be removed. For the case in which the dielectric is not homogenous, an expression equal to (3.19) can be obtained in which the variable x has been replaced by a new variable u, and the constant β by a new constant β'. The variable u and the constant β' are defined by the relation

$\beta' u = \int_0^x \beta(x) dx$. Introducing this variable u amounts to measuring the distance along the line in wavelengths.

To simplify the form of expression (3.19), a normalized coordinate, y, is introduced:

$$y \equiv \frac{x}{\ell} \qquad (3.20)$$

where ℓ is the length of the nonuniform line.

A new frequency variable, s, is introduced also:

$$s \equiv \frac{\ell}{\lambda} \qquad (3.21)$$

It is readily seen that s is a dimensionless variable, proportional to frequency. With these new variables (3.19) can now be written in its final form.

$$\Gamma(s) = \int_0^1 \rho(y)\, e^{-j4\pi sy}\, dy \qquad (3.22)$$

This, then, is the well-known Fourier transform approximation for the input reflection coefficient. The limits of integration on the Fourier transform are $-\infty$ and $+\infty$. Because, however, the function $\rho(y)$ is identically zero outside the region of the nonuniform line, i.e., $\rho(y) = 0$ outside the interval (0,1), Eq. 3.22 does represent the Fourier transform of the reflection-distribution function $\rho(y)$.

It should be pointed out that (3.22) can also be derived by a more direct approach. One can argue that the reflection generated by a portion, dx, of the line, located at the point x, can be written as

$$\frac{Z_0(x+dx) - Z_0(x)}{Z_0(x+dx) + Z_0(x)} = \rho(x)\, dx$$

A portion of length dx of the line, located at the point x, therefore contributes an amount $d\Gamma$ to the total reflection at the input of the line, which can be written as:

$$d\Gamma = \rho(x)\, dx\, e^{-j2\beta x}$$

The total input reflection is then found by integrating the contributions from all points along the line. The result is Eq. (3.19).

The Fourier transform has become the most widely used and most convenient tool in the synthesis of nonuniform lines. Due to its restrictions, small reflections and matched load, the synthesis effort has been confined to nonuniform lines acting as impedance transformers between real impedances. Some of the representative procedures will be reviewed in section 3.3.

3.2.2 The Approximate Solution in the Presence of Load Reflections. Recently Orlov (Ref. 9) and Sharpe (Ref. 11) have developed an approximate solution that is valid for the case in which an arbitrary mismatch exists at the receiving end of the line.

Orlov considers the behavior of a line varying in small discrete steps. By letting the number of steps go to infinity, Orlov obtains a solution for a continuously varying line which has an arbitrary mismatch at the receiving end.

Let the reflection at the load be equal to Γ_2:

$$\Gamma_2 = \frac{Z_2 - Z_{02}}{Z_2 + Z_{02}} \qquad (3.23)$$

where Z_2 is the value of the load impedance and Z_{02} is the value of the characteristic impedance of the line at its receiving end.

The load reflection can be referred back to the input of the line. This defines the quantity Γ_o, which can be written as:

$$\Gamma_o = \Gamma_2 e^{-j2\beta\ell} = \Gamma_2 e^{-j4\pi s} \qquad (3.24)$$

Orlov obtains a complicated expression for the input reflection coefficient, which in first approximation can be written as follows:

$$\Gamma(s) = \frac{\Gamma_o + \int_0^1 \rho(y)\, e^{-j4\pi sy} dy}{1 + \Gamma_o \int_0^1 \rho(y)\, e^{j4\pi sy} dy} \qquad (3.25)$$

When the line is matched at the receiving end ($\Gamma_o = 0$), this expression reduces to the familiar Fourier transform (3.22).

Sharpe treats the problem of the nonuniform line, terminated in a mismatch, as a one-dimensional scattering problem, using perturbation techniques. An expression for the input reflection coefficient is then obtained in the form of a Fredholm series expansion. The first-order approximation to this series expansion agrees with Orlov's result (3.25). According to Orlov it is required that $|\rho(y)|_{max} \ll 1$. According to Sharpe the condition $|\rho(y)|_{max} \ll 2\pi s$ should be satisfied.

From (3.25) both Orlov and Sharpe derive a synthesis formula, which gives the relationship between the reflection-distribution function $\rho(y)$ and the reflection coefficients at the input and output of the nonuniform line:

$$\int_0^1 \rho(y)\, e^{-j4\pi sy}\, dy = \frac{\Gamma(1 - |\Gamma_o|^2) - \Gamma_o(1 - |\Gamma|^2)}{1 - |\Gamma|^2\, |\Gamma_o|^2} \qquad (3.26)$$

This expression will be the basis for the discussion in Chapter IV, where the synthesis of matching sections is developed.

3.3 Applications of the Fourier Transform

As was mentioned above, the synthesis of nonuniform lines has been restricted so far to impedance transformers (between real impedances) because of the requirement that the line be properly terminated at the receiving end. In this part of the literature review several of these synthesis methods and their results will be discussed.

3.3.1 The Synthesis of Impedance Transformers. A method to synthesize impedance transformers was presented by Willis and Sinha (Refs. 13, 14). The same method was described by Baur (Ref. 1), who obtained essentially the same results.

In this method the reflection-distribution function $\rho(y)$ is expanded in a trigonometric series containing either odd sine terms or even cosine terms. The input reflection coefficient $|\Gamma(s)|$ is then determined using Eq. 3.22.

Let $\rho(y)$ be expanded in a sum of odd sine terms:

$$\rho(y) = \sum_{n=1}^{N} b_n \sin n\pi y \qquad (3.27)$$

where n is an odd integer and the b_n's are constants. Evaluating the integral (3.22) one finds the input reflection coefficient of the line:

$$|\Gamma(s)| = \left| \sum_{n=1}^{N} b_n \frac{2n \cos 2\pi s}{\pi[(4s)^2 - n^2]} \right| \quad (n \text{ odd}) \qquad (3.28)$$

The reason given by Willis and Sinha for choosing only odd values of n is the fact that only odd sine terms contribute to impedance transformation. The amount of impedance transformation is found by integrating $\rho(y)$. It follows from the definition of $\rho(y)$, Eq. 3.13, that

$$\int_0^1 \rho(y)\, dy = \frac{1}{2} \ln \frac{Z_0(1)}{Z_0(0)} \tag{3.29}$$

Therefore the amount of impedance transformation caused by even sine terms is zero because

$$\int_0^1 \sin n\pi y\, dy = 0 \qquad \text{when n is even.}$$

The amount of impedance transformation caused by the odd sine terms is equal to:

$$\int_0^1 \sin n\pi y\, dy = \frac{-1}{n\pi} \cos n\pi y \Big|_0^1 = \frac{2}{n\pi} \qquad \text{for n odd.} \tag{3.30}$$

There is, however, a more important reason for using only odd values of n, which is not mentioned by Willis and Sinha. The contributions to the input reflection coefficient by the odd sine terms are all in phase and 90 degrees out of phase with the contributions from the even sine terms. Using only odd values of n therefore has the advantage that the input reflection coefficient $|\Gamma(s)|$ can be written in the form of a simple addition as expressed in Eq. 3.28.

Willis and Sinha also consider the case in which $\rho(y)$ is written as the sum of even cosine terms. Written in this form:

$$\rho(y) = \sum_{n=0}^{N} a_n \cos n\pi y \qquad \text{(n even)} \tag{3.31}$$

The input reflection coefficient $|\Gamma(s)|$ resulting from this reflection-distribution function becomes

$$|\Gamma(s)| = \left| \sum_{n=0}^{N} a_n \frac{8s \sin 2\pi s}{\pi[(4s)^2 - n^2]} \right| \quad \text{(n even)} \qquad (3.32)$$

Using either (3.28) or (3.32) Willis and Sinha then proceed to determine the coefficients b_n and a_n to obtain a high-pass characteristic for $|\Gamma(s)|$, with a minimum amount of reflection in the pass band. The coefficients in their method are determined by trial and error.

Figures 3.1 and 3.2 show two reflection patterns obtained by this method. Figure 3.1 shows the reflection pattern for an impedance transformer that is 0.75 λ long at the lowest frequency of the pass band. The corresponding reflection-distribution function is equal to

$$\rho(y) = k(1 - 0.636 \cos 2\pi y) \qquad (3.33)$$

Figure 3.2 shows the reflection pattern for a line which is 1 λ long at the lowest frequency of the pass band. The $\rho(y)$ producing this pattern is proportional to:

$$\rho(y) = k(1 - 0.899 \cos 2\pi y + 0.0112 \cos 4\pi y) \qquad (3.34)$$

The maximum height of the side lobes is 0.031 for the 0.75 λ line and 0.0056 for the 1 λ line, for the case in which $k = 1$.

The proportionality factor k is selected to give the correct amount of impedance transformation. For lines characterized by (3.31), where only the constant term gives impedance transformation, the factor k is determined by:

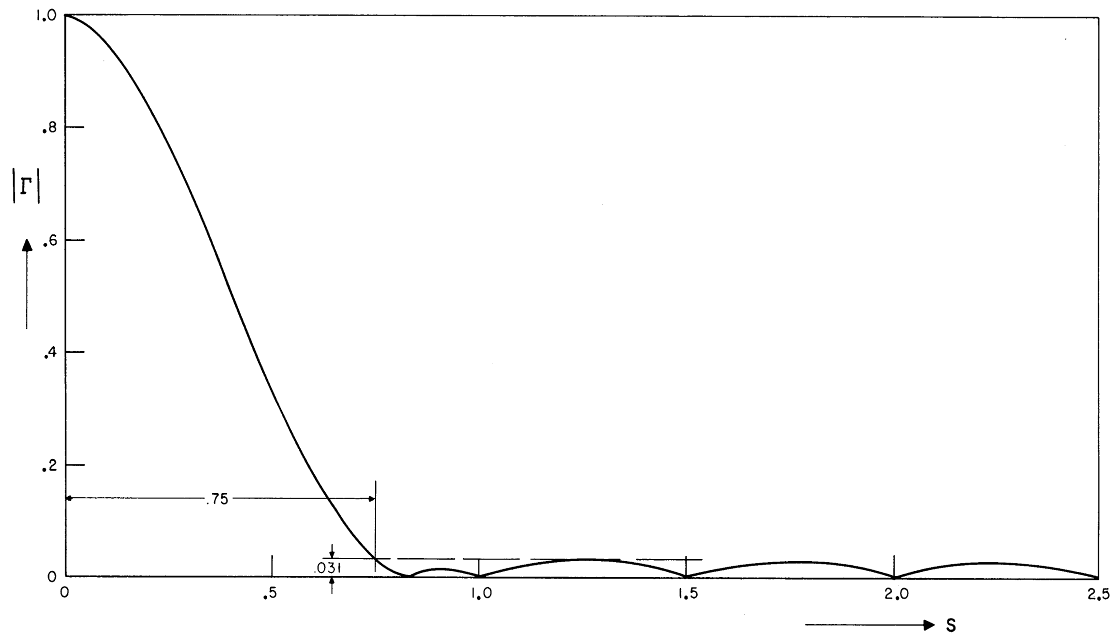

Fig. 3.1 Reflection pattern for $\rho(y) = 1 - 0.636 \cos 2\pi y$.

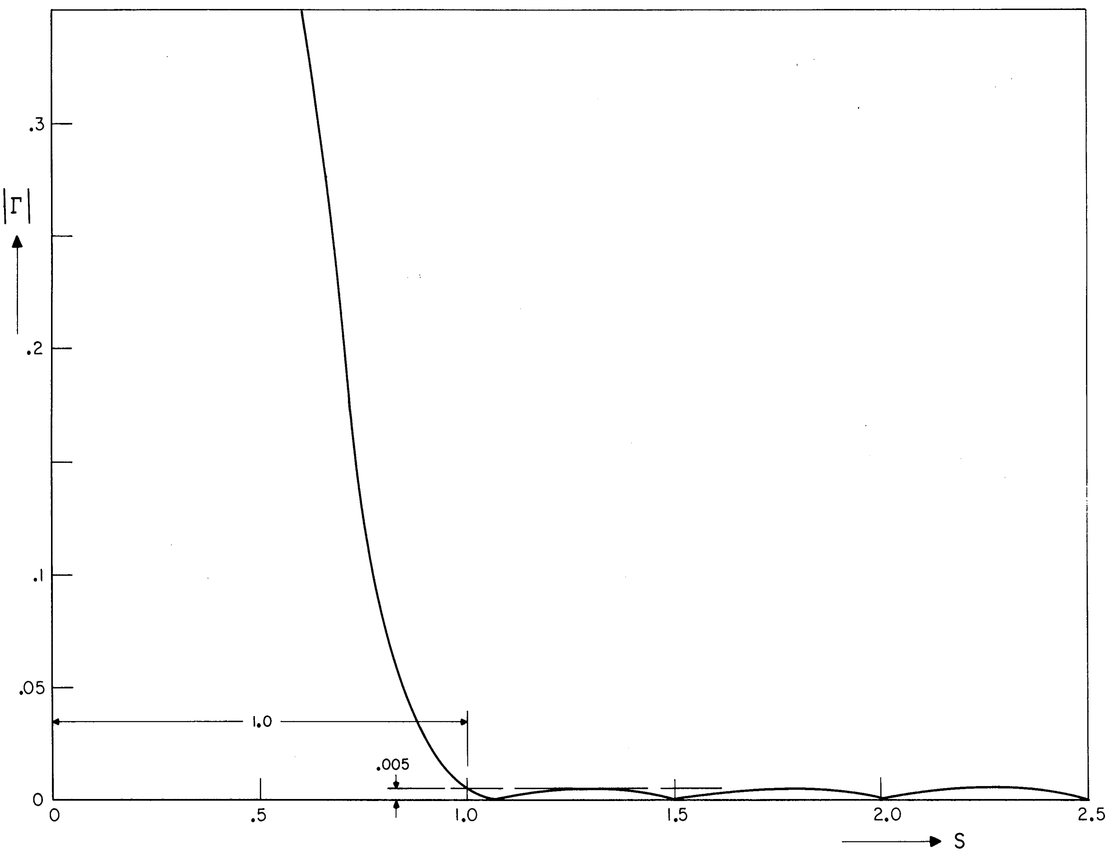

Fig. 3.2 Reflection pattern for
$\rho(y) = 1 - 0.889 \cos 2\pi y + 0.0112 \cos 4\pi y$.

$$k = \frac{1}{2} \ln \frac{Z_0(1)}{Z_0(0)} \tag{3.35}$$

Everywhere in the present discussion k will be taken equal to k = 1. When the line is expanded in odd sine terms, (3.27), k' must be determined from the following equation, using the result (3.30):

$$\frac{1}{2} \ln \frac{Z_0(1)}{Z_0(0)} = k' \sum_{n=1}^{N} \frac{b_n}{n\pi} \quad (n \text{ odd}) \tag{3.36}$$

Measurements were taken by Willis and Sinha on impedance transformers synthesized by this method. Excellent agreement between theory and experiment is reported in Ref. 14.

Very similar work was reported by Feldshtein (Ref. 6), although his method is less general. He considers a nonuniform line for which the reflection-distribution function $\rho(y)$ can be written in the form

$$\rho(y) = k(1 + a_2 \cos 2\pi y) \tag{3.37}$$

Somewhat arbitrarily he defines this function to represent an optimum smooth transition. Feldshtein then proceeds in a manner similar to that of Willis and Sinha and finds that minimum reflection in the pass band is obtained for $a_2 = -0.632$ in the case of a 0.75 λ line and $a_2 = -0.840$ in the case of a 1 λ line. The height of the side lobes thus obtained is equal to 0.032 for the 0.75 λ line and 0.0082 for the 1 λ line. Feldshtein's result for the 0.75 λ line is essentially equal to that obtained by Willis and Sinha. His 1 λ line gives slightly more reflection in the pass band, because only two terms are used in the expansion for $\rho(y)$.

Another example of the use of the Fourier transform has been demonstrated by Klopfenstein (Ref. 7). He reminds his readers of the analogy between the uses of the Fourier transform in transmission lines and in antenna pattern design, an analogy that was already pointed out by Bolinder (Ref. 3). Klopfenstein then adapts to transmission lines the work of Taylor, who studied the synthesis of Dolph-Chebyshev patterns using continuously illuminated apertures. This procedure leads to an input reflection pattern consisting of a main lobe and infinitely many side lobes, all of equal height. The taper that produces this reflection pattern is characterized by two step discontinuities in $Z(y)$, one at the beginning of the line and one at the end.

The height of the side lobes equals $\frac{1}{\cosh (2\pi s_o)}$, where s_o is the value of $s = \frac{\ell}{\lambda}$ at the lowest frequency of the pass band. Thus it is found that the maximum reflection in the pass band equals 0.018 for the 0.75 λ line and 0.0037 for the 1 λ line, results that are significantly better than those obtained by Willis and Sinha.

<u>3.3.2 Nonuniform Lines as Filters.</u> Feldshtein (Ref. 5) studied and experimentally measured the behavior of lines whose tapers vary rather violently. He shows that these lines exhibit the properties of a band-rejection filter. In the theoretical derivations, the Fourier transform approximation is used.

Consider a line for which the reflection-distribution function is:

$$\rho(y) = 3.3 \cos 2\pi y \qquad (3.38)$$

The ratio between the maximum and minimum value of the characteristic impedance $[Z_0(y)]$ of such a line is equal to 8.

When the Fourier transform of $\rho(y)$ is taken an input reflection $|\Gamma(s)|$ is obtained, whose absolute value exceeds unity. Since it is impossible for a reflection coefficient to exceed unity, Feldshtein defines $|\Gamma(s)|$ to be equal to one for those values of s, for which the absolute value of the Fourier transform of $\rho(y)$ exceeds this value. The reflection coefficients obtained by this process are shown in Fig. 3.3 and in Fig. 3.4 for the case in which $\rho(y) = 6.6 \cos 4\pi y$.

The broken lines in these figures show the results that were obtained experimentally by Feldshtein. It is interesting to note that the experimental values are reasonably close to the calculated values. This is unexpected because the Fourier transform approximation is valid only when $|\Gamma|^2 \ll 1$, a condition that is by no means fulfilled in this example.

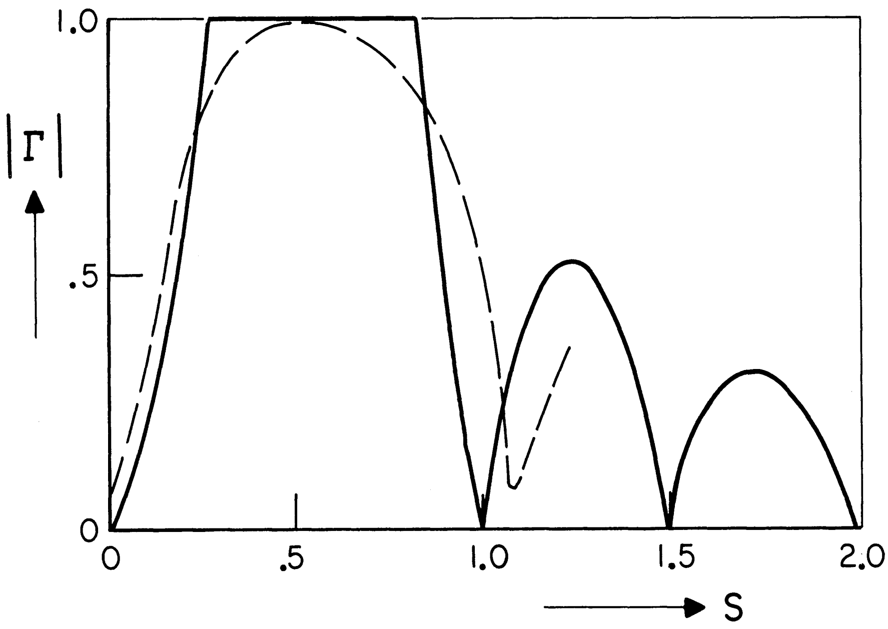

Fig. 3.3 Theoretical and experimental filter behavior for $\rho(y) = 3.3 \cos 2\pi y$

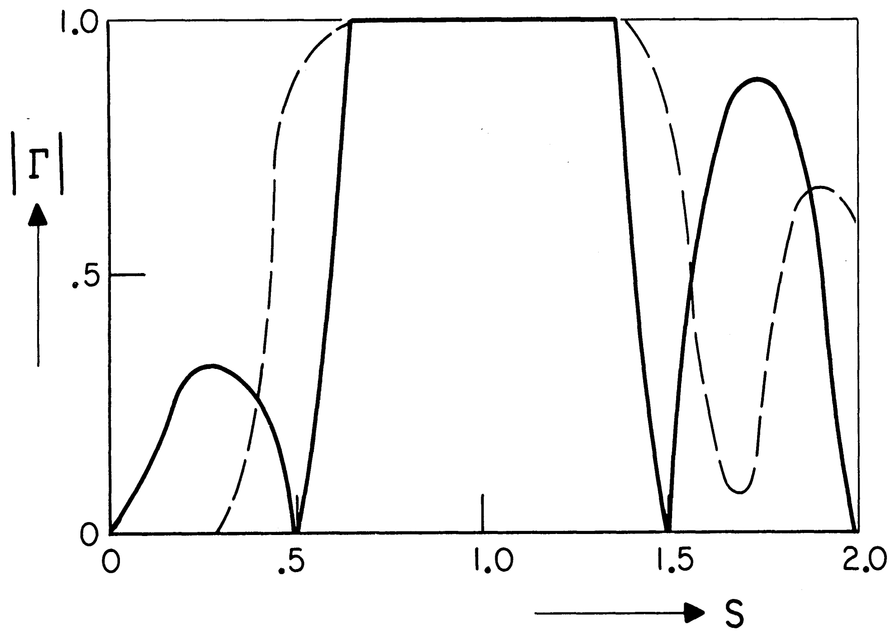

Fig. 3.4 Theoretical and experimental filter behavior for $\rho(y) = 6.6 \cos 4\pi y$.

CHAPTER IV

THE SYNTHESIS OF MATCHING SECTIONS

4.1 Introduction

In this chapter the synthesis of matching sections will be developed. By matching section is meant a nonuniform line which can be inserted between a generator and a load, to obtain maximum energy transfer over a certain range of frequencies. Both the output impedance of the generator and the load impedance are complex quantities, which are functions of frequency.

The synthesis method developed in this chapter is based on the approximate solution developed by Orlov (Ref. 9) and Sharpe (Ref. 11). The matching sections that are synthesized with this method are sections of nonuniform line, which have the following properties:

a. The length of the nonuniform line, ℓ, is finite

b. The nonuniform line is lossless and has a homogeneous dielectric

c. The taper is continuous.

It will first be shown how the formula developed by Orlov can be used to reduce the matching problem to the problem of finding a complex function $\Gamma_1(s)$ whose inverse Fourier transform is the function $\rho(y)$, the reflection-distribution function. The requirement that the nonuniform line be lossless and of finite length puts restrictions on the function $\rho(y)$. $\rho(y)$ must be a real function and identically zero outside the interval (0,1). In general the inverse Fourier transform of $\Gamma_1(s)$ will not meet these conditions. A complex valued function G(s) will therefore be sought which approximates $\Gamma_1(s)$. The inverse Fourier transform of G(s) will be a real function, zero outside the interval

(0,1). The error in the approximation will cause a slight mismatch, giving rise to undesired reflections. In the synthesis procedure these undesired reflections are minimized.

4.2 The General Synthesis Problem

When a complex load impedance is connected to a lossless transmission line, there will always be a reflection from this load, because the characteristic impedance of the line is real. Specifically, if the line is a lossless nonuniform line, an approximate solution is required that is valid in the case that such load reflections exist. As was discussed in Chapter III, such an approximate solution has been recently developed by Orlov (Ref. 9) and Sharpe (Ref. 11) (see section 3.2.2):

$$F[\rho(y)] = \int_0^1 \rho(y)\, e^{-j4\pi sy} dy = \frac{\Gamma(1-|\Gamma_o|^2) - \Gamma_o(1-|\Gamma|^2)}{1-|\Gamma|^2|\Gamma_o|^2} \qquad (4.1)$$

where $\rho(y)$ is the reflection-distribution function:

$$\rho(y) \equiv \frac{1}{2}\, \frac{d \ln Z_0(y)}{dy} \qquad (4.2)$$

Γ, which is a function of the frequency variable s, is the input reflection coefficient of the nonuniform line and Γ_o is the load reflection (Γ_2) referred to the input of the line:

$$\Gamma_o \equiv \Gamma_2\, e^{-j4\pi s} \qquad (4.3)$$

Furthermore the frequency variable s equals: $s = \frac{\ell}{\lambda}$, i.e., the ratio between the length of the nonuniform line and the wavelength. Therefore s is proportional to frequency.

The function $\Gamma_1(s)$ will next be defined. It is equal to the right-hand side of Eq. 4.1:

$$\Gamma_1(s) = \frac{\Gamma(1 - |\Gamma_o|^2) - \Gamma_o(1 - |\Gamma|^2)}{1 - |\Gamma|^2 |\Gamma_o|^2} \tag{4.4}$$

The circuit configuration is illustrated in Fig. 4.1. Maximum power transfer is to be obtained from the generator, whose internal impedance is equal to Z_1, to the load (Z_2). This matching is to be achieved using a nonuniform line of finite length. The line extends from y = 0 to y = 1, as shown in Fig. 4.1.

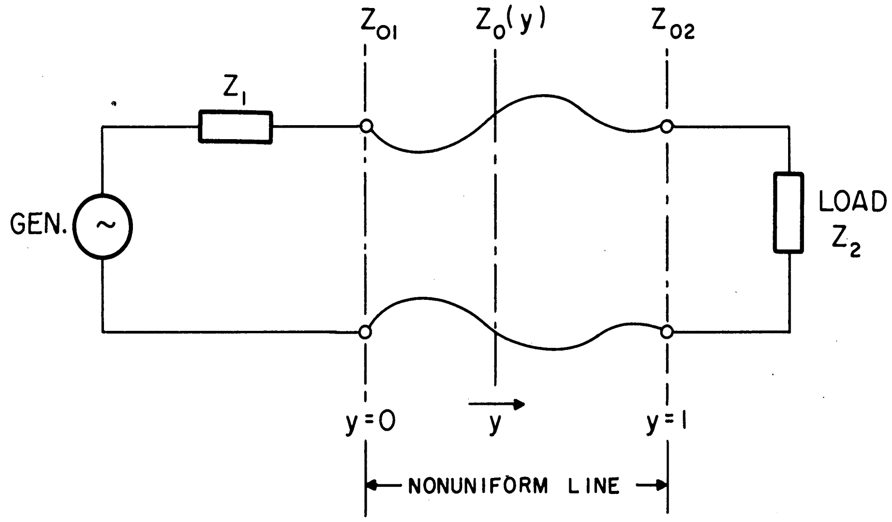

Fig. 4.1 Circuit configuration for the matching problem.

The condition necessary to obtain maximum power flow at the input of the line is well-known. The input impedance of the nonuniform line, terminated in Z_2, must equal Z_1^*, the complex conjugate of Z_1. This, of course, guarantees maximum power flow only across the point y = 0. However, together with the fact that the line is lossless, it also implies maximum power transfer from generator to load.

Let the characteristic impedance at the input of the nonuniform line be equal to $Z_0(0) \equiv Z_{01}$. When the input impedance of the line is equal to Z_1^*, the input reflection coefficient (Γ) will be:

$$\Gamma = \frac{Z_1^* - Z_{01}}{Z_1^* + Z_{01}} \tag{4.5}$$

Let the characteristic impedance at the receiving end of the line be equal to $Z_0(1) \equiv Z_{02}$. The load reflection (Γ_2) is then equal to

$$\Gamma_2 = \frac{Z_2 - Z_{02}}{Z_2 + Z_{02}} \tag{4.6}$$

Γ_o is the load reflection referred back to the input of the line, so that

$$\Gamma_o = \frac{Z_2 - Z_{02}}{Z_2 + Z_{02}} e^{-j4\pi s} \tag{4.7}$$

where Γ, Γ_2, Γ_o, Z_1, and Z_2 are in general functions of frequency.

The discussion can now be continued as follows. Let the configuration of Fig. 4.1 be given where the characteristic impedance of the nonuniform line at the input and at the receiving end is equal to Z_{01} and Z_{02}, respectively. The function $\Gamma_o(s)$ is then determined, and the function $\Gamma(s)$, necessary for maximum power transfer, can be determined from Eq. 4.5.

The functions $\Gamma(s)$ and $\Gamma_o(s)$ completely determine the function $\Gamma_1(s)$, as defined by Eq. 4.4. It seems that by taking the inverse Fourier transform of $\Gamma_1(s)$, one could then find the reflection-distribution function $\rho(y)$:

$$\rho(y) = F^{-1}[\Gamma_1(s)] = 2 \int_{-\infty}^{\infty} \Gamma_1(s)\, e^{j4\pi sy} ds \qquad (4.8)$$

Unfortunately the situation is not that simple because of the restrictions imposed on $\rho(y)$. These restrictions arise from the requirement that $\rho(y)$ be the reflection-distribution function of a lossless line of finite length. $\rho(y)$ must be a real function which is identically zero outside the interval (0,1). In general the inverse Fourier transform of $\Gamma_1(s)$ will not yield such a function $\rho(y)$.

In the following sections a method will be developed by which a function G(s) can be found which approximates the function $\Gamma_1(s)$ in a Chebyshev sense over a given interval of frequency, at a number of discrete sampling points. The most important property of G(s) is, that its inverse Fourier transform is a function $\rho(y)$ which is real and identically zero outside the interval (0,1). The error E(s) made in this approximation process will be:

$$E(s) = \Gamma_1(s) - G(s) \qquad (4.9)$$

E(s) equals the amount of undesired reflection between generator and load when they are connected by the matching section.

4.2.1 The Case of a Real Generator Impedance. Before the discussion is continued it will be of interest to consider as a special matching problem the case in which the generator has a real internal impedance ($Z_1 = R_1$), independent of frequency (see Fig. 4.1). By choosing the characteristic impedance of the line at its input equal to R_1 ($Z_{01} = Z_0(0) = R_1$), maximum power transfer is obtained when $\Gamma = 0$.

The synthesis formula (4.1) then reduces to:

$$F[\rho(y)] = -\Gamma_o \qquad (4.10)$$

It is interesting to note the physical significance of this result, which can also be arrived at using the following consideration.

Consider a nonuniform line matching a generator to a load impedance Z_2 as shown in Fig. 4.1. The characteristic impedance (Z_{01}) of the line at the input equals R_1, the internal impedance of the generator. For maximum power transfer, the input reflection coefficient of the line must be zero.

The total input reflection of the line is principally generated by two sources. One is the reflection from the load (Γ_2). Referred back to the input of the line, the load contributes an amount, equal to Γ_o, to the input reflection coefficient (Eq. 4.3). The other source is the integrated reflection from the nonuniformities of the line. This contribution can be expressed by $F[\rho]$, the Fourier transform of the reflection-distribution function.

The input reflection coefficient of the line will be zero when these two reflections exactly cancel each other. This occurs when they are equal in magnitude and 180 degrees out of phase. In other words, matching is achieved when

$$F[\rho(y)] = -\Gamma_o \qquad (4.10)$$

which is exactly the same result that was obtained above.

<u>4.2.2 Synthesis of Driving Point Impedances.</u> The synthesis of driving point impedances can be reduced to the same problem that arises in the synthesis of matching sections, i.e., to find a real function $\rho(y)$ which vanishes outside the interval (0,1) such that its

Fourier transform approximates a given complex valued function $\Gamma(s)$.

To synthesize an impedance that behaves like a given impedance $Z(s)$ over a certain band of frequencies, it is sufficient to synthesize a function $\Gamma(s)$ such that

$$\Gamma(s) = \frac{Z(s) - Z_{01}}{Z(s) + Z_{01}} \tag{4.11}$$

where $\Gamma(s)$ if the input reflection coefficient of a nonuniform line of finite length, terminated at the receiving end.

The problem then becomes to find a function $G(s)$ that approximates $\Gamma(s)$. The inverse Fourier transform of $G(s)$ is a real function $\rho(y)$ that is identically zero outside the interval $(0,1)$.

4.3 The Determination of the Reflection-Distribution Function

In the previous sections the synthesis problem was reduced to the problem of finding a function $G(s)$, which is the Fourier transform of the reflection distribution function $\rho(y)$ for a lossless nonuniform line of finite length.

From the assumption that the line is lossless, it follows that $\rho(y)$ is a real function, because the characteristic-impedance function $Z_0(y)$ is real. From the requirement that the line be of finite length, it follows that the function $\rho(y)$ must be identically zero outside the interval $(0,1)$.

In this section the nature of the function $G(s)$ will be determined. It will be shown that a convenient expression for $G(s)$ is obtained when the function $\rho(y)$ is expanded in a trigonometric series:

$$\rho(y) = \sum_{n=0}^{N} [a_n \cos n\pi y + b_n \sin n\pi y] \tag{4.12}$$

One is in general interested in letting the matching section provide a match over only a finite range of frequencies, and therefore a finite number of terms will be used in the expansion (4.12). Of course $\rho(y)$ is identically zero outside the interval (0,1).

The function G(s) is the Fourier transform of $\rho(y)$ and is therefore equal to:

$$G(s) = F[\rho] = \sum_{n=0}^{N} \left\{ a_n F[\cos n\pi y] + b_n F[\sin n\pi y] \right\} \qquad (4.13)$$

The coefficients a_n and b_n must then be determined in such a way that the function G(s) is a good approximation to the function $\Gamma_1(s)$.

In the following paragraphs the Fourier transforms of cos nπy and sin nπy will be evaluated.

<u>4.3.1 Evaluation of F[cos nπy].</u> The Fourier transform of cos nπy is equal to

$$F[\cos n\pi y] = \int_0^1 \cos n\pi y \, e^{-j4\pi sy} dy \qquad (4.14)$$

This integral can be calculated by first integrating by parts twice.

$$\int_0^1 \cos n\pi y \, e^{-j4\pi sy} dy =$$

$$\frac{1}{n\pi} \sin n\pi y \, e^{-j4\pi sy} \Big|_0^1 + \frac{j4s}{n} \int_0^1 \sin n\pi y \, e^{-j4\pi sy} dy =$$

$$\frac{1}{n\pi} \sin n\pi y \, e^{-j4\pi sy} \Big|_0^1 - \frac{j4s}{n^2\pi} \cos n\pi y \, e^{-j4\pi sy} \Big|_0^1 + \frac{16s^2}{n^2} \int_0^1 \cos n\pi y \, e^{-j4\pi sy} dy$$

This expression can be reduced to

$$[1 - \frac{(4s)^2}{n^2}] \int_0^1 \cos n\pi y \; e^{-j4\pi sy} dy = \frac{j4s}{n^2\pi} [1 - \cos n\pi \; e^{-j4\pi s}]$$

Because $\cos n\pi = (-1)^n$ this can be written as:

$$\int_0^1 \cos n\pi y \; e^{-j4\pi sy} dy = \frac{j\,4s}{[n^2 - (4s)^2]\pi} [1 - (-1)^n e^{-j4\pi s}]$$

A factor $e^{-j2\pi s}$ can be brought outside the brackets in the right-hand side of this equation, resulting in:

$$\int_0^1 \cos n\pi y \; e^{-j4\pi sy} dy = \frac{j4s \; e^{-j2\pi s}}{\pi[n^2 - (4s)^2]} [e^{j2\pi s} - (-1)^n \; e^{-j2\pi s}] \qquad (4.15)$$

For even values of n, the term inside the brackets reduces to

$$[e^{j2\pi s} - e^{-j2\pi s}] = 2j \sin 2\pi s$$

For odd values of n, the same term reduces to:

$$[e^{j2\pi s} + e^{-j2\pi s}] = 2 \cos 2\pi s$$

This completes the evaluation of the integral (4.14) which can now be written in final form:

$$\int_0^1 \cos n\pi y \; e^{-j4\pi sy} dy = \begin{cases} \dfrac{-8s \sin 2\pi s}{\pi[n^2 - (4s)^2]} \; e^{-j2\pi s} & \text{for n even} \\ \dfrac{j8s \cos 2\pi s}{\pi[n^2 - (4s)^2]} \; e^{-j2\pi s} & \text{for n odd} \end{cases} \qquad (4.16)$$

To achieve convenient expressions in the following paragraphs, the following notation will be introduced. Even values of n are designated by ne, odd values by no.

The functions $C_{ne}(s)$ and $C_{no}(s)$ are now defined:

$$C_{ne}(s) \equiv \frac{-8s \sin 2\pi s}{\pi[n^2 - (4s)^2]} \qquad \text{(n even)} \qquad (4.17)$$

$$C_{no}(s) \equiv \frac{8s \cos 2\pi s}{\pi[n^2 - (4s)^2]} \qquad \text{(n odd)} \qquad (4.18)$$

Or written in slightly different form:

$$C_{ne}(s) = \frac{\sin 2\pi s}{\pi}\left[\frac{1}{4s + n} + \frac{1}{4s - n}\right] \qquad \text{(n even)} \qquad (4.19)$$

$$C_{no}(s) = \frac{-\cos 2\pi s}{\pi}\left[\frac{1}{4s + n} + \frac{1}{4s - n}\right] \qquad \text{(n odd)} \qquad (4.20)$$

Using the notation adopted above, the Fourier transform of a trigonometric cosine series can now be written as follows:

$$F\left[\sum_{n=0}^{N} a_n \cos n\pi y\right] = \left[\sum_{n=0}^{N} a_{ne} C_{ne}(s) + j \sum_{n=1}^{N} a_{no} C_{no}(s)\right] e^{-j2\pi s} \qquad (4.21)$$

As an example, the function $C_4(s)$ is plotted in Fig. 4.2.

The functions $C_{ne}(s)$ equal zero whenever 4s assumes an even integer value, different from ne. When 4s approaches ne, $C_{ne}(s)$ approaches a limit that can be evaluated using de l'Hôpital's rule.

$$\lim_{4s \to n} C_{ne}(s) = \lim_{4s \to n} \frac{-8s \sin 2\pi s}{\pi[n^2 - (4s)^2]}$$

$$= \lim_{4s \to n} \frac{-8 \sin 2\pi s - 16\pi s \cos 2\pi s}{-32\pi s}$$

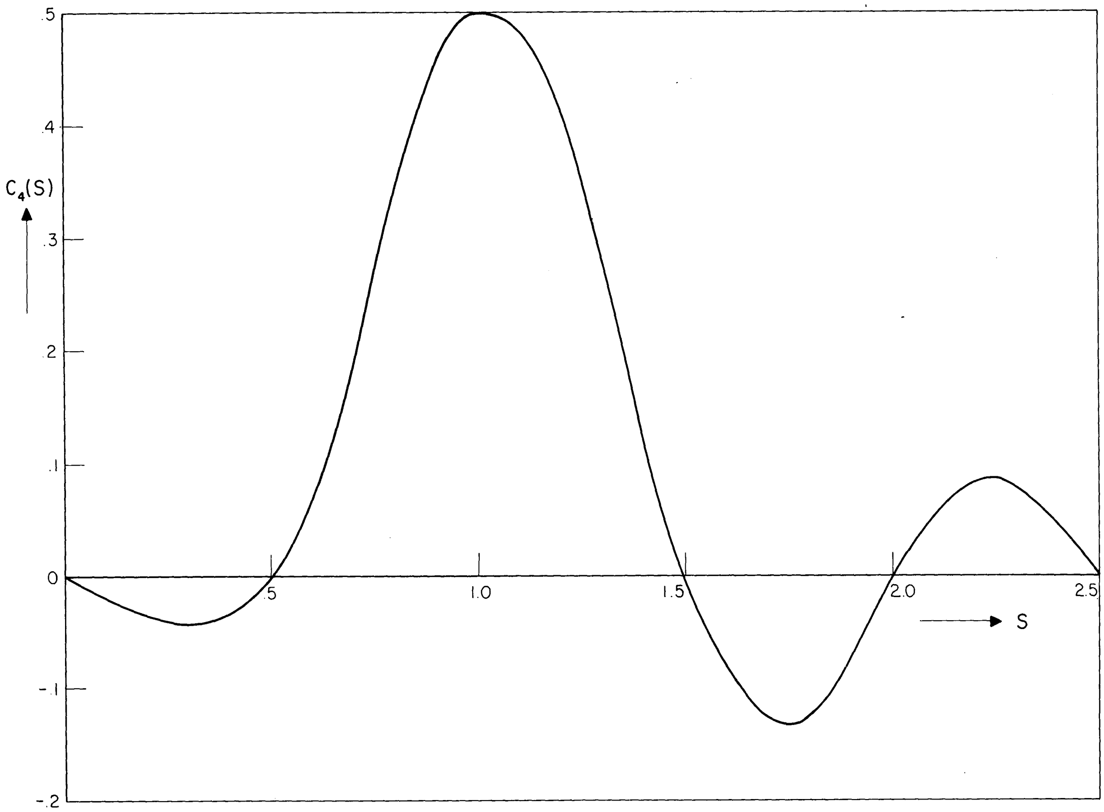

Fig. 4.2 Plot of the function $C_4(s)$.

The denominator is non-zero when 4s approaches n, provided that $n \neq 0$.

Because n is even: $\sin 2\pi s = 0$ when $4s = n$

$$\text{and} \qquad \cos 2\pi s = \cos \frac{n\pi}{2} = (-1)^{n/2} \text{ when } 4s = n.$$

Therefore

$$\lim_{4s \to n} C_{ne}(s) = \frac{1}{2}(-1)^{n/2} \qquad \text{for ne} \neq 0 \tag{4.22}$$

For the case in which n = 0, de l'Hôpital's rule has to be applied a second time.

$$\lim_{4s \to 0} C_0(s) = \lim_{4s \to 0} \frac{-8 \sin 2\pi s - 16\pi s \cos 2\pi s}{-32\pi s} =$$

$$\lim_{4s \to 0} \frac{-16 \cos 2\pi s - 16\pi \cos 2\pi s + 32\pi^2 s \sin 2\pi s}{-32\pi}$$

Therefore

$$\lim_{4s \to 0} C_0(s) = 1 \tag{4.23}$$

Similarly, the function $C_{no}(s) = 0$ whenever 4s assumes an odd integer value different from no. When 4s approaches no, $C_{no}(s)$ approaches a limit whose value can be found using de l'Hôpital's rule.

$$\lim_{4s \to n} C_{no}(s) = \lim_{4s \to n} \frac{8s \cos 2\pi s}{\pi[n^2 - (4s)^2]} =$$

$$\lim_{4s \to n} \frac{8 \cos 2\pi s - 16\pi s \sin 2\pi s}{-32\pi s}$$

The denominator does not vanish when 4s approaches n. Because n is odd: $\cos 2\pi s = 0$ when $4s = n$

$$\text{and} \quad \sin 2\pi s = \sin \frac{n\pi}{2} = (-1)^{(n-1)/2} \quad \text{when } 4s = n.$$

Therefore

$$\lim_{4s \to n} C_{no}(s) = \frac{1}{2} (-1)^{(n-1)/2} \tag{4.24}$$

<u>4.3.2 Evaluation of F[sin nπy].</u> The Fourier transform of sin nπy is equal to

$$F[\sin n\pi y] = \int_0^1 \sin n\pi y \, e^{-j4\pi sy} dy \tag{4.25}$$

Like F[cos nπy] it can be evaluated by integrating by parts twice.

$$\int_0^1 \sin n\pi y \, e^{-j4\pi sy} dy =$$

$$\frac{-1}{n\pi} \cos n\pi y \, e^{-j4\pi sy} \Big|_0^1 - \frac{j4s}{n} \int_0^1 \cos n\pi y \, e^{-j4\pi sy} dy =$$

$$\frac{-1}{n\pi} \cos n\pi y \, e^{-j4\pi sy} \Big|_0^1 - \frac{j4s}{n^2\pi} \sin n\pi y \, e^{-j4\pi sy} \Big|_0^1 + \frac{16s^2}{n^2} \int_0^1 \sin n\pi y \, e^{-j4\pi sy} dy$$

This expression can be reduced to:

$$[1 - \frac{(4s)^2}{n^2}] \int_0^1 \sin n\pi y \, e^{-j4\pi sy} dy = \frac{1}{n\pi} [1 - \cos n\pi \cdot e^{-j4\pi s}]$$

Because $\cos n\pi = (-1)^n$, this can be written as:

$$\int_0^1 \sin n\pi y \, e^{-j4\pi sy} dy = \frac{n}{\pi[n^2 - (4s)^2]} [1 - (-1)^n \, e^{-j4\pi s}]$$

Proceeding as before, a term $e^{-j2\pi s}$ is eliminated from the brackets on the right-hand side, giving:

$$\int_0^1 \sin n\pi y \, e^{-j4\pi sy} dy = \frac{n \, e^{-j2\pi s}}{\pi[n^2 - (4s)^2]} [e^{j2\pi s} - (-1)^n \, e^{-j2\pi s}] \qquad (4.26)$$

For even values of n, the expression inside the brackets becomes

$$[e^{j2\pi s} - e^{-j2\pi s}] = 2j \sin 2\pi s$$

For odd values of n, the same expression reduces to

$$[e^{j2\pi s} + e^{-j2\pi s}] = 2 \cos 2\pi s$$

This completes the evaluation of the integral (4.25). Written in final form:

$$\int_0^1 \sin n\pi y \; e^{-j4\pi s y} dy = \begin{cases} \dfrac{j2n \sin 2\pi s}{\pi[n^2 - (4s)^2]} \, e^{-j2\pi s} & \text{for n even} \\ \dfrac{2n \cos 2\pi s}{\pi[n^2 - (4s)^2]} \, e^{-j2\pi s} & \text{for n odd} \end{cases} \qquad (4.27)$$

Using again the notation ne and no for even and odd values of n, respectively, the functions $S_{ne}(s)$ and $S_{no}(s)$ can be defined:

$$S_{ne}(s) \equiv \frac{2n \sin 2\pi s}{\pi[n^2 - (4s)^2]} \qquad \text{(n even)} \qquad (4.28)$$

$$S_{no}(s) \equiv \frac{2n \cos 2\pi s}{\pi[n^2 - (4s)^2]} \qquad \text{(n odd)} \qquad (4.29)$$

These functions can also be written in slightly different form:

$$S_{ne}(s) = \frac{\sin 2\pi s}{\pi} \left[\frac{1}{4s + n} - \frac{1}{4s - n} \right] \qquad \text{(n even)} \qquad (4.30)$$

$$S_{no}(s) = \frac{\cos 2\pi s}{\pi} \left[\frac{1}{4s + n} - \frac{1}{4s - n} \right] \qquad \text{(n odd)} \qquad (4.31)$$

With this notation, the Fourier transform of a trigonometric sine series can be written in the following form:

$$F\left[\sum_{n=1}^{N} b_n \sin n\pi y \right] = \left[\sum_{n=1}^{N} b_{no} S_{no}(s) + j \sum_{n=2}^{N} b_{ne} S_{ne}(s) \right] e^{-j2\pi s} \qquad (4.32)$$

The function $S_5(s)$ is plotted in Fig. 4.3 as an example. As is apparent, the functions $S_n(s)$ behave essentially like the functions $C_n(s)$.

The functions $S_{ne}(s)$ equal zero whenever 4s assumes an even

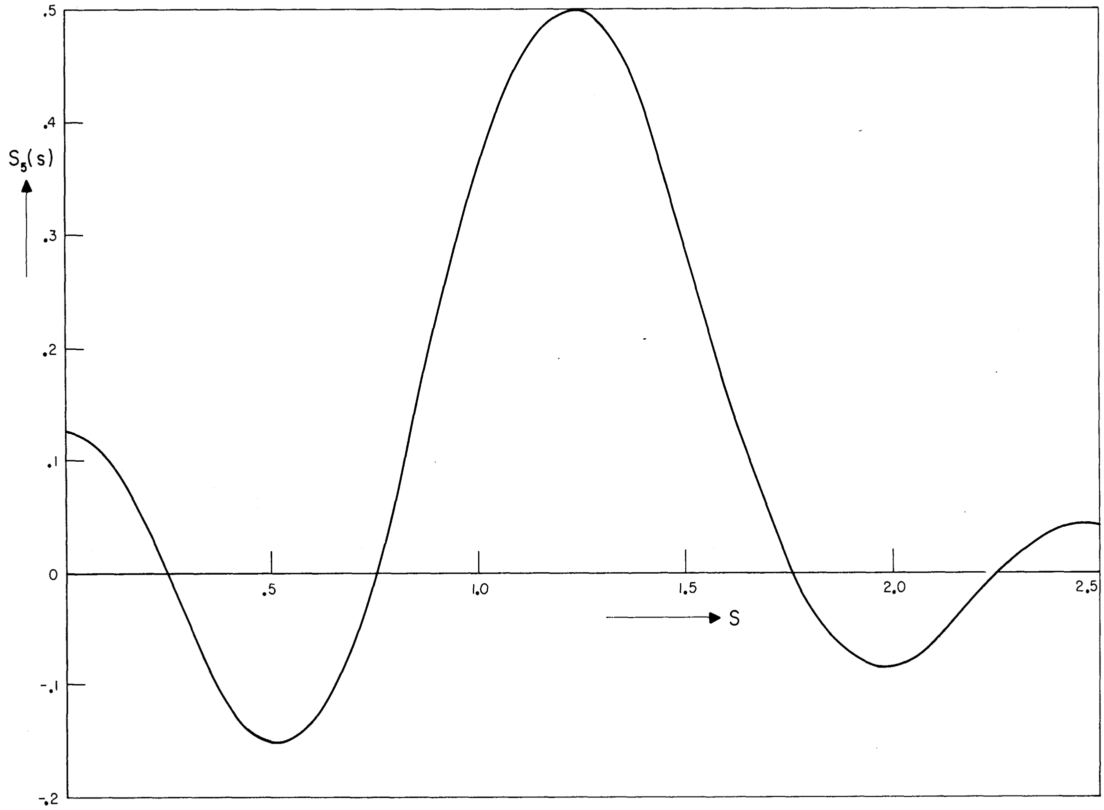

Fig. 4.3 Plot of the function $S_5(s)$.

integer value, and the functions $S_{no}(s)$ are zero when 4s assumes an odd integer value, except in the case that 4s = ne and 4s = no, respectively.

The limit of $S_n(s)$ when 4s approaches n can again be evaluated using the rule of de l'Hôpital.

$$\lim_{4s \to n} S_{ne}(s) = \lim_{4s \to n} \frac{2n \sin 2\pi s}{\pi[n^2 - (4s)^2]} =$$

$$\lim_{4s \to n} \frac{4\pi n \cos 2\pi s}{-32\pi s} = -\frac{1}{2} \cos \frac{n\pi}{2} \qquad (4.33)$$

$$= \frac{1}{2} (-1)^{1+(n/2)} \quad \text{(n even)}$$

Similarly

$$\lim_{4s\to n} S_{no}(s) = \lim_{4s\to n} \frac{2n \cos 2\pi s}{\pi[n^2 - (4s)^2]} =$$

$$\lim_{4s\to n} \frac{-4\pi n \sin 2\pi s}{-32\pi s} = \frac{1}{2} \sin \frac{n\pi}{2} \qquad (4.34)$$

$$= \frac{1}{2} (-1)^{(n-1)/2} \quad (n \text{ odd})$$

4.3.3 Impedance Transformation. Because the functions $\Gamma(s)$ and $\Gamma_o(s)$, which arise in the synthesis problem, depend on the value of the characteristic impedance of the nonuniform line at its terminals, it is of importance to know how Z_{01} and Z_{02} are affected by the reflection-distribution function $\rho(y)$.

From the definition of $\rho(y)$:

$$\rho(y) = \frac{d \ln Z_0(y)}{2\, dy} \qquad (4.2)$$

it follows immediately by integration that

$$\int_0^1 \rho(y)\, dy = \frac{1}{2} \ln \frac{Z_0(1)}{Z_0(0)} \equiv \frac{1}{2} \ln \frac{Z_{02}}{Z_{01}} \qquad (4.35)$$

When $\rho(y)$ is expanded in a trigonometric series,

$$\rho(y) = \sum_{n=0}^{N} [a_n \cos n\pi y + b_n \sin n\pi y] \qquad (4.12)$$

the amount of impedance transformation along the line can be expressed in terms of the coefficients a_n and b_n.

First the amount of impedance transformation caused by the cosine terms, is evaluated:

$$\int_0^1 \cos n\pi y \, dy = \frac{1}{n\pi} \sin n\pi y \Big|_0^1 = \frac{\sin n\pi}{n\pi} \begin{cases} = 0 & \text{for } n \neq 0 \\ = 1 & \text{for } n = 0 \end{cases} \quad (4.36)$$

Thus it is seen that cosine terms do not contribute to impedance transformation, except when n = 0, which corresponds to the constant term in the expansion.

Secondly the amount of impedance transformation contributed by the sine terms is evaluated:

$$\int_0^1 \sin n\pi y \, dy = \frac{-1}{n\pi} \cos n\pi y \Big|_0^1 = \frac{1 - \cos n\pi}{n\pi} = \quad (4.37)$$

$$\frac{1 - (-1)^n}{n\pi} = \begin{cases} 0 \text{ for } n \text{ even} \\ \frac{2}{n\pi} \text{ for } n \text{ odd} \end{cases}$$

The amount of impedance transformation resulting from a reflection-distribution function $\rho(y)$ given by Eq. 4.12 , is equal to:

$$\frac{1}{2} \ln \frac{Z_{02}}{Z_{01}} = a_0 + \sum_{n=1}^{N} b_{no} \frac{2}{no\pi} \quad (4.38)$$

<u>4.3.4 Summary of Results.</u> The results obtained in Sections 4.3.1 and 4.3.2 can now be combined into a single equation. When the reflection-distribution function $\rho(y)$ is given by the trigonometric expansion (4.12), its Fourier transform $F[\rho]$ can be written as follows:

$$F[\rho] = G(s) = \int_0^1 \rho(y) \, e^{-j4\pi sy} dy =$$

$$\left\{ \sum_{n=0}^{N} [a_{ne} C_{ne}(s) + b_{no} S_{no}(s)] + j \sum_{n=1}^{N} [a_{no} C_{no}(s) + b_{ne} S_{ne}(s)] \right\} e^{-j2\pi s} \quad (4.39)$$

There now remains the determination of the coefficients a_n and b_n. The first step will be the separation of the real and imaginary parts by rewriting the identity (4.39) as follows:

$$\mathrm{Re}\left\{G(s)\ e^{j2\pi s}\right\} = \sum_{n=0}^{N} [a_{ne} C_{ne}(s) + b_{no} S_{no}(s)] \qquad (4.40)$$

$$\mathrm{Im}\left\{G(s)\ e^{j2\pi s}\right\} = \sum_{n=1}^{N} [a_{no} C_{no}(s) + b_{ne} S_{ne}(s)] \qquad (4.41)$$

In Chapter V it will be shown that the theory of discrete Chebyshev approximations provides a means by which the coefficients a_n and b_n can be determined. By this method the coefficients a_{ne} and b_{no} are determined such that $\mathrm{Re}\left\{G(s)\ e^{j2\pi s}\right\}$ approximates $\mathrm{Re}\left\{\Gamma_1(s)\ e^{j2\pi s}\right\}$ in a Chebyshev sense at a discrete number of sampling points in the frequency interval of interest. By Chebyshev sense is meant that the maximum deviation of $\mathrm{Re}\left\{G(s)\ e^{j2\pi s}\right\}$ from $\mathrm{Re}\left\{\Gamma_1(s)\ e^{j2\pi s}\right\}$ is minimum. Similarly the coefficients a_{no} and b_{ne} are determined such that $\mathrm{Im}\left\{G(s)\ e^{j2\pi s}\right\}$ approximates $\mathrm{Im}\left\{\Gamma_1(s)\ e^{j2\pi s}\right\}$ in a Chebyshev sense at a discrete number of sampling points in the frequency interval of interest.

It will be recalled that the function $\Gamma_1(s)$ depends on $\Gamma(s)$ and $\Gamma_o(s)$, which in turn depend on the characteristic impedance of the line at its terminals, Z_{01} and Z_{02}. The whole synthesis procedure depends on a knowledge of $\Gamma(s)$ and $\Gamma_o(s)$. It is therefore important to control Z_{01} and Z_{02} carefully. This means that the equation (4.38)

$$\frac{1}{2} \ln \frac{Z_{02}}{Z_{01}} = a_0 + \sum_{n=1}^{N} b_{no} \frac{2}{no\ \pi} \qquad (4.42)$$

must be satisfied exactly. It will be shown in Chapter V that the theory of discrete Chebyshev approximation can be used to satisfy some equations exactly, while distributing the approximation error equally over the remaining equations.

A special synthesis case arises in the construction of impedance transformers. With the tools developed in this chapter the work of Willis and Sinha (Ref. 13) (see also paragraph 3.3.1) can be extended further. Willis and Sinha considered the case in which $\rho(y)$ is either of the form

$$\rho(y) = \sum_{n=0}^{N} a_{ne} \cos ne\pi y \tag{4.43}$$

or of the form

$$\rho(y) = \sum_{n=1}^{N} b_{no} \sin no\pi y \tag{4.44}$$

Better results can be obtained by considering functions $\rho(y)$ of the form:

$$\rho(y) = \sum_{n=0}^{N} [a_{ne} \cos ne\pi y + b_{no} \sin no\pi y] \tag{4.45}$$

The input reflection coefficient of the line now becomes

$$\Gamma(s) = \left\{ \sum_{n=0}^{N} [a_{ne}C_{ne}(s) + b_{no}S_{no}(s)] \right\} e^{-j2\pi s} \tag{4.46}$$

The coefficients a_{ne} and b_{no} can be determined to give a high-pass character to $\Gamma(s)$ while at the same time minimizing the amount of reflection in the pass band. The theory of discrete Chebyshev approximation provides a very convenient tool to determine the coefficients while at the same time controlling the amount of impedance transformation in the impedance transformer exactly. The amount of impedance transformation is again given by Eq. 4.38. A synthesis example will be given in Chapter VI.

This section will be concluded by a short summary of the steps to be taken in the synthesis of a matching section.

Given are the internal impedance of a generator as a function of frequency and a load impedance as a function of frequency. It is desired to obtain a match, i.e., maximum power transfer, between generator and load over a band of frequencies which is also given.

Step 1. Select the length ℓ of the matching section. A suitable choice might be to make the nonuniform line one wavelength long at the lowest frequency of the band of interest. This will determine the frequency variable s.

Step 2. Choose Z_{01} and Z_{02}, the characteristic impedances of the line at its terminals. The magnitudes of the reflection functions $\Gamma(s)$ and $\Gamma_o(s)$ depend on the values chosen for Z_{01} and Z_{02}. It is, therefore, logical to choose Z_{01} and Z_{02} in such a manner that the maximum values of the magnitudes of the functions $\Gamma(s)$ and $\Gamma_o(s)$ are kept as small as possible.

Step 3. Calculate the functions $\Gamma(s)$ (Eq. 4.5) and $\Gamma_o(s)$ (Eq. 4.7). These functions are then used to calculate $\Gamma_1(s)$ (Eq. 4.4).

Step 4. Separate the real and imaginary part of $\Gamma_1(s)\ e^{j2\pi s}$ as outlined in the discussion following Eqs. 4.40 and 4.41.

Step 5. Determine the coefficients a_n and b_n such that the real and imaginary parts of $G(s)\ e^{j2\pi s}$ give the best approximation to the corresponding parts of $\Gamma_1(s)\ e^{j2\pi s}$. (Eqs. 4.40 and 4.41) This process also involves the selection of the terms that are to be used in the trigonometric expansion (4.12) of the reflection-distribution function $\rho(y)$. The coefficients a_n and b_n must be determined in such a way that Eq. 4.38, which gives the amount of impedance transformation, is satisfied exactly.

The determination of the coefficients a_n and b_n essentially concludes the synthesis procedure. When these coefficients are known, the reflection-distribution function $\rho(y)$ is known exactly. The steps necessary to construct a nonuniform line when $\rho(y)$ is known, are outlined in the next paragraph.

4.4 The Determination of the Characteristic-Impedance Function

The characteristic-impedance function $Z_0(y)$ can be found immediately by integrating the reflection-distribution function $\rho(y)$.

$$\int_0^y \rho(\eta)\, d\eta = \frac{1}{2} \ln \frac{Z_0(y)}{Z_0(0)} = \frac{1}{2} \ln \frac{Z_0(y)}{Z_{01}} \tag{4.47}$$

Therefore:

$$Z_0(y) = Z_{01} \exp\left[2 \int_0^y \rho(\eta)\, d\eta\right] \tag{4.48}$$

With $\rho(y)$ given by a trigonometric expansion (4.12), this becomes

$$Z_0(y) = Z_{01} \exp\left\{2\left[a_0 y + \sum_{n=1}^{N} a_n \frac{\sin n\pi y}{n\pi} + \sum_{n=1}^{N} b_n \frac{1 - \cos n\pi y}{n\pi}\right]\right\} \tag{4.49}$$

In other words, when the reflection-distribution function is known, the characteristic impedance is known at every point in the nonuniform line, as given by (4.48) and (4.49).

The physical dimensions of the line are determined directly by $Z_0(y)$. In the case of a coaxial structure, for instance, $Z_0(y)$ determines the ratio between outer and inner conductor at every point in the line. For a coaxial structure:

$$\frac{d_o}{d_i}(y) = \exp\left[\sqrt{\epsilon}\,\frac{Z_0(y)}{60}\right] \tag{4.50}$$

where:

d_o is the inside diameter of the outer conductor

d_i is the diameter of the center conductor

ϵ is the relative dielectric constant of the dielectric in the line, whose relative permeability is assumed to be unity.

CHAPTER V

THE THEORY OF DISCRETE CHEBYSHEV APPROXIMATION

5.1 Introduction

In Chapter IV the problem of synthesizing matching sections was reduced to the problem of approximating two given functions, of the variable s, by linear combinations of the functions $C_n(s)$ and $S_n(s)$, which will be called the approximating functions. This is expressed in Eqs. 4.40 and 4.41. In this chapter it will be shown how this problem can be solved by means of the theory of discrete Chebyshev approximation. This theory provides a numerical method by which the coefficients a_n and b_n can be determined. It will also be possible to control the amount of impedance transformation, as given by Eq. 4.38, exactly.

The theory of discrete Chebyshev approximation has also proved useful in the synthesis of networks having a prescribed impulse response. This method was recently developed by Ruston (Ref. 10).

The treatment in this chapter will follow along lines similar to those used by Stiefel in his recent publication on the theory of discrete Chebyshev approximation (Ref. 12).

5.2 Reduction to an Overdetermined System of Linear Equations

A function F(s) is to be approximated by a function f(s), which is a linear combination of n approximating functions $f_j(s)$ $(1 \leq j \leq n)$:

$$f(s) = x_1f_1(s) + x_2f_2(s) + \ldots + x_nf_n(s) \tag{5.1}$$

where the coefficients x_j $(1 \leq j \leq n)$ are to be determined such that the maximum value of the magnitude of the approximation error h(s),

$$\max |h(s)| = \max |F(s) - f(s)| \qquad (5.2)$$

is minimum.

The theory of discrete Chebyshev approximation provides a solution to this problem. The functions F(s) and f(s) are sampled at an arbitrarily large number, m, of sampling points, where $m > n$. Let the m sampling points be s_k $(1 \leq k \leq m)$, then the function F(s) has to be approximated, at the sampling points, by the function f(s), such that the maximum of the values

$$\max |h(s_k)| = \max |F(s_k) - f(s_k)| \qquad (5.3)$$

is minimum.

If the approximation of the function F(s) by the function f(s) can be achieved without error at the sampling points, m linear equations E_k $(1 \leq k \leq m)$ can be written, one for each sampling point:

$$E_k: \quad x_1 f_1(s_k) + x_2 f_2(s_k) + \ldots + x_n f_n(s_k) - F(s_k) = 0 \quad (5.4)$$

To simplify the notation, the quantities a_{kj} and c_k will be defined:

$$a_{kj} = f_j(s_k) \qquad (5.5)$$

$$c_k = -F(s_k) \qquad (5.6)$$

For the error $h(s_k)$ at the sampling point s_k, the notation h_k will be used.

The m equations (5.4) can now be written in the form:

$$E_k: \quad a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n + c_k = 0 \quad (1 \le k \le m) \qquad (5.7)$$

The set of equations (5.4) or (5.7) can be interpreted to represent m planes in n-dimensional Euclidean space. Finding a set of coefficients $(x_1, x_2, \ldots, x_n)$ such that the approximation (5.4) is valid is equivalent to finding the coordinates $(x_1, x_2, \ldots, x_n)$ of a point in n-dimensional Eculidean space.

If the number (m) of equations equals the dimension (n) of the space, the system (5.4) can, in general, be solved exactly. This means that coefficients $(x_1, x_2, \ldots, x_n)$ can be found such that the function f(s) (Eq. 5.1) equals the function F(s) at the m sampling points. In terms of the n-dimensional Euclidean space it means that the m planes (m=n) intersect in a point whose coordinates are $(x_1, x_2, \ldots, x_n)$.

In the case under study, however, the number (m) of sampling points exceeds the number (n) of approximating functions. In that case the system (5.4) becomes overdetermined and it is no longer possible to find a set of coefficients $(x_1, x_2, \ldots, x_n)$ such that all m equations (5.4) are satisfied at once. Or, in other words, when there are m planes in n-dimensional space $(m > n)$, they do, in general, not intersect in one single point whose coordinates are $(x_1, x_2, \ldots, x_n)$. This case will be studied in the remainder of this chapter.

It will be assumed throughout this chapter, that no two planes are parallel.

5.3 Theory of Overdetermined Systems of Linear Equations

If a solution exists to the system (5.7), it means that all m planes in the n-dimensional space intersect in a single point x, whose coordinates are $(x_1, x_2, \ldots, x_n)$. In general this will not be the case. Consider, therefore, a point P, with coordinates $(x'_1, x'_2, \ldots, x'_n)$, that does not lie on the plane represented by equation E_k. Substitution of the coordinates of P into the equation E_k, will result in an error, or residue, h_k, given by:

$$a_{k1}x'_1 + a_{k2}x'_2 + \ldots + a_{kn}x'_n + c_k = h_k \tag{5.8}$$

The approximation problem then becomes that of finding a point T, in n-dimensional space, such that max $|h_k|$ $(1 \leq k \leq m)$ is minimum. This point T will be called the Chebyshev point of the overdetermined system (5.7).

In the special case that the normal vectors, $\bar{n}_k$, to the m planes

$$\bar{n}_k = (a_{k1}, a_{k2}, \ldots, a_{kn}) \tag{5.9}$$

have unit length, the residue h_k represents the distance from point P to the plane E_k, and the approximation problem then becomes the determination of a point T, whose largest distance to any of the m planes is minimum.

To find the Chebyshev point, it will be shown that a point P exists, such that the residues h_k of a number (n+1) of the m equations (5.7), have equal magnitude $|h_k| = |h|$. When the selection of these (n+1) planes is such that the magnitude of the error to the remaining equations is less than $|h|$, the Chebyshev point has been found.

In the following paragraphs it will first be shown how the error to a set of (n+1) equations can be determined. Secondly, it will be shown how a selection of (n+1) planes can be reached to yield the Chebyshev point.

A reference will now be defined. A reference is a set of (n+1) equations out of the set (5.7). Without loss of generality, the first (n+1) equations of (5.7) can be taken to constitute the first reference. The (n+1) planes represented by these equations have (n+1) normals $\bar{n}_k$. These (n+1) vectors in n-dimensional space must be dependent, and therefore coefficients λ_k exist such that

$$\lambda_1\bar{n}_1 + \lambda_2\bar{n}_2 + \ldots + \lambda_{n+1}\bar{n}_{n+1} = 0 \tag{5.10}$$

This will be called the characteristic equation of the reference. Because of the assumption that no two planes are parallel, the coefficients λ_k are nonzero constants.

The characteristic equation (5.10) can also be written as a set of simultaneous linear equations:

$$\begin{aligned} a_{11}\lambda_1 + a_{21}\lambda_2 + \ldots + a_{(n+1)1}\lambda_{n+1} &= 0 \\ a_{12}\lambda_1 + a_{22}\lambda_2 + \ldots + a_{(n+1)2}\lambda_{n+1} &= 0 \\ \vdots \qquad\qquad\qquad & \vdots \\ a_{1n}\lambda_1 + a_{2n}\lambda_2 + \ldots + a_{(n+1)n}\lambda_{n+1} &= 0 \end{aligned} \tag{5.11}$$

When the coordinates of a point $P(x_1', x_2', \ldots, x_n')$ are substituted into the (n+1) equations of the reference, (n+1) equations result:

$$E_k: \quad a_{k1}x_1' + a_{k2}x_2' + \ldots + a_{kn}x_n' + c_k = h_k \quad (1 \leq k \leq n+1) \tag{5.12}$$

By definition, the point P will be called a reference point, when the same condition either

$$\operatorname{sgn} h_k = \operatorname{sgn} \lambda_k \quad (1 \leq k \leq n+1) \tag{5.13}$$

or

$$\operatorname{sgn} h_k = - \operatorname{sgn} \lambda_k \quad (1 \leq k \leq n+1) \tag{5.14}$$

is satisfied by all h_k belonging to the reference.

The equations E_k of (5.12) are added together after having been multiplied by their corresponding λ_k's. Because of (5.10) and (5.11), the result of this process is:

$$\sum_{k=1}^{n+1} \lambda_k c_k = \sum_{k=1}^{n+1} \lambda_k h_k \tag{5.15}$$

Because for a reference point either (5.13) or (5.14) is valid, there are now two possibilities. When (5.13) is valid,

$$\sum_{k=1}^{n+1} \lambda_k c_k = \sum_{k=1}^{n+1} |\lambda_k| \, |h_k| \tag{5.16}$$

When, however, (5.14) is the case,

$$\sum_{k=1}^{n+1} \lambda_k c_k = - \sum_{k=1}^{n+1} |\lambda_k| \, |h_k| \tag{5.17}$$

The significance of the reference point, as defined by (5.13) or (5.14), can be understood using the following argument. First it is observed that, when two points are taken, each lying on different sides of a reference plane, and if one of these points is a reference point, the other is not. This is true because the corresponding error h_k has a different sign for the two points, and if condition (5.13) or (5.14) is satisfied for one of the points, it cannot hold for the other. Therefore, the set of reference points forms a continuum, bounded by the planes of the reference set and possibly by infinity.

Secondly, because the quantities c_k are finite constants, and because the coefficients λ_k are finite nonzero constants, it follows from (5.16) and (5.17) that for all reference points $|h_k|$ is bounded. This proves that the reference points lie inside the volume enclosed by the (n+1) planes of the reference.

5.3.1 Determination of the Center of a Reference. The center of a reference, by definition, is that reference point for which all values $|h_k|$ are equal. It follows from (5.16) and (5.17) that the reference error, h, equals:

$$h = \frac{\lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_{n+1} c_{n+1}}{|\lambda_1| + |\lambda_2| + \cdots + |\lambda_{n+1}|} \tag{5.18}$$

This equation combines Eqs. 5.13 and 5.14. It is easily verified that, when (5.13) is valid, sgn h = +1, and when (5.14) is valid, sgn h = -1. It also follows, using (5.13) and (5.14), that

$$h_k = h \operatorname{sgn} \lambda_k \tag{5.19}$$

When the reference error (h) has been determined, the coordinates of the reference center can be calculated by solving the following system of linear equations:

$$E_k: \quad a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n + c_k = h \operatorname{sgn} \lambda_k \qquad (1 \le k \le n+1) \tag{5.20}$$

In case the normals $\bar{n}_k$ have unit length, the reference center corresponds to the center of the n-dimensional sphere that can be inscribed in the (n+1) reference planes.

Of course, the question remains whether a solution exists to the (n+1) equations of (5.20). From (5.11) and (5.15), it follows that the (n+1) equations of (5.20) are linearly dependent, and therefore the existence of the reference center is proved. Because of the assumption that no two planes are parallel, any n equations of the set (5.20) will be independent, and therefore the center of the reference is a unique point.

The left-hand side of Eq. 5.16 has a constant value. Therefore, a point P for which one of the errors $|h_k|$ is smaller than $|h|$ must have at least one other error $|h_k|$ which is larger than $|h|$. From this it follows immediately that the center of the reference is also the Chebyshev point for the (n+1) planes of the reference.

5.3.2 Overdetermined Systems with Constraints. As was mentioned in Chapter IV, it is necessary to control the amount of impedance transformation, from one end of the matching section to the other, exactly. This implies that Eq. 4.38 must be satisfied exactly. Equation 4.38 also has the general form of Eq. 5.4, and therefore the requirement that it be satisfied exactly means that the equation E_0 be satisfied exactly:

$$E_0: \quad a_{01}x_1 + a_{02}x_2 + \cdots + a_{0n}x_n + c_0 = 0 \tag{5.21}$$

The equation (5.21) acts as a constraint on the overdetermined system (5.7). It is now required that a point P $(x_1, x_2, \ldots, x_n)$ be found such that max $|h_k|$ is minimum and such that the point P lies on the plane represented by equation E_0,(5.21).

Again a reference is defined. It is a selection of n equations from the set (5.7). The first n equations of (5.7) will be chosen to constitute the first reference.

The characteristic equation becomes

$$\lambda_0 \bar{n}_0 + \lambda_1 \bar{n}_1 + \lambda_2 \bar{n}_2 + \ldots + \lambda_n \bar{n}_n = 0 \qquad (5.22)$$

and because the (n+1) normal vectors in n-dimensional space must be dependent (no two are parallel), a set of nonzero coefficients λ_k $(0 \leq k \leq n)$ exists such that (5.22) is satisfied.

The point P, by definition, is a reference point, when either

$$\operatorname{sgn} h_k = \operatorname{sgn} \lambda_k \quad (1 \leq k \leq n) \qquad (5.23)$$

or

$$\operatorname{sgn} h_k = -\operatorname{sgn} \lambda_k \quad (1 \leq k \leq n) \qquad (5.24)$$

is true for all h_k belonging to the reference.

The n equations of the reference and equation E_0 are added together after having been multiplied by their corresponding λ_k's. Because of (5.22) and because $h_0 = 0$, the result is:

$$\lambda_0 c_0 + \sum_{k=1}^{n} \lambda_k c_k = \sum_{k=1}^{n} \lambda_k h_k \qquad (5.25)$$

Because of (5.23) and (5.24) this can be written as follows:

$$\lambda_0 c_0 + \sum_{k=1}^{n} \lambda_k c_k = \pm \sum_{k=1}^{n} |\lambda_k| \; |h_k| \qquad (5.26)$$

where the positive sign is valid in case Eq. 5.23 holds, and the negative sign when (5.24) is true.

The center of a reference is that reference point for which all values $|h_k|$ are equal. The reference error h is found from (5.26):

$$h = \frac{\lambda_0 c_0 + \lambda_1 c_1 + \lambda_2 c_2 + \ldots + \lambda_n c_n}{|\lambda_1| + |\lambda_2| + \ldots + |\lambda_n|} \tag{5.27}$$

and the errors to the individual equations equal:

$$h_k = h \operatorname{sgn} \lambda_k \tag{5.28}$$

The coordinates of the reference center can now be found by solving the following set of simultaneous linear equations:

$$\begin{aligned} E_0: &\quad a_{01}x_1 + a_{02}x_2 + \ldots + a_{0n}x_n + c_0 = 0 \\ E_k: &\quad a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n + c_k = h_k \\ &\qquad\qquad (1 \le k \le n) \end{aligned} \tag{5.29}$$

As is apparent from the discussion above, the constraint does not basically alter the procedure for finding the center of a reference. The only difference, as was to be expected, is in the determination of the reference error. Compare Eq. 5.27 with Eq. 5.18.

5.3.3 The Replacement Procedure. After the center of the reference, consisting of the first (n+1) equations of (5.7), has been determined, its coordinates can be substituted into the remaining equations. This substitution gives a set of residues h_k $(n+2 \le k \le m)$. When the magnitude of these residues is smaller than that of the reference error, the reference center is the Chebyshev point for the overdetermined system (5.7).

Suppose, however, that one of the residues, designated by h_i, has a magnitude larger than $|h|$. A new reference must then be selected. This second reference will be formed by replacing one of the equations of the old reference by the equation E_i, whose residue is h_i. When the correct equation is replaced, a new reference is generated whose reference error is larger than the old error $|h|$. It will be shown next that this is the case when the new set of (n+1) reference planes is selected such that the center (A) of the old reference is a reference point of the new reference. In other words, the (n+1) planes of the new reference must be selected such that the old reference center (A) lies inside the volume enclosed by the new reference planes.

That this choice leads to a larger reference error is proved as follows. The new reference consists of n planes of the old reference and the plane E_i. Assume that the plane E_r has been replaced. When the coordinates of (A) are substituted into these (n+1) equations, a number n of the residues will have the magnitude $|h|$, the remaining one will have the magnitude $|h_i|$, where $|h_i| > |h|$. When (A) is a reference point of the new reference, Eq. 5.16 or 5.17 holds. Substituting (5.16) and (5.17) into (5.18), one finds the following value for the magnitude of the new reference error h':

$$|h'| = \frac{|\lambda_i|\ |h_i| + |h| \sum_{k=1;\ k\neq r}^{n+1} |\lambda_k|}{|\lambda_i| + \sum_{k=1;\ k\neq r}^{n+1} |\lambda_k|} \qquad (5.30)$$

It follows immediately from (5.30), that

$$|h| < |h'| < |h_i| \qquad (5.31)$$

This completes the proof that the new reference error (h') is larger in magnitude than the old one (h).

It now remains to be shown that it is indeed possible to replace one equation by E_i in such a way that the point (A) is again a reference point. By replacing the (n+1) planes of the old reference, one at a time, one obtains (n+1) new references, from which a choice must be made. For each of the (n+1) new references a characteristic equation can be written which is of the form (5.10).

The characteristic equation for the old reference is:

$$\lambda_1\bar{n}_1 + \lambda_2\bar{n}_2 + \cdots + \lambda_r\bar{n}_r + \cdots + \lambda_{n+1}\bar{n}_{n+1} = 0 \qquad (5.32)$$

The (n+1) normal vectors of the old reference and the normal $\bar{n}_i$ of the plane E_i which is to replace one of these, form a set of (n+2) vectors in n-dimensional space. Therefore, coefficients μ_k, not all equal to zero, must exist such that

$$\mu_1\bar{n}_1 + \mu_2\bar{n}_2 + \cdots + \mu_r\bar{n}_r + \cdots + \mu_{n+1}\bar{n}_{n+1} + \bar{n}_i = 0 \qquad (5.33)$$

By eliminating $\bar{n}_r$ $(1 \leq r \leq n+1)$ from (5.32) and (5.33), a set of (n+1) characteristic equations is obtained:

$$\left[\frac{\mu_1}{\lambda_1} - \frac{\mu_r}{\lambda_r}\right]\lambda_1\bar{n}_1 + \left[\frac{\mu_2}{\lambda_2} - \frac{\mu_r}{\lambda_r}\right]\lambda_2\bar{n}_2 + \cdots + 0\,\bar{n}_r + \cdots + \left[\frac{\mu_{n+1}}{\lambda_{n+1}} - \frac{\mu_r}{\lambda_r}\right]\lambda_{n+1}\bar{n}_{n+1} + \bar{n}_i = 0 \qquad (5.34)$$

where $1 \leq r \leq n+1$.

These are the (n+1) characteristic equations of all possible references formed by replacing one plane (E_r) of the old reference by the plane E_i. Let primes denote quantities belonging to the new reference. Then (5.34) can be written in the form:

$$\lambda_1'\bar{n}_1 + \lambda_2'\bar{n}_2 + \ldots + 0\,\bar{n}_r + \ldots + \lambda_{n+1}'\bar{n}_{n+1} + \lambda_i'\bar{n}_i = 0 \qquad (5.35)$$

which is a linear combination of a total of (n+1) vectors.

Because the point (A) is a reference point of the old reference, the following condition is satisfied. From (5.19):

$$\operatorname{sgn}\lambda_k = \operatorname{sgn} h \ \operatorname{sgn} h_k \qquad (1 \leq k \leq n+1) \qquad (5.36)$$

When the point (A) is also a reference point of the new reference, the following condition must also hold:

$$\operatorname{sgn}\lambda_k' = \operatorname{sgn} h' \ \operatorname{sgn} h_k' \qquad (5.37)$$

where k assumes the following values

$$k = 1, 2, \ldots, n+1, i \ ; k \neq r \qquad (5.38)$$

For the point (A), $h_k' = h_k$ for $k = 1, 2, \ldots, n+1; k \neq r$. Two possibilities now exist. First consider the case in which $\operatorname{sgn} h_i = \operatorname{sgn} h$ and observe that $\lambda_i' = +1$. The condition (5.37) is fulfilled for all values k of the new reference, as given by (5.38), when $\operatorname{sgn}\lambda_k = \operatorname{sgn}\lambda_k'$. This condition will be fulfilled when that plane E_r is replaced for which the ratio $\frac{\mu_r}{\lambda_r}$ is minimum. This follows from the fact that, when

$$\frac{\mu_r}{\lambda_r} < \frac{\mu_k}{\lambda_k} \qquad (k = 1, 2, \ldots, n+1 \ ; k \neq r) \qquad (5.39)$$

then:

$$\left[\frac{\mu_k}{\lambda_k} - \frac{\mu_r}{\lambda_r}\right] > 0 \tag{5.40}$$

Secondly, when sgn h_i = - sgn h, condition (5.37) will be satisfied when sgn λ_k' = - sgn λ_k. This can be achieved by replacing that plane E_r for which the ratio $\frac{\mu_r}{\lambda_r}$ is maximum. In that case,

$$\frac{\mu_r}{\lambda_r} > \frac{\mu_k}{\lambda_k} \qquad (k = 1, 2, \ldots, n+1 \; ; \; k \neq r) \tag{5.41}$$

and

$$\left[\frac{\mu_k}{\lambda_k} - \frac{\mu_r}{\lambda_r}\right] < 0 \tag{5.42}$$

This completes the replacement rule, which can be summarized as follows:

$$\begin{aligned} &\text{when sgn } h_i = \text{sgn } h\text{, replace the plane for which } \frac{\mu_k}{\lambda_k} \text{ is minimum} \\ &\text{when sgn } h_i = -\text{sgn } h\text{, replace the plane for which } \frac{\mu_k}{\lambda_k} \text{ is maximum} \end{aligned} \tag{5.43}$$

After replacement, a new reference is obtained whose center can be determined with the procedure discussed in paragraph 5.3.1. The coordinates of the new reference center are then substituted into the remaining equations of the overdetermined system (5.7). When any of the resulting errors are larger than the reference error $|h'|$, the replacement process must be undertaken a second time.

Every time the replacement process takes place, the new reference error is larger than the previous one. After a finite number of steps, the replacement process must terminate, because there are only a finite number (m) of equations. Also, the same reference will never be arrived at twice in this process, because the reference error $|h|$ increases monotonically. The center of the last reference is the Chebyshev point for the overdetermined system (5.7).

When the overdetermined system is subject to a constraint, the replacement process remains unaltered, except for the fact that the constraint equation E_0 is not eligible for replacement.

5.3.4 Example. To conclude Chapter V, an example of an overdetermined system will be given.

Consider the overdetermined system

$$\begin{aligned} E_1: &\quad x_1 + 2\,x_2 + 5 = 0 \\ E_2: &\quad 3\,x_1 + x_2 + 2 = 0 \\ E_3: &\quad 2\,x_1 + 3\,x_2 + 7 = 0 \end{aligned} \tag{5.44}$$

subject to the constraint that the equation E_0:

$$E_0: \quad x_1 + x_2 - 3 = 0 \tag{5.45}$$

be satisfied exactly.

The equations E_k represent planes (lines) in two-dimensional space. The normals to these lines are:

$$\begin{aligned} \bar{n}_0 &= 1,\ 1 \\ \bar{n}_1 &= 1,\ 2 \\ \bar{n}_2 &= 3,\ 1 \\ \bar{n}_3 &= 2,\ 3 \end{aligned} \tag{5.46}$$

Let the first reference consist of the equations E_1 and E_2. Then the characteristic equation can be written according to (5.22):

$$\lambda_0\bar{n}_0 + \lambda_1\bar{n}_1 + \lambda_2\bar{n}_2 = 0 \tag{5.47}$$

The coefficients λ_k are determined by solving the following set of simultaneous linear equations, corresponding to (5.47):

$$\begin{aligned} \lambda_0 + \lambda_1 + 3\lambda_2 &= 0 \\ \lambda_0 + 2\lambda_1 + \lambda_2 &= 0 \end{aligned} \tag{5.48}$$

The set of λ's satisfying these equations is:

$$\lambda_0 = -5; \quad \lambda_1 = 2; \quad \lambda_2 = 1.$$

With these values, the reference error h can be calculated using (5.27).

$$h = \frac{(-5)(-3) + (2)(5) + (1)(2)}{|2| + |\cancel{3}\,1|} = 9 \tag{5.49}$$

The errors to the reference equations are $h_k = h \operatorname{sgn} \lambda_k$. Substituting these errors into the set of reference equations gives:

$$\begin{aligned} E_0: &\quad x_1 + x_2 - 3 = 0 \\ E_1: &\quad x_1 + 2\,x_2 + 5 = 9 \\ E_2: &\quad 3\,x_1 + x_2 + 2 = 9 \end{aligned} \tag{5.50}$$

The center of the reference is found by solving this set of equations. One finds that:

$$x_1 = 2\;;\; x_2 = 1 \tag{5.51}$$

It is easily verified that these valus for x_1 and x_2 satisfy all three equations (5.50).

The error of the remaining equation E_3 is next determined. Substituting the values (5.51) into E_3, one finds that the error h_3 equals 14, which is obviously larger than the reference error. A new reference must now be chosen, using the replacement process.

First, coefficients μ are determined such that

$$\mu_0\bar{n}_0 + \mu_1\bar{n}_1 + \mu_2\bar{n}_2 + \bar{n}_3 = 0 \tag{5.52}$$

One of these coefficients μ can be chosen arbitrarily. Let $\mu_0 = 0$. Then the following system of linear equations has to be solved:

$$\begin{aligned} \mu_1 + 3\,\mu_2 + 2 &= 0 \\ 2\,\mu_1 + \mu_2 + 3 &= 0 \end{aligned} \tag{5.53}$$

The following values of μ satisfy (5.53):

$$\mu_1 = -\frac{7}{5}\,; \quad \mu_2 = -\frac{1}{5} \tag{5.54}$$

The ratio $\frac{\mu_k}{\lambda_k}$ can now be determined for the two planes of the old reference.

$$\frac{\mu_1}{\lambda_1} = -\frac{7}{10}\,; \quad \frac{\mu_2}{\lambda_2} = -\frac{1}{5} \tag{5.55}$$

Because $h = 9$ (5.49) and $h_3 = 14$, $\operatorname{sgn} h = \operatorname{sgn} h_3$. According to rule (5.43), that plane has to be replaced for which $\frac{\mu_k}{\lambda_k}$ is minimum. From (5.55) it is found that equation E_1 has to be replaced. The new reference then consists of equations E_2 and E_3, subject to the condition E_0.

The characteristic equation for the new reference is written in the form of two linear equations as follows:

$$\begin{aligned} \lambda_0 + 3\,\lambda_2 + 2\,\lambda_3 &= 0 \\ \lambda_0 + \lambda_2 + 3\,\lambda_3 &= 0 \end{aligned} \tag{5.56}$$

The following values for λ_k satisfy these equations:

$$\lambda_0 = -7; \quad \lambda_2 = 1; \quad \lambda_3 = 2 \tag{5.57}$$

The new reference error h is calculated:

$$h = \frac{(-7)(-3) + (1)(2) + (2)(7)}{|1| + |2|} = \frac{37}{3} \tag{5.58}$$

The new reference center is found by solving the following set of equations:

$$\begin{aligned} E_0{:}\quad x_1 + x_2 - 3 &= 0 \\ E_2{:}\quad 3\,x_1 + x_2 + 2 &= 12\tfrac{1}{3} \\ E_3{:}\quad 2\,x_1 + 3\,x_2 + 7 &= 12\tfrac{1}{3} \end{aligned} \tag{5.59}$$

and the coordinates of the reference center are:

$$x_1 = \frac{11}{3}; \quad x_2 = -\frac{2}{3} \tag{5.60}$$

As a final check, these coordinates are substituted into equation E_1, which gives $h_1 = 7\frac{1}{3}$. This is smaller than the reference error, and the Chebyshev point for the system therefore has the coordinates (5.60).

CHAPTER VI

EXAMPLES

6.1 Introduction

In this chapter some examples will be given of the synthesis of nonuniform lines, using the tools developed in Chapters IV and V. In Section 6.2 the theory of discrete Chebyshev approximation will be applied to the synthesis of impedance transformers. In Section 6.3 a matching section will be synthesized to provide a match between a generator, with an internal impedance of 50 ohms, and a 100-ohm load with stray capacitance.

When a given function is approximated by a linear combination of approximating functions, using the theory of discrete Chebyshev approximation, the maximum error at the sampling points will be minimum. One has essentially no control over the behavior of the function between the sampling points. However, by choosing a sufficiently large number of sampling points, spaced closely together, one can be confident that the error between sampling points will not exceed the error at the points by any appreciable amount, so that, for all practical purposes, a true Chebyshev approximation is obtained. In the examples given in this chapter the sampling points will be chosen at integral values of the independent variable 4s. The approximating functions $C_n(s)$ and $S_n(s)$ either have the value zero at these points, or reach an extreme value in the close vicinity of these sampling points (see Figs. 4.2 and 4.3). Because of the smooth behavior of the functions $C_n(s)$ and $S_n(s)$, good results are obtained using these sampling points.

The functions $C_n(s)$, evaluated at integral values of the variable 4s, are given in Table I. The functions $S_n(s)$ are given in Table

4s	$c_0(s)$	$c_1(s)$	$c_2(s)$	$c_3(s)$	$c_4(s)$	$c_5(s)$	$c_6(s)$	$c_7(s)$	$c_8(s)$	$c_9(s)$	$c_{10}(s)$	$c_{11}(s)$	$c_{12}(s)$	$c_{13}(s)$	$c_{14}(s)$	$c_{15}(s)$
0	1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0.63662	0.5	-0.21221	0	-0.04244	0	-0.01819	0	-0.01010	0	-0.00644	0	-0.00445	0	-0.00326	0
2	0	0.42441	-0.5	-0.25456	0	-0.06063	0	-0.02829	0	-0.01654	0	-0.01089	0	-0.00771	0	-0.00576
3	-0.21221	0	-0.38197	-0.5	0.27284	0	0.07073	0	0.03473	0	0.02099	0	0.01415	0	0.01021	0
4	0	-0.16976	0	-0.36378	0.5	0.28294	0	0.07717	0	0.03918	0	0.02425	0	0.01665	0	0.01218
5	0.12732	0	0.15157	0	0.35368	0.5	-0.28938	0	-0.08162	0	-0.04244	0	-0.02675	0	-0.01862	0
6	0	0.10913	0	0.14147	0	0.34724	-0.5	-0.29383	0	-0.08488	0	-0.04494	0	-0.02872	0	-0.02021
7	-0.09094	0	-0.09903	0	-0.13503	0	-0.34279	-0.5	0.29709	0	0.08738	0	0.04691	0	0.03031	0
8	0	-0.08084	0	-0.09259	0	-0.13058	0	-0.33953	0.5	0.29959	0	0.08935	0	0.04850	0	0.03163
9	0.07074	0	0.07440	0	0.08814	0	0.12732	0	0.33703	0.5	-0.30156	0	-0.09094	0	-0.04982	0
10	0	0.06430	0	0.06995	0	0.08488	0	0.12482	0	0.33506	-0.5	-0.30315	0	-0.09226	0	-0.05093
11	-0.05786	0	-0.05985	0	-0.06669	0	-0.08238	0	-0.12285	0	-0.33347	-0.5	0.30447	0	0.09337	0
12	0	-0.05341	0	-0.05659	0	-0.06419	0	-0.08041	0	-0.12126	0	-0.33215	0.5	0.30558	0	0.09431
13	0.04896	0	0.05015	0	0.05409	0	0.06222	0	0.07882	0	0.11994	0	0.33104	0.5	-0.30652	0
14	0	0.04570	0	0.04765	0	0.05212	0	0.06063	0	0.07750	0	0.11883	0	0.33010	-0.5	-0.30733
15	-0.04244	0	-0.04320	0	-0.04568	0	-0.05053	0	-0.05931	0	-0.07639	0	-0.11789	0	-0.32929	-0.5

TABLE I. The functions $C_{ne}(s)$ and $C_{no}(s)$ for integral values of (4s).

4s	$S_1(s)$	$S_2(s)$	$S_3(s)$	$S_4(s)$	$S_5(s)$	$S_6(s)$	$S_7(s)$	$S_8(s)$	$S_9(s)$	$S_{10}(s)$	$S_{11}(s)$	$S_{12}(s)$	$S_{13}(s)$	$S_{14}(s)$	$S_{15}(s)$
0	0.63662	0	0.21221	0	0.12732	0	0.09094	0	0.07074	0	0.05786	0	0.04896	0	0.04244
1	0.5	0.42441	0	0.16976	0	0.10913	0	0.08084	0	0.06430	0	0.05341	0	0.04570	0
2	0.21221	0.5	-0.38197	0	-0.15157	0	-0.09903	0	-0.07440	0	-0.05985	0	-0.05015	0	-0.04320
3	0	0.25465	-0.5	-0.36378	0	-0.14147	0	-0.09259	0	-0.06995	0	-0.05659	0	-0.04765	0
4	-0.04244	0	-0.27284	-0.5	0.35368	0	0.13503	0	0.08814	0	0.06669	0	0.05409	0	0.04568
5	0	-0.06063	0	-0.28294	0.5	0.34724	0	0.13058	0	0.08488	0	0.06419	0	0.05212	0
6	0.01819	0	0.07073	0	0.28938	0.5	-0.34279	0	-0.12732	0	-0.08238	0	-0.06222	0	-0.05053
7	0	0.02829	0	0.07717	0	0.29383	-0.5	-0.33953	0	-0.12482	0	-0.08041	0	-0.06063	0
8	-0.01010	0	-0.03473	0	-0.08162	0	-0.29709	-0.5	0.33703	0	0.12285	0	0.07882	0	0.05931
9	0	-0.01654	0	-0.03918	0	-0.08488	0	-0.29959	0.5	0.33506	0	0.12126	0	0.07750	0
10	0.00644	0	0.02099	0	0.04244	0	0.08738	0	0.30156	0.5	-0.33347	0	-0.11994	0	-0.07639
11	0	0.01089	0	0.02425	0	0.04494	0	0.08935	0	0.30315	-0.5	-0.33215	0	-0.11883	0
12	-0.00445	0	-0.01415	0	-0.02675	0	-0.04691	0	-0.09094	0	-0.30447	-0.5	0.33104	0	0.11789
13	0	-0.00771	0	-0.01665	0	-0.02872	0	-0.04850	0	-0.09226	0	-0.30558	0.5	0.33010	0
14	0.00326	0	0.01021	0	0.01862	0	0.03031	0	0.04982	0	0.09337	0	0.30652	0.5	-0.32929
15	0	0.00576	0	0.01218	0	0.02021	0	0.03163	0	0.05093	0	0.09431	0	0.30733	-0.5

TABLE II. The functions $S_{ne}(s)$ and $S_{no}(s)$ for integral values of (4s).

II. As can be seen from these tables and also from Figs. 4.2 and 4.3, the functions $C_n(s)$ and $S_n(s)$ consist of a main lobe around the point $4s = n$ and smaller side lobes. The farther the side lobe from the main lobe, the smaller its amplitude.

When a large number of functions $C_n(s)$ and $S_n(s)$ are used in the approximation process, the amount of computation could be reduced if some means could be found to let the amplitude of the side lobes approach zero more rapidly. Danielson and Lanczos have developed a transformation by which the set of functions $C_n(s)$ and $S_n(s)$ is transformed into a new set of functions whose side lobes approach zero more rapidly than those of the original functions. Their method can be modified to apply to the present problem. This process is treated in the appendix.

6.2 Synthesis of Impedance Transformers

The first two problems to be considered will be the synthesis of two impedance transformers, one 0.75 λ long at the lowest frequency of the pass band, the other 1 λ long.

Results are obtained that represent an improvement over those obtained by Willis and Sinha (paragraph 3.3.1), partly because of the more general synthesis formula (Eq. 4.45 and 4.46), and partly because of the use of the theory of discrete Chebyshev approximation. This theory, developed in Chapter V, provides a very powerful tool, far superior to the trial-and-error method used by Willis and Sinha. A large number of terms in the expansion (4.45) can be handled conveniently, while, at the same time, the amount of impedance transformation (4.38) is controlled directly.

6.2.1 0.75 λ Transformer.

To determine which terms should be used in the trigonometric expansion (4.12) of the reflection-distribution function $\rho(y)$, a qualitative argument can be used, involving Parseval's theorem and the equation giving the amount of impedance transformation (4.35). Parseval's theorem, applied to the present problem, states that:

$$2 \int_{-\infty}^{\infty} |\Gamma(s)|^2 \, ds = \int_0^1 |\rho(y)|^2 \, dy \qquad (6.1)$$

When the amount of impedance transformation is given, the following integral (4.35) is determined:

$$\int_0^1 \rho(y) \, dy = \frac{1}{2} \ln \frac{Z_{02}}{Z_{01}} \qquad (6.2)$$

where $\rho(y)$, of course, is a real function. One can state that qualitatively the amount of impedance transformation, in first approximation, determines the value of the integral (6.1).

If $\Gamma(s)$ has a high-pass character, it immediately follows from (6.1) that the area of the main lobe must increase when the reflections in the pass band decrease. To increase the value of $|\Gamma(s)|$ outside the pass band, functions $C_{ne}(s)$ and $S_{no}(s)$ can be selected such that their main lobes fall outside the pass band. A suitable choice for a 0.75 λ transformer would be the functions $C_0(s)$, $C_2(s)$, and $S_1(s)$.

The reflection-distribution function is then expanded as follows:

$$\rho(y) = a_0 + a_2 \cos 2\pi y + b_1 \sin \pi y \qquad (6.3)$$

It follows from (4.46) that

$$\Gamma(s) \, e^{j2\pi s} = a_0 C_0(s) + a_2 \, C_2(s) + b_1 \, S_1(s) \qquad (6.4)$$

The input reflection $\Gamma(s)$ should approximate the value zero for all values of the variable s equal to and greater than 0.75.

Using Tables I and II, one finds the following equations (E_j) at the sampling points, where the subscripts correspond to the value of 4s at the sampling point.

The equation E_0 is the equation corresponding to Eq. 4.38, determined by the amount of impedance transformation. This equation has to be satisfied exactly and the error has to be distributed over the remaining equations. The amount of impedance transformation will be chosen equal to $\frac{Z_{02}}{Z_{01}} = e^2$ so that:

$$\frac{1}{2} \ln \frac{Z_{02}}{Z_{01}} = 1 \tag{6.5}$$

A direct comparison is then possible with impedance transformers given in the literature and discussed in paragraph 3.3.1.

$$\begin{array}{llllll}
E_0: & 1.0\ a_0 & & +0.6366\ b_1 & -1.0 & = 0 \\
E_3: & -0.2112\ a_0 & -0.3820\ a_2 & & & = 0 \\
E_4: & & & -0.0424\ b_1 & & = 0 \\
E_5: & 0.1273\ a_0 & +0.1516\ a_2 & & & = 0 \\
E_6: & & & 0.0182\ b_1 & & = 0 \\
E_7: & -0.0909\ a_0 & -0.0990\ a_2 & & & = 0 \\
E_8: & & & -0.0101\ b_1 & & = 0 \\
E_9: & 0.0707\ a_0 & +0.0744\ a_2 & & & = 0 \\
E_{10}: & & & 0.0064\ b_1 & & = 0 \\
E_{11}: & -0.0579\ a_0 & -0.0598\ a_2 & & & = 0
\end{array}$$

$$
\begin{aligned}
E_{12}: &\quad & -0.0044\, b_1 &= 0 \\
E_{13}: &\quad 0.0490\, a_0 + 0.0502\, a_2 & &= 0 \\
E_{14}: &\quad & 0.0033\, b_1 &= 0 \\
E_{15}: &\quad -0.0424\, a_0 - 0.0432\, a_2 & &= 0
\end{aligned}
\tag{6.6}
$$

Obviously the set of equations (6.6) cannot be satisfied simultaneously. A reference is chosen consisting of equations E_0, E_3, E_4, and E_5. Four equations are taken because the space is three-dimensional. Next, a set of λ_j's are calculated such that

$$\lambda_0\bar{n}_0 + \lambda_3\bar{n}_3 + \lambda_4\bar{n}_4 + \lambda_5\bar{n}_5 = 0 \tag{6.7}$$

When these λ_j's are known, the Chebyshev error for the chosen reference can be determined. Equation 6.7 can be written in the form of a set of simultaneous linear equations as follows:

$$
\begin{aligned}
1.0\ \lambda_0 - 0.2122\ \lambda_3 + 0.1273\ \lambda_5 &= 0 \\
-0.3820\ \lambda_3 + 0.1516\ \lambda_5 &= 0 \\
0.6366\ \lambda_0 - 0.0424\ \lambda_4 &= 0
\end{aligned}
\tag{6.8}
$$

The set of λ_j satisfying these equations is:

$$\lambda_0 = 1; \quad \lambda_3 = -9.2036; \quad \lambda_4 = 15.0005; \quad \lambda_5 = -23.1935$$

The Chebyshev error is determined next (5.27):

$$h = \frac{\lambda_0 c_0 + \lambda_3 c_3 + \lambda_4 c_4 + \lambda_5 c_5}{|\lambda_3| + |\lambda_4| + |\lambda_5|} = -0.0211 \tag{6.9}$$

The error of the jth equation, h_j, equals

$$h_j = h \operatorname{sgn} \lambda_j = -0.0211 \operatorname{sgn} \lambda_j \tag{6.10}$$

This error is applied to the equations of the reference, which leads to a set of equations that can now be solved.

$$\begin{aligned}
E_0:&\quad 1.0\ a_0 \qquad\qquad\qquad +0.6366\ b_1 - 1.0 &= 0\\
E_3:&\quad -0.2112\ a_0 - 0.3820\ a_2 &= 0.0211\\
E_4:&\quad -0.0424\ b_1 &= -0.0211\\
E_5:&\quad 0.1273\ a_0 + 0.1516\ a_2 &= 0.0211
\end{aligned} \tag{6.11}$$

The following set of coefficients satisfies these four equations:

$$a_0 = 0.6835; \quad a_2 = -0.4350; \quad b_1 = 0.4972 \tag{6.12}$$

These values for the coefficients are substituted in the equations E_6 through E_{15} to determine the error at the other sampling points. One finds the following values for these errors:

$$\begin{aligned}
h_6 &= 0.0090 & h_7 &= -0.0191 & h_8 &= -0.0050 & h_9 &= 0.0160\\
h_{10} &= 0.0032 & h_{11} &= -0.0135 & h_{12} &= -0.0022 & h_{13} &= 0.0117\\
h_{14} &= 0.0016 & h_{15} &= -0.0102
\end{aligned} \tag{6.13}$$

It is easily verified that all these errors are smaller than the reference error (6.9). This, therefore, concludes the determination of the coefficients a_0, a_2, and b_1.

The reflection-distribution function for this impedance transformer is equal to

$$\rho(y) = 0.6835 - 0.4350 \cos 2\pi y + 0.4972 \sin \pi y \qquad (6.14)$$

The reflection coefficient of the line is plotted in Fig. 6.1. As was found above (Eq. 6.9), the maximum reflection in the pass band is 0.0211. This value can be compared with the results obtained by Willis

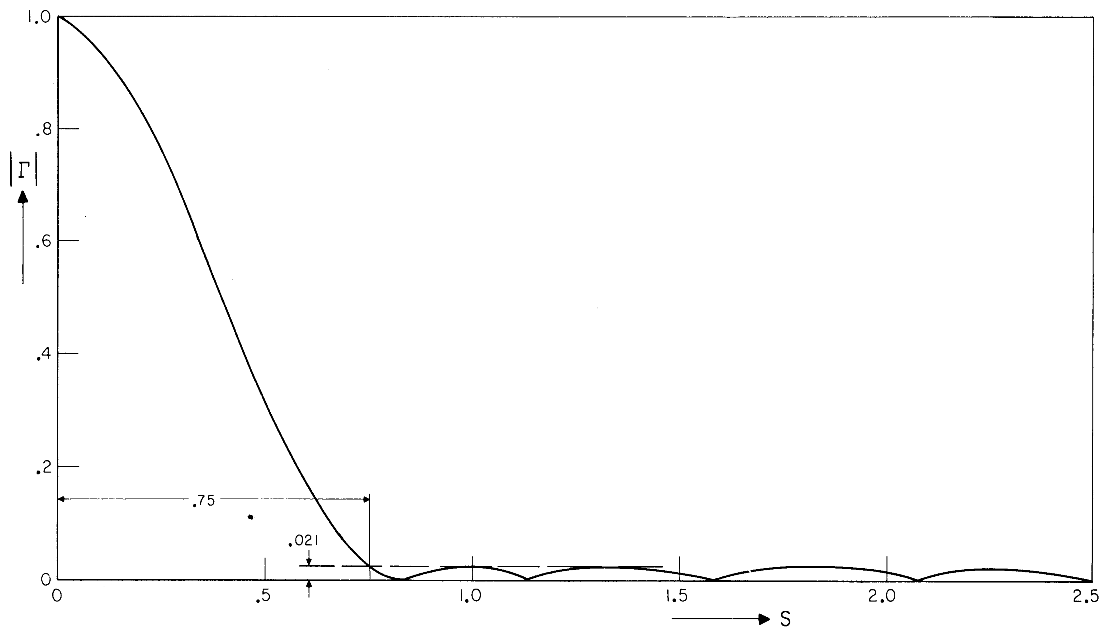

Fig. 6.1 Reflection pattern for a 0.75 λ transformer.

and Sinha and by Klopfenstein. The reflection in the pass band in the example of Willis and Sinha, for an impedance transformer of the same length, equals 0.031 (see Fig. 3.1). The present design compares favorably with this and is almost as good as the Chebyshev taper which gives reflections in the pass band equal to 0.018. The Chebyshev taper has the disadvantage of having discrete impedance steps at the two ends of the line. The line designed in this paragraph is smooth everywhere.

From the reflection-distribution function the characteristic impedance everywhere in the line can be calculated using Eq. 4.48. The reflection-distribution function $\rho(y)$ is plotted in Fig. 6.2. The

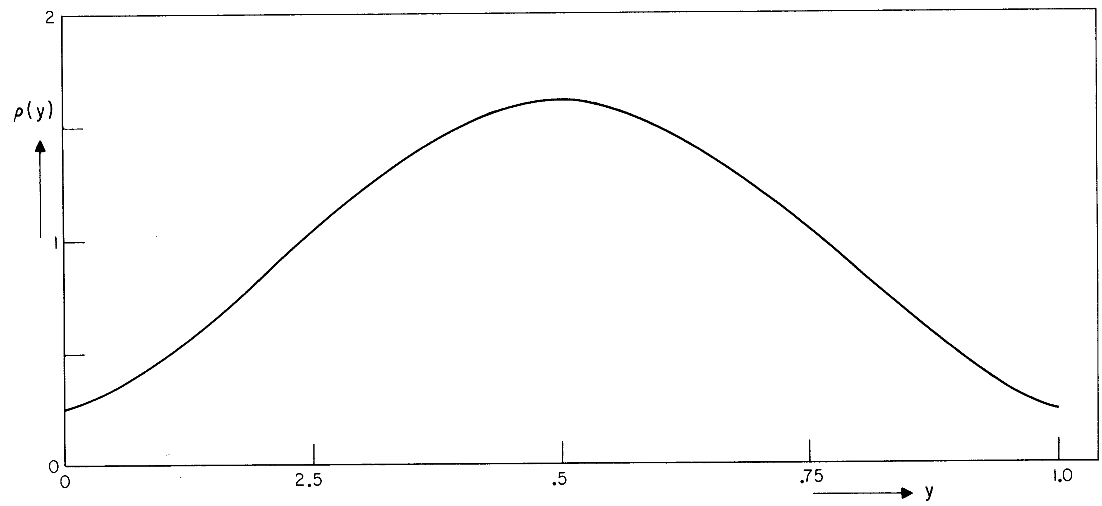

Fig. 6.2 Reflection-distribution function for a 0.75 λ transformer.

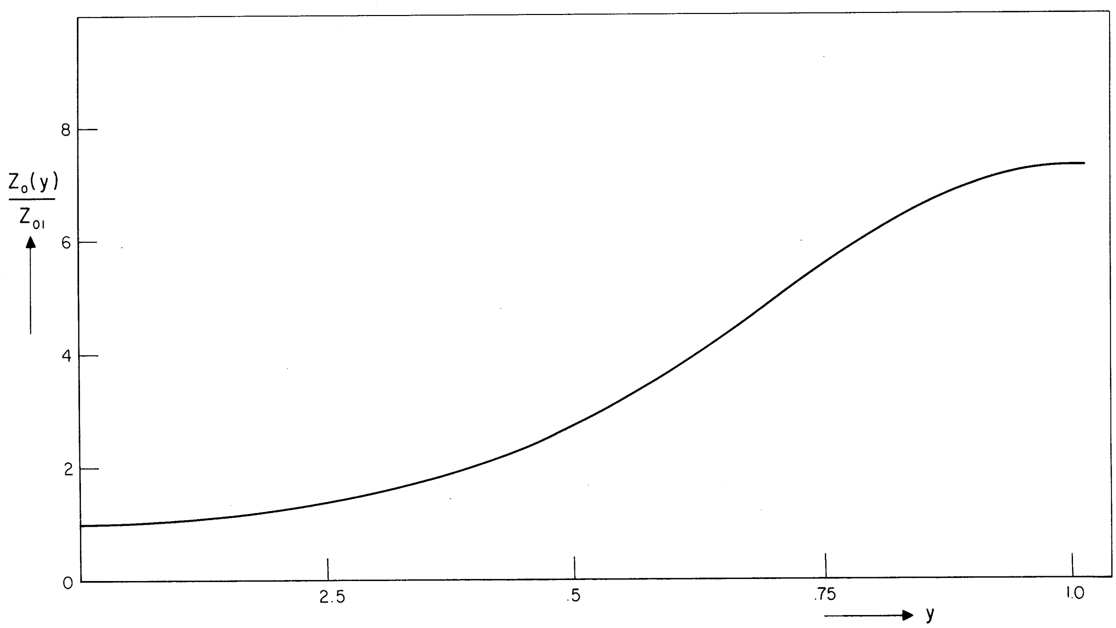

Fig. 6.3 Characteristic impedance function for a 0.75 λ transformer.

characteristic-impedance function for the whole line is plotted in Fig. 6.3.

6.2.2 1 λ Transformer. The next example of synthesis will be an impedance transformer which has a length of one wavelength at the lowest frequency of the pass band. In this example the functions $C_0(s)$, $C_2(s)$, $C_4(s)$, and $C_6(s)$ will be chosen to synthesize a reflection coefficient Γ with high-pass character. This implies that the reflection-distribution function $\rho(y)$ is of the form:

$$\rho(y) = a_0 + a_2 \cos 2\pi y + a_4 \cos 4\pi y + a_6 \cos 6\pi y \tag{6.15}$$

The corresponding input reflection coefficient can be written according to Eq. 4.46:

$$\Gamma(s)\, e^{j2\pi s} = a_0 C_0(s) + a_2 C_2(s) + a_4 C_4(s) + a_6 C_6(s) \tag{6.16}$$

The only term in the expansion of $\rho(y)$ that contributes to impedance transformation is the constant term (a_0). The restriction placed on the overdetermined system by the amount of impedance transformation now becomes a trivial one. In this example, therefore, an overdetermined system will be considered that is not subject to constraints.

The coefficient a_0 is directly determined by the amount of impedance transformation [Eq. 4.38]:

$$a_0 = \frac{1}{2} \ln \frac{Z_{02}}{Z_{01}} = 1 \tag{6.17}$$

Because only cosine terms are used in the expansion (6.15), the function $\Gamma(s)$ will be zero for even integral values of 4s larger than 6. The corresponding equations, therefore, do not appear among the ones listed below. The following set of equations can now be written, using Table I.

$$
\begin{array}{lrrrrl}
E_4: & & 0.5000\ a_4 & & & = 0 \\
E_5: & 0.1516\ a_2 & +0.3537\ a_4 & -0.2894\ a_6 & +0.1273 & = 0 \\
E_6: & & & -0.5000\ a_6 & & = 0 \\
E_7: & -0.0990\ a_2 & -0.1350\ a_4 & -0.3428\ a_6 & -0.0909 & = 0 \\
E_9: & 0.0744\ a_2 & +0.0881\ a_4 & +0.1273\ a_6 & +0.0707 & = 0 \\
E_{11}: & -0.0598\ a_2 & -0.0667\ a_4 & -0.0824\ a_6 & -0.0579 & = 0 \\
E_{13}: & 0.0502\ a_2 & +0.0541\ a_4 & +0.0622\ a_6 & +0.0490 & = 0 \\
E_{15}: & -0.0432\ a_2 & -0.0457\ a_4 & -0.0505\ a_6 & -0.0424 & = 0
\end{array}
\qquad (6.18)
$$

A reference is chosen out of this set, consisting of equations E_4, E_5, E_6, and E_7. Four equations are taken because the space is three-dimensional. A set of λ_j's for this reference is calculated to satisfy:

$$\lambda_4\bar{n}_4 + \lambda_5\bar{n}_5 + \lambda_6\bar{n}_6 + \lambda_7\bar{n}_7 = 0 \qquad (6.19)$$

This can be written in the form of a set of simultaneous linear equations as follows:

$$
\begin{array}{rrrrl}
 & 0.1516\ \lambda_5 & & -0.0990\ \lambda_7 & = 0 \\
0.5000\ \lambda_4 & +0.3537\ \lambda_5 & & -0.1350\ \lambda_7 & = 0 \\
 & -0.2894\ \lambda_5 & -0.5000\ \lambda_6 & -0.3428\ \lambda_7 & = 0
\end{array}
\qquad (6.20)
$$

The following set of values satisfies these equations:

$$\lambda_4 = -0.1921 \quad \lambda_5 = 0.6534 \quad \lambda_6 = -1.0637 \quad \lambda_7 = 1.0 \tag{6.21}$$

The reference error is next determined (5.18):

$$h = \frac{\lambda_4 c_4 + \lambda_5 c_5 + \lambda_6 c_6 + \lambda_7 c_7}{|\lambda_4| + |\lambda_5| + |\lambda_6| + |\lambda_7|} = -0.0027 \tag{6.22}$$

The error to the jth equation (h_j), equals $h_j = h \operatorname{sgn} \lambda_j$. This error is added to the equations of the reference, which leads to a set of equations that can now be solved.

$$\begin{aligned}
E_4&: & & 0.5000\, a_4 & & & = & \; 0.0027 \\
E_5&: & 0.1516\, a_2 & +0.3537\, a_4 & -0.2894\, a_6 & +0.1273 & = & -0.0027 \\
E_6&: & & & -0.5000\, a_6 & & = & \; 0.0027 \\
E_7&: & -0.0990\, a_2 & -0.1350\, a_4 & -0.3428\, a_6 & -0.0909 & = & -0.0027
\end{aligned} \tag{6.23}$$

The following set of coefficients satisfies these equations:

$$a_0 = 1.0; \quad a_2 = -0.8598; \quad a_4 = 0.0053; \quad a_6 = -0.0053 \tag{6.24}$$

These values for the coefficients can be substituted in the remaining equations to determine the error at the other sampling points. One finds:

$$h_9 = 0.0066; \quad h_{11} = -0.0063; \quad h_{13} = 0.0058; \quad h_{15} = -0.0053 \tag{6.25}$$

All these errors happen to be larger than the reference error. The synthesis problem would now be solved if one were interested only in obtaining an impedance transformer over a frequency range corresponding to a range of values for s between one and two. In the problem considered in Section 6.3 such an impedance transformation will be needed, and the coefficients given in Eq. 6.24 will be used.

In the present problem, however, a high-pass characteristic is desired. The replacement process will then be necessary to determine which of the equations of the reference (6.23) must be replaced. The largest of the errors (6.25) is h_9. The equation E_9 will therefore be used to replace one of the equations of the old reference. Replacement by the equation possessing the largest error does not guarantee success in the next attempt, but it is the most appropriate choice under the circumstances.

The replacement process starts by determining the coefficients μ_j that can be found by solving the equation:

$$\mu_4\bar{n}_4 + \mu_5\bar{n}_5 + \mu_6\bar{n}_6 + \mu_7\bar{n}_7 + \bar{n}_9 = 0 \qquad (6.26)$$

This equation can be written as a set of simultaneous linear equations.

$$\begin{aligned} 0.1516\,\mu_5 \qquad\qquad -0.0990\,\mu_7 + 0.0744 &= 0 \\ 0.5000\,\mu_4 + 0.3537\,\mu_5 \qquad\qquad -0.1350\,\mu_7 + 0.0881 &= 0 \\ -0.2894\,\mu_5 - 0.5000\,\mu_6 - 0.3428\,\mu_7 + 0.1273 &= 0 \end{aligned} \qquad (6.27)$$

The following set of μ_j's is one of the infinitely many that satisfy Eq. 6.27

$$\mu_4 = 0.0116; \quad \mu_5 = 0; \quad \mu_6 = -0.2604; \quad \mu_7 = 0.7513 \qquad (6.28)$$

Using these values, the ratio $\frac{\mu_j}{\lambda_j}$ can be determined:

$$\frac{\mu_4}{\lambda_4} = -0.0603; \quad \frac{\mu_5}{\lambda_5} = 0; \quad \frac{\mu_6}{\lambda_6} = 0.2450; \quad \frac{\mu_7}{\lambda_7} = 0.7513 \qquad (6.29)$$

Because $h < 0$ (6.22) and $h_9 > 0$ (6.25), that equation from the old reference must be replaced, for which the ratio $\frac{\mu_j}{\lambda_j}$ is maximum, according to rule (5.43). Therefore equation 7 has to be replaced so that the new reference consists of E_4, E_5, E_6, and E_9.

Again coefficients λ_j are determined such that

$$\lambda_4\bar{n}_4 + \lambda_5\bar{n}_5 + \lambda_6\bar{n}_6 + \lambda_9\bar{n}_9 = 0 \qquad (6.30)$$

This can be written as a set of simultaneous linear equations as follows:

$$\begin{aligned} 0.1516\,\lambda_5 \qquad\qquad\qquad + 0.0744\,\lambda_9 &= 0 \\ 0.5000\,\lambda_4 + 0.3537\,\lambda_5 \qquad\qquad\qquad + 0.0881\,\lambda_9 &= 0 \\ -0.2894\,\lambda_5 - 0.5000\,\lambda_6 + 0.1273\,\lambda_9 &= 0 \end{aligned} \qquad (6.31)$$

The following λ_j's satisfy these equations

$$\lambda_4 = 0.1709; \quad \lambda_5 = -0.4909; \quad \lambda_6 = 0.5387; \quad \lambda_9 = 1.0 \qquad (6.32)$$

These values are used to calculate the new reference error:

$$h = \frac{\lambda_4 c_4 + \lambda_5 c_5 + \lambda_6 c_6 + \lambda_9 c_9}{|\lambda_4| + |\lambda_5| + |\lambda_6| + |\lambda_9|} = 0.0037 \qquad (6.33)$$

The set of reference equations can now be solved by introducing the error into the equations:

$$
\begin{aligned}
E_4&: & 0.5000\,a_4 & & & & &= 0.0037 \\
E_5&: & 0.1516\,a_2 & +0.3537\,a_4 & -0.2894\,a_6 & +0.1273 & &= -0.0037 \\
E_6&: & & & -0.5000\,a_6 & & &= 0.0037 \\
E_9&: & 0.0744\,a_2 & +0.0881\,a_4 & +0.1273\,a_6 & +0.0707 & &= 0.0037
\end{aligned}
\qquad (6.34)
$$

The solution of this set yields the following values for the coefficients:

$$a_2 = -0.8964; \quad a_4 = 0.0075; \quad a_6 = -0.0075 \qquad (6.35)$$

These values are substituted into the remaining equations to determine the error at the other sampling points. One finds:

$$h_7 = -0.0006; \quad h_{11} = -0.0041; \quad h_{13} = 0.0039; \quad h_{15} = -0.0037 \quad (6.36)$$

It can be seen that h_{11} exceeds the reference error, and therefore the replacement process has to be undertaken a second time. Again coefficients μ_j are determined such that

$$\mu_4\bar{n}_4 + \mu_5\bar{n}_5 + \mu_6\bar{n}_6 + \mu_9\bar{n}_9 + \bar{n}_{11} = 0 \qquad (6.37)$$

Written as a set of simultaneous linear equations this becomes:

$$
\begin{aligned}
& & 0.1516\,\mu_5 & & +0.0744\,\mu_9 & -0.0598 &= 0 \\
& 0.5000\,\mu_4 & +0.3537\,\mu_5 & & +0.0881\,\mu_9 & -0.0667 &= 0 \\
& & -0.2894\,\mu_5 & -0.5000\,\mu_6 & +0.1273\,\mu_9 & -0.0824 &= 0
\end{aligned}
\qquad (6.38)
$$

The following coefficients μ_j satisfy these equations:

$$\mu_4 = -0.0084; \quad \mu_5 = 0; \quad \mu_6 = 0.0401; \quad \mu_9 = 0.8044 \qquad (6.39)$$

The ratios $\frac{\mu_j}{\lambda_j}$ are then determined using (6.39) and (6.32):

$$\frac{\mu_4}{\lambda_4} = -0.0492; \quad \frac{\mu_5}{\lambda_5} = 0; \quad \frac{\mu_6}{\lambda_6} = 0.0744; \quad \frac{\mu_9}{\lambda_9} = 0.8044 \qquad (6.40)$$

Because $h > 0$ (6.33) and $h_{11} < 0$ (6.36), that plane of the reference must be replaced for which the ratio $\frac{\mu_j}{\lambda_j}$ is maximum. Therefore equation 9 is replaced by equation 11, and the new reference consists of equations E_4, E_5, E_6, and E_{11}.

First the coefficients λ_j are determined again such that

$$\lambda_4\bar{n}_4 + \lambda_5\bar{n}_5 + \lambda_6\bar{n}_6 + \lambda_{11}\bar{n}_{11} = 0 \qquad (6.41)$$

This leads to the following set of equations:

$$\begin{aligned} 0.1516\,\lambda_5 \qquad\qquad\qquad -0.0598\,\lambda_{11} &= 0 \\ 0.5000\,\lambda_4 + 0.3537\,\lambda_5 \qquad\qquad\qquad -0.0667\,\lambda_{11} &= 0 \\ -0.2894\,\lambda_5 - 0.5000\,\lambda_6 - 0.0834\,\lambda_{11} &= 0 \end{aligned} \qquad (6.42)$$

with the solution:

$$\lambda_4 = -0.1459; \quad \lambda_5 = 0.3949; \quad \lambda_6 = -0.3933; \quad \lambda_{11} = 1.0 \qquad (6.43)$$

Using these values, the reference error is determined:

$$h = \frac{\lambda_4 c_4 + \lambda_5 c_5 + \lambda_6 c_6 + \lambda_{11} c_{11}}{|\lambda_4| + |\lambda_5| + |\lambda_6| + |\lambda_{11}|} = -0.0039 \qquad (6.44)$$

The reference set is now written as follows, including the error:

$$
\begin{aligned}
E_4:&\quad 0.5000\, a_4 &&= 0.0039\\
E_5:&\quad 0.1516\, a_2 + 0.3537\, a_4 - 0.2893\, a_6 + 0.1273 &&= -0.0039\\
E_6:&\quad -0.5000\, a_6 &&= 0.0039\\
E_{11}:&\quad -0.0598\, a_2 - 0.0667\, a_4 - 0.0824\, a_6 - 0.0579 &&= -0.0039
\end{aligned}
\qquad (6.45)
$$

The solution is:

$$a_0 = 1.0; \quad a_2 = -0.8991; \quad a_4 = 0.0078; \quad a_6 = -0.0078 \qquad (6.46)$$

These values are then substituted in the remaining equations to determine the error at the different sampling points:

$$h_7 = -0.0003; \quad h_9 = 0.0035; \quad h_{13} = 0.0038; \quad h_{15} = -0.0036 \qquad (6.47)$$

All these individual errors are smaller than the reference error and therefore the process is completed.

The reflection-distribution function for the impedance transformer is now equal to:

$$\rho(y) = 1.0 - 0.8991 \cos 2\pi y + 0.0078 \cos 4\pi y - 0.0078 \cos 6\pi y \qquad (6.48)$$

The reflection pattern for this line is plotted in Fig. 6.4. According to Eq. 6.44 the maximum reflection in the pass band equals 0.0039. This value can be compared with the results obtained by Willis and Sinha and by Klopfenstein (see paragraph 3.3.1). For a 1 λ transformer Willis and Sinha find a reflection in the pass band of 0.0056, and the reflections for the Dolph-Chebyshev taper are 0.0037. It is seen that the line synthesized above is almost as good as the Dolph-Chebyshev line, while it does not have the disadvantage of the discrete impedance jumps

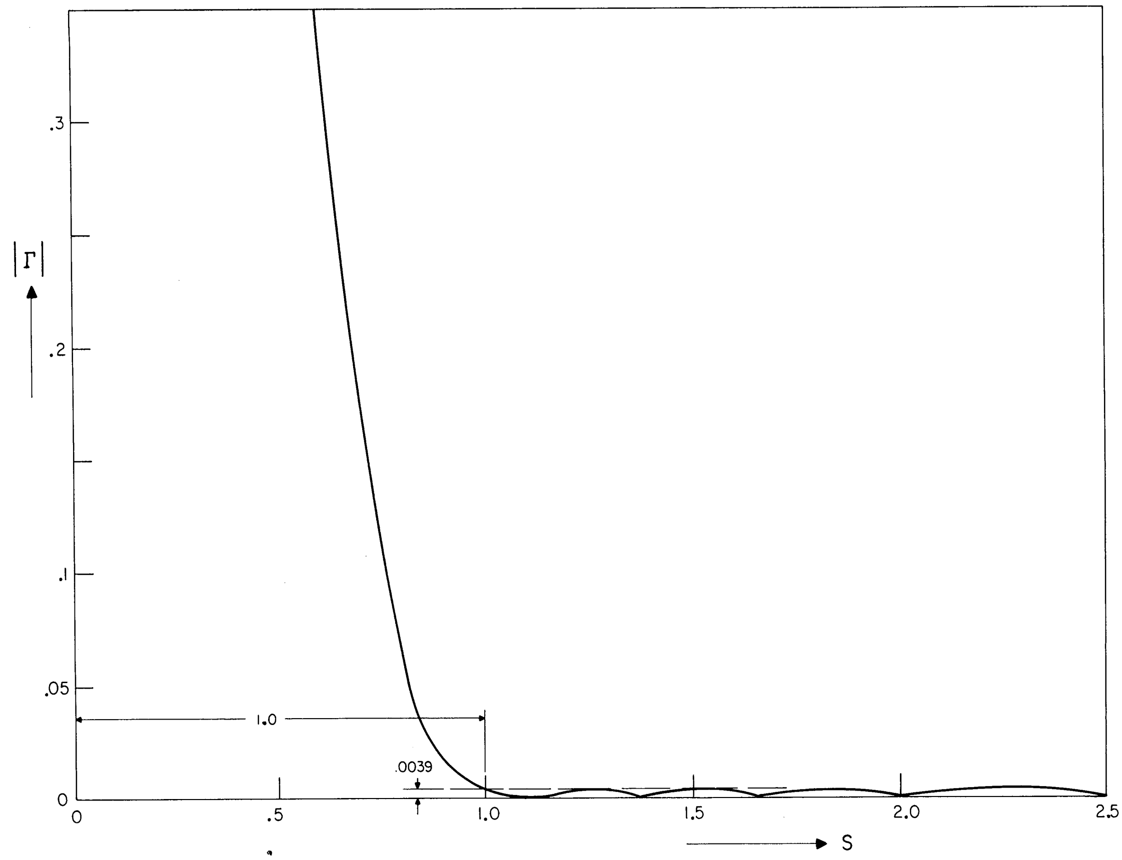

Fig. 6.4 Reflection pattern for a 1 λ transformer.

at the ends of the line.

The reflection-distribution function (Eq. 6.48) for the line is plotted in Fig. 6.5. The characteristic-impedance function Eq. 4.48 is plotted in Fig. 6.6.

6.3 Synthesis of a Matching Section

As a third example a matching section will be synthesized that matches a generator, with real internal impedance (50 ohms), to a complex load. The load is a 100-ohm resistor with 1 pf stray parallel capacitance. A uniform line can be used to connect the generator to the matching section, as indicated in Fig. 6.7. The matching section will provide a match over a frequency range extending from 500 to 1000 Mc, and the length of the matching section will be 60 cm, which

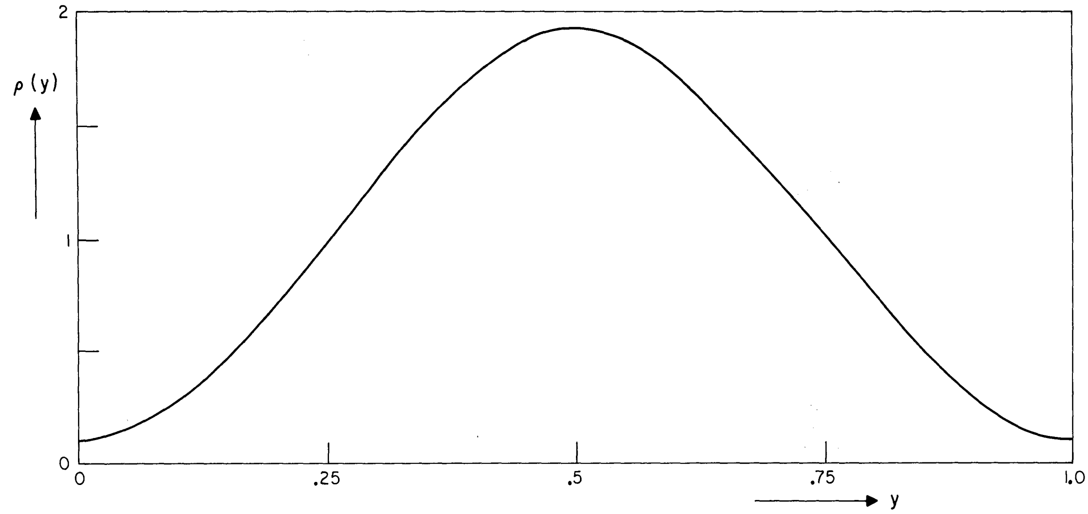

Fig. 6.5 Reflection distribution function for a 1 λ transformer.

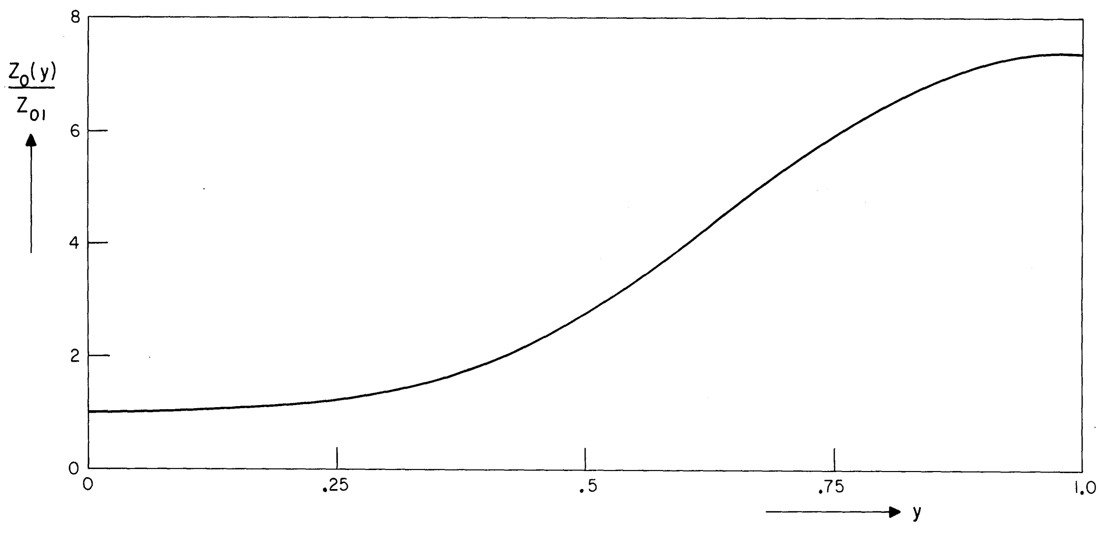

Fig. 6.6 Characteristic-impedance function for a 1 λ transformer.

corresponds to one wavelength at 500 Mc, the lowest frequency of the pass band.

When the load Z_2, as shown in Fig. 6.7, is connected directly to the terminals of the uniform line, omitting the matching section, it will cause a reflection of magnitude

$$|\Gamma| = \left| \frac{Z_2 - 50}{Z_2 + 50} \right| \leq 0.39 \qquad (6.49)$$

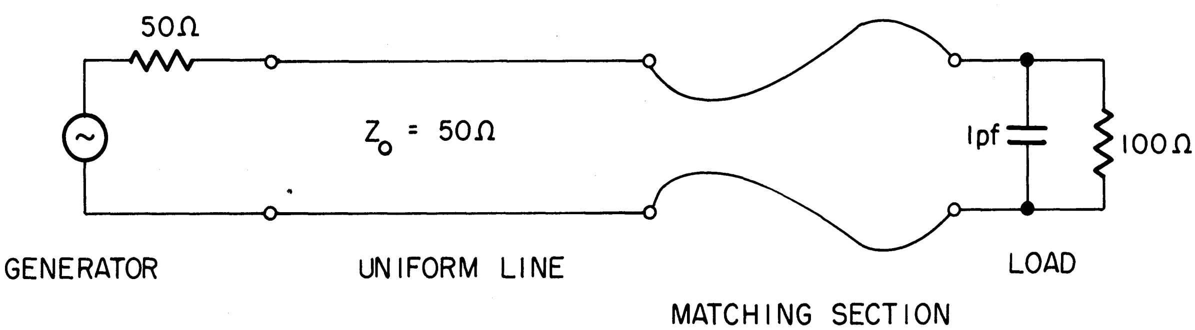

Fig. 6.7 Circuit diagram for the matching example.

which corresponds to a voltage standing wave ratio (VSWR) of 2.26 in the 50-ohm line. If, instead of the matching section, a 1 λ impedance transformer is used, best results are obtained when the impedance transformer transforms from 50 to 84.7 ohms. (This fact will be discussed further on in this section.) Using this optimum transformer, the VSWR in the 50-ohm line will equal 1.81.

In this section two matching sections will be synthesized, giving standing wave ratios equal to 1.07 and 1.10, respectively.

The first step in the synthesis procedure is to choose the length of the matching section. In this example the matching section will be one wavelength long at the lowest frequency, 500 Mc. The matching section will therefore by 60 cm long. The frequency variable s is then determined. s varies linearly with frequency, and s = 1 for 500 Mc, and s = 2 for 1000 Mc (Eq. 3.21).

The second step is to choose the value of the characteristic-impedance function at the two terminals of the matching section, Z_{01} and Z_{02}. Z_{01} will be chosen equal to 50 ohms, so that the section matches the generator. Z_{02} will be chosen such that the maximum value reached by $|\Gamma_2(s)|$, as defined by Eq. 4.6, is minimum. $\Gamma_2(s)$ can be written as follows:

$$\Gamma_2(s) = \frac{\frac{1}{Z_{02}} - \frac{1}{Z_2}}{\frac{1}{Z_{02}} + \frac{1}{Z_2}} = \frac{\frac{1}{Z_{02}} - \frac{1}{R} - j\omega C}{\frac{1}{Z_{02}} + \frac{1}{R} + j\omega C} = \frac{\frac{1}{Z_{02}} - 0.01 - j\,0.0031\,s}{\frac{1}{Z_{02}} + 0.01 + j\,0.0031\,s} \qquad (6.50)$$

$|\Gamma_2(s)|$ reaches its maximum value at the highest frequency of the pass band, when s = 2. Setting s = 2 in Eq. 6.50, it can be shown that the value of $|\Gamma_2(2)|$ reaches a minimum when

$$Z_{02} = 84.7 \text{ ohms} \qquad (6.51)$$

This value of Z_{02} will be used in the example.

The functions $\Gamma(s)$ and $\Gamma_o(s)$, as defined in Eqs. 4.5 and 4.7, can now be calculated. Obviously $\Gamma(s) = 0$, and $\Gamma_o(s)$ is given by (compare Eqs. 6.50 and 6.51):

$$\Gamma_o(s) = \frac{0.0018 - j\,0.0031\,s}{0.0218 + j\,0.0031\,s}\, e^{-j4\pi s} \qquad (6.52)$$

$\Gamma_1(s)$, as defined in Eq. 4.4, becomes (4.10):

$$\Gamma_1(s) = -\Gamma_o(s) \tag{6.53}$$

In the synthesis procedure, the real and imaginary parts of $\Gamma_1(s)\, e^{j2\pi s}$ are needed. They can be calculated using (6.52) and (6.53) and are tabulated below for integral values of 4s, which are the sampling points, in the frequency range of interest.

$$\Gamma_1(s)\, e^{j2\pi s} = -\frac{0.0018 - j\,0.0031\, s}{0.0218 + j\,0.0031\, s}\, e^{-j2\pi s} \tag{6.54}$$

(6.55)

4s	$Re\left\{\Gamma_1(s)\, e^{j2\pi s}\right\}$	$Im\left\{\Gamma_1(s)\, e^{j2\pi s}\right\}$
4	-0.0610	0.1528
5	0.1888	0.0490
6	0.0347	-0.2236
7	-0.2567	-0.0183
8	0.0	0.2881

A function G(s) is to be found that approximates $\Gamma_1(s)$, where G(s) is the Fourier transform of a real function $\rho(y)$, which is zero outside the interval (0,1). First, the real part of $G(s)\, e^{j2\pi s}$ will be synthesized.

According to Eq. 4.40:

$$Re\left\{G(s)\, e^{j2\pi s}\right\} = \sum_{n=0}^{N} \left[a_{ne} C_{ne}(s) + b_{no} S_{no}(s)\right] \tag{6.56}$$

It can be seen from the table (6.55) that $Re\left\{\Gamma_1(s)\, e^{j2\pi s}\right\}$ is small for even values of 4s. Because of this, the functions $S_3(s)$, $S_5(s)$, $S_7(s)$, and $S_9(s)$, will be used for the approximation process.

There are five sampling points, and therefore the four chosen functions lead to an overdetermined system. There is also the Eq. 4.38 which controls the amount of impedance transformation along the line. This equation has to be satisfied exactly. When this overdetermined system, with constraint, is solved, the error will prove to be too large. Therefore more approximating functions, for instance $C_2(s)$, $C_4(s)$, etc., are needed, leading to an overdetermined system of higher dimension. To avoid the tremendous amount of computation involved in solving this system, a slightly different approach will be taken in this example. First, the coefficients b_3, b_5, b_7, and b_9 will be determined, without regard to impedance transformation. The error in the impedance transformation will then be corrected using the functions $C_0(s)$, $C_2(s)$, $C_4(s)$, and $C_6(s)$, without essentially disturbing the approximation.

The approximation process is started by writing the equations for the sampling points, using Table II and (6.55).

$$\begin{array}{llllllll}
E_4: & -0.2728\, b_3 & +0.3537\, b_5 & +0.1350\, b_7 & +0.0881\, b_9 & +0.0610 & = & 0 \\
E_5: & & 0.5000\, b_5 & & & -0.1888 & = & 0 \\
E_6: & 0.0707\, b_3 & +0.2894\, b_5 & -0.3428\, b_7 & -0.1273\, b_9 & -0.0347 & = & 0 \\
E_7: & & & -0.5000\, b_7 & & +0.2567 & = & 0 \\
E_8: & -0.0347\, b_3 & -0.0816\, b_5 & -0.2971\, b_7 & +0.3370\, b_9 & & = & 0
\end{array} \qquad (6.57)$$

To determine the Chebyshev error for this system, coefficients λ_j must be determined such that

$$\lambda_4\bar{n}_4 + \lambda_5\bar{n}_5 + \lambda_6\bar{n}_6 + \lambda_7\bar{n}_7 + \lambda_8\bar{n}_8 = 0 \qquad (6.58)$$

This leads to the following system of simultaneous equations.

$$\begin{array}{llllll}
-0.2728\,\lambda_4 & & +0.0707\,\lambda_6 & -0.0347\,\lambda_8 & = & 0 \\
0.3537\,\lambda_4 & +0.5000\,\lambda_5 & +0.2894\,\lambda_6 & -0.0816\,\lambda_8 & = & 0
\end{array}$$

$$0.1350\,\lambda_4 \qquad -0.3428\,\lambda_6 -0.5000\,\lambda_7 -0.2971\,\lambda_8 = 0$$

$$0.0881\,\lambda_4 \qquad -0.1273\,\lambda_6 \qquad +0.3370\,\lambda_8 = 0 \tag{6.59}$$

This set is satisfied by the following set of values for λ_j:

$$\lambda_4 = 0.6812;\quad \lambda_5 = -2.1236;\quad \lambda_6 = 3.1187;\quad \lambda_7 = -2.5483;\quad \lambda_8 = 1.0 \tag{6.60}$$

Using these values for λ_j, the Chebyshev error for the system (6.57) can be determined.

$$h = \frac{\lambda_4 c_4 + \lambda_5 c_5 + \lambda_6 c_6 + \lambda_7 c_7 + \lambda_8 c_8}{|\lambda_4| + |\lambda_5| + |\lambda_6| + |\lambda_7| + |\lambda_8|} = -0.0338 \tag{6.61}$$

The error h_j of equation E_j is then determined by $h_j = h \operatorname{sgn} \lambda_j$, so that the following set of equations results, that can now be solved.

$$E_4:\quad -0.2728\,b_3 +0.3537\,b_5 +0.1350\,b_7 +0.0881\,b_9 +0.0610 = -0.0338$$

$$E_5:\quad 0.5000\,b_5 -0.1888 = 0.0338$$

$$E_6:\quad 0.0707\,b_3 +0.2894\,b_5 -0.3428\,b_7 -0.1273\,b_9 -0.0347 = -0.0338$$

$$E_7:\quad -0.5000\,b_7 +0.2567 = 0.0338$$

$$E_8:\quad -0.0347\,b_3 -0.0816\,b_5 -0.2971\,b_7 +0.3370\,b_9 = -0.0338 \tag{6.62}$$

The coefficients that satisfy these equations are:

$$b_3 = 1.3181;\quad b_5 = 0.4451;\quad b_7 = 0.4458;\quad b_9 = 0.5363 \tag{6.63}$$

The amount of impedance transformation was not controlled in the above procedure. The amount of impedance transformation is given by Eq. 4.38, which becomes:

$$0.2122\ b_3 + 0.1273\ b_5 + 0.0909\ b_7 + 0.0707\ b_9 = 0.4149 \qquad (6.64)$$

The correct amount of impedance transformation is given by

$$\frac{1}{2} \ln \frac{Z_{02}}{Z_{01}} = \frac{1}{2} \ln \frac{84.7}{50} = 0.2653 \qquad (6.65)$$

A correction to Eq. 6.64, equal to -0.1496, is needed to obtain the correct amount of impedance transformation in the line. The first approximation to the impedance transformer synthesized in paragraph 6.2.2 is ideally suited for the purpose of providing the necessary correction terms. One will recall that the coefficients for this transformer, as written in Eq. 6.24, provide minimum reflection inside the frequency band $1 \leq s \leq 2$, while the reflections outside this band are larger. By multiplying the coefficients, given in (6.24), by the factor -0.1496, the following coefficients are obtained that will give the correct amount of correction to Eq. 6.64.

$$a_0 = -0.1496; \quad a_2 = 0.1286; \quad a_4 = -0.0008; \quad a_6 = 0.0008 \qquad (6.66)$$

By making this correction, an additional error is introduced in the equations (6.62). This additional error is equal to 0.1496 x 0.0027 = 0.0004. This additional error is small compared to the error h = -0.0338 (6.61) and will therefore be neglected.

This concludes the approximation for the real part of $\Gamma_1\ e^{j2\pi s}$. No replacement process is necessary because there were only five equations in the four-dimensional space.

Next, the imaginary part of $\Gamma_1 e^{j2\pi s}$ must be approximated.

From the tabulation (6.55) it can be seen that this imaginary part is very small for odd values of the variable 4s. According to Eq. 4.40:

$$\text{Im}\left\{G(s)\ e^{j2\pi s}\right\} = \sum_{n=1}^{N} \left[a_{no}C_{no}(s) + b_{ne}S_{ne}(s)\right] \qquad (6.67)$$

The functions $S_2(s)$, $S_4(s)$, $S_6(s)$, and $S_8(s)$ will be chosen for the approximation process. The equations for the five sampling points can then be written using Table II:

$$\begin{aligned}
E_4:&\quad & -0.5000\,b_4 & & & -0.1528 = 0\\
E_5:&\quad -0.0606\,b_2 & -0.2829\,b_4 & +0.3472\,b_6 & +0.1306\,b_8 & -0.0490 = 0\\
E_6:&\quad & & 0.5000\,b_6 & & +0.2236 = 0\\
E_7:&\quad 0.0283\,b_2 & +0.0772\,b_4 & +0.2938\,b_6 & -0.3395\,b_8 & +0.0183 = 0\\
E_8:&\quad & & & -0.5000\,b_8 & -0.2881 = 0
\end{aligned} \qquad (6.68)$$

To determine the Chebyshev error for this system, one must first find coefficients λ_j such that:

$$\lambda_4\bar{n}_4 + \lambda_5\bar{n}_5 + \lambda_6\bar{n}_6 + \lambda_7\bar{n}_7 + \lambda_8\bar{n}_8 = 0 \qquad (6.69)$$

This leads to the following set of simultaneous linear equations:

$$\begin{aligned}
& -0.0606\,\lambda_5 & & +0.0283\,\lambda_7 & & = 0\\
-0.5000\,\lambda_4 & -0.2829\,\lambda_5 & & +0.0772\,\lambda_7 & & = 0\\
& 0.3472\,\lambda_5 & +0.5000\,\lambda_6 & +0.2938\,\lambda_7 & & = 0\\
& 0.1306\,\lambda_5 & & -0.3395\,\lambda_7 & -0.5000\,\lambda_8 & = 0
\end{aligned} \qquad (6.70)$$

The following values for λ_j satisfy these equations:

$$\lambda_4 = -0.1097; \quad \lambda_5 = 0.4666; \quad \lambda_6 = -0.9117; \quad \lambda_7 = 1.0; \quad \lambda_8 = -0.5572 \tag{6.71}$$

Using these values the Chebyshev error can be calculated:

$$h = \frac{\lambda_4 c_4 + \lambda_5 c_5 + \lambda_6 c_6 + \lambda_7 c_7 + \lambda_8 c_8}{|\lambda_4| + |\lambda_5| + |\lambda_6| + |\lambda_7| + |\lambda_8|} = -0.0102 \tag{6.72}$$

The error h_j of equation E_j is found from $h_j = h \text{ sgn } \lambda_j$, and the following set of equations must then be solved to find the coefficients:

$$\begin{array}{llllllll}
E_4: & & -0.5000\, b_4 & & & -0.1528 & = & 0.0102 \\
E_5: & -0.0606\, b_2 & -0.2829\, b_4 & +0.3472\, b_6 & +0.1306\, b_8 & -0.0490 & = & -0.0102 \\
E_6: & & & 0.5000\, b_6 & & +0.2236 & = & 0.0102 \\
E_7: & 0.0283\, b_2 & +0.0772\, b_4 & +0.2938\, b_6 & -0.3395\, b_8 & +0.0183 & = & -0.0102 \\
E_8: & & & & -0.5000\, b_8 & -0.2881 & = & 0.0102
\end{array} \tag{6.73}$$

The following values for the coefficients satisfy these equations:

$$b_2 = -2.8464; \quad b_4 = -0.3261; \quad b_6 = -0.4267; \quad b_8 = -0.5966 \tag{6.74}$$

This completes the synthesis of the matching section. The complete reflection-distribution function can now be written as follows:

$$\begin{aligned}
\rho(y) = & - 0.1496 + 0.1286 \cos 2\pi y - 0.0008 \cos 4\pi y + 0.0008 \cos 6\pi y \\
& - 2.8464 \sin 2\pi y + 1.3181 \sin 3\pi y - 0.3261 \sin 4\pi y \\
& + 0.4451 \sin 5\pi y - 0.4267 \sin 6\pi y + 0.4458 \sin 7\pi y \\
& -0.5966 \sin 8\pi y + 0.5363 \sin 9\pi y
\end{aligned} \tag{6.75}$$

This reflection-distribution function is plotted in Fig. 6.8. Using Eqs. 4.49 and 4.50, the characteristic-impedance function and the ratio of outer and inner conductor for a coaxial structure can be computed. The characteristic-impedance function $Z_0(y)$ is plotted in Fig. 6.9. Figure 6.10 shows a cross section of a matching section which has a reflection-distribution function equal to (6.75).

As can be seen from Fig. 6.8, the reflection-distribution function has a rather large magnitude at a few points in the line. Also, the general behavior of $\rho(y)$ is such that the characteristic impedance along the line varies between 8.5 and 85 ohms. The coefficient $b_2 = 2.8464$ in the expansion (6.75) is largely responsible for this behavior. By choosing different approximating functions to form the imaginary part of $G(s)\ e^{j2\pi s}$, a design can be obtained in which the characteristic impedance along the line does not vary over such a large range. This will be demonstrated below.

The matching section, synthesized above, does not provide an exact match over the pass band, as is apparent from the Chebyshev errors found in Eqs. 6.61 and 6.72. These are the errors in the real and imaginary part of $\Gamma_1(s)\ e^{j2\pi s}$, respectively. To calculate the total reflection from the matching section, these two errors must be added vectorially. The total reflection is therefore equal to $\sqrt{0.0338^2 + 0.0102^2} = 0.0353$, which corresponds to a VSWR equal to 1.07, in the uniform line connecting the generator to the matching section. See Fig. 6.7.

A general remark should be made regarding the total reflection error. In the example above, the total reflection error is equal at all five sampling points. This is a consequence of the fact that the number of sampling points exceeded the number of approximating functions

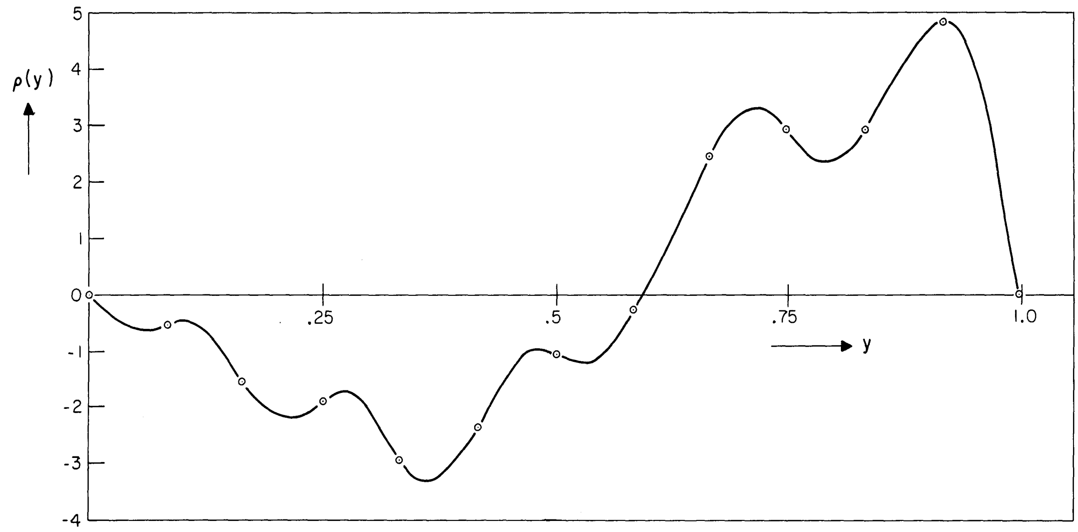

Fig. 6.8 Reflection-distribution function for matching section No. 1.

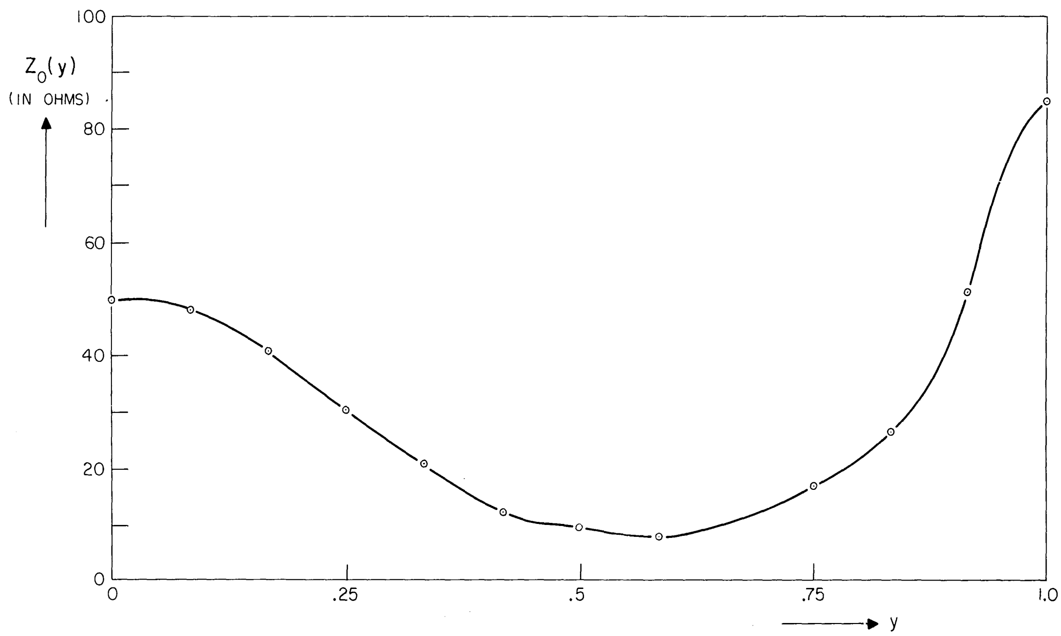

Fig. 6.9 Characteristic-impedance function for matching section No. 1.

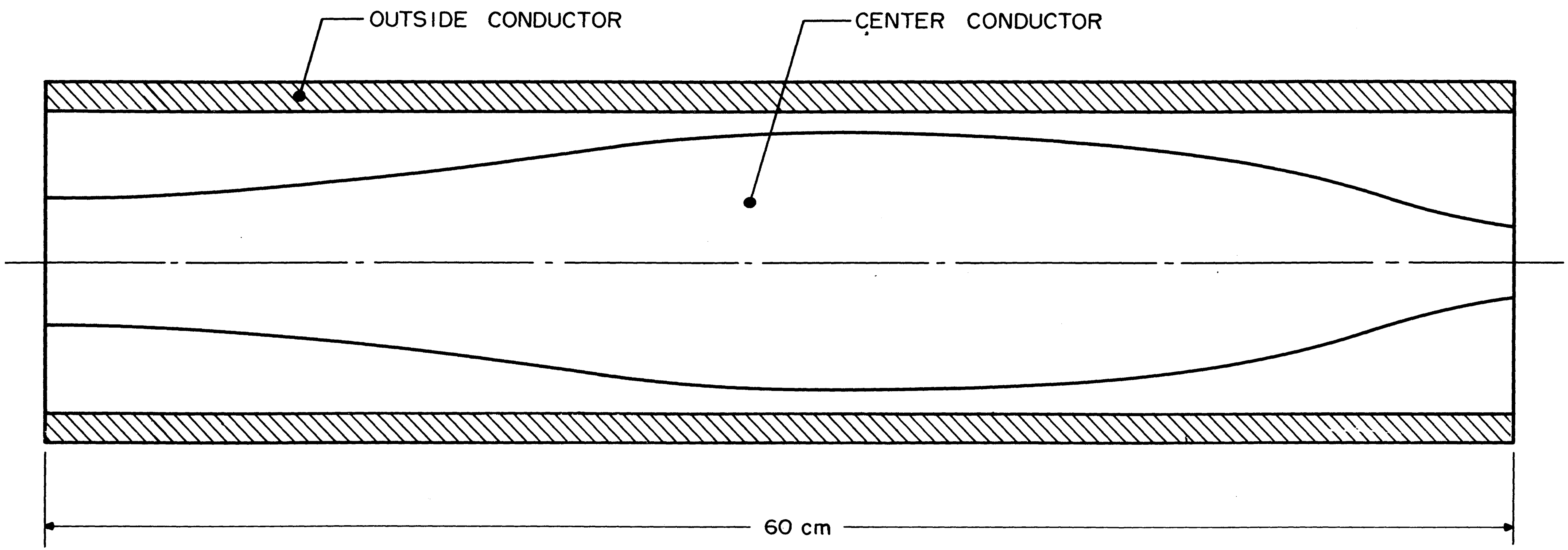

Fig. 6.10 Cross section of matching section No. 1.

by exactly one. In general, when a large number of sampling points is taken, the maximum error in the real part will not necessarily occur at the same sampling points at which the error in the imaginary part is located. The only conclusion that can be drawn in such a case is that the total reflection error does not exceed the vectorial sum of the real and imaginary errors anywhere.

To conclude this chapter, the imaginary part of $\Gamma_1 e^{j2\pi s}$ will be constructed again, using a different set of approximating functions. The approximating functions will be $S_4(s)$, $S_6(s)$, $S_8(s)$, and $C_5(s)$. Using Table I and Table II, the following equations can now be written for the sampling points:

$$
\begin{aligned}
E_4:&\quad -0.5000\, b_4 && && && +0.2829\, a_5 && -0.1528 &&= 0\\
E_5:&\quad -0.2829\, b_4 && +0.3472\, b_6 && +0.1306\, b_8 && +0.5000\, a_5 && -0.0490 &&= 0\\
E_6:&\quad && 0.5000\, b_6 && && +0.3472\, a_5 && +0.2236 &&= 0\\
E_7:&\quad 0.0772\, b_4 && +0.2938\, b_6 && -0.3395\, b_8 && && +0.0183 &&= 0\\
E_8:&\quad && && -0.5000\, b_8 && -0.1306\, a_5 && -0.2881 &&= 0
\end{aligned}
\qquad (6.76)
$$

To find the Chebyshev error to this system, the set of λ_j must first be determined such that

$$\lambda_4\bar{n}_4 + \lambda_5\bar{n}_5 + \lambda_6\bar{n}_6 + \lambda_7\bar{n}_7 + \lambda_8\bar{n}_8 = 0 \qquad (6.77)$$

The coefficients λ_j can be found by solving the following set of equations:

$$
\begin{aligned}
-0.5000\, \lambda_4 -0.2829\, \lambda_5 \qquad\qquad +0.0772\, \lambda_7 &= 0\\
0.3472\, \lambda_5 +0.5000\, \lambda_6 +0.2938\, \lambda_7 &= 0
\end{aligned}
$$

$$0.1306\,\lambda_5 \qquad -0.3395\,\lambda_7 - 0.5000\,\lambda_8 = 0$$

$$0.2829\,\lambda_4 + 0.5000\,\lambda_5 + 0.3472\,\lambda_6 \qquad -0.1306\,\lambda_8 = 0 \tag{6.78}$$

The following values of λ_j satisfy these equations:

$$\lambda_4 = 1.2165; \quad \lambda_5 = -2.8504; \quad \lambda_6 = 3.4892; \quad \lambda_7 = -2.5689; \quad \lambda_8 = 1.0 \tag{6.79}$$

Using these values, the Chebyshev error for the system (6.76) can be found:

$$h = \frac{\lambda_4 c_4 + \lambda_5 c_5 + \lambda_6 c_6 + \lambda_7 c_7 + \lambda_8 c_8}{|\lambda_4| + |\lambda_5| + |\lambda_6| + |\lambda_7| + |\lambda_8|} = 0.0358 \tag{6.80}$$

The error h_j of equation E_j equals $h_j = h \operatorname{sgn} \lambda_j$. The errors can be substituted in the equations (6.76), resulting in:

$$\begin{aligned}
E_4&: -0.5000\,b_4 \qquad\qquad +0.2829\,a_5 - 0.1528 = 0.0358\\
E_5&: -0.2829\,b_4 + 0.3472\,b_6 + 0.1306\,b_8 + 0.5000\,a_5 - 0.0490 = -0.0358\\
E_6&: \qquad 0.5000\,b_6 \qquad +0.3472\,a_5 + 0.2236 = 0.0358\\
E_7&: 0.0772\,b_4 + 0.2938\,b_6 - 0.3395\,b_8 \qquad +0.0183 = -0.0358\\
E_8&: \qquad -0.5000\,b_8 - 0.1306\,a_5 - 0.2881 = 0.0358
\end{aligned} \tag{6.81}$$

The solution of these equations yields the following values for the coefficients:

$$b_4 = 0.6851; \quad b_6 = -1.6794; \quad b_8 = -1.1382; \quad a_5 = 1.8776 \tag{6.82}$$

This completes the synthesis of matching section No. 2. Using Eqs. 6.63 and 6.66, the reflection-distribution function can be written as:

$$\rho(y) = -0.1496 + 0.1286 \cos 2\pi y - 0.0008 \cos 4\pi y + 1.8776 \cos 5\pi y + 0.0008 \cos 6\pi y + 1.3181 \sin 3\pi y + 0.6851 \sin 4\pi y + 0.4451 \sin 5\pi y - 1.6794 \sin 6\pi y + 0.4458 \sin 7\pi y - 1.1382 \sin 8\pi y + 0.5363 \sin 9\pi y \quad (6.83)$$

In Fig. 6.11 and 6.12, the real and imaginary parts of $\Gamma_1 e^{j2\pi s}$ and those of the approximation $G(s) e^{j2\pi s}$ are plotted. The approximation errors are those given by Eqs. 6.61 and 6.80. The total reflection at the input of matching section No. 2 is equal to the vectorial sum of these errors: $\sqrt{0.0338^2 + 0.0358^2} = 0.0492$. This corresponds to a VSWR of 1.10 in the uniform line connecting the generator in Fig. 6.7 to the load. This standing wave ratio is slightly higher than that obtained with matching section No. 1. Matching section No. 2, however, exhibits much smaller variations in the characteristic impedance along the line.

The reflection-distribution function for this matching section is plotted in Fig. 6.13. Figure 6.14 shows the characteristic-impedance function. A cross section of the coaxial structure is shown in Fig. 6.15.

It is of interest to determine the mechanical precision with which the nonuniform line shown in Fig. 6.15 has to be manufactured to produce the predicted result. An estimate of this tolerance can be made by considering the deviation allowed in the individual coefficients in the expansion (6.83) such that the resulting error in the input reflection of the line is an order of magnitude smaller than the input reflection itself. The input reflection coefficient of the line of Fig. 6.15 is equal to 0.05. One can then determine the deviation in the individual terms of the expansion (6.83) that would cause an error in the

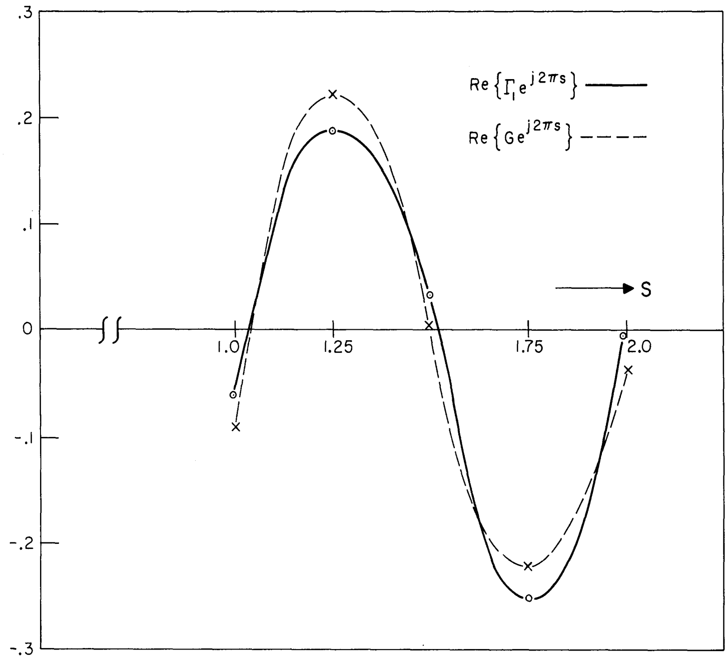

Fig. 6.11 Plot of the function Re $\{\Gamma_1 \; e^{j2\pi s}\}$ and its approximation.

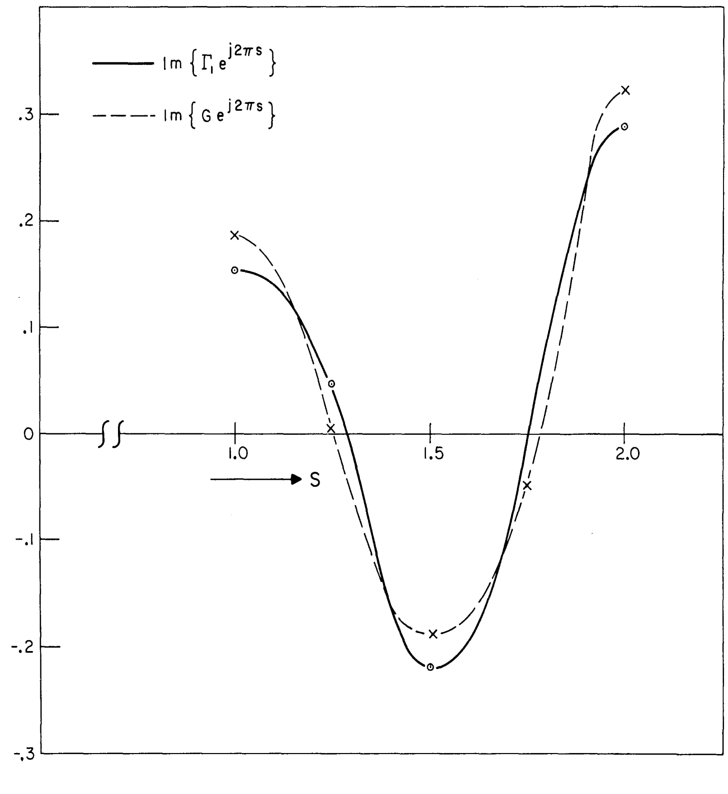

Fig. 6.12 Plot of the function Im $\{\Gamma_1 \; e^{j2\pi s}\}$ and its approximation.

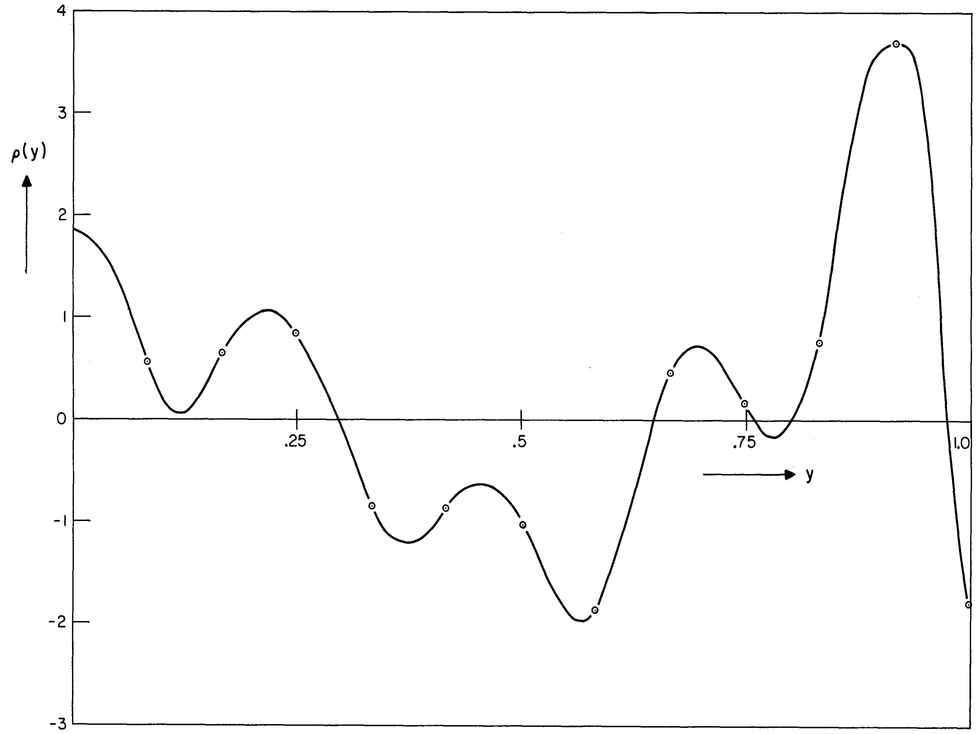

Fig. 6.13 Reflection-distribution function for matching section No. 2.

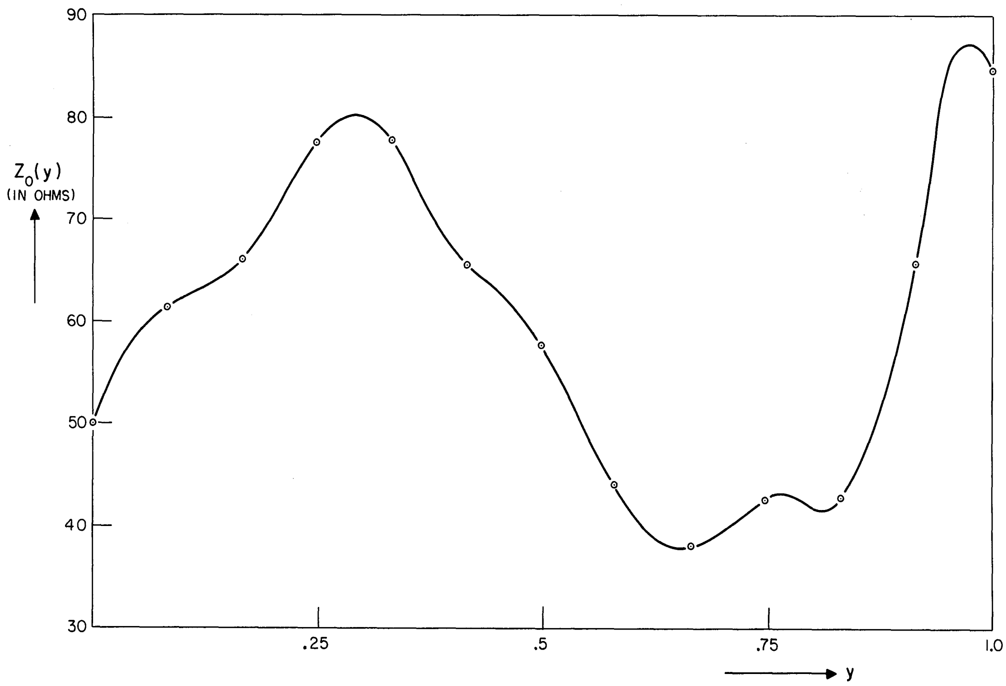

Fig. 6.14 Characteristic-impedance function for matching section No. 2.

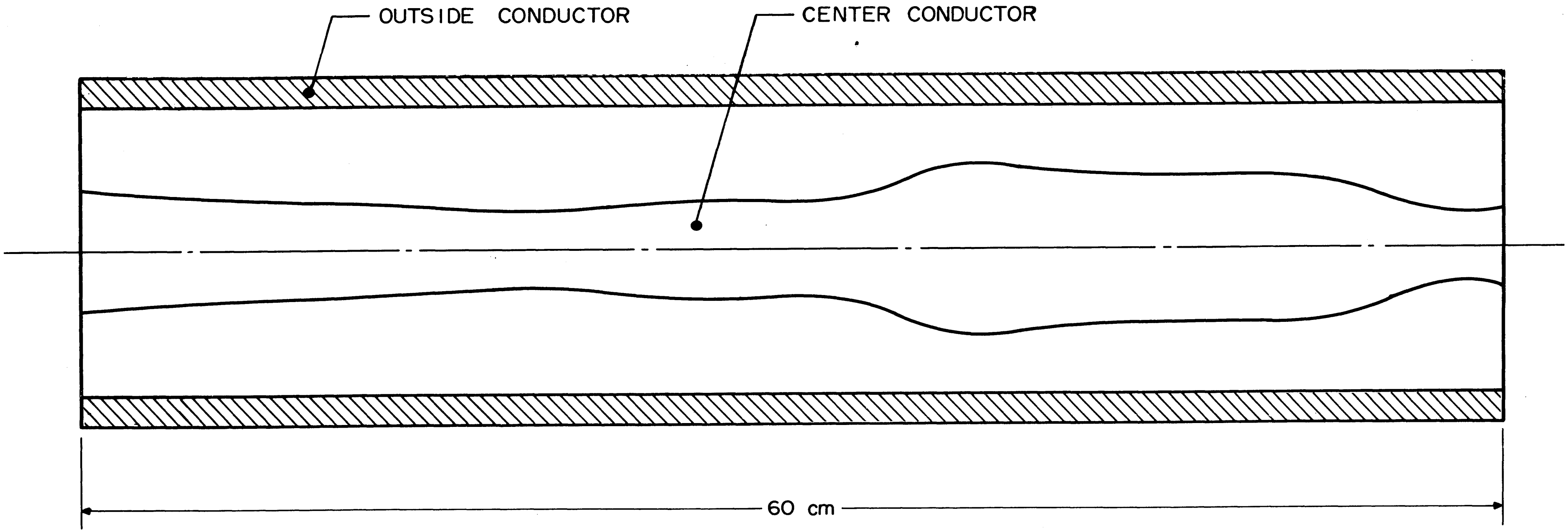

Fig. 6.15 Cross section of matching section No. 2.

input reflection coefficient equal to 0.005.

An error in the coefficients of the expansion (6.83) will cause an error in the diameter of the center conductor of the line. When the inside diameter of the outside conductor (d_o) is equal to 1 inch, the deviations of the diameter of the center conductor, caused by the error in the coefficients of (6.83), are of the order of a few mills. For good results, therefore, the dimensions of the center conductor of the line have to be accurate to within a few thousandths of an inch.

CHAPTER VII

CONCLUSIONS

A general synthesis procedure has been developed for the synthesis of matching sections. The matching section provides a match between a generator, with complex internal impedance, and a complex load impedance, such that maximum power transfer is obtained over a given range of frequencies. Because the method is essentially a numerical one, the internal impedance of the generator and the load impedance can be given either in equation form or in the form of measurements.

A special case of a matching section is the impedance transformer which matches two real impedances of different values. The synthesis procedure can also be used, without essential modifications, for the synthesis of driving point impedances that must exhibit a certain behavior over a given band of frequencies.

Nonuniform lines are synthesized which have the following properties:

(a) The nonuniform line is of finite length

(b) The line is lossless and has a homogeneous dielectric

(c) The taper is continuous.

The synthesis procedure is based on the approximate solution to the nonuniform-line equations that was developed by Orlov (Ref. 9) and Sharpe (Ref. 11). Using their approximate solution, the synthesis problem is first reduced to the problem of constructing a real function, identically zero outside a prescribed interval, whose Fourier transform approximates a given complex function. It is shown that this problem can be put into a convenient mathematical form when the reflection-distribution function is expanded in a trigonometric series. This results in an approximation problem in which given functions have to be approximated by a linear combination of approximating functions. To solve this approximation problem the theory of discrete Chebyshev approximation is introduced and is shown to be a very powerful tool, excellently suited to the problem.

Several examples have been given which demonstrate how impedance transformers and matching sections can be synthesized.

It appears that the general synthesis procedure can be extended to the synthesis of nonuniform transmission lines, behaving as filters. A study of the cut-off characteristics of such filters would present an interesting area for further investigation. The magnitude of the reflection coefficient for filters, however, is no longer small compared to unity, and therefore methods must be developed by which the approximation errors can be evaluated.

Another question that arises for possible further investigation is whether the approximate solution to the nonuniform-line equations can be extended to cover the case of lossy lines. If this were the case, the synthesis procedure developed in the present investigation might be extended to construct lossy nonuniform transmission lines also.

APPENDIX

A METHOD FOR IMPROVED CONVERGENCE

As was mentioned in the introduction to Chapter VI, it is possible to transform the set of functions $C_n(s)$ and $S_n(s)$ into a new set of functions having improved convergence. By improved convergence is meant that the side lobes of the new functions approach zero faster than those of the old functions $C_n(s)$ and $S_n(s)$, as the point s moves away from the main lobe. Use of the new functions will reduce the amount of computation, in case a large number of functions is used in the summation (4.39).

Danielson and Lanczos (Refs. 4 and 8) have published a transformation method that can be adapted to the present problem. The four sets of functions, $C_{ne}(s)$, $C_{no}(s)$, $S_{ne}(s)$, and $S_{no}(s)$, have to be considered separately. It will be recalled that the symbol ne indicates only even values of n, and the symbol no stands for only odd values of n.

First the summation

$$\sum_{n=0}^{Ne} a_{ne} C_{ne}(s) \tag{A.1}$$

is considered. It will be shown that the summation (A.1) equals the summation

$$\sum_{n=0}^{Ne} a_{ne} C_{ne}(s) = \sum_{n=0}^{Ne} u_{ne} U_{ne}(s) \tag{A.2}$$

when the coefficients a_{ne} and u_{ne} are related by the following transformation:

$$\begin{aligned} u_0 &= a_0 \\ u_2 &= a_2 + u_0 \\ &\vdots \\ u_{ne} &= a_{ne} + u_{ne-2} \\ &\vdots \\ u_{Ne} &= a_{Ne} + u_{Ne-2} \end{aligned} \qquad (A.3)$$

The transformation (A.3) can also be written in the following, equivalent form:

$$\begin{aligned} a_0 &= u_0 \\ a_2 &= u_2 - u_0 \\ &\vdots \\ a_{ne} &= u_{ne} - u_{ne-2} \\ &\vdots \\ a_{Ne} &= u_{Ne} - u_{Ne-2} \end{aligned} \qquad (A.4)$$

Substituting these values for a_{ne} into (A.1), one finds:

$$\sum_{n=0}^{Ne} a_{ne}C_{ne}(s) =$$

$$u_0C_0(s) + (u_2 - u_0)C_2(s) + \dots + (u_{ne}-u_{ne-2})C_{ne}(s) + \dots$$

$$\dots + (u_{Ne} - u_{Ne-2})C_{Ne}(s) =$$

$$u_0[C_0(s)-C_2(s)] + \dots + u_{ne}[C_{ne}(s)-C_{ne+2}(s)] + \dots \qquad (A.5)$$

$$\dots + u_{Ne}C_{Ne}(s)$$

By equating the corresponding terms in equations (A.2) and (A.5), it follows that the functions $U_{ne}(s)$ are defined by the following relationship:

$$U_{ne}(s) \equiv [C_{ne}(s) - C_{ne+2}(s)] \qquad 0 \leq ne < Ne$$
$$U_{Ne}(s) \equiv C_{Ne}(s) \tag{A.6}$$

The functions $U_{ne}(s)$ can be evaluated using equations (A.6) and (4.19).

$$U_{ne}(s) = \frac{\sin 2\pi s}{\pi}\left[\frac{1}{4s+n} + \frac{1}{4s-n} - \frac{1}{4s+n+2} - \frac{1}{4s-n-2}\right]$$
$$= \frac{\sin 2\pi s}{\pi}\left[\frac{2}{(4s+n)(4s+n+2)} - \frac{2}{(4s-n)(4s-n-2)}\right] \quad \text{(n even)} \tag{A.7}$$

It is apparent from (A.7) that the functions $U_{ne}(s)$ converge faster than the functions $C_{ne}(s)$.

The functions $C_{no}(s)$, $S_{ne}(s)$, and $S_{no}(s)$ can be transformed in exactly the same manner as the functions $C_{ne}(s)$ above. The functions $C_{no}(s)$ are transformed into the functions $U_{no}(s)$ such that

$$\sum_{n=1}^{No} a_{no} C_{no}(s) = \sum_{n=1}^{No} u_{no} U_{no}(s) \tag{A.8}$$

where:

$$\begin{aligned} u_1 &= a_1 \\ u_3 &= a_3 + u_1 \\ &\vdots \\ u_{no} &= a_{no} + u_{no-2} \\ &\vdots \\ u_{No} &= a_{No} + u_{No-2} \end{aligned} \tag{A.9}$$

and the functions

$$U_{no}(s) \equiv \left[C_{no}(s) - C_{no+2}(s)\right] \qquad 1 \leq no < No$$

$$U_{No}(s) \equiv C_{No}(s) \tag{A.10}$$

It follows from (A.10) and (4.20)

$$U_{no}(s) = \frac{-\cos 2\pi s}{\pi}\left[\frac{2}{(4s+n)(4s+n+2)} - \frac{2}{(4s-n)(4s-n-2)}\right]_{(n\ odd)} \tag{A.11}$$

For the functions $S_{ne}(s)$, it is found that:

$$\sum_{n=2}^{Ne} b_{ne} S_{ne}(s) = \sum_{n=2}^{Ne} v_{ne} V_{ne}(s) \tag{A.12}$$

where:

$$\begin{aligned} v_2 &= b_2 \\ v_4 &= b_4 + v_2 \\ &\vdots \\ v_{ne} &= b_{ne} + v_{ne-2} \\ &\vdots \\ v_{Ne} &= b_{Ne} + v_{Ne-2} \end{aligned} \tag{A.13}$$

and the functions

$$V_{ne}(s) \equiv \left[S_{ne}(s) - S_{ne+2}(s)\right] \qquad 2 \leq ne < Ne$$

$$V_{Ne}(s) \equiv S_{Ne}(s) \tag{A.14}$$

Therefore, from (A.14) and (4.30),

$$V_{ne}(s) = \frac{\sin 2\pi s}{\pi}\left[\frac{1}{4s+n} - \frac{1}{4s+n+2} - \frac{1}{4s-n} + \frac{1}{4s-n-2}\right]$$
$$= \frac{\sin 2\pi s}{\pi}\left[\frac{2}{(4s+n)(4s+n+2)} + \frac{2}{(4s-n)(4s-n-2)}\right] \quad (n \text{ even}) \tag{A.15}$$

And finally, the functions $S_{no}(s)$ are transformed as follows:

$$\sum_{n=1}^{No} b_{no} S_{no}(s) = \sum_{n=1}^{No} v_{no} V_{no}(s) \tag{A.16}$$

where

$$\begin{aligned} v_1 &= b_1 \\ v_3 &= b_3 + v_1 \\ &\vdots \\ v_{no} &= b_{no} + v_{no-2} \\ &\vdots \\ v_{No} &= b_{No} + v_{No-2} \end{aligned} \tag{A.17}$$

and the functions $V_{no}(s)$ equal

$$V_{no}(s) \equiv \left[S_{no}(s) - S_{no+2}(s)\right] \quad 1 \leq no < No$$
$$V_{No}(s) \equiv S_{No}(s) \tag{A.18}$$

And it follows from (A.18) and (4.31), that

$$U_{no}(s) = \frac{\cos 2\pi s}{\pi}\left[\frac{2}{(4s+n)(4s+n+2)} + \frac{2}{(4s-n)(4s-n-2)}\right] \tag{A.19}$$

(n odd)

LIST OF REFERENCES

1. E. Baur, "Beitrag zur Transformation mit inhomogenen Leitungen," Archiv der Elektrischen Übertragung, XIII, No. 3, pp. 114-120, (March 1959).

2. E. F. Bolinder, "Fourier Transforms in the Theory of Inhomogeneous Transmission Lines," Proceedings IRE, XXXVIII, No. 11, p. 1354, (November 1950).

3. E. F. Bolinder, "Fourier Transforms in the Theory of Inhomogeneous Transmission Lines," Acta Polytechnica, Electrical Engineering Series, III, No. 12, pp. 1-83, (1951).

4. G. C. Danielson and C. Lanczos, "Some Improvements in Practical Fourier Analysis and their Application to X-Ray Scattering from Liquids," Journal of the Franklin Institute, CCXXXIII, Nos. 4 and 5, pp. 365-380 and 435-452, (April and May 1942).

5. A. L. Feldshtein, "Nonuniform Lines as Filters," Radiotekhnika, VIII, No. 3, pp. 31-35, (1953).

6. A. L. Feldshtein, "The Design of an Optimum Smooth Transition," Radiotekhnika, XIV, No. 3, pp. 40-46, (1959).

7. R. W. Klopfenstein, "A Transmission Line Taper of Improved Design," Proceedings IRE, XLIV, No. 1, pp. 31-35, (January 1956).

8. C. Lanczos, Applied Analysis, Prentice Hall, 1956.

9. S. I. Orlov, "Concerning the Theory of Inhomogeneous Transmission Lines," Zhurnal Tekhnicheskoi Fiziki, XXVI, No. 10, pp. 2361-2372, (October 1956).

10. H. Ruston, "Synthesis of R-L-C Networks by Discrete Tschebyscheff Approximations in the Time Domain," Ph.D. Thesis, Department of Electrical Engineering, The University of Michigan, April 1960.

11. C. B. Sharpe, "The Scattering Approach to the Synthesis of Non-uniform Lines," paper given at the North East Electronics Research and Engineering Meeting, Boston, Mass., November 14-17, 1960. An abstract is published in the Conference Digest.

12. E. Stiefel, "Über diskrete und lineare Tschebyscheff-Approximationen," Numerische Mathematik, I, No. 1, pp. 1-28, (January 1959).

13. J. Willis and N. K. Sinha, "Non-uniform Transmission Lines as Impedance Transformers," Proceedings IEE, CIII (part B), pp. 166-172, (March 1956).

14. J. Willis and N. K. Sinha, "Impedance Transformers," Wireless Engineer, XXXIII, pp. 204-208, (September 1956).

DISTRIBUTION LIST

Copy No.	
1-2	Commanding Officer, U.S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey, Attn: Senior Scientist, Countermeasures Division
3	Commanding General, U.S. Army Electronic Proving Ground, Fort Huachuca, Arizona, Attn: Director, Electronic Warfare Department
4	Chief, Research and Development Division, Office of the Chief Signal Officer, Department of the Army, Washington 25, D.C., Attn: SIGEB
5	Commanding Officer, Signal Corps Electronics Research Unit, 9560th USASRU, P.O. Box 205, Mountain View, California
6	U.S. Atomic Energy Commission, 1901 Constitution Avenue, N.W., Washington 25, D.C., Attn: Chief Librarian
7	Director, Central Intelligence Agency, 2430 E Street, H.W., Washington 25, D.C., Attn: OCD
8	Signal Corps Liaison Officer, Lincoln Laboratory, Box 73, Lexington 73, Mass., Attn: Col. Clinton W. James
9-18	Commander, Armed Services Technical Information Agency, Arlington Hall Station, Arlington 12, Virginia
19	Commander, Air Research and Development Command, Andrews Air Force Base, Washington 25, D.C., Attn: RDTC
20	Directorate of Research and Development, USAF, Washington 25, D.C., Attn: Chief Electronic Division
21-22	Commander, Wright Air Development Center Wright Patterson Air Force Base, Ohio, Attn: WCOSI-3
23	Commander, Wright Air Development Center, Wright Patterson Air Force Base, Ohio, Attn: WCLGL-7
24	Commander, Air Force Cambridge Research Center, L. G. Hanscom Field, Bedford, Massachusetts, Attn: CROTLR-2
25	Commander, Home Air Development Center, Griffiss Air Force Base, New York, Attn: RCSSLD
26	Commander, Air Proving Ground Center, Attn: Adj/Technical Report Branch, Eglin Air Force Base, Florida
27	Commander, Special Weapons Center, Kirtland Air Force Base, Albuquerque, New Mexico

Copy No.	
28	Chief, Bureau of Naval Weapons, Code RRR-E, Department of the Navy, Washington 25, D.C.
29	Chief of Naval Operations, EW Systems Branch, OP-35, Department of the Navy, Washington 25, D.C.
30	Chief, Bureau of Ships, Code 691C, Department of the Navy, Washington 25, D.C.
31	Chief, Bureau of Ships, Code 684, Department of the Navy, Washington 25, D.C.
32	Chief, Bureau of Naval Weapons, Code RAAV-33, Department of the Navy, Washington 25, D.C.
33	Commander, Naval Ordnance Test Station, Inyokern, China Lake, California, Attn: Test Director-Code 30
34	Director, Naval Research Laboratory, Countermeasures Branch, Code 5430, Washington 25, D.C.
35	Director, Naval Research Laboratory, Washington 25, D.C., Attn: Code 2021
36	Director, Air University Library, Maxwell Air Force Base, Alabama, Attn: CR-4987
37	Commanding Officer-Director, U.S. Naval Electronic Laboratory, San Diego 52, California
38	Office of the Chief of Ordnance, Department of the Army, Washington 25, D.C., Attn: ORDTU
39	Chief, West Coast Office, U.S. Army Signal Research and Development Laboratory, Bldg. 6, 75 S. Grand Avenue, Pasadena 2, California
40	Commanding Officer, U.S. Naval Ordnance Laboratory, Silver Springs 19, Maryland
41-42	Chief, U.S. Army Security Agency, Arlington Hall Station, Arlington 12, Virginia, Attn: LADEV
43	President, U.S. Army Defense Board, Headquarters, Fort Bliss, Texas
44	President U.S. Army Airborne and Electronics Board, Fort Bragg, North Carolina
45	U.S. Army Antiaircraft Artillery and Guided Missile School, Fort Bliss, Texas
46	Commander, USAF Security Service, San Antionio, Texas, Attn: CLR
47	Chief, Naval Research, Department of the Navy, Washington 25, D.C., Attn Code 931

Code No.	
48	Commanding Officer, U.S. Army Security Agency, Operations Center, Fort Huachuca, Arizona
49	President, U.S. Army Security Agency Board, Arlington Hall Station, Arlington 12, Virginia
50	Operations Research Office, John Hopkins University, 6935 Arlington Road, Betheseda 14, Maryland, Attn: U.S. Army Liaison Officer
51	The John Hopkins University, Radiation Laboratory, 1315 St. Paul Street, Baltimore 2, Maryland, Attn: Librarian
52	Stanford Electronics Laboratories, Stanford University, Stanford, California, Attn: Applied Electronics Laboratory Document Library
53	HRB-Singer, Inc., Science Park, State College, Pennsylvania, Attn: R. A. Evans, Manager, Technical Information Center
54	ITT Laboratories, 500 Washington Avenue, Nutley 10, New Jersey, Attn: Mr. L. A. DeRosa, Div. R-15 Lab.
55	The Rand Corporation, 1700 Main Street, Santa Monica, California, Attn: Dr. J. L. Hult
56	Stanford Electronics Laboratories, Stanford University, Stanford, California, Attn: Dr. R. C. Cumming
57	Willow Run Laboratories, The University of Michigan, P.O. Box 2008, Ann Arbor, Michigan, Attn: Dr. Boyd
58	Stanford Research Institute, Menlo Park, California, Attn: Dr. Cohn
59-60	Commanding Officer, U.S. Army Signal Missile Support Agency, White Sands Missile Range, New Mexico, Attn: SIGWS-EW and SIGWS-FC

Code No.	
61	Commanding Officer, U.S. Naval Air Development Center, Johnsville, Pennsylvania, Attn: Naval Air Development Center Library
62	Commanding Officer, U.S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey, Attn: U.S. Marine Corps Liaison Office, Code AO-4C
63	President, U.S. Army Signal Board, Fort Monmouth, New Jersey
64-73	Commanding Officer, U.S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey Attn: 1 Copy - Director of Research 1 Copy - Technical Documents Center ADT/E 1 Copy - Chief, Countermeasures Systems Branch, Countermeasures Division 1 Copy - Chief, Detection and Location Branch, Countermeasures Division 1 Copy - Chief, Jamming and Deception Branch, Countermeasures Division 1 Copy - File Unit No. 2, Mail and Records, Countermeasures Division 1 Copy - Chief, Interference Reduction Branch, Electromagnetic Environment Division 3 Copies - Chief, Security Division (for retransmittal to BJSM)
74	Director, National Security Agency, Fort George G. Meade, Maryland, Attn: TEC
75	Dr. H. W. Farris, Director, Cooley Electronics Laboratory, University of Michigan Research Institute, Ann Arbor, Michigan
76-79	Cooley Electronics Laboratory Project File, University of Michigan Research Institute, Ann Arbor, Michigan
100	Project File, University of Michigan Research Institute, Ann Arbor, Michigan

Above distribution is effected by Countermeasures Division, Surveillance Department, USASRDL, Evans Area, Belmar, New Jersey. For further information contact Mr. I. O. Myers, Senior Scientist, Telephone PRospect 5-3000, Ext. 61252